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GEOMETRY
PLANE TRIGONOMETRY
CHAIN SURVEYING
COMPASS SURVEYING
TRANSIT SURVEYING

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CONTENTS

GEOMETRY

(PART 1)

PRELIMINARY DEFINITIONS

NOTE.—The study of Geometry is a process of systematic and orderly reasoning rather than a matter of memory. The student is advised to study the principles and propositions stated until he understands them thoroughly and sees their relation one to another, and, when a proposition is accompanied by an explanation in small type, to read over the explanation carefully one or more times, until he clearly understands the matter, following out the references to the figure when a figure is given. If he will do this he will find Geometry to be of great benefit and assistance to him in his subsequent studies. But he is not required to commit to memory the explanations or any part of the text except a few of the more important principles and propositions, such as those to which the Examination Questions relate.

1. Every material body possesses two general properties without regard to any other condition, namely: **form,** or **shape,** which is due to the relative positions of its parts; and **magnitude,** or **size,** which is due to the distance of its parts from one another.

The form and magnitude of a body can be described by the relative positions of *points*, *lines*, and *surfaces*.

2. A **point** has position without magnitude. A dot is commonly used to represent a point; but a dot, no matter how small, has length, breadth, and thickness, while a theoretical point has position only.

3. A **line** is the path of a point in motion; it has one dimension—length. Thus, if a point is moved from the position *A*, Fig. 1, to the position *B*, its path, or trace, is the line *A B*.

FIG. 1

FIG. 2

4. A straight line, or right line, Fig. 2, is a line that does not change its direction.

5. The **distance** between two points is the length of the straight line joining them.

FIG. 3

6. A curved line, Fig. 3, is a line that changes its direction at every point.

FIG. 4

7. A broken line, Fig. 4, is a line that changes its direction at only certain points. It is made up wholly of different straight lines.

The word *line*, when not qualified by any other word, is understood to mean a straight line.

8. A **surface** is the path of a line when moved in a direction other than its length. Thus, if a line is moved from the position AB, Fig. 5, to the position CD, the line describes the surface $ABDC$.

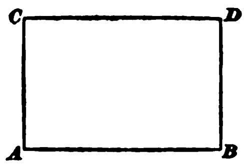


FIG. 5

9. A flat surface, plane surface, or simply a **plane,** is a surface such that a straight line between any two of its points lies wholly in the surface. If a straightedge is laid on a plane surface in any direction, every point of the straightedge will touch the surface.

10. A **figure** is any combination of points and lines. A figure that lies entirely in one plane is a **plane figure.**

In referring to a figure, a point is designated by a letter placed conveniently near it; thus, in Fig. 1, the left end of the line is referred to as the point A. The entire line is referred to as "the line AB," the letters A and B designating two points, usually the ends of the line. If a line is broken or curved, as many points are named as are considered necessary to designate the line.

11. Geometry is that branch of mathematics that treats of the construction and properties of figures.

12. To **produce** a line is to prolong it or to increase its length. A straight line can be prolonged or produced to any extent in either direction. Thus, in Fig. 6, the straight line AB is produced to the points C and D.

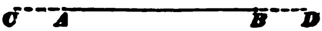


FIG. 6

13. To **bisect** any given magnitude is to divide it into two equal parts. Thus, the straight line AB, Fig. 7, is bisected at the point C if AC is equal to CB. When a given magnitude is bisected, each of the parts into which it is divided is one-half the given magnitude.

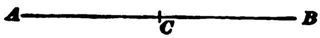


FIG. 7

STRAIGHT-LINE FIGURES

ANGLES AND PERPENDICULARS

14. An **angle**, Fig. 8, is the opening between two straight lines that meet in a point. The two straight lines are the **sides**, and the point where the lines meet is the **vertex**, of the angle. Thus, in Fig. 8, the straight lines OA and OB form an angle at the point O; the lines OA and OB are the sides of this angle, and the point O is its vertex.

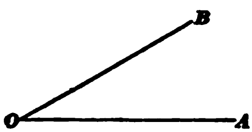


FIG. 8

An angle is usually referred to by naming a letter on each of its sides and a third letter at the vertex, the letter at the vertex being placed between the other two. Thus, the angle in Fig. 8 is called angle AOB or angle BOA.

An angle may also be designated by a letter placed between its sides near the vertex. Thus, the two angles XCY and YCZ, Fig. 9, may be referred to as the angles A and B, respectively.

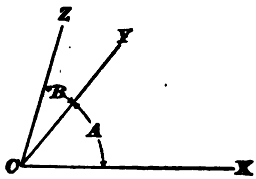


FIG. 9

An isolated angle, that is, an angle whose vertex is not the vertex of any other angle, may be designated by naming the letter at its vertex. For example, the angle in Fig. 8 may be called the angle O.

15. Two angles, as A and B, Fig. 9, having the same vertex and a common side $C\,Y$, are called **adjacent angles.**

16. Two angles are equal when one can be placed on the other so that they will coincide. Thus, in Fig. 10, the angles $A\,O\,B$ and $A'\,O'\,B'$ are equal, because $A'\,O'\,B'$ can be superimposed on $A\,O\,B$, so that with O' upon O and $A'\,O'$ along $A\,O$, $B'\,O'$ will take the direction of $B\,O$ and coincide with it.

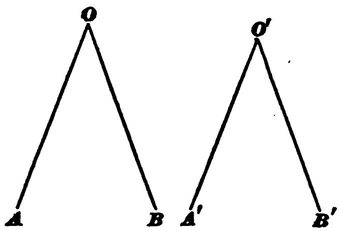


FIG. 10

17. Any angle may be thought of as being formed, or **generated,** by a line turning about the vertex as a pivot, from the position of one side to the position of the other. Thus, the angle $A\,O\,B$, Fig. 8, may be conceived as generated by a line turning about O from the position $O\,A$ to the position $O\,B$. The size of the angle does not depend on the length of the sides, which are supposed to be of indefinite length, but on the opening between the sides; or, what is the same thing, on the amount of turning necessary to bring one side to the position of the other.

18. If a straight line, as $A\,B$, Fig. 11, meets another straight line, as $C\,D$, so as to make with it two equal adjacent angles, each of these angles is a **right angle,** and the first line is said to be **perpendicular** to the second. The point where the first line meets the second is called the **foot** of the perpendicular. It is evident that all right angles are equal.

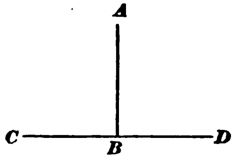


FIG. 11

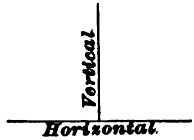


FIG. 12

19. A **horizontal line** is a line parallel to the horizon, or to the surface of still water.

20. A **vertical line** is a line perpendicular to a horizontal line, and having, therefore, the direction of a plumb-line. See Fig. 12.

21. An **oblique angle** is any angle that is not a right angle. An **acute angle** is an oblique angle that is less than a right angle. An **obtuse angle** is an oblique angle that is greater than a right angle. In Fig. 13, $B\,O\,C$ and $A\,O\,C$ are oblique angles, $B\,O\,C$ being an acute angle, and $A\,O\,C$ an obtuse angle.

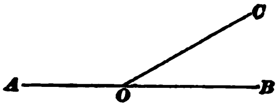


FIG. 13

22. Two angles are said to be **complementary** when their sum is equal to one right angle. Each of two complementary angles is called the **complement** of the other. Thus, in Fig. 14, in which $A\,B$ is perpendicular to $B\,D$, the angles M and N are complementary, their sum being equal to the right angle $A\,B\,D$.

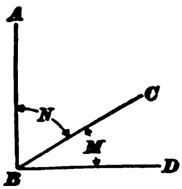


FIG. 14

23. Two angles are said to be **supplementary** when their sum is equal to two right angles. Each of two supplementary angles is called the **supplement** of the other. In Fig. 15, $A\,O\,D$ and $D\,O\,B$ are supplementary angles, their sum being evidently equal to the sum of the two right angles $P\,O\,B$ and $P\,O\,A$.

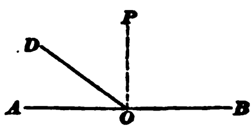


FIG. 15

It will be seen from this illustration that two adjacent angles whose non-common sides are in the same straight line are always supplementary. Conversely, if two adjacent angles are supplementary, their non-common sides are in the same straight line.

24. At a given point in a straight line, one perpendicular to the line and only one can be drawn.

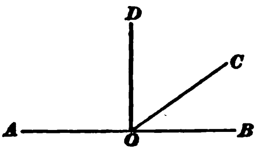


FIG. 16

Let O, Fig. 16, be the given point in the line $O\,B$. Suppose that with the point O fixed, the line $O\,C$ starts from the position $O\,B$ and revolves about O. In any position, as $O\,C$, it makes two angles with the line $A\,B$; one $A\,O\,C$, the other $B\,O\,C$. As $O\,C$ revolves from the position $O\,B$ to the position $O\,A$, the angle $B\,O\,C$ will continually increase, and the angle AOC will continually decrease. There will therefore be one position, as OD, where the two angles are equal, and there can evidently be but one such position.

25. The sum of all the angles formed on the same side of a straight line about the same point in the line is equal to two right angles.

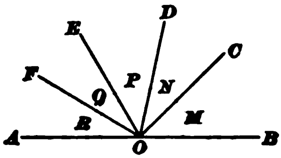


FIG. 17

In Fig. 17, the sum of the three angles M, N, and P is evidently equal to the angle BOE, and the sum of the angles Q and R is equal to the angle EOA. But, by Art. **23**, $BOE + EOA$ is equal to two right angles. Hence, $M + N + P + Q + R =$ two right angles.

26. The sum of all the angles formed in the same plane about one point is equal to four right angles. Thus, in Fig. 18, $M + N + P + Q + R + S + T + U =$ four right angles.

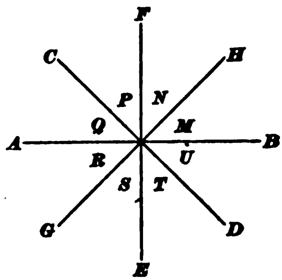


FIG. 18

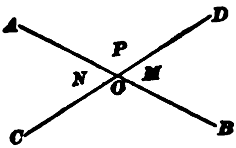


FIG. 19

27. When two lines, as AB and CD, Fig. 19, cut or cross each other, they are said to **intersect.** Their common point O is called their **point of intersection,** or simply their **intersection.**

28. Two intersecting straight lines determine four angles having a common vertex. Any one of these angles and the angle on the opposite side of both lines, as the angles M and N, Fig. 19, are called **vertical angles** with respect to each other. Vertical angles may also be defined as those having a common vertex and in which the sides of the one are the prolongations of the sides of the other.

Since M and N are each the supplement of P, they are equal to each other. Any angle is equal to its vertical angle.

29. If two straight lines intersect and one of the angles is a right angle, the other three angles are right angles, and the lines are perpendicular to each other.

30. Two oblique lines drawn from the same point in a perpendicular to a line, and cutting off on that line equal distances from the foot of the perpendicular, are equal.

Let PO and PQ, Fig. 20, be two oblique lines drawn from the point P in the perpendicular AB, and let BO and BQ be equal. Then, by turning the right side of the figure about AB, it will coincide with the left side; O will fall on Q, and PO will coincide with PQ. Hence, PO is equal to PQ.

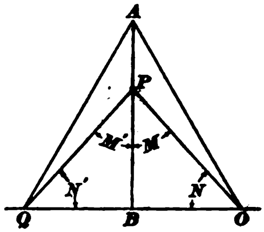


FIG. 20

31. Every point in the perpendicular at the middle point of a straight line is equally distant from the ends of the line. Thus, in Fig. 20, P, which may be any point in the perpendicular AB at the middle point B of OQ is equally distant from Q and O.

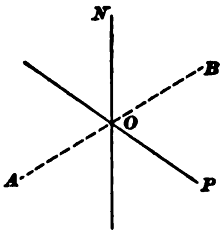
32. Two equal oblique lines drawn from the same point in the perpendicular to a straight line make equal angles with the straight line and with the perpendicular.

Since, when PBO, Fig. 20, is brought to coincide with PBQ, PO coincides with PQ and BO with BQ, the angle M = angle M', and angle N = angle N'.

33. A line that divides an angle into two equal angles is called the **bisector** of that angle. In Fig. 20, PB is the bisector of OPQ, since $M = M'$.

34. Two points, each of which is equally distant from the two extremities of a line, determine a perpendicular bisecting the line. Thus, in Fig. 20, A and P are two points equally distant from Q and O and determine the perpendicular bisecting the line OQ.

EXAMPLES FOR PRACTICE



1. Show that the bisectors of two vertical angles are in the same straight line.

SUGGESTION.—In Fig. 21, show that the sum of the angles on one side of the bisector AB of the angle NOP is equal to the sum of the angles on the other side.

FIG. 21

2. Show that the bisectors of two supplementary adjacent angles are perpendicular to each other.

SUGGESTION.—In Fig. 22, show that the angle EOF is one-half of two right angles.

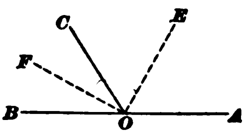


FIG. 22

PARALLELS

35. Parallel lines, Fig. 23, are straight lines that lie in the same plane and never meet, however far they are produced. Any two parallel lines have the same direction and are everywhere equally distant from each other.

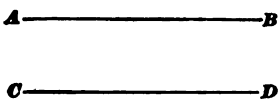


FIG. 23

36. When two parallel lines, as PQ and RS, Fig. 24, are cut by a third line, as XY, the cutting line XY is called **a secant line** or a **transversal.**

The eight angles thus formed are named as follows: The angles a, A, d, and D are **exterior angles.** The angles b, B, c, and C are **interior angles.** The pairs of angles a and d or A and D are **alternate-exterior angles.** The pairs of angles b and c or B and C are **alternate-interior angles.** The pairs of angles a and c, A and C, b and d, or B and D are **exterior-interior** or **corresponding angles.**

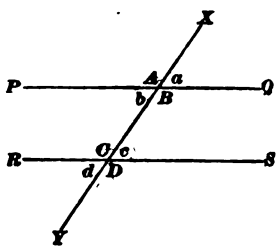


FIG. 24

37. When two parallel lines are cut by a transversal, the alternate-interior angles are equal.

Let *C D* and *E F*, Fig. 25, be the parallel lines and *A B* the transversal. The angles *M* and *M'* have their sides *G D* and *H F* parallel and *A G* and *G H* in the same line; hence, the turning in changing from the direction *H F* to the direction *H G* is equal to the turning in changing from the direction *G D* to the direction *G A*. That is, angle *A G D*, or *M*, is equal to the angle *G H F*, or *M'*, Art. **17**. But angle *M* is equal to angle *N*, Art. **28**; therefore, angle *N* is equal to angle *M'*. In like manner, it can be shown that the angle *D G H* is equal to the angle *G H E*.

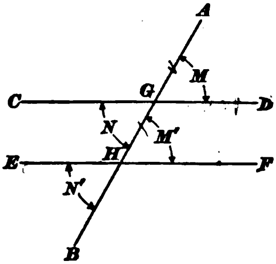


FIG. 25

38. It follows from the preceding article that the alternate-exterior angles are equal; also, the exterior-interior angles. Thus, in Fig. 24, we have $a = d$, $A = D$; $B = D$, $b = d$; $B = C$, $b = c$.

39. In Fig. 24, the angle *a* and the angle *A* are supplementary adjacent angles, and their sum is, therefore, equal to two right angles. From this, and from the principle stated in the preceding article, it follows that any angle in Fig. 24 marked by a capital letter and any angle marked by a small letter are together equal to two right angles.

The principles stated in this and in the two preceding articles may be summed up as follows: When two parallel lines are cut by an oblique transversal, the four obtuse angles are equal to one another; the four acute angles are equal to one another; and any of the obtuse angles is the supplement of any of the acute angles.

40. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

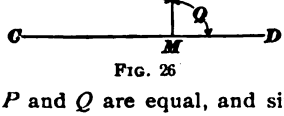


FIG. 26

In Fig. 26, *A B* and *C D* are parallel, and *L M* is drawn perpendicular to *A B*. Then, since the alternate-interior angles *P* and *Q* are equal, and since *P* is a right angle, *Q* must be a right angle also; that is, *L M* is perpendicular to *C D*.

41. The distance between two parallel lines is the length intercepted by the two parallels on any line perpendicular to them. Thus, *L M*, Fig. 26, is the distance between *A B* and *C D*.

42. If two straight lines *A B* and *C D*, Fig. 27, are cut by a third straight line *E F* so that the exterior-interior angles *M* and *N* are equal, the two straight lines are parallel.

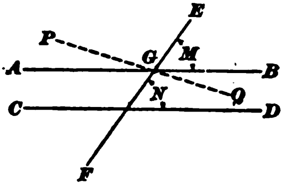


FIG. 27

If *A B* were not parallel to *C D*, we might draw through *G* a line *P Q* that was parallel to *C D*. But then the exterior-interior angles *N* and *E G Q* would be equal (Art. **38**), which is obviously inconsistent with the supposition that *N* is equal to *M*.

43. If two lines, as *A B* and *C D*, Fig. 28, are parallel to a third line, as *E F*, they are parallel to each other.

Draw a transversal *G H*. Then, since *A B* is parallel to *E F*, the alternate-interior angles *M* and *N* are equal; and, since *C D* is parallel to *E F*, the alternate-interior angles *P* and *N* are equal. We have, therefore, $N = M$, $N = P$, and, consequently, $M = P$. As *M* and *P* are exterior-interior angles, it follows, from Art. **42**, that *A B* and *C D* are parallel.

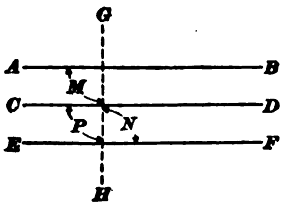


FIG. 28

44. Two angles whose sides are respectively parallel and lie in the same or opposite directions from their vertexes are equal.

In Fig. 29 (*a*), *B A* and *E D* are parallel and extend in the same direction; also, *B C* and *E F* are parallel and extend in the same direction from the vertexes. Let *O* be the point of intersection of the sides *B C* and *E D* produced. Then, since *B Q* and *E F* are parallel, the exterior-interior angles *E* and *M* are equal; and, since *B A* and *E G* are parallel, the exterior-interior angles *B* and *M* are equal. Therefore, the angles *B* and *E*, being each equal to *M*, are equal to each other.

In Fig. 29 (*b*), *B A* and *E F* are parallel and extend in opposite directions; also, *B C* and *E D* are parallel and extend in opposite directions from the vertexes. Producing *F E* and *D E*, we have, by the preceding case, $B = D'E F'$. As *D E F* and *D' E F'* are vertical angles, they are equal, and, therefore, B, which is equal to $D'EF'$, is also equal to DEF.

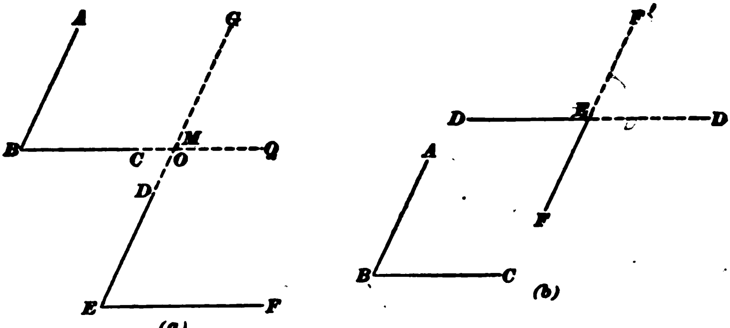


FIG. 29

45. If one side of an angle is parallel to one side of another angle, the two extending in the same direction from the vertexes, and if the other sides of the two angles are also parallel, but extend in opposite directions from the vertexes, the two angles are supplementary.

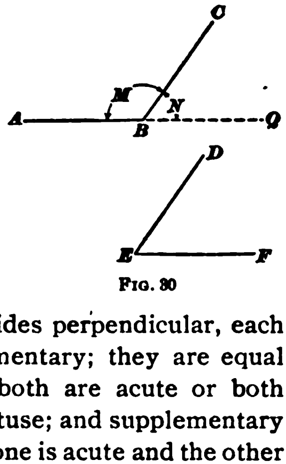


FIG. 30

In Fig. 30, BC and ED are parallel and extend in the same direction, while BA and EF are parallel and extend in opposite directions from the vertexes. Producing AB, we have, by Art. **44**, $N = E$. Now, $M + N =$ two right angles; therefore, $M + E =$ two right angles.

46. Two angles that have their sides perpendicular, each to each, are either equal or supplementary; they are equal if both are acute or both obtuse; and supplementary if one is acute and the other obtuse.

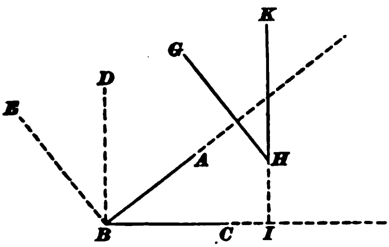


FIG. 31

In Fig. 31, let GH be perpendicular to AB, and KH perpendicular to BC. Draw BD parallel to KH, and BE parallel to GH. Then, by Art. **44**, DBE is equal to KHG. Since EBA and DBC are right angles, by taking DBA

from each of them *E B D* is seen to be equal to *A B C*. Hence, the acute angle *A B C* is equal to the acute angle *K H G*. Also, when one angle is the acute angle *A B C* and the other is the obtuse angle *G H I*, since *G H I* is the supplement of *K H G*, it must be the supplement of *A B C*.

POLYGONS

DEFINITIONS

47. A **polygon** is a portion of a plane bounded by straight lines. The boundary lines are the **sides** of the polygon. The angles formed by the sides are the **angles** of the polygon. The vertexes of the angles of the polygon are the **vertexes** of the polygon. The broken line that bounds it, or the whole distance around it, is the **perimeter** of the polygon. Thus, *A B C D E*, Fig. 32, is a polygon; the sides of this polygon are *A B*, *B C*, *CD*, *DE*, and *E A*; its angles are *A B C*, *B C D*, *C D E*, *D E A*, and *E A B*; and its vertexes are *A*, *B*, *C*, *D*, and *E*.

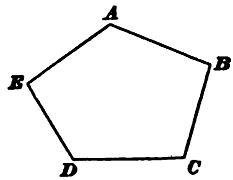


FIG. 32

48. The number of vertexes of a polygon is the same as the number of sides.

49. The least number of sides that a polygon can have is three, since two straight lines cannot enclose space.

50. Polygons are classified in various manners. One of these classifications is based on the number of sides. A polygon of three sides is a **triangle**; a polygon of four sides, a **quadrilateral**; a polygon of five sides, a **pentagon**; a polygon of six sides, a **hexagon**; a polygon of seven sides, a **heptagon**; a polygon of eight sides, an **octagon**; a polygon of nine sides, a **nonagon**; a polygon of ten sides, a **decagon**; a polygon of twelve sides, a **dodecagon**.

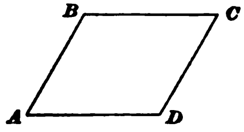


FIG. 33

51. An **equilateral polygon** is a polygon whose sides are all equal. Thus, in Fig. 33, $AB = BC = CD = DA$; hence, *A B C D* is an equilateral polygon.

52. An **equiangular polygon** is a polygon whose angles are all equal. Thus, in Fig. 34, angle A = angle B = angle D = angle C; hence, $ABDC$ is an equiangular polygon.

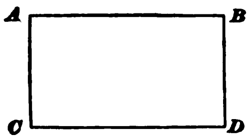


Fig. 34

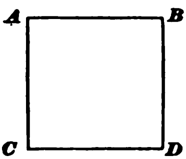


Fig. 35

53. A **regular polygon** is a polygon in which all the sides and all the angles are equal. Thus, in Fig. 35, $AB = BD = DC = CA$; and angle A = angle B = angle D = angle C; hence, $ABDC$ is a regular polygon. Some regular polygons are shown in Fig. 36.

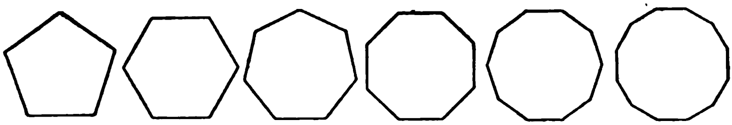

Pentagon Hexagon Heptagon Octagon Decagon Dodecagon

Fig. 36

54. A **reentrant angle** of a polygon is an angle whose sides if produced through the vertex will enter the surface bounded by the perimeter of the polygon. Thus, BCD, Fig. 37, is a reentrant angle.

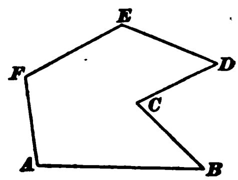


Fig. 37

TRIANGLES

55. Triangles are classified with regard to their sides into *scalene*, *isosceles*, and *equilateral* triangles.

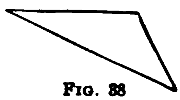


Fig. 38

56. A **scalene triangle**, Fig. 38, is a triangle that has no two of its sides equal.

57. An **isosceles triangle**, Fig. 39, is a triangle that has two of its sides equal.

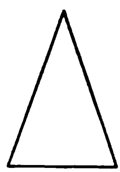


Fig. 39

58. An **equilateral triangle**, Fig. 40, is a triangle that has its three sides equal. An equilateral triangle is a particular kind of isosceles triangle. Thus, the triangle *A B C*, Fig. 40, may be regarded as an isosceles triangle whose equal sides are *A B* and *A C*, as an isosceles triangle whose equal sides are *B A* and *B C*, or as an isosceles triangle whose equal sides are *C A* and *C B*. All the statements made with regard to isosceles triangles are, therefore, true of equilateral triangles.

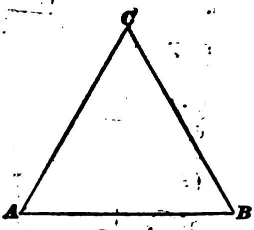


Fig. 40

59. Triangles are classified with regard to their angles into *right-angled*, *obtuse-angled*, and *acute-angled triangles*. See Fig. 41.

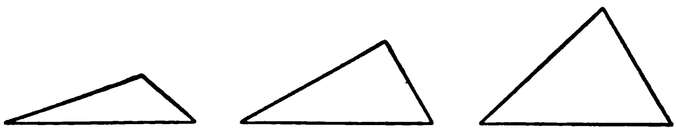


Fig. 41

60. A **right-angled triangle**, or a **right triangle**, is a triangle having a right angle. The **hypotenuse** of a right triangle is the side opposite the right angle. The **legs** of a right triangle are the sides that include the right angle.

61. An **obtuse-angled triangle** is a triangle having an obtuse angle.

62. An **acute-angled triangle** is a triangle all the angles of which are acute.

63. An **oblique triangle** is a triangle that has no right angle. The class oblique triangles includes all obtuse-angled and acute-angled triangles.

64. An **equiangular triangle** is a triangle whose three angles are equal.

65. The **base** of a triangle is the side on which the angle is supposed to stand. In a scalene triangle, any

may be considered as the base. In an isosceles triangle, the unequal side is usually, though not necessarily, taken as the base.

The angle opposite the base of a triangle is sometimes called the **vertical angle** of the triangle. In Figs. 42 and 43, *A C* is the base.

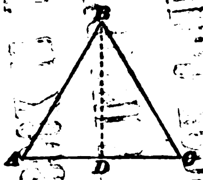


FIG. 42

66. The **altitude** of a triangle is the length of a line drawn from the vertex of the angle opposite the base perpendicular to the base. Thus, in Figs. 42 and 43, the length of *B D* is the altitude.

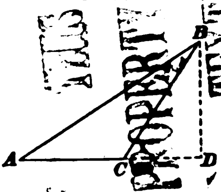


FIG. 43

67. An **exterior angle** of a triangle is an angle formed by a side and the prolongation of another side. Thus, in Figs. 43 and 44, the angle *B C D*, formed by the side *B C* and the prolongation of the side *A C*, is an exterior angle of the triangle *A B C*. The angle *B C A* is adjacent to the exterior angle *B C D*. The angles *A* and *B* are **opposite-interior angles** to the angle *B C D*.

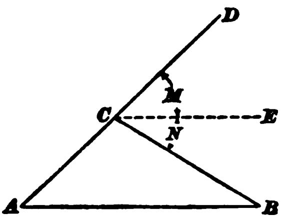


FIG. 44

68. In any triangle, an exterior angle is equal to the sum of the opposite-interior angles.

Let *D C B*, Fig. 44, be an exterior angle of the triangle *A B C*. Draw *C E* through *C* parallel to *A B*. Then, the angles *M* and *A*, being exterior-interior angles, are equal. Also, *N* and *B*, being alternate-interior angles, are equal. Hence, angle *M* plus angle *N*, that is, the exterior angle *D C B*, is equal to angle *A* plus angle *B*, or the sum of the opposite-interior angles.

69. The sum of the interior angles of a triangle is equal to two right angles.

In Fig. 44, the angles *B C D* and *B C A*, being supplementary adjacent angles, are together equal to two right angles. But, by the preceding article, the angle *B C D* is equal to the sum of the angles *A* and *B*. Hence, the sum of the three interior angles *A*, *B*, and *B C A* is equal to two right angles.

70. The following important propositions are immediate consequences of that stated in Art. **69**:

1. If two angles of a triangle are known, or if their sum is known, the third angle can be found by subtracting their sum from two right angles.

2. If two angles of a triangle are equal, respectively, to two angles of another triangle, the third angle of the first-mentioned triangle is equal to the third angle of the other triangle.

3. A triangle can have but one right angle, or one obtuse angle.

4. In any right triangle, the two acute angles are complementary.

5. Each angle of an equiangular triangle is equal to one-third of two right angles, or two-thirds of one right angle.

6. From a point without a line, only one perpendicular to the line can be drawn.

EXAMPLES FOR PRACTICE

1. If one acute angle of a right triangle is one-third of a right angle, what is the value of the other? Ans. Two-thirds of a right angle

2. If one angle of a triangle is one-half of a right angle, and another is five-sixths of a right angle, what is the third angle?

Ans. Two-thirds of a right angle

3. The exterior angle of a triangle is $1\frac{2}{5}$ right angles, and one of the opposite-interior angles is one-fourth of a right angle; what are the other angles of the triangle?

Ans. { Other opposite-interior angle = $\frac{23}{20}$ = 1.15 right angles
Angle adjacent to exterior angle = three-fifths of a right angle

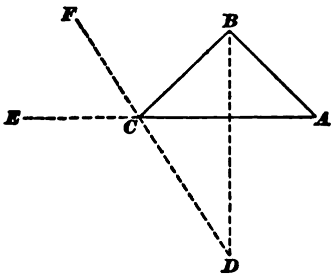


FIG. 45

4. Show that in the triangle ABC, Fig. 45, the bisector of the right angle ABC forms with the bisector of the exterior angle at C an angle that is equal to one-half of the angle A.

SUGGESTION.—Let BD be the bisector of ABC and FD the bisector of BCE. Then BCF is equal to CBD plus CDB, or CDB is equal to BCF minus CBD. Also, ECB is equal to CBA plus A, or A is equal to ECB minus CBA. Furthermore, ECB is equal to twice BCF and CBA is equal to twice CBD.

5. One angle of a triangle is one-half of a right angle: (*a*) What are the remaining two angles, if one is twice as large as the other? (*b*) What kind of triangle is this?

Ans. { (*a*) One-half of a right angle and one right angle
(*b*) An isosceles right triangle

---

71. Two plane figures are **equal** when one can be placed on the other so that they will coincide in all their parts.

Thus, the triangles ABC and $A'B'C'$, Fig. 46, are equal, because if $A'B'C'$ is imagined to be lifted off the paper, moved over and placed on ABC, the sides $A'B'$, $B'C'$, and $C'A'$ can be made to coincide

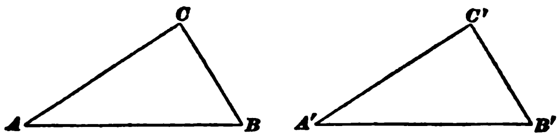


FIG. 46

with AB, BC, and CA, respectively, and the angles A', B', and C' to coincide with the angles A, B, and C. It is evident, from the figure, that if the vertexes of the two triangles coincide, the triangles will coincide throughout, and are, therefore, equal.

The polygons $ABCDE$ and $A'B'C'D'E'$, Fig. 47, are equal, because $A'B'C'D'E'$ can be imagined to be lifted, turned over, and placed on $ABCDE$ so as to make the two polygons coincide in all their parts.

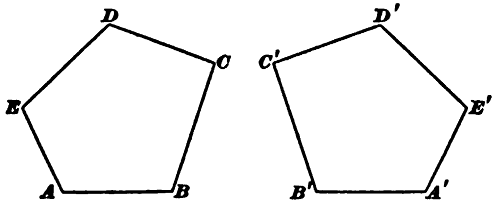


FIG. 47

72. Two triangles are equal when a side and two adjacent angles of one are equal to a side and two adjacent angles of the other.

Let $A'B'$, Fig. 48, equal AB, the angle A' equal the angle A, and the angle B' equal the angle B. Now, if $A'B'C'$ is placed on ABC so that $A'B'$ coincides with its equal AB, with A' on A and B' on B, $A'C'$ will take the direction AC; since the angle A' is equal to the angle A, and as B' is equal to B, $B'C'$ will take the direction BC.

Now, the point C' will fall somewhere on the line $A\,C$, and also somewhere on the line $B\,C$, and since two lines can intersect in only one point, C' must fall at the intersection of $A\,C$ and $B\,C$, or at C. Hence, the vertexes of the triangles coincide and the triangles are equal.

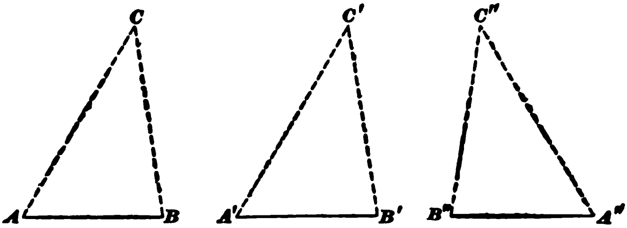


Fig. 48

The same reasoning applies to the triangles $A\,B\,C$ and $A''\,B''\,C''$, in which $A\,B = A''B''$, and $A = A''$, $B = B''$; but the triangle $A''\,B''\,C''$ must be imagined to be lifted and turned over before it can be placed on $A\,B\,C$.

73. The following important principles are consequences of the preceding proposition:

1. Two triangles are equal when one side and any two angles of one are equal, respectively, to one side and the two similarly situated angles of the other.

2. Two right triangles are equal when one side and one acute angle of the one are equal, respectively, to one side and the similarly situated acute angle of the other.

74. Two triangles are equal when two sides and the included angle of one are equal to two sides and the included angle of the other.

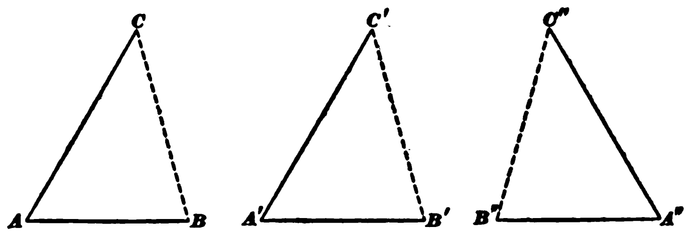


Fig. 49

In Fig. 49, $A\,B = A'\,B' = A''\,B''$, $A\,C = A'\,C' = A''\,C''$, and $A = A' = A''$. If $A'\,B'\,C'$ is placed on $A\,B\,C$, so that A' will coincide with A. and $A'\,B'$ with $A\,B$, the rest of the triangles will evidently coincide; for since $A' = A$, $A'C'$ will take the direction AC, and since $A'C' = AC$, C' will coincide with C. The same reasoning applies to $A''B''C''$, after the latter triangle has been turned over.

75. Two triangles are equal when the three sides of the one are equal, respectively, to the three sides of the other.

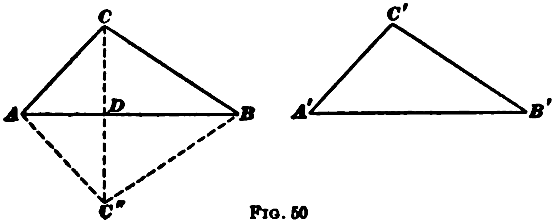


FIG. 50

In Fig. 50, let $A'B'$, $B'C'$, and $C'A'$ be equal, respectively, to AB, BC, and CA. Place $A'B'C'$ in the position ABC'', with its longest side $A'B'$ coinciding with AB, and C'' on the opposite side of AB from C; then join C and C''. Now, AC is equal to AC'', and BC is equal to BC''; hence, A and B determine a perpendicular to CC'' at its mid-point (Art. **34**). Then, by Art. **32**, the angle CAD is equal to the angle $C''AD$, or to $C'A'B'$, and by Art. **74** the triangles CAB and $C'A'B'$ are equal.

76. In an isosceles triangle, the angles opposite the equal sides are equal.

Let ABC, Fig. 51, be an isosceles triangle in which $AB = BC$. Draw the bisector BD of the angle B. Then, by Art. **74**, the triangles ABD and CBD are equal. Therefore, $A = C$.

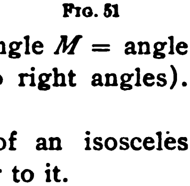


FIG. 51

77. The equality of the triangles ABD and CBD, Fig. 51, gives $AD = DC$, and angle M = angle N = one right angle (since $M + N$ = two right angles). Hence,

1. The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to it.

2. Conversely, the perpendicular bisecting the base of an isosceles triangle passes through the vertex of the opposite angle and bisects that angle.

3. Also, the perpendicular drawn from the vertical angle of an isosceles triangle to the base, bisects both the base and the vertical angle.

78. If two angles of a triangle are equal, the sides opposite these two angles are equal, and the triangle is therefore isosceles.

In Fig. 51, let $A = C$. Draw BD perpendicular to AC. The right triangles BDC and BDA have the common side BD, and acute angle $A = C$. Therefore (Art. **73**), they are equal, and their hypotenuses BA and BC are equal.

79. It follows from Art. **76** that an equilateral triangle is also equiangular, and from the preceding article that an equiangular triangle is also equilateral.

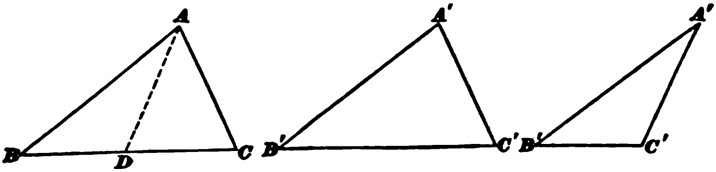


FIG. 52

80. If two sides of a triangle are equal, respectively, to two sides of another triangle, and the angle opposite one of these two sides in the first triangle is equal to the corresponding angle in the second triangle, the angles opposite the other two equal sides are either equal or supplementary.

In Fig. 52, let $A'C' = AC$, $A'B' = AB$, and the angle $B' = B$. Place $A'B'C'$ on ABC so that $A'B'$ coincides with AB. Then since $B' = B$, $B'C'$ will take the direction BC, and since $A'C'$ joins $B'C'$, C' must fall on BC, at either C or D. If C' falls at C, the triangles are equal and the angle $C' = C$; but if C' falls at D, ADB is the angle C', and ADB, the supplement of ADC, is the supplement of C, since, by Art. **76**, $ADC = C$.

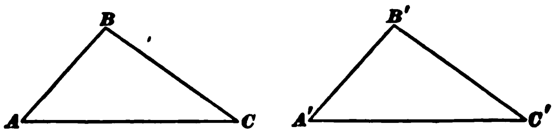


FIG 53

81. If two triangles have two sides of the one equal to two sides of the other, and the angles opposite one pair of the equal sides are right angles or equal obtuse angles, the triangles are equal.

Since a triangle can have but one right or one obtuse angle, when the angles B and B', Fig. 53, are obtuse, the angle C' cannot be the supplement of C, hence C' must equal C.

82. Of two sides of a triangle, that is greater which is opposite the greater angle.

In the triangle ABC, Fig. 54, let the angle C be greater than the angle B. Draw CD, making with CB an angle BCD equal to the angle B. Then BCD is an isosceles triangle, and $CD = DB$. Therefore, $AD + DB$, or AB, is the same as $AD + DC$, which is evidently greater than AC.

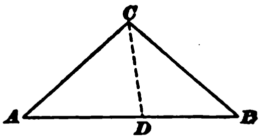


FIG. 54

83. Of two angles of a triangle, that is greater which is opposite the greater side.

Let A and B be two angles of a triangle, a the side opposite A, and b the side opposite B. Suppose that a is greater than b. If A were equal to B, the triangle would be isosceles, and $a = b$. If B were greater than A, then by the preceding article, b would be greater than a. Therefore, since B cannot be equal to or greater than A, it must be less, or A must be greater than B.

84. If from a point O, Fig. 55, without a line AB, a perpendicular OP to the line is drawn, and also two oblique lines OL and OL', the oblique line OL, whose foot L is farther from the foot P of the perpendicular, is the greater of the two oblique lines.

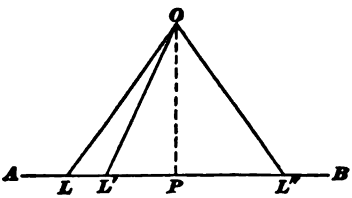


FIG. 55

Suppose the two oblique lines OL and OL' to be on the same side of the perpendicular. Since $OL'P$ is a right triangle and OPL' the right angle, the angle $OL'P$ is acute; also, the angle $OL'L$ is obtuse, since it is the supplement of $OL'P$. As the triangle OLL' can have but one obtuse angle, $OL'L$ is greater than OLP, and, therefore (Art. **82**), OL is greater than OL'. If OL lies on the opposite side of the perpendicular from OL', as in the position OL'', and if $PL'' = PL$, which is greater than PL', then, by Art. **30**, $OL'' = OL$, which is greater than OL'.

85. If the hypotenuse, as AB, Fig. 56, and one leg, as BC, of a right triangle are equal, respectively, to the hypotenuse and one leg of another right triangle, as $A' B' C'$, the two triangles are equal.

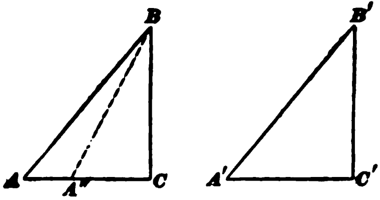


FIG. 56

Place $A' B' C'$ on $A B C$, so that $B' C'$ will coincide with its equal $B C$. Since $B' C' A'$ is a right angle, $C' A'$ will take the direction $C A$; and, since $B' A' = B A$, A' must fall on A; for if it fell to the right of A, as at A'', the hypotenuse $B A''$, or $B' A'$, would be less than $B A$ (Art. 84); and, if A' fell on the left of A, the hypotenuse $B' A'$ would be greater than $B A$.

EXAMPLES FOR PRACTICE

1. Show that, if two intersecting lines, as $A B$ and $D C$, Fig. 57, bisect each other, the lines $A C$ and $D B$ are parallel.

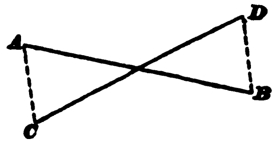


FIG. 57

2. If the value of the unequal or vertical angle of an isosceles triangle is two-fifths of a right angle, what is the value of each of the base angles? Ans. Four-fifths of a right angle

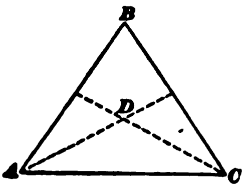


FIG. 58

3. Show that the bisectors of the base angles of an isosceles triangle form with the base an isosceles triangle; or that, $A D C$, Fig. 58, is an isosceles triangle.

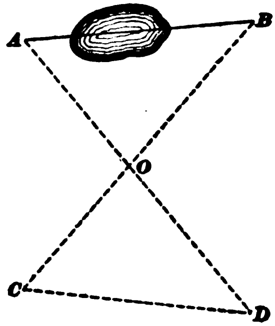


FIG. 59

4. Show that the length of the inaccessible line $A B$, Fig. 59, can be found by measuring $A O$ and $B O$, then making $O D = O B$ and $O C = O A$, and finally measuring $C D$.

QUADRILATERALS

86. There are three kinds of quadrilaterals: the *parallelogram*, the *trapezoid*, and the *trapezium*.

87. A **parallelogram** is a quadrilateral whose opposite sides are parallel. There are four kinds of parallelograms: the *rectangle*, the *square*, the *rhomboid*, and the *rhombus*.

88. A **rectangle**, Fig. 60, is a parallelogram whose angles are all right angles.

FIG. 60

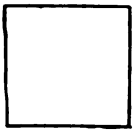


FIG. 61

89. A **square**, Fig. 61, is a rectangle whose sides are equal.

90. A **rhomboid**, Fig. 62, is a quadrilateral whose opposite sides are parallel, and whose angles are not right angles.


FIG. 62

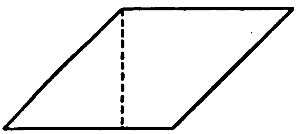


FIG. 63

91. A **rhombus**, Fig. 63, is a rhomboid having equal sides.

92. A **trapezoid**, Fig. 64, is a quadrilateral that has only two of its sides parallel.

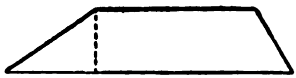


FIG. 64

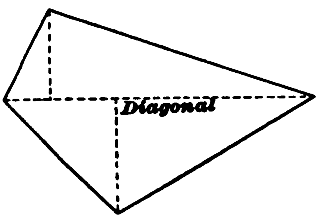


FIG. 65

93. A **trapezium**, Fig. 65, is a quadrilateral having no two sides parallel.

94. The **altitude** of a parallelogram, or of a trapezoid, is the length of the perpendicular distance between the parallel sides. See dotted line in Figs. 62, 63, and 64.

95. A **diagonal** of a quadrilateral is a straight line drawn from the vertex of any angle of the quadrilateral to the vertex of the angle opposite. A diagonal divides a quadrilateral into two triangles. See Figs. 60 and 65.

96. In a parallelogram, as $ABCD$, Fig. 66, the opposite sides and opposite angles are equal; that is, $AB = DC, AD = BC$, angle A = angle C, angle B = angle D.

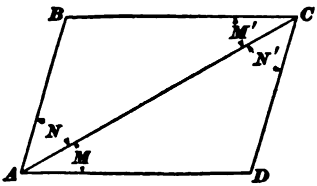


FIG. 66

Draw the diagonal AC. Then, angle M = angle M', and $N = N'$ (Art. 37). The triangles ADC and ABC, having the common side AC and the adjacent angles M and N' equal, respectively, to M' and N, are equal (Art. 72). Therefore, $AD = BC$, $AB = DC$, and angle B = angle D. Also, since $M = M'$ and $N = N'$, it follows that $M + N$, or BAD, is equal to $M' + N'$, or BCD.

97. The diagonal of a parallelogram divides the parallelogram into two equal triangles.

98. Parallel lines intercepted between parallel lines are equal. Thus, if the parallels AB and CD, Fig. 67, are cut by the parallels EF, GH, IJ, KL, we have, from Art. **96,** $MN = OP = QR = ST$.

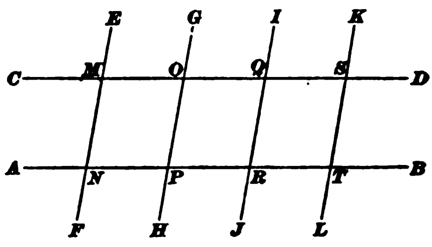


FIG. 67

99. The diagonals of a parallelogram, as AC and BD, Fig. 68, bisect each other; that is, denoting by O the point of intersection of the diagonals, $OA = OC$ and $OB = OD$.

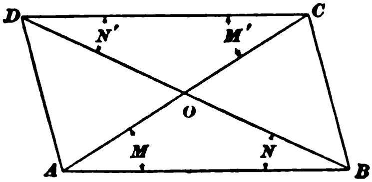


FIG. 68

In the triangles AOB and DOC, $AB = DC$ (Art. 96), $M = M'$ and $N = N'$ (Art. 37). Therefore, the triangles are equal (Art. 72) and $OA = OC$, $OB = OD$.

EXAMPLES FOR PRACTICE

1. Show that if the diagonals of a quadrilateral bisect each other the figure is a parallelogram.

SUGGESTION.—In Fig. 68, assume that $OA = OC$, $OB = OD$. Then show that triangle BOC = triangle AOD, and triangle AOB = triangle DOC.

2. Show that the diagonals of a rectangle are equal.

SUGGESTION.—Show that in any rectangle $ABCD$ the triangle ABC = triangle ABD.

3. Show that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

SUGGESTION.—Draw the diagonal. Then, by Art. 75, the triangles formed are equal.

4. Show that if two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

5. Show that if one angle of a parallelogram is a right angle, the parallelogram is a rectangle.

ADDITIONAL PROPERTIES OF TRIANGLES

100. The bisectors of the three angles of a triangle meet in a point.

In the triangle ABC, Fig. 69, draw the bisectors of the angles A and B and let them meet at O. Join C and O, and draw the perpendiculars from O to the sides of the triangle. Then, in the right triangles BOF and BOE, BO is common and the angle OBF = angle OBE. Hence, by Art. **73**, these triangles are equal. Therefore, $OF = OE$. In a similar manner it can be shown that $OD = OE$. Therefore, $OD = OF$. The right triangles OFC and ODC, having $OD = OF$ and OC common, are equal (Art. **85**). Hence, angle OCF = angle OCD; that is, OC, which meets the bisectors AO and BO in O, is the bisector of the angle C.

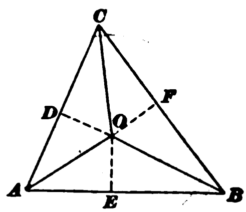


FIG. 69

101. Any point in the bisector of an angle is equally distant from the sides of the angle. For it has just been shown that, in Fig. 69, $OF = OE$.

102. The perpendiculars erected at the middle points of the three sides of a triangle meet in a point equally distant from the vertexes of the triangle.

In Fig. 70, draw the perpendiculars to CB and AC at their mid-points D and E, and let O be the point in which these perpendiculars meet. Now, O, being in OD, is equally distant from C and B (Art. **31**), that is, $OB = OC$: and being in OE, is equally distant from A and C; that is, $OA = OC$. From these two equalities it follows that $OB = OA$. Therefore, the perpendicular to AB at its middle point F passes through O. (Art. **77**).

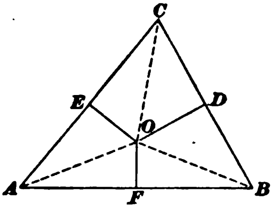


FIG. 70

103. If several parallel lines intercept equal distances on one transversal, they intercept equal distances on any other transversal.

In Fig. 71, let the parallels AE, BG, CI, DK intercept the equal distances AB, BC, and CD on the transversal PQ, and let RS be any other transversal. Draw EF, GH, IJ, parallel to AB. Then, by Art. **98**, $EF = AB$, $GH = BC$, $IJ = CD$. Hence, $EF = GH = IJ$. In the triangles EFG, GHI, and IJK, angle E = angle G = angle I (Art. **38**), and angle F = angle H = angle J (Art. **44**). Hence, by Art. **72**, these triangles are eoual, and, therefore, $EG = GI = IK$.

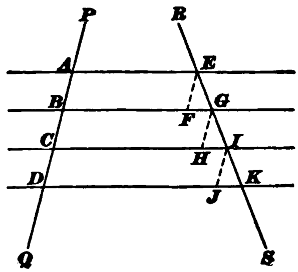


FIG. 71

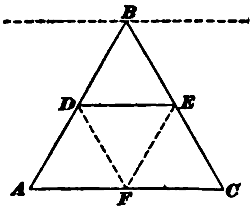


FIG. 72

104. A line parallel to one of the sides of a triangle and bisecting one of the other sides, bisects the third side also.

In Fig. 72, let DE bisect AB and be parallel to AC. Draw a line through B parallel to AC. Then since the three parallels intercept equal parts on AB, they intercept equal parts on BC; that is, $BE = EC$.

105. A line joining the middle points of two sides of a triangle is parallel to the third side and equal to one-half of that third side.

In Fig. 72, let DE join D and E, the middle points of AB and BC. The first part of this proposition follows at once from the preceding article. Let F be the middle point of AC, and draw FE. This line is parallel to AB, and, therefore, $ADEF$ is a parallelogram. Consequently (Art. 96), $DE = AF = \frac{1}{2} AC$.

106. The lines joining the middle points of the three sides of a triangle divide it into four equal triangles.

The diagonal DF, Fig. 72, divides the parallelogram $ADEF$ into two equal triangles AFD and DFE. Likewise, the diagonal EF divides $DECF$ into two equal triangles DFE and EFC; and the diagonal DE divides the parallelogram $BDFE$ into the two equal triangles DFE and BDE. Hence, triangle AFD = triangle DFE = triangle EFC = triangle BDE.

107. Any of the parallelograms $ADEF$, $FCED$, $DBEF$, Fig. 72, is equal to one-half the given triangle, since it contains two of the four equal triangles into which the given triangle is divided.

108. A line, as EF, Fig. 73, parallel to the bases AB and DC of a trapezoid and passing through the middle point E of one of the non-parallel sides, passes through the middle point of the other non-parallel side and is equal to one-half the sum of the parallel sides or bases.

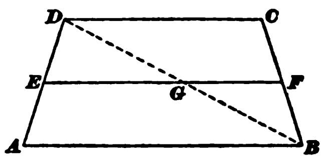


FIG. 73

Since the parallels AB, EF, and DC intercept equal parts on AD, they intercept equal parts on BC (Art. 103); that is, $BF = FC$.

Draw BD, meeting EF in G. Then, by Art. 105, in the triangle DCB, FG is one-half CD. Also, in the triangle ADB, GE is one-half BA. Hence, $FG + GE$, or FE, $= \frac{1}{2}(CD + BA)$.

109. The **medians** of a triangle are the lines drawn from the vertexes to the middle points of the opposite sides.

110. The medians of a triangle meet in a point whose distance from any vertex is two-thirds the length of the median from that vertex.

In Fig. 74, AD, BE, CF are the median lines of the triangle ABC; they meet at O, and $AO = \frac{2}{3} AD$, $BO = \frac{2}{3} BE$ and $CO = \frac{2}{3} CF$.

Let AD and CF meet at O. Join I and H, the mid-points of CO and AO, respectively; also join D and F. Then, in the triangle AOC, IH is parallel to AC and equal to one-half AC (Art. **105**). Also, in triangle ABC, DF is parallel to AC and equal to one-half AC. Hence, IH and DF are equal and parallel. It follows that the triangles DOF and HOI are equal, and that, therefore, $HO = OD$. But, by construction, $AH = HO$. Hence, $AH = HO = OD$, whence, $AO = \frac{2}{3} AD$. Similarly $CO = \frac{2}{3} CF$. That is, one median cuts off on the other median two-thirds of the distance from the vertex to the opposite side.

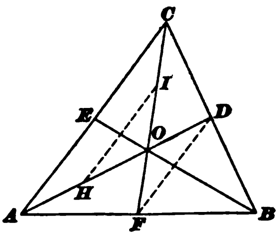


FIG. 74

POLYGONS IN GENERAL

111. Two polygons are equal when they can be divided into the same number of triangles equal each to each and similarly placed. Thus, the polygons shown at (a) and (b) in Fig. 75 are composed of the same number of triangles equal each to each and similarly placed, and it is evident that one polygon can be placed on the other so that they will coincide throughout; hence, they are equal.

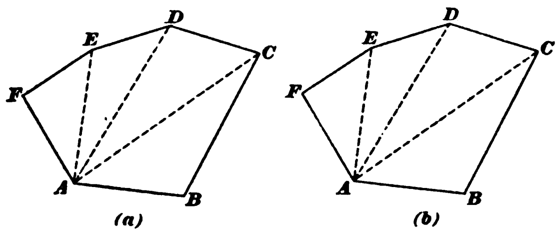


FIG. 75

112. An **exterior angle** of a polygon is an angle formed by any side and the prolongation of an adjacent side. In Fig. 76, the angles M and N are exterior angles of the polygon $ABCDE$.

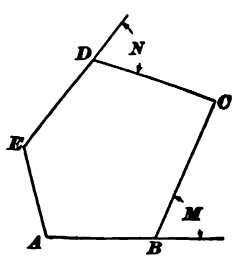


FIG. 76

113. A **diagonal** of a polygon is any line joining two vertexes not adjacent to the same side of the polygon. Thus, in Fig. 75, AC, AD, and AE are diagonals of the polygon $ABCDEF$.

114. The sum of the interior angles of any polygon is equal to two right angles multiplied by a number that is two less than the number of sides of the polygon.

Let (*a*), Fig. 75, be any polygon. Draw the diagonals from one vertex and thus divide the polygon into triangles. It is seen that the first triangle ABC and the last triangle AFE, each contains two sides of the polygon, while each of the other triangles contains but one side of the polygon. Thus, the number of triangles formed is two less than the number of the sides of the polygon. Hence (Art. **69**), the sum of the angles of the triangles, or of the polygon, is two right angles multiplied by a number that is two less than the number of sides of the polygon.

115. Let n = number of sides of a polygon;
S = sum of interior angles of the polygon, expressed in right angles.

Then,
$$S = 2(n-2) = 2n - 4$$

If $n = 4$, then $S = 2 \times 4 - 4 = 4$ right angles; that is, the sum of the angles of a quadrilateral is equal to four right angles.

EXAMPLE 1.—What is the value of one of the interior angles of an equiangular hexagon?

SOLUTION.—The number of sides of a hexagon is six; hence, applying the formula, $S = 2 \times (6-2) = 8$ right angles, that is, the sum of the interior angles of a hexagon is equal to eight right angles. Since the hexagon is equiangular, one of the angles is equal to one-sixth of eight right angles, or $1\frac{1}{3}$ right angles. Ans.

EXAMPLE 2.—If one of the interior angles of an equiangular polygon is equal to $1\frac{3}{7}$ right angles, what is the name of the polygon?

SOLUTION.—If one of the interior angles is equal to $1\frac{3}{7}$ or $\frac{10}{7}$ right angles, their sum S is equal to $\frac{10}{7} \times n = \frac{10n}{7}$. But from the formula, $S = 2n - 4$. Therefore, $\frac{10n}{7} = 2n - 4$; whence, $n = 7$. A polygon of seven sides is a heptagon; therefore, the polygon is a heptagon. Ans.

EXAMPLES FOR PRACTICE

1. Show that if two angles of a quadrilateral are supplementary the other two angles are supplementary.

2. In a triangle *A B C*, the angle *C* is twice the angle *B*. Show that the line that bisects the angle *C* meets the line *A B* at a point *D* so that *C D* = *B D*.

SUGGESTION.—Half the angle *C* = angle *B*. Then in the triangle *C D B*, angle *B C D* = angle *C B D*.

3. What is the value of one of the interior angles of an equiangular octagon? Ans. $1\frac{1}{2}$ right angles

4. (*a*) What is the value of one of the interior angles of an equiangular quadrilateral? (*b*) What kind of quadrilateral is it?

Ans. { (*a*) One right angle; (*b*) Rectangle }

5. If one of the interior angles of an equiangular polygon is equal to $1\frac{5}{9}$ right angles, what is the name of the polygon? Ans. Nonagon

THE CIRCLE

DEFINITIONS AND GENERAL PROPERTIES

116. A **circle**, Fig. 77, is a plane figure bounded by a curved line every point of which is equally distant from a point within called the **center.**

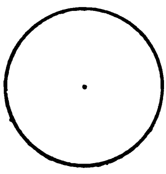


FIG. 77

117. The **circumference** of a circle is the line that bounds the circle. The term *circle* is often used in the sense of *circumference*.

118. The **diameter** of a circle is a straight line drawn through the center and terminated at both ends by the circumference. Thus, *A E*, Fig. 78, is a diameter of the circle whose center is *O*.

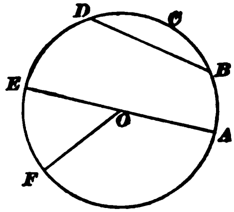


FIG. 78

119. The **radius** of a circle is any straight line drawn from the center to the circumference. The plural of *radius* is *radii*. Thus, OA, OE, and OF, Fig. 78, are radii of the circle whose center is O.

120. The distance from the center to the circumference is, by the definition of a circle, the same for all points in the same circle; hence, all radii are equal.

121. When any two radii, as OA and OE, Fig. 78, are in the same straight line, they form a diameter. Hence, the length of the diameter is twice the length of the radius.

122. An **arc** of a circle is any part of its circumference, as DCB, Fig. 78.

123. An arc equal to one-half the circumference is a **semi-circumference**; and an arc equal to one-fourth the circumference is a **quadrant.**

124. A **chord** is a straight line, as BD, Fig. 78, joining any two points in a circumference, or it is a line joining the extremities of an arc.

125. The longest chord that can be drawn in a circle is a chord that passes through the center and is, therefore, a diameter.

126. An arc of a circle is said to be **subtended** by its chord. Thus, the arc BCD, Fig. 78, is subtended by the chord BD.

Every chord in a circle subtends two arcs. Thus, BD subtends both the arcs BCD and $BAFED$.

When an arc and its chord are spoken of, the arc less than a semi-circumference is meant, unless the contrary is stated. The shorter arc is usually referred to by naming the letters at its extremities; thus, the arc BCD is called the arc BD.

127. A **segment** of a circle is a part of the circle enclosed by an arc and its chord. In Fig. 78, the part of the circle between the chord BD and the arc BD is a segment.

A segment equal to one-half the circle is a **semicircle.**

128. A **sector** of a circle is the space included between an arc and the two radii drawn to the extremities of the arc. In Fig. 78, the space included between the arc FE and the radii OF and OE is a sector.

129. Two circles are equal when the radius or diameter of one is equal to the radius or diameter of the other.

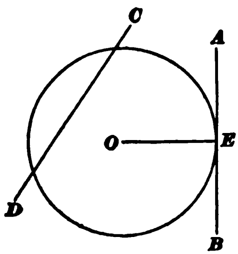


FIG. 79

130. A **tangent** to a circle is a line that touches the circumference in only one point. In Fig. 79, AB is tangent to the circle whose center is O. The point E at which the tangent touches the circumference is the **point of contact,** or **point of tangency.**

131. Two circles are tangent when they touch each other in one point only, as in Fig. 80. When two circles are tangent, they are tangent to the same straight line at the point of tangency.

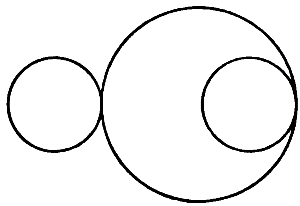


FIG. 80

132. A **secant,** as the term is used in geometry, is a line that intersects the circumference of a circle in two points. In Fig. 79, CD is a secant to the circle whose center is O.

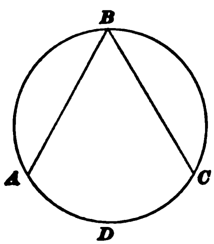


FIG. 81

133. An **inscribed angle** is an angle whose vertex lies on the circumference of a circle, and whose sides are chords. In Fig. 81, ABC is an inscribed angle.

134. A **central angle,** or an **angle at the center,** is an angle whose vertex is at the center of a circle and whose sides are radii. Thus, in Fig. 82, AOB is a central angle.

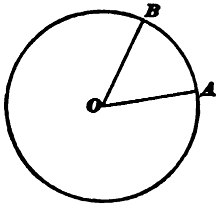


FIG. 82

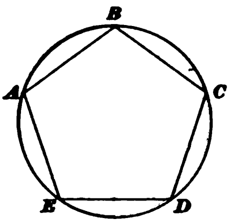


FIG. 83

135. An **inscribed polygon** is a polygon each of whose vertexes lies on the circumference of a circle, as in Fig. 83. The circle is said to be **circumscribed** about the polygon.

136. An **inscribed circle** is a circle whose circumference touches but does not intersect each of the sides of a polygon, as in Fig. 84. The polygon is said to be **circumscribed** about the circle.

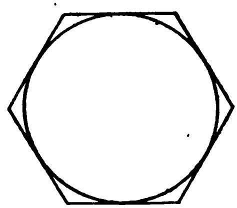


FIG. 84

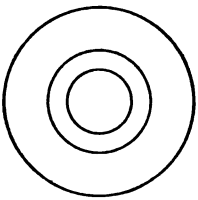


FIG. 85

137. **Concentric circles** are circles having the same center. See Fig. 85.

138. Every diameter of a circle bisects the circle and its circumference. Thus, in Fig. 86, both the arc and the portion of the circle on one side of the diameter AB are equal, respectively, to the arc and the portion of the circle on the other side.

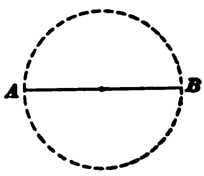


FIG. 86

139. In the same circle, or equal circles, equal angles at the center intercept equal arcs on the circumference.

Let O and O', Fig. 87, be equal circles, and $A\,O\,B$ and $A'\,O'\,B'$ equal angles. Place the circle O' on O so that the point O' coincides with O and the line $O'\,B'$ takes the direction $O\,B$. Then, since $O\,B$ and $O'\,B'$ are equal, being radii of equal circles, B' will fall on B, and, since the angle O' is equal to the angle O, the line $O'\,A'$ will take the direction of $O\,A$, and, being equal to $O\,A$, its extremity A' will fall on A. Hence, the arcs $A\,B$ and $A'\,B'$ will coincide and are equal.

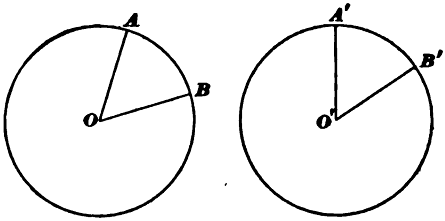


FIG. 87

140. In the same circle, or equal circles, equal arcs are intercepted by equal angles at the center.

Let O and O', Fig. 87, be equal circles, and $A\,B$ and $A'\,B'$ equal arcs. Place the circle O' on the circle O, with the points O' and A' on O and A, respectively. Then, since the arc $A'\,B'$ is equal to the arc $A\,B$, B' will fall on B. Then the angle O' is equal to the angle O, as the vertex and the sides of the angles coincide.

141. In the same circle or equal circles, equal chords subtend equal arcs.

Let $A\,B$ and $C\,D$, Fig. 88, be equal chords. Draw the radii $A\,O$, $B\,O$, $C\,O$, and $D\,O$, joining A, B, C and D to O. Then the triangles $A\,O\,B$ and $C\,O\,D$, having three sides of one equal to three sides of the other, are equal. Hence, the angle $A\,O\,B$ is equal to the angle $C\,O\,D$, and, therefore (Art. **139**), the arc $A\,B$ is equal to the arc $C\,D$.

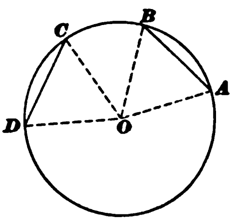


FIG. 88

142. In the same circle, or equal circles, equal arcs are subtended by equal chords.

143. A perpendicular from the center of a circle to a chord bisects the chord and the arc subtended by it.

Let OM, Fig. 89, be drawn from O perpendicular to the chord AB. Join O to A and B. The triangle AOB is isosceles, since the two sides OA and OB are radii of the same circle. Therefore (Art. 77), $AM = MB$. Also, $AOM = MOB$ (Art. 77); therefore (Art. 139), arc AC = arc CB.

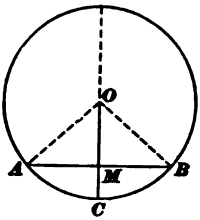


FIG. 89

144. The perpendicular erected at the middle of a chord passes through the center of the circle and bisects the arc subtended by the chord.

145. Through any three points not in a straight line a circumference can be passed.

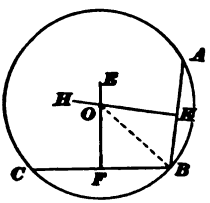


FIG. 90

Let A, B, and C, Fig. 90, be any three points. Draw AB and BC. At the middle point of AB draw KH perpendicular to AB; at the middle point of CB draw FE perpendicular to BC and meeting KH at O. As O is a point in the perpendiculars at the middle points of AB and BC, it is equally distant from A, B, and C. Therefore, a circle with O as center and OB as radius will pass through A, B, and C.

146. A straight line perpendicular to a radius at its extremity is tangent to the circle.

Let AB, Fig. 91, be perpendicular to OH at its extremity H. As OH is perpendicular to AB it is shorter than any other line, as OM, drawn from O to AB. Hence, M is without the circle, and any point in AB other than H is without the circle. Therefore, AB touches the circle in only the point H, and is, consequently, tangent to the circle.

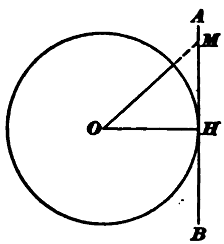


FIG. 91

147. A perpendicular to a tangent at the point of tangency passes through the center of the circle.

148. A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

149. If two circles intersect, the line joining their centers bisects at right angles the line joining the points where the circles intersect.

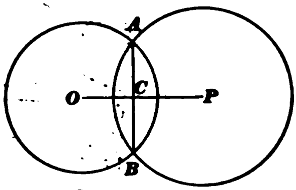


FIG. 92

Let the two circles whose centers are O and P, Fig. 92, intersect at A and B. The point P, being the center of a circle, is equally distant from A and B, points on the circumference. Similarly, O is equally distant from A and B. Hence, by Art. **34**, O and P determine the perpendicular bisecting AB.

150. The two tangents from a point to a circle are equal.

Let PA and PB, Fig. 93, be tangents from P to the circle whose center is O. Draw OA, OP, OB. Then the triangles POB and POA are right triangles (Art. **148**). In these triangles, PO is common and OA is equal to OB. Hence, the triangles are equal, and $PA = PB$.

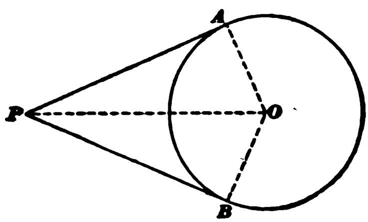


FIG. 93

151. The line joining an external point to the center of a circle bisects the angle made by the two tangents drawn from the point to the circle. Thus, the angle OPA, Fig. 93, is equal to the angle OPB.

EXAMPLES FOR PRACTICE

1. Show that the line joining the intersection of two tangents to the center of the circle bisects the chord joining the points of tangency.

2. Show that the bisector of the angle between two tangents passes through the center of the circle.

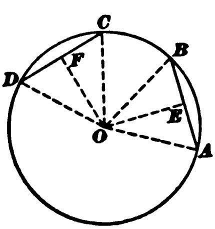


FIG. 94

3. Show that in the same circle, or equal circles, equal chords are equally distant from the center.

SUGGESTION.—Draw OE and OF, Fig. 94, perpendicular to the equal chords AB and CD. Then what is true of the triangles AEO and DOF?

4. Show that the tangents to a circle at the extremities of a diameter are parallel.

5. Show that in any circle a chord parallel to a tangent is bisected by the diameter drawn to the point of contact.

MEASUREMENT OF ANGLES

152. The **ratio** of one quantity to another of the same kind is the number of times that the first contains the second. When both quantities are represented by numbers, their ratio is the same as the quotient obtained by dividing one of the numbers by the other.

153. In the same circle, or equal circles, two central angles have the same ratio as their intercepted arcs; that is, in Fig. 95, angle AOB : angle COD = arc AB : arc CD.

Suppose the arc AB to be three-fifths of the arc CD. Divide AB into three equal parts, and CD into five equal parts, as shown, and join the points of division with the center.

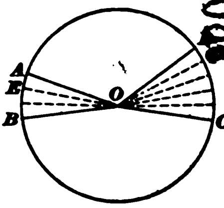


Fig. 95

Since $AB : CD = 3 : 5$, or $\frac{AB}{CD} = \frac{3}{5}$, it follows that one-third of AB is one-fifth of CD; that is, arc AE = arc DF, and, therefore, angle AOE = angle DOF. We have, therefore, angle $AOB = 3 \times$ angle AOE, angle $COD = 5 \times$ angle $DOF = 5 \times$ angle AOE; whence,

$$\frac{\text{angle } AOB}{\text{angle } COD} = \frac{3 \times \text{angle } AOE}{5 \times \text{angle } AOE} = \frac{3}{5} = \frac{\text{arc } AB}{\text{arc } CD}.$$

154. Since the angle at the center and its intercepted arc increase and decrease in the same ratio, it is said that *an angle at the center is measured by its intercepted arc.*

155. The whole circumference of a circle is divided into 360 equal parts, called **degrees**. A degree is divided into 60 equal parts, called **minutes**; and a minute is divided into 60 equal parts, called **seconds**. Degrees, minutes, and seconds of arc are used as units for measuring circular arcs. Since the circumference of every circle contains 360 degrees, the length of a degree differs in different circles. Thus, if AOB, Fig. 96, is an angle of 1°, AB is an arc of 1° in the larger circle and CD is also and arc of 1° in the smaller concentric circle. A degree of the earth's equator is a little more than 69 miles

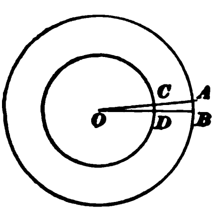


Fig. 96

long; and a degree of the circumference of a circle whose diameter is 360 inches is 3.1416 inches long.

Degrees, minutes, and seconds are indicated by °, ′, ″. Thus, 25° 3′ 10″ means 25 degrees, 3 minutes, and 10 seconds.

Since a right angle intercepts one quarter of a circumference, the number of degrees measuring it is $360 \div 4 = 90°$. The number of degrees measuring an angle equal to one-half of a right angle is $90° \div 2 = 45°$.

Usually, the magnitude of an angle is expressed by stating the number of degrees that it subtends. Thus, a right angle is referred to as an angle of 90°; one-third of a right angle, as an angle of 30°, etc.

156. An inscribed angle is measured by one-half the intercepted arc. Thus, in Fig. 97, the angle ABC is measured by one-half the arc ADC.

Draw the diameter BOD and the radii OC and OA. The angle COD, the exterior angle of the triangle OBC, is equal to the angle OBC plus the angle OCB. But the angle OCB is equal to the angle OBC, as they are opposite the equal sides of an isosceles triangle. Hence, the angle COD, which is measured by the arc CD, is equal to $2 \times OBC$. Therefore, OBC is measured by one-half the arc CD. Similarly, the angle OBA is measured by one-half the arc AD. Therefore, the angle ABC is measured by one-half the arc AD plus one-half the arc DC; that is, by one-half the arc AC.

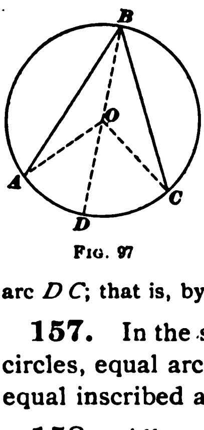


FIG. 97

157. In the same circle, or equal circles, equal arcs are intercepted by equal inscribed angles.

158. All angles inscribed in the same segment are equal.

159. Any angle inscribed in a semicircle is a right angle.

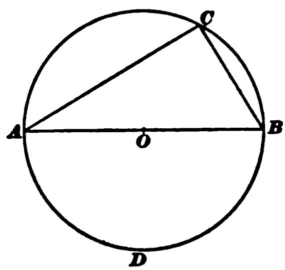


FIG. 98

The angle ACB, Fig. 98, is measured by one-half the arc ADB, which is a semi-circumference. As a semi-circumference contains 180°, the angle ACB is measured by one-half of 180°, or 90°, and is, therefore, a right angle.

160. The vertexes of all the angles of a given magnitude whose sides pass through two fixed points, lie on a circle that passes through the two fixed points and any one of the vertexes.

In Fig. 99, let APB be an angle of the given magnitude and A and B the fixed points. Through A, B, and P, pass a circle. Now, any angle, as AP_1B or AP_2B, whose sides pass through A and B and whose vertex lies on the arc APB is (Art. **158**) equal to the given angle APB.

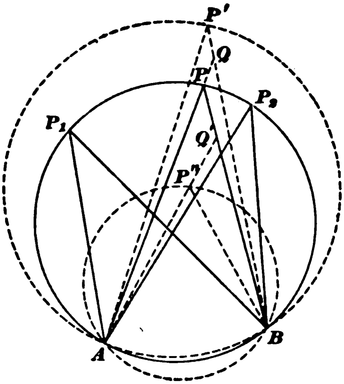


FIG. 99

Again, any angle, as $AP'B$, whose sides pass through A and B and whose vertex lies without the arc APB is less than the angle APB. For if AP is produced to meet BP' at Q, the angle APB being an exterior angle of the triangle BPQ, is equal to $PQB + PBQ$ and is therefore greater than PQB, and, as PQB is greater than $AP'B$ (since $PQB = AP'B + QAP'$), it follows that APB is greater than $AP'B$.

In like manner it can be shown that any angle, as $AP''B$, whose sides pass through A and B and whose vertex lies within the arc APB, is greater than the given angle APB.

161. An angle formed by a tangent, as TM, Fig. 100, and a chord, as TP, is measured by one-half the intercepted arc TEP.

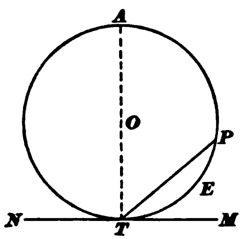


FIG. 100

Draw the diameter TOA. Then MTA is a right angle and is, therefore, measured by one-half the semi-circumference $TEPA$. The angle PTA is measured by one-half the arc PA. Hence, the angle MTP, equal to MTA minus PTA, is measured by one-half the difference between the semi-circumference and PA; that is, by one-half the arc TEP.

EXAMPLES FOR PRACTICE

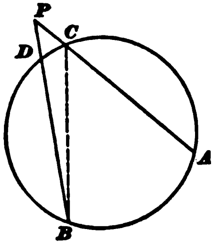


FIG. 101

1. Prove that the angle BPA, Fig. 101, formed by two secants intersecting without the circumference is measured by one-half the difference of the intercepted arcs AB and CD; that is, by $\frac{1}{2}(AB - CD)$.

SUGGESTION.—Join C and B. Then angle BCA is an exterior angle of the triangle BCP, and angle BPC is equal to angle ACB minus angle DBC.

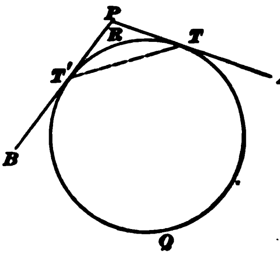


FIG. 102

2. Show that the angle APB, Fig. 102, formed by two tangents PT and PT' is measured by one-half the difference of the intercepted arcs TQT' and TRT'.

SUGGESTION.—Join T and T'. Then ATT' is an exterior angle of triangle $TT'P$, while ATT' and $PT'T$ are angles formed by a tangent and a chord.

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162. The **angle of intersection** of two tangents is the angle formed by one tangent with the prolongation of the other tangent. Thus, the angle APT', Fig. 103, is the angle of intersection of the two tangents TP and PT'.

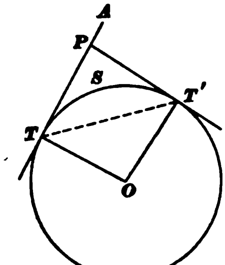


FIG. 103

163. The angle of intersection of two tangents is equal to the central angle whose sides pass through the points of tangency.

In Fig. 103, join TT'. Then, the angle APT' is equal to the sum of the equal angles PTT' and $PT'T$. But each of these angles is made by a tangent and a chord and is, therefore, measured by one-half of the arc TST'. Hence, the angle APT' is measured by the arc TST'. The central angle O is also measured by this arc; therefore, the angle O is equal to the angle APT'.

164. The opposite angles of an inscribed quadrilateral are supplementary; that is, their sum is equal to two right angles or 180°.

In Fig. 104, the angle B is measured by one-half the arc $A D C$, and the opposite angle D is measured by one-half the arc $A B C$. The sum of the arcs $A D C$ and $A B C$ is a circumference, or 360°. Hence, the sum of the angles $A D C$ and $A B C$ is measured by one-half of 360°, or 180°.

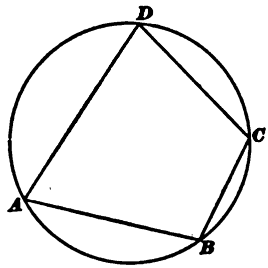


FIG. 104

165. If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.

EXAMPLE 1.—What is the number of degrees in each angle of an equilateral triangle?

SOLUTION.—The sum of the three angles of the triangle is two right angles, or 180°. Since the three angles are equal, each angle is one-third of 180°, or $\frac{180°}{3} = 60°$. Ans.

EXAMPLE 2.—The unequal angle of an isosceles triangle is 75° 32′ 10″; what is the magnitude of each of the equal angles?

SOLUTION.—Since the sum of the three angles is 180°, the sum of the two equal angles is 180° minus the other angle, or $180° - 75° 32' 10'' = 104° 27' 50''$, and each of them is one-half of this sum, or $(104° 27' 50'') \div 2 = 52° 13' 55''$. Ans.

EXAMPLE 3.—The exterior angle of a triangle is 124° 3′ 40″, and one of the opposite-interior angles is 60°; find the other two angles of the triangle.

SOLUTION.—Let the given exterior angle be denoted by A, the given interior angle by B, the other opposite-interior angle by C, and the third angle of the triangle by A'. (Let the student draw the triangle and mark these angles.) Then, $A = B + C$; whence, $C = A - B = 124° 3' 40'' - 60° = 64° 3' 40''$. Ans. Also, $A + A' = 180°$; whence, $A' = 180° - A = 180° - 124° 3' 40'' = 55° 56' 20''$. Ans.

EXAMPLES FOR PRACTICE

1. Show that the only parallelogram that can be inscribed in a circle is a rectangle.

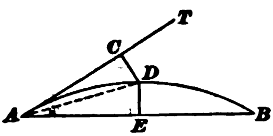


FIG. 105

2. Show that if from a point A, Fig. 105, on the arc of a circle a chord $A B$ and a tangent $A T$ are drawn, the perpendiculars $D C$ and $D E$ drawn to them from the middle point D of the subtended arc are equal.

3. The angle of intersection of two tangents is 100°; find the number of degrees in each angle formed by the tangents and the chord through the points of contact. Ans. 50°

4. One of the acute angles of a right triangle is 50°; what is the magnitude of the other acute angle? Ans. 40°

5. Each of the equal angles of an isosceles triangle is 45°; show that the triangle is right-angled.

6. Two angles of a triangle are 37° 41′ 33″ and 86° 51′ 2″; what is the value of the other angle? Ans. 55° 27′ 22″

GEOMETRY

(PART 2)

PROPORTION

DEFINITIONS AND GENERAL PRINCIPLES

1. A **proportion** is an equality of ratios or of fractions. Thus, the fractions $\frac{4}{5}$ and $\frac{8}{10}$, being equal, form a proportion. In general, if $\frac{a}{b}$ is equal to $\frac{c}{d}$, these two ratios or fractions form a proportion, which may be written in any of the following forms: $\frac{a}{b} = \frac{c}{d}$, $a : b = c : d$, $a : b :: c : d$. When written in either of the last two forms, the proportion is read *a* is to *b* as *c* is to *d*.

2. Properties of Proportions.—The first and the fourth term of a proportion are called the **extremes**; the second and the third, the **means**. Thus, in the proportion $a : b = c : d$, the extremes are a and d, and the means, b and c.

3. If any four quantities are in proportion, the product of the extremes is equal to the product of the means. This principle follows at once from the definition of a proportion, as will be explained presently. If a, b, c, and d, are in proportion, then, by the definition,

$$\frac{a}{b} = \frac{c}{d} \qquad (1)$$

This equation may be treated the same as any other algebraic equation. Both members of the equation may be multiplied or divided by the same quantity, or the same quantity may be added to or subtracted from both members, and the proportion may thus be changed to a great number of forms without destroying the equality of the ratios. Different names are applied to these changes, some of the most common of which are given in the following articles.

In order to show that the product of the means is equal to the product of the extremes, multiply both members of equation (1) by $b\,d$ to clear of fractions; the equation then becomes

$$a\,d = b\,c \qquad (2)$$

4. It is evident that if two fractions are equal, their reciprocals are also equal. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$; that is, if $a : b = c : d$, we have also $b : a = d : c$.

Taking the reciprocal of a fraction is called **inverting** the fraction. The operation of inverting the two fractions of a proportion is called **inversion.**

5. If both members of equation (2), Art. **3**, are divided by $c\,d$, there results

$$\frac{a}{c} = \frac{b}{d}, \text{ or } a : c = b : d$$

Or, if both members of equation (2) are divided by $b\,a$, the result is

$$\frac{d}{b} = \frac{c}{a}, \text{ or } d : b = c : a$$

Therefore, either the means or the extremes of a proportion can be interchanged. This operation is called **alternation.**

6. If 1 is added to each member of equation (1), Art. **3**, the equation becomes

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

Reducing each member to an improper fraction,

$$\frac{a+b}{b} = \frac{c+d}{d}, \text{ or } a+b : b = c+d : d \qquad (1)$$

In a similar manner it can be shown that

$$a+b : a = c+d : c \qquad (2)$$

The proportions (1) and (2) are said to be derived from the original proportion by **composition.**

7. If 1 is subtracted from each member of equation (1), Art. **3**, the equation becomes

$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

Reducing each member to an improper fraction,

$$\frac{a-b}{b} = \frac{c-d}{d}, \text{ or } a - b : b = c - d : d \qquad (1)$$

In a similar manner it can be shown that

$$a - b : a = c - d : c \qquad (2)$$

The proportions (1) and (2) are said to be derived from the original proportion by **division.**

LINES DIVIDED PROPORTIONALLY

8. Two straight lines are **divided proportionally** when the corresponding segments or parts are in proportion; or when the ratio of the two segments of one is the same as the ratio of the two segments of the other. Thus, the lines AB and CD, Fig. 1, are divided proportionally in the points E and F if $AE : EB = CF : FD$.

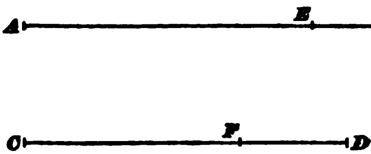


FIG. 1

9. A line parallel to one of the sides of a triangle divides the other two sides proportionally. Thus, in Fig. 2, where DE is parallel to BC, $AD : DB = AE : EC$.

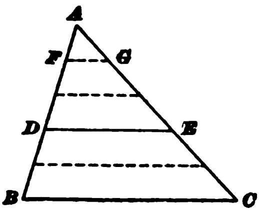


FIG. 2

Suppose that the ratio of AD to DB is as 3 to 2; that is, let $\frac{AD}{DB} = \frac{3}{2}$. Divide AB into five equal parts, and through the points of division draw lines parallel to BC. These lines will intercept equal distances on AC (see *Geometry*, Part 1). As the ratio of AD to DB is that of 3 to 2, AD will contain three, and DB will contain two, of the equal parts into which AB is divided.

Also, AE will contain three and EC two of the equal parts into which AE is divided; so that $AE = 3 \times AG$, and $EC = 2 \times AG$; whence

$$\frac{AE}{EC} = \frac{3 \times AG}{2 \times AG} = \frac{3}{2} = \frac{AD}{DB}$$

or,

$$AE : EC = AD : DB$$

10. Any two sides of a triangle are to each other as the segments into which they are divided by any line parallel to the third side. Thus, in Fig. 2, $AB : AC = AD : AE = DB : EC$.

From the preceding article, we have

$$AD : DB = AE : EC$$

whence (Art. 6),

$$AD + DB : DB = AE + EC : EC$$

that is,

$$AB : DB = AC : EC$$

and, interchanging the means (Art. 5),

$$AB : AC = DB : EC$$

In the same manner it may be shown that

$$AB : AC = AD : AE$$

11. If a line divides two sides of a triangle proportionally, it is parallel to the third side. Thus, if DE, Fig. 3, divides AB and AC so that $AD : DB = AE : EC$, then DE is parallel to BC.

If DE were not parallel to BC, a line DE' could be drawn through D parallel to BC. Then, by Art. 9, we should have $\frac{AD}{DB} = \frac{AE'}{E'C}$; whence, since we have assumed that $\frac{AD}{DB} = \frac{AE}{EC}$,

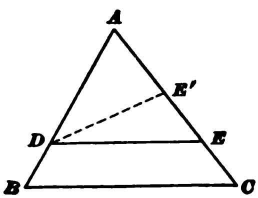


FIG. 3

$$\frac{AE}{EC} = \frac{AE'}{E'C}$$

By interchanging the means of this proportion, we obtain

$$\frac{AE}{AE'} = \frac{EC}{E'C}$$

This equality is evidently absurd, since AE is greater than AE', whereas EC is less than $E'C$. Therefore, no other line than DE can pass through D and be parallel to BC.

EXAMPLE 1.—Find the length of the line AB, Fig. 4, of which the end B is inaccessible.

SOLUTION.—There are several ways of solving this problem in practice. The one illustrated in the figure is as follows: Any convenient distance AC is measured and the angle C observed with a transit or compass. From C, a distance CE is measured, and at E an angle AED equal to C is turned off. The point D where the line of sight ED meets AB is marked, and the distances AD and AE are measured. Then, since AED equals C, the lines ED and CB are parallel (see *Geometry*, Part 1) and, therefore (Art. 9),

$$AB : AC = AD : AE$$

whence (Art. 3),

$$AB \times AE = AC \times AD$$

and, dividing by AE,

$$AB = \frac{AC \times AD}{AE}. \quad \text{Ans.}$$

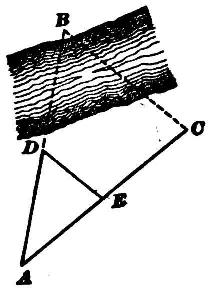


FIG. 4

EXAMPLE 2.—Divide a line AB, Fig. 1, of given length into two parts AE and EB whose ratio shall be the same as that of two given numbers m and n; that is, so that $AE : EB = m : n$.

SOLUTION.—Since $AE : EB = m : n$, we must have (Art. 6),

$$\frac{AE + EB}{EB} = \frac{m+n}{n}, \text{ or, } \frac{AB}{EB} = \frac{m+n}{n},$$

whence, solving for EB,

$$EB = \frac{n \times AB}{m+n}. \quad \text{Ans.}$$

AE can be found in a similar manner, or by subtracting the value of EB from AB.

EXAMPLES FOR PRACTICE

1. If the measured distances in Fig. 4 are AC = 100 feet, AE = 45.2 feet, AD = 48.36 feet, what are the distances AB and DB?

Ans. $\begin{cases} AB = 106.99 \text{ ft.} \\ DB = 58.63 \text{ ft.} \end{cases}$

2. If, in Fig. 3, AD = 75 feet, DB = 16.25 feet, and AC = 80 feet, find AE and EC.

Ans. $\begin{cases} AE = 65.75 \text{ ft.} \\ EC = 14.25 \text{ ft.} \end{cases}$

3. If AB, Fig. 1, is equal to 125 feet, find the distances AE and EB so that the line will be divided at E in the ratio of 5 to 2.

Ans. $\begin{cases} AE = 89.286 \text{ ft.} \\ EB = 35.714 \text{ ft.} \end{cases}$

12. If two lines, as AB and CD, Fig. 5, are cut by any number of parallel lines, as EM, GN, IO, etc., the corresponding intercepts are proportional; that is, $EG : GI = MN : NO$; $GI : IK = NO : OP$, or, by interchanging the means, $EG : MN = GI : NO = IK : OP$, etc.

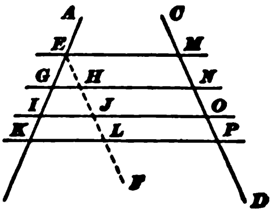


FIG. 5

Through E, draw EF parallel to CD. Then, $EH = MN$, $HJ = NO$, $JL = OP$. (See *Geometry*, Part 1.) Also, by Arts. **9** and **10**,

$$\frac{EK}{EL} = \frac{EG}{EH} = \frac{GI}{HJ} = \frac{IK}{JL}$$

that is,

$$\frac{EK}{MP} = \frac{EG}{MN} = \frac{GI}{NO} = \frac{IK}{OP}$$

or, $$EK : MP = EG : MN = GI : NO = IK : OP$$

13. In any triangle ABC, Fig. 6, the bisector BD of an angle divides the side opposite proportionally to the including sides; that is, $AB : BC = AD : DC$.

Draw CE parallel to BD and meeting AB produced in E. Then, in the triangle AEC, by Art. **9**,

$$AB : BE = AD : DC \qquad (1)$$

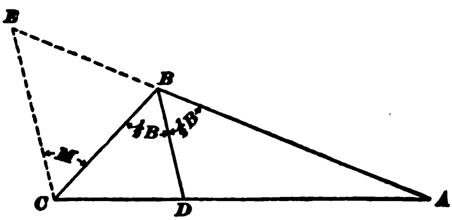


FIG. 6

The angles DBC and M, being alternate-interior angles, are equal; that is, $M = \frac{1}{2}B$. The angles DBA and E, being exterior-interior angles, are equal; that is, $E = \frac{1}{2}B$. Therefore, $E = M$, and $BE = BC$. (See *Geometry*, Part 1.) Substituting, in equation (1), BC for its equal BE,

$$AB : BC = AD : DC$$

POLYGONS

SIMILAR POLYGONS

SIMILAR TRIANGLES

14. Similar polygons are those whose corresponding angles are equal and whose corresponding sides are proportional.

In order that two polygons may be similar, it is manifestly necessary that each angle of the one shall be equal to the corresponding angle of the other. But this is not sufficient; the corresponding sides must be proportional. For example,

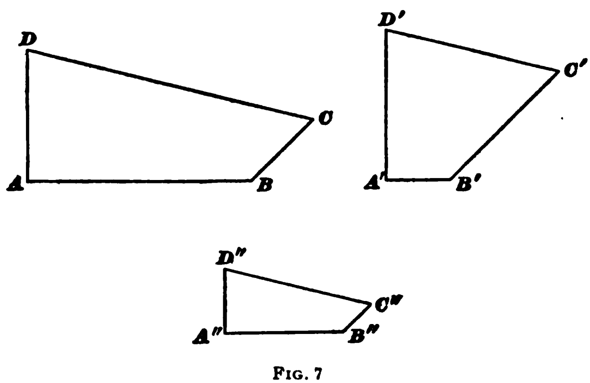


FIG. 7

the quadrilaterals $ABCD$ and $A'B'C'D'$, Fig. 7, have their corresponding angles equal, but they are not similar, because their corresponding sides are not proportional. The quadrilaterals $ABCD$ and $A''B''C''D''$ have their corresponding angles equal and their corresponding sides proportional, and are, therefore, similar.

The corresponding sides of similar polygons are called **homologous sides.**

15. Two triangles are similar when the angles of one are equal to the angles of the other.

In Fig. 8, let the angles of the triangle ABC be equal, respectively, to those of the triangle $A'B'C'$. Place the triangle $A'B'C'$ upon ABC, so that the angle A' will coincide with its equal A. Then B' will fall along AB and C' along AC, as at B'' and C'', respectively, and $B'C'$ will take the position $B''C''$. The angle B'', which is equal to B', is equal to B, and the angle C'', which is equal to C', is equal to C; hence, $B''C''$ is parallel to BC (see *Geometry*, Part 1). Then, by Art. **10**,

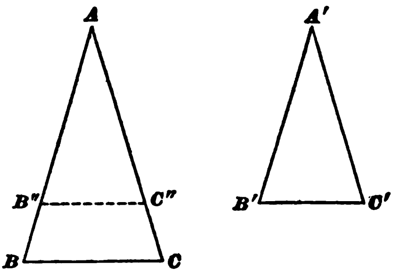


FIG. 8

$$AB : AB'' = AC : AC''$$

Substituting $A'B'$ and $A'C'$ for their respective equals AB'' and AC'',

$$AB : A'B' = AC : A'C'$$

In like manner, it can be proved that

$$AB : A'B' = BC : B'C'$$

Therefore, the triangles, having their angles equal and their corresponding sides proportional, are similar.

16. Two triangles are similar when two angles of the one are equal respectively to two angles of the other.

17. Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.

18. A triangle is similar to any triangle formed by a line parallel to one of its sides and the segments it intercepts on the other two sides or the other two sides prolonged.

19. Two triangles are similar when the three sides of one are either parallel or perpendicular to the three sides of the other.

20. Two triangles are similar when their corresponding sides are proportional.

In Fig. 9, $AB : A'B' = AC : A'C' = BC : B'C'$

On AB, lay off AD equal to $A'B'$; on AC, lay off AE equal to $A'C'$, and join DE. Then, since $AB : AD = AC : AE$, DE is parallel to BC. Hence, by Art. **18**, triangles ABC and ADE are similar, and, consequently, triangles ABC and $A'B'C'$ are similar if it can be shown that $DE = B'C'$. Now,

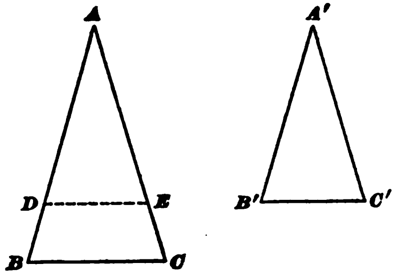


FIG. 9

$$AB : AD = BC : DE, \text{ or } AB : A'B' = BC : DE$$

But, $$AB : A'B' = BC : B'C'$$

The last two proportions are the same, term for term, excepting the last term; hence, DE is equal to $B'C'$, and the triangles ADE and $A'B'C'$ are equal. Therefore, the triangles ABC and $A'B'C'$ are similar.

21. Two triangles are similar when an angle of the one is equal to an angle of the other and the including sides are proportional.

22. In two similar triangles, corresponding altitudes have the same ratio as any two corresponding sides.

Let CD and $C'D'$, Fig. 10, be the corresponding altitudes of the two similar triangles ABC and $A'B'C'$. The right triangles ACD and $A'C'D'$, having angle A equal to the angle A', are similar; hence,

$$CD : C'D' = AC : A'C'$$

But, since the triangles ABC and $A'B'C'$ are similar,

$$AC : A'C' = AB : A'B' = BC : B'C'$$

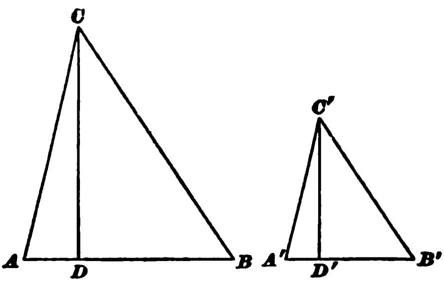


FIG. 10

Therefore,

$$CD : C'D' = AC : A'C' = AB : A'B' = BC : B'C'$$

23. As stated in Art. **14**, two polygons are similar when their corresponding angles are equal and their corresponding sides are proportional. It has now been shown that, in triangles, either of these conditions includes the other. This could have been expected from the fact that either the three angles or the three sides of a triangle fix its shape. This is not true of a polygon of more than three sides, as the angles can be changed without altering the sides, or the proportions of the sides can be changed without altering the angles.

EXAMPLE 1.—In the triangles ABC and $A'B'C'$, Fig. 11, angle A = angle A', angle B = angle B', and angle C = angle C', and the

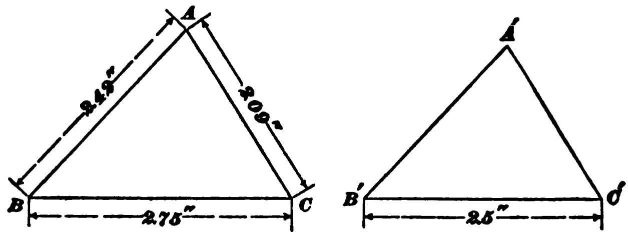


FIG. 11

sides BC, CA, AB, and $B'C'$ have the dimensions that are marked on them; find the lengths of the sides $C'A'$ and $A'B'$.

SOLUTION.—Since the two triangles are equiangular, they are similar, and hence the value of $A'B'$ and that of $A'C'$ are conveniently found as follows:

$$2.75 : 2.42 = 2.5 : A'B'$$

whence $$A'B' = \frac{2.42 \times 2.5}{2.75} = 2.2 \text{ in. Ans.}$$

$$2.75 : 2.09 = 2.5 : A'C'$$

whence $$A'C' = \frac{2.09 \times 2.5}{2.75} = 1.9 \text{ in. Ans.}$$

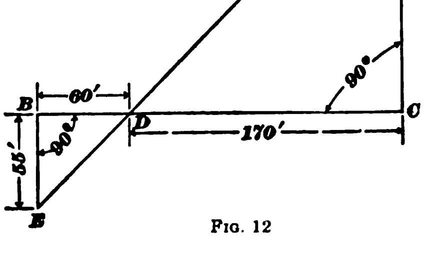


FIG. 12

EXAMPLE 2.—In Fig. 12, CD is perpendicular to AC and is 170 feet long; DB is 60 feet; and BE, perpendicular to BD, is 55 feet long; find the distance AC.

SOLUTION.—The right triangles ACD and DBE have the angles ADC and BDE equal; hence, they are similar, and

$AC : CD = BE : BD$,
or $AC : 170 = 55 : 60$

whence,

$$AC = \frac{170 \times 55}{60} = 155.83 \text{ ft.} \quad \text{Ans.}$$

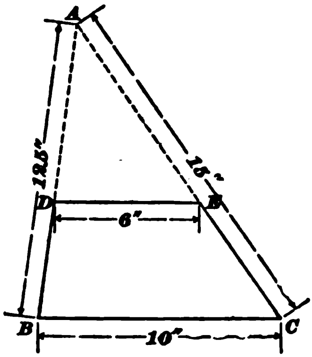


FIG. 13

EXAMPLE 3.—It is required to cut from a triangular plate $A\,B\,C$, Fig. 13, having the dimensions shown, a trapezoidal plate $B\,D\,E\,C$ whose upper base $D\,E$ shall be 6 inches; find the distances $A\,D$ and $A\,E$ that must be cut off.

SOLUTION.—The similar triangles $A\,D\,E$ and $A\,B\,C$ give,

$$\frac{AD}{AB} = \frac{DE}{BC};\; AD = \frac{AB \times DE}{BC} = \frac{12.5 \times 6}{10} = 7.5 \text{ in.} \quad \text{Ans.}$$

$$\frac{AE}{AC} = \frac{DE}{BC};\; AE = \frac{AC \times DE}{BC} = \frac{15 \times 6}{10} = 9 \text{ in.} \quad \text{Ans.}$$

EXAMPLE 4.—In order to measure the height $R\,H$ of a pier $A_1\,P\,Q\,R$, Fig. 14, whose base and top are, respectively, 22 feet and 12 feet square and whose sides all have the same inclination, a transit was set at a point O distant 250 feet from the side A_1 of the pier; that is, so that $O\,M_1$ = 250 feet. $A_1\,B_1$ was a rod on which the horizontal line of sight $O\,M_1$ intercepted a distance $A_1\,M_1$ = 4.5 feet. The same rod was

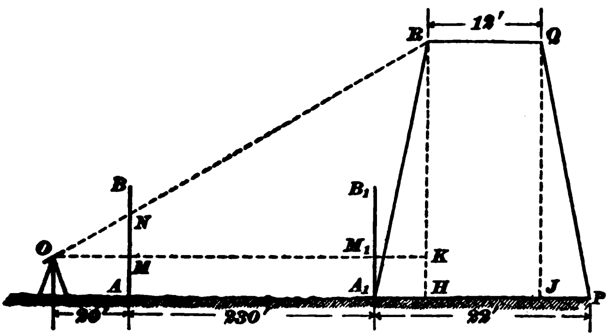


FIG. 14

held at a distance $O\,M$ = 20 feet from the instrument, and the height $A\,M$ above the ground noted. Then the telescope of the transit was directed to the top R of the pier, and, with the rod still held at A, the height $A\,N$ was read on the rod. By subtracting $A\,M$ from $A\,N$, the distance $M\,N$ intercepted between the lines $O\,M_1$ and $O\,R$ was found to be 6 feet. What was the height $R\,H$?

SOLUTION.—The inclination of $A_1 R$ and that of PQ being equal, we have, $A_1 H = JP$, and $HJ = RQ = 12$ ft. Now,

$$A_1 P = A_1 H + HJ + JP = 2\,A_1 H + 12$$

whence, $$A_1 H = \frac{A_1 P - 12}{2} = \frac{22 - 12}{2} = 5 \text{ ft.}$$

The similar triangles OMN and OKR give

$$\frac{OM}{MN} = \frac{OK}{KR}; \; KR = \frac{OK \times MN}{OM}$$

$$= \frac{(OM_1 + M_1 K) \times MN}{OM} = \frac{(OM_1 + A_1 H) \times MN}{OM}$$

$$= \frac{(250 + 5) \times 6}{20} = \frac{255 \times 6}{20} = 76.5 \text{ ft.}$$

Finally,

$RH = RK + KH = RK + M_1 A_1 = 76.5 + 4.5 = 81$ ft. Ans.

EXAMPLES FOR PRACTICE

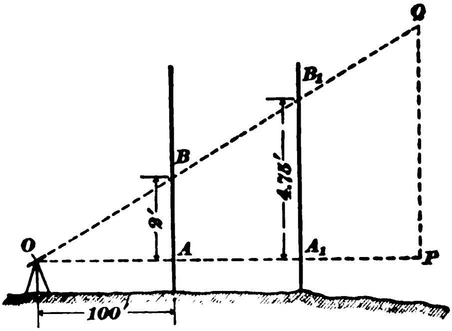


FIG. 15

1. In Fig. 15, the lines of sight OP and OQ of a transit intercept on a rod distances, $AB = 2$ feet and $A_1 B_1 = 4.75$ feet; if the distance OA is 100 feet, what is the distance OA_1? Ans. 237.5 ft.

2. In order to find the stress in the member BD, Fig. 16, by the method of moments, it is necessary to find the distance DO from D to

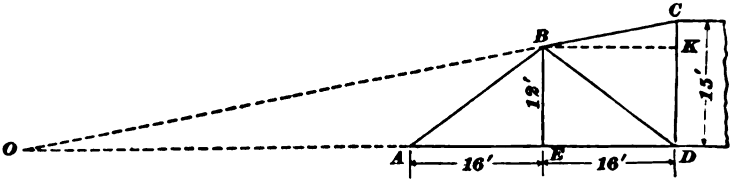


FIG. 16

the point of intersection O of DA and CB, both produced; the dimensions being as shown, what is that distance? Ans. $DO = 80$ ft.

3. $ABCD$, Fig. 17, is a trapezoid whose non-parallel sides produced meet at O; the line MN is parallel to the bases AD and BC; the dimensions of AD, BC, BM, and MA being as shown, find OB and MN.

Ans. $\begin{cases} OB = 315 \text{ ft.} \\ MN = 85.714 \text{ ft.} \end{cases}$

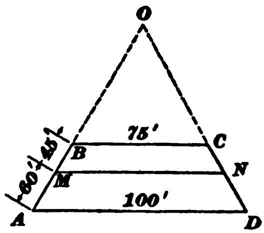


FIG. 17

4. In a triangle ABC, side $AB = 32$ feet, $BC = 34$ feet, and $AC = 48$ feet; if side $A'B'$ of a similar triangle $A'B'C'$ is 72 feet long, what are the lengths of the other two sides?

Ans. $\begin{cases} A'C' = 108 \text{ ft.} \\ B'C' = 76.5 \text{ ft.} \end{cases}$

5. The base of a right triangle is 24 inches, and its altitude 72 inches; at what distance from the top is the triangle 16 inches wide?

Ans. 48 in.

IMPORTANT CONSEQUENCES OF THE THEORY OF SIMILAR TRIANGLES

24. When the first of three quantities is to the second as the second is to the third, the three quantities are in **continued proportion**; the second is a **mean proportional** between the first and third; and the third is a **third proportional** to the first and second. Thus, if $a : b = b : c$, the three quantities a, b, and c are in continued proportion; b is a mean proportional between a and c; and c is a third proportional to a and b.

25. In a right triangle, as ABC, Fig. 18, the perpendicular CD drawn from the vertex of the right angle to the hypotenuse, divides the triangle into two triangles ACD and CDB that are similar to the whole triangle and to each other.

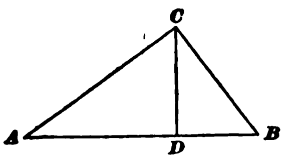


FIG. 18

The right triangles ABC and ACD are similar, by Art. **17**, as the angle A is common. Also, the triangles ABC and CBD, having angle B in common, are similar. Again, the triangles ACD and CBD, being each similar to ABC, are similar to each other.

26. In a right triangle, the perpendicular to the hypotenuse from the vertex of the right angle is a mean proportional between the two parts or segments into which it divides the hypotenuse; that is, Fig. 18, $AD:CD = CD:DB$.

As the triangles BCD and ABC are similar, and the angle B is common, the angle BCD must equal the angle A, and similarly, the angle ACD must equal the angle B. The triangles ACD and BCD are similar, hence the sides opposite equal angles are in proportion; that is,

$$\frac{AD \text{ (side opposite } ACD)}{CD \text{ (side opposite } B)} = \frac{CD \text{ (side opposite } A)}{DB \text{ (side opposite } BCD)}$$

Or, $$AD:CD = CD:DB$$

27. The side AC, Fig. 18, is a mean proportional between the whole hypotenuse and the segment AD on the same side of CD as the side AC; that is, $AB:AC = AC:AD$. Similarly, $AB:BC = BC:BD$.

The triangles ABC and ACD are similar, hence the sides opposite equal angles are proportional; that is,

$$\frac{AB \text{ (opposite right angle)}}{AC \text{ (opposite right angle)}} = \frac{AC \text{ (opposite } B)}{AD \text{ (opposite } ACD)}$$

Or, $$AB:AC = AC:AD$$

Example 1.—In the right triangle ABC, Fig. 19, find the length of the perpendicular CD.

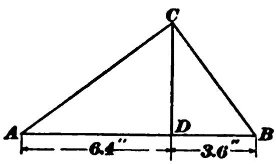


Fig. 19

Solution.—The perpendicular is a mean proportional between the parts AD and DB into which it divides the hypotenuse; therefore,

$$6.4:CD = CD:3.6$$

whence, $$\overline{CD}^2 = 6.4 \times 3.6$$

and $CD = \sqrt{6.4 \times 3.6} = 4.8$ in. Ans.

Example 2.—Find the length of the sides of the right triangle ABC, Fig. 20, in which CD is the perpendicular from the vertex of the right angle to the hypotenuse.

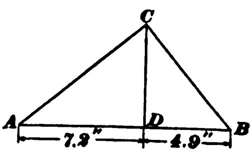


Fig. 20

Solution.—The hypotenuse is 7.2 in. + 4.9 in. = 12.1 in. The side CB is a mean proportional between the hypotenuse AB and the part DB; therefore,

$$12.1:CB = CB:4.9$$

$$\overline{CB}^2 = 12.1 \times 4.9$$

$$CB = \sqrt{12.1 \times 4.9} = 7.7 \text{ in. Ans.}$$

The leg AC is a mean proportional between AB and AD; that is,

$$AB : AC = AC : AD$$
$$AC = \sqrt{AB \times AD}$$
$$= \sqrt{12.1 \times 7.2} = 9.34 \text{ in. Ans.}$$

28. Since an angle inscribed in a semicircle is a right angle, it follows from Arts. **26** and **27**, that:

(*a*) A perpendicular CD, Fig. 21, drawn from any point on the circumference of a circle to a diameter AB, is a mean proportional between the segments into which it divides the diameter; that is,

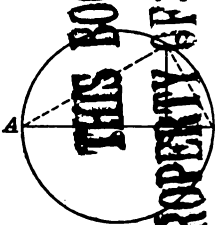


FIG. 21

$$AD : CD = CD : DB$$

(*b*) A chord CA drawn from a point in a circumference to the end of a diameter is a mean proportional between the whole diameter and the adjacent segment AD; that is,

$$AB : AC = AC : AD$$

29. If from a point without a circle, a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and the exterior segment; that is, in Fig. 22, $PB : PT = PT : PA$.

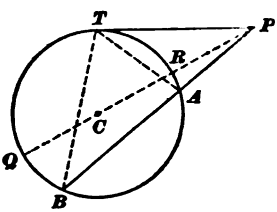


FIG. 22

In the triangles BPT and APT, the angle P is common. The angle B, an inscribed angle, and the angle PTA, an angle formed by a tangent and a chord, are equal, since each is measured by one-half the same arc AT. Hence, the triangles are similar by Art. **16**, and

$$\frac{PB \text{ (opposite angle } PTB)}{PT \text{ (opposite angle } PAT)} = \frac{PT \text{ (opposite angle } B)}{PA \text{ (opposite angle } PTA)}$$

$$\overline{PT}^2 = PB \times PA$$
$$PT = \sqrt{PB \times PA}$$

30. If from a point without a circle any two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

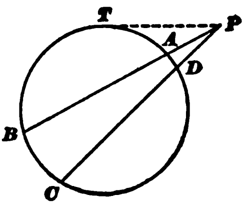


FIG. 23

In Fig. 23, PB and PC are secants. Draw the tangent PT. Then, from Art. **29**,

$$\overline{PT}^2 = PA \times PB$$

and $$\overline{PT}^2 = PC \times PD$$

hence, $$PA \times PB = PC \times PD$$

31. If any two chords be drawn through a point within a circle, the product of the segments of one is equal to the product of the segments of the other.

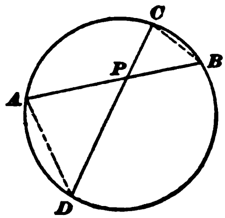


FIG. 24

In Fig. 24, the angles D and B, being measured by one-half the arc AC, are equal. The angles BPC and DPA, being vertical angles, are equal. Hence, by Art. **16**, the triangles CBP and ADP are similar. Therefore,

$$\frac{AP}{CP} = \frac{PD}{PB}$$

and $$AP \times PB = CP \times PD$$

EXAMPLES FOR PRACTICE

1. The perpendicular from the vertex of the right angle of a right triangle divides the hypotenuse into parts of 23.04 inches and 1.96 inches. Find: (*a*) the length of the perpendicular; (*b*) the length of the two sides of the triangle. Ans. $\begin{cases} (a)\ 6.72 \text{ in.} \\ (b)\ 24 \text{ in. and } 7 \text{ in.} \end{cases}$

2. If, in Fig. 22, the distance CP of the point P from the center of the circle is 65 feet, and the radius CR is 25 feet, what is the length of the tangent PT? Ans. 60 ft.

3. The chord of the arc of a segment is 14 inches long and the height of the segment is 2 inches; what is the radius? Ans. 13¼ in.

OTHER SIMILAR POLYGONS

32. Two polygons are similar when they are composed of the same number of triangles similar each to each and similarly placed.

Thus, in Fig. 25, the polygons $ABCDE$ and $A'B'C'D'E'$ are composed of the same number of similar triangles similarly placed.

Since the triangle AED is similar to the triangle $A'E'D'$, angle E = angle E' and angle ADE = angle $A'D'E'$. Also, in the similar triangles ADC and $A'D'C'$, angle ADC = angle $A'D'C'$. Hence, the sum of the angles ADE and ADC, or the angle EDC, is equal to the sum of the angles $A'D'E'$ and $A'D'C'$, or the angle $E'D'C'$. In like manner, angle DCB = angle $D'C'B'$,

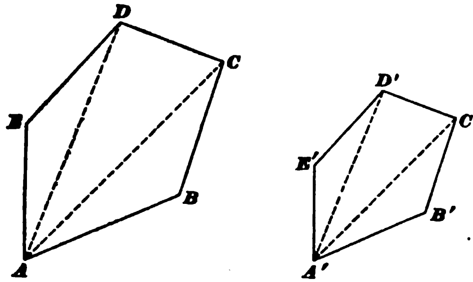


FIG. 25

angle B = angle B', and angle BAE = angle $B'A'E'$. Since the triangles are similar,

$$ED : E'D' = AD : A'D' \text{ and } AD : A'D' = DC : D'C'$$

hence, $$ED : E'D' = DC : D'C'$$

In like manner,

$$DC : D'C' = CB : C'B' = BA : B'A' = AE : A'E'$$

Therefore, as the angles of the one polygon are equal to the corresponding angles of the other and the sides of the one polygon are proportional to the sides of the other, the polygons are similar.

33. Two similar polygons can be divided into the same number of similar triangles similarly placed.

34. The perimeters of two similar polygons are in the same ratio as any two homologous sides.

In Fig. 25, let P be the perimeter of the polygon $ABCDE$, and P' the perimeter of the polygon $A'B'C'D'E'$. Since the polygons are similar

$$\frac{AE}{A'E'} = \frac{ED}{E'D'} = \frac{DC}{D'C'} = \frac{CB}{C'B'} = \frac{BA}{B'A'} \qquad (1)$$

Let each of these equal ratios be denoted by R; that is, let

$$\frac{AE}{A'E'} = R, \frac{ED}{E'D'} = R, \frac{DC}{D'C'} = R, \frac{CB}{C'B'} = R, \frac{BA}{B'A'} = R.$$

From these equations we obtain,

$$AE = R \times A'E', ED = R \times E'D', DC = R \times D'C',$$
$$CB = R \times C'B', BA = R \times B'A'$$

Adding the sides of these equalities,

$$AE + ED + DC + CB + BA$$
$$= R \times A'E' + R \times E'D' + R \times D'C' + R \times C'B' + R \times B'A'$$
$$= R(A'E' + E'D' + D'C' + C'B' + B'A')$$

whence $\dfrac{AE+ED+DC+CB+BA}{A'E'+E'D'+D'C'+C'B'+B'A'} = R$

But $R = \dfrac{AE}{A'E'} = \dfrac{ED}{E'D'} = \dfrac{DC}{D'C'}$, etc.;

therefore, $\dfrac{P}{P'} = \dfrac{AE}{A'E'} = \dfrac{ED}{E'D'} = \dfrac{DC}{D'C'}$, etc.

35. Equation (1) of the preceding article is a series of equal ratios, of which the numerators are the antecedents and the denominators the consequents. The general truth was shown in that article, that in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

AREAS OF POLYGONS

36. Definitions.—The **area** of a surface is the superficial space included within its boundary lines. Area is expressed by the ratio of the surface to a surface of fixed value chosen as a unit and called the **unit of area.**

37. A square whose side is equal in length to the unit of length is usually taken as the unit of area, and its area is called the **square unit.** For example, if the unit of length is 1 inch, the unit of area, or square inch, is the square whose sides measure 1 inch, and the area of any surface is expressed by the number of square inches that the surface contains. If the unit of length were 1 foot, the unit of area would measure 1 foot on each side, and the area of the surface would be expressed in square feet. Square inch and square foot are abbreviated to sq. in. and sq. ft., respectively, and are often indicated by the symbols □″ and □′.

38. Two surfaces are **equivalent** when their areas are equal.

39. Comparison of the Areas of Two Rectangles. The areas of two rectangles $ABCD$ and $A'B'C'D'$, Fig. 26, having equal altitudes are to each other as their bases; that is, area $ABCD$: area $A'B'C'D' = AB : A'B'$.

Suppose that $A'B'$ is four-fifths of AB, or that $AB : A'B' = 5 . 4$. Divide AB into five equal parts AE, EF, etc., and $A'B'$ into four equal parts $A'E'$, $E'F'$, etc. It is evident that $A'E' = AE$, for, since AB is to $A'B'$ in the ratio of 5 to 4, any quantity, as AE, that is contained five times in AB must be contained four times in $A'B'$. Through the points of division E, F, E', F', etc., draw perpendiculars

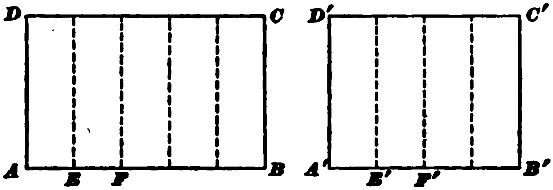


FIG. 26

to AB and $A'B'$. Each large rectangle is thus divided into small rectangles, all of which are equal. As $ABCD$ contains five, and $A'B'C'D'$ contains four, of the small rectangles, the ratio of the two large rectangles is that of 5 to 4, which is also the ratio of their bases.

40. Since any of the sides of a rectangle can be considered as the base, it follows that the area of two rectangles having equal bases are to each other as their altitudes.

41. The areas of any two rectangles are to each other as the products of their bases by their altitudes.

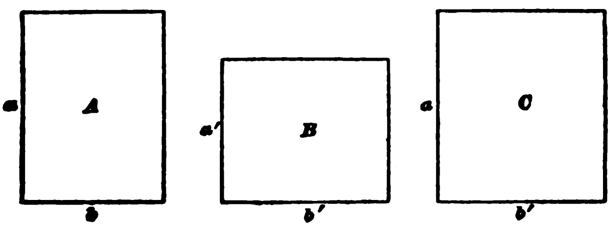


FIG. 27

Let A and B, Fig. 27, be two rectangles whose altitudes are a and a' and whose bases are b and b', respectively. Construct a rectangle C with an altitude a and a base b'. Then, by Arts. **39** and **40**,

$$A : C = b : b' \qquad (1)$$

and

$$C : B = a : a' \qquad (2)$$

Multiplying equation (1) by equation (2),

$$AC : BC = ab : a'b' \qquad (3)$$

Dividing the terms of the first member of equation (3) by C,

$$A : B = ab : a'b'$$

42. Area of a Rectangle.—The area of a rectangle is equal to its base multiplied by its altitude; that is, in Fig. 28,

$$A = bh.$$

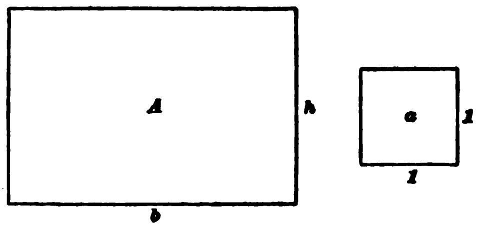


Fig. 28

Construct a unit square a. Then (Art. 41),

$$A : a = h \times b : 1 \times 1;$$

or, $$\frac{A}{a} = \frac{h \times b}{1 \times 1}$$

But a is a unit square, and its area is therefore equal to 1; hence,

$$A = bh$$

43. Area of a Triangle.—The area of a right triangle is equal to one-half the product of the two legs of the triangle; that is, in Fig. 29, area $ABC = \frac{1}{2}ab$.

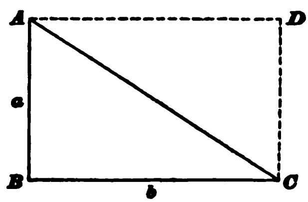


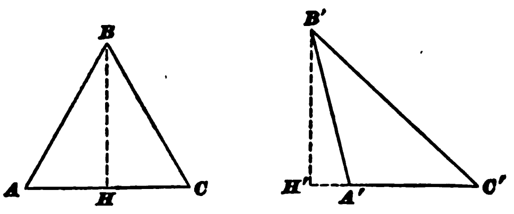
Fig. 29

For the triangle ABC is one-half the rectangle $ABCD$ and the area of the latter is ab.

44. The area of any triangle is equal to one-half the product of its base and altitude.

In Fig. 30 (a), let AC be the base and BH the altitude of the triangle ABC. The area ABC is equal to the sum of the right triangles AHB and CHB, which, by the last article, is

$$\tfrac{1}{2}BH \times AH + \tfrac{1}{2}BH \times HC = \tfrac{1}{2}BH \times (AH + HC) = \tfrac{1}{2}BH \times AC$$



(a) (b)

Fig. 30

In Fig. 30 (b), the area $A'B'C'$ is the difference between the areas of the right triangles $B'H'C'$ and $B'H'A'$; that is,

$$\tfrac{1}{2}B'H' \times H'C' - \tfrac{1}{2}B'H' \times H'A' = \tfrac{1}{2}B'H' \times (H'C' - H'A')$$
$$= \tfrac{1}{2}B'H' \times A'C'$$

Let b be the base, h the altitude, and A the area of any triangle; then,

$$A = \frac{1}{2} b h$$

45. Two triangles having the same base are to each other as their altitudes, and two triangles having the same altitude are to each other as their bases.

46. Two triangles having the same base and the same altitude are equivalent.

It should be borne in mind that any side of a triangle can be taken as the base, the altitude being the perpendicular to that side from the opposite vertex.

47. To find the area of a triangle from the lengths of its three sides, apply the following:

Rule.—*From half the sum of the three sides subtract each side separately; multiply together the half sum and the three remainders and extract the square root of the product.*

Let a, b, and c be the three sides of a triangle, and A the area; let

$$s = \frac{1}{2}(a + b + c)$$

Then $$A = \sqrt{s(s-a)(s-b)(s-c)}$$

The geometrical proof of this rule is very laborious, and will not be given here. A proof will be found in *Trigonometry.*

EXAMPLE.—What is the area of a triangle having two sides 19.8 feet long, and one side 28 feet long?

SOLUTION.—It is immaterial which side is called a, b, or c. $s = \frac{a+b+c}{2} = \frac{28 + 19.8 + 19.8}{2} = 33.8$; taking b and c as the short sides, $s - a = 33.8 - 28 = 5.8$, and $s - b$ and $s - c$ are each $33.8 - 19.8 = 14$. Then, applying the formula $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{33.8 \times 5.8 \times 14 \times 14} = 196$ sq. ft., nearly. Ans.

48. A triangle equivalent to any given polygon may be constructed as follows:

Let $ABCDEF$, Fig. 31, be the given polygon. Produce any of the sides, as AF, in both directions, as indicated by XY. This line will be referred to as the base. Starting from one of the ends of AF, as A, draw a diagonal AC forming a triangle with AB and BC. Draw BB_1 parallel to CA, meeting the base at B_1, and join C to B_1.

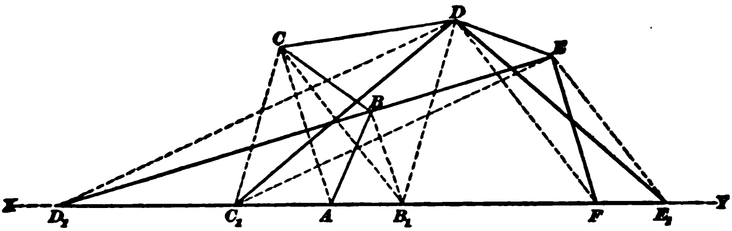


FIG. 31

The polygon B_1CDEF has one side less than the given polygon, and is equivalent to it. For

$$ABCDEF = B_1BCDEF + \text{triangle } B_1BA$$
$$B_1CDEF = B_1BCDEF + \text{triangle } B_1BC$$

The two triangles B_1BA and B_1BC are equivalent, for they have the common base B_1B, and their altitudes, being each equal to the distance between the parallels AC and B_1B, are equal. Proceeding with the polygon B_1CDEF as with the original polygon, draw the diagonal B_1D, forming a triangle with B_1C and CD. Draw CC_1 parallel to DB_1, and join D and C_1. It can be shown as before that the polygon C_1DEF is equivalent to B_1CDEF, and, therefore, to the original polygon. Finally, draw the diagonal C_1E, and DD_1 parallel to it, meeting the base at D_1. Then will the triangle D_1EF be the required triangle equivalent to the given polygon.

In practice, it is more convenient, as well as more accurate, to reduce about one-half of the polygon on one side of A and the rest on the other side of F. Thus, having reduced the polygon to the quadrilateral C_1DEF, the diagonal FD is drawn from F; EE_1 is drawn through E parallel to DF, and E_1 joined to D. This gives C_1DE_1 as the required triangle.

49. Area of a Parallelogram.—The area of a parallelogram is equal to its base multiplied by its altitude; that is, in Fig. 32, area $ABCD = AD \times MN$.

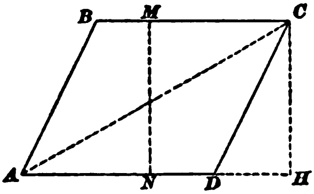


FIG. 32

For $ABCD$ is equal to the sum of the equal triangles ABC and ADC, or to twice either of them, as ADC; that is, $ABCD = 2 \times \frac{1}{2} AD \times CH = AD \times CH = AD \times MN$.

EXAMPLE 1.—What is the cost of paving a street 1,800 feet long and 36 feet wide with asphalt, the price being $2 per square yard?

SOLUTION.—The surface to be covered is a rectangle whose sides are 36 ft. and 1,800 ft., or 12 yd. and 600 yd., and whose area is, therefore, 12 × 600 = 7,200 sq. yd. The cost of paving is, then, 2 × 7,200 = $14,400. Ans.

EXAMPLE 2.—One side of a triangular plot of land is 125 feet long and the perpendicular distance from the opposite vertex to this side is 174.24 feet; it is desired to find a side of a rectangle that has the same area as the triangle and one side 75 feet long.

SOLUTION.—The area of the triangle is $\frac{1}{2} \times 125 \times 174.24 = 10{,}890$ sq. ft. Then the other side of the rectangle is $10{,}890 \div 75 = 145.2$ ft. Ans.

EXAMPLE 3.—Divide a triangular plot of land into any number of equal parts by lines from a vertex to the opposite side.

SOLUTION.—Divide the side opposite the vertex through which the lines are to be run into the required number of equal parts and run lines from the vertex of the triangle to the points of division. Then, since the triangles thus formed have equal bases and their vertexes in the same point, they are equivalent. Ans.

EXAMPLE 4.—Divide a given triangle into parts proportional to any given numbers by lines run through a vertex.

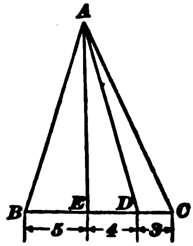


FIG. 33

SOLUTION.—Let the given triangle be ABC, Fig. 33, and let it be required to divide it into parts proportional to 3, 4, and 5, by lines drawn from the vertex A.

The base BC is divided into parts proportional to the numbers 3, 4, and 5, by dividing it into $3 + 4 + 5 = 12$ equal parts, and then marking the third and the seventh points of division. From the points thus marked, lines are run to the vertex A. Then, by Art. 45,

$$CAD : ADE = 3 : 4$$

and,

$$ADE : ABE = 4 : 5 \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find the area of a square whose side is 5 feet 9 inches. Ans. 33.062 sq. ft.

2. Find the area of a rhombus whose length is 12.5 feet, and whose height is 9.25 feet. Ans. 115.62 sq. ft.

3. One side of a room is 16 feet long; if the floor contains 240 square feet, what is the length of the other side? Ans. 15 ft.

4. In a trapezium two not adjacent sides are 16 and 14 inches, respectively. A diagonal divides the trapezium into two triangles whose altitudes from their vertexes to the given sides as bases are 17 inches and 3 inches, respectively; what is the area of the trapezium? Ans. 157 sq. in.

5. The base BC of a triangle is 150 chains and the perpendicular from the opposite vertex A to BC is 45 chains; it is desired to divide the triangle into two parts equal in area by a line from A to BC; how far from B is D, the intersection of this line with BC? Ans. 75 ch.

6. From the mid-point E of the side AB of a parallelogram $ABCD$, lines are drawn to the vertexes D and C and to the mid-point of the side CD; show that these lines divide the parallelogram into four triangles that are equal in area.

7. Find the area of a triangle whose three sides are 13, 14, and 15 feet. Ans. 84 sq. ft.

8. Find the area of a right triangle whose hypotenuse is 50 feet and one of whose legs is 40 feet. Ans. 600 sq. ft.

---

50. Area of a Trapezoid.—The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude; that is, in Fig. 34, area of trapezoid $ABCD = \frac{1}{2}(AB + DC) \times MN$.

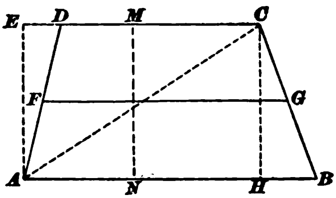


FIG. 34

The area of the trapezoid is equal to the sum of the areas of the two triangles ABC and ADC; hence,

$$\begin{aligned} ABCD &= \tfrac{1}{2} AB \times CH + \tfrac{1}{2} DC \times AE \\ &= \tfrac{1}{2} AB \times MN + \tfrac{1}{2} DC \times MN \\ &= \tfrac{1}{2}(AB + DC) \times MN \end{aligned}$$

Let b_1 = length of lower base;
b_2 = length of upper base;
h = altitude.

Then, the area A of the trapezoid $ABCD$ is

$$A = \frac{1}{2}(b_1 + b_2)h$$

51. Since the median line FG, Fig. 34, joining the mid-points of the non-parallel sides is equal to $\frac{1}{2}(AB + DC)$, the area of a trapezoid is equal to the product of the median line by the altitude.

EXAMPLE.—Divide a plot of ground in the form of a trapezoid into any number of equal parts by lines intersecting the two bases.

SOLUTION.—Divide each of the bases into the same number of equal parts into which the trapezoid is to be divided and run lines through the corresponding points of division. The trapezoids thus formed have equal bases and the same altitude and are, therefore, equal in area. Ans.

EXAMPLES FOR PRACTICE

1. The parallel sides of a trapezoid are 321.51 and 214.24 feet, and the perpendicular distance between them is 171.16 feet; what is the area of the trapezoid? Ans. 45,849 sq. ft.

2. Find the area of a trapezoid whose parallel sides are 20.5 and 12.25 chains, the perpendicular distance between them being 10.75 chains. Ans. 17.603 A.

3. The parallel sides of a trapezoidal plot of ground are 400 feet and 360 feet long; the distance between the parallel sides is 100 feet. It is desired to divide this plot into five lots by lines intersecting the parallel sides; what will be the length of the front and the rear of one of the lots? Ans. 80 ft. and 72 ft.

4. How many square feet are there in a board 12 feet long, 18 inches wide at one end, and 12 inches wide at the other end? Ans. 15 sq. ft.

---

52. Area of Any Polygon.—The area of any polygon can be found by dividing the polygon into triangles, determining the area of each triangle, and adding the results.

53. Comparison of the Areas of Similar Polygons. The areas of two similar triangles are to each other as the squares of their homologous sides.

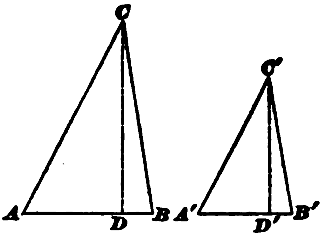


FIG. 35

In Fig. 35,

Area $ABC = \frac{1}{2} AB \times CD$ (1)

Area $A'B'C' = \frac{1}{2} A'B' \times C'D'$ (2)

Dividing equation (1) by equation (2), $\frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{CD}{C'D'}$ (3)

but, by Art. 22, $\frac{CD}{C'D'} = \frac{AB}{A'B'}$;

hence, substituting in (3)

$$\frac{AB}{A'B'} \text{ for } \frac{CD}{C'D'},\ \frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$$

that is, $ABC : A'B'C' = \overline{AB}^2 : \overline{A'B'}^2$

54. The areas of two similar triangles are to each other as the squares of any two homologous lines.

55. The areas of two similar polygons are to each other as the squares of their homologous lines.

By Art. **33**, two similar polygons can be divided into the same number of similar triangles. The sums of these triangles will, by Art. **35**, be to each other as any triangle of one polygon is to the corresponding triangle of the other. But these triangles are to each other as the squares of any two homologous lines. Hence, the sum of the triangles, or the polygons, are to each other as the squares of any two homologous lines.

EXAMPLE 1.—Divide a given triangle by a line parallel to the base into parts such that the given triangle shall be to the triangle cut off as $m : n$.

SOLUTION.—Let ABC, Fig. 36, be the given triangle, and ADE be the triangle cut off so that $ABC : ADE = m : n$. By Art. **18**, ADE and ABC are similar; hence, by Art. **53**,

$$ABC : ADE = \overline{AB}^2 : \overline{AD}^2$$

But by the conditions of the problem,

$$ABC : ADE = m : n$$

Therefore, $\overline{AB}^2 : \overline{AD}^2 = m : n$;

whence, $AD = AB\sqrt{\frac{n}{m}}$. Ans.

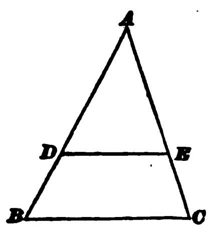


FIG. 36

When the triangle ABC is to be divided into two equal parts,

$$AD = AB\sqrt{\frac{1}{2}} = .70711\, AB$$

EXAMPLE 2.—Let the length AB of example 1 be 32 chains and the area of ABC be 25.6 acres; what is the length of AD, if it is desired to make the triangle ADE contain 15 acres?

SOLUTION.—The area ABC is to be to the area of ADE as 25.6 : 15; hence, $m : n = 25.6 : 15$.

Then, $$AD = 32\sqrt{\frac{15}{25.6}} = 24.495 \text{ ch. Ans.}$$

EXAMPLE 3.—Divide a given triangle ABC, by lines parallel to the base, into n equal parts.

SOLUTION.—Let ABC, Fig. 37, be the triangle, and DE, FG, HI, etc., divide it into n equal parts. Then ADE is one part, AFG is two parts, and so on. Hence,

$$ABC : ADE = n : 1$$
$$ABC : AFG = n : 2; \text{ etc.}$$

Then, by example 1,

$$AD = AB\sqrt{\frac{1}{n}};\ AF = AB\sqrt{\frac{2}{n}};$$

$$AH = AB\sqrt{\frac{3}{n}}; \text{ etc. Ans.}$$

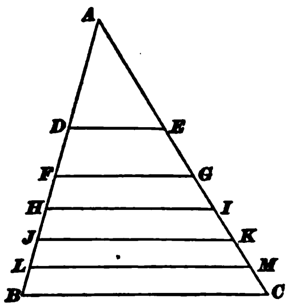


FIG. 37

EXAMPLE 4.—Two triangles ABC and $A'B'C'$ are similar. The sides of the triangle ABC are: $AB = 10$ inches, $BC = 21$ inches, $AC = 17$ inches, and in the triangle $A'B'C'$ the side $B'C' = 42$ inches; what is the area of the triangle ABC?

SOLUTION.—In the triangle ABC, $s = \frac{10 + 17 + 21}{2} = 24$. Then $s - a = 3$, $s - b = 7$, $s - c = 14$, and the area is $\sqrt{24 \times 3 \times 7 \times 14} = 84$ sq. in. By the principle of Art. **53**,

$$\text{area of } A'B'C' : \text{area of } ABC = \overline{B'C'}^2 : \overline{BC}^2;$$

that is, $$\text{area of } A'B'C' : 84 = 42^2 : 21^2$$

But $$42^2 : 21^2 = 4 : 1$$

hence, $$\text{area of } A'B'C' : 84 = 4 : 1$$

whence, $$\text{area of } A'B'C' = 4 \times 84 = 336 \text{ sq. in. Ans.}$$

EXAMPLES FOR PRACTICE

1. Suppose that the sides of the triangle $A'B'C'$ in example 4 of Art. **55** are $A'B' = 20$ inches, $B'C' = 42$ inches, and $C'A' = 34$ inches; show that the answer that is given to the example is correct.

2. The triangles ABC and $A'B'C'$ are similar; being given $BC = 13$ inches, $CA = 14$ inches, $AB = 15$ inches, and $B'C' = 19.5$ inches; find the area of the triangle $A'B'C'$. Ans. 189 sq. in.

3. Let AB, one side of a triangle ABC, be 60 chains long, and let it be required to divide, by lines parallel to BC, the triangle ABC into five equal parts. (*a*) What are the lengths of the lines AD, AF, AH, and AT? (*b*) Let the area of ABC be 120 acres; by means of Art. **53**, prove your results.

Ans. $\begin{cases} AD = 26 \text{ ch. } 83.3 \text{ l.} \\ AF = 37 \text{ ch. } 94.7 \text{ l.} \\ AH = 46 \text{ ch. } 47.6 \text{ l.} \\ AT = 53 \text{ ch. } 66.6 \text{ l.} \end{cases}$

4. Find the lengths of AD and AF when the triangle of example 3 is divided into three parts, whose areas shall be proportional to the numbers 3, 4, and 5.

Ans. $\begin{cases} AD = 30 \text{ ch.} \\ AF = 45 \text{ ch. } 82.6 \text{ l.} \end{cases}$

HINT.—This is the same as if the triangle were divided into 3 + 4 + 5 equal parts and ADE contained three, and AFG, seven of these equal parts.

---

56. The Theorem of Pythagoras.—In any right triangle, the square described on the hypotenuse is equivalent to the sum of the squares described on the other two sides.

Let ABC, Fig. 38, be a right triangle. Draw an equal triangle in the position $CB'C'$, so that CB' will be in the prolongation of BC. Construct the squares $ABDE$ and $B'C'FD$ on AB and $B'C'$, respectively. Since $M + N_1$ ($= M + N$) is a right angle, ACC' is also a right angle. Produce EF to A', making $FA' = BA = DE$. Then, since EF is the difference between DF and DE, or BC and AB, $EA' = BC$. Draw AA' and $C'A'$. Each of the right triangles T_2 and T_3 is equal to T, since their legs are respectively equal. The quadrilateral $ACC'A'$, having all its sides equal and a right angle C, is a square—the square on the hypotenuse AC. This square is equal to the shaded figure plus the sum of the triangles T_2 and T_3; or to the shaded figure plus twice the triangle T. The sum of the squares $ABDE$ and $B'C'FD$ is equal to the shaded figure plus the sum of the triangles T and T_1, or to the shaded figure plus twice the triangle T. Therefore, square $ACC'A' =$ square $ABDE +$ square $B'C'FD$.

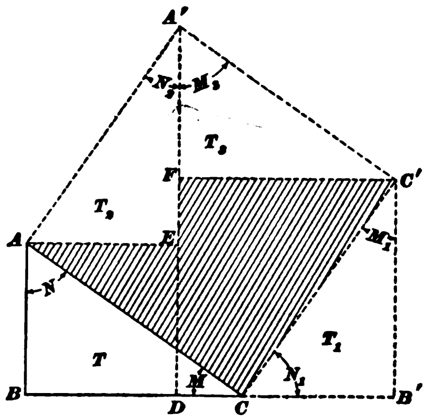


FIG. 38

A particular case of the proposition just proved is shown in Fig. 39.

Let c be the hypotenuse, and a and b the other two sides of any right triangle.

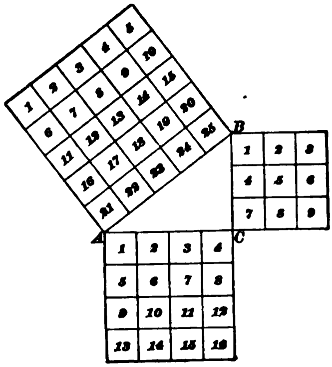


FIG. 39

Then,

$$c^2 = a^2 + b^2 \qquad (1)$$

$$c = \sqrt{a^2 + b^2} \qquad (2)$$

$$a = \sqrt{c^2 - b^2} \qquad (3)$$

Formula **3** may be written

$$a = \sqrt{(c - b)(c + b)} \qquad (4)$$

EXAMPLE 1.—If $AB = 3$ inches and $BC = 4$ inches, what is the length of the hypotenuse AC, Fig. 38?

SOLUTION.—

$$AC = \sqrt{\overline{AB}^2 + \overline{BC}^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ in.} \quad \text{Ans.}$$

EXAMPLE 2.—The side given is 3 inches ($= b$, say), the hypotenuse is 5 inches ($= c$); what is the length of the other side?

SOLUTION.—Applying formula **4**, Art. **56**,

$$a = \sqrt{(5 - 3)(5 + 3)} = \sqrt{16} = 4 \text{ in.} \quad \text{Ans.}$$

Also, $a = \sqrt{c^2 - b^2} = \sqrt{5^2 - 3^2} = 4$ in. Ans.

EXAMPLE 3.—If, from a church steeple that is 150 feet high, a rope is to be attached at the top and to a stake in the ground 85 feet from its foot (the ground being supposed to be level), what must be the length of the rope?

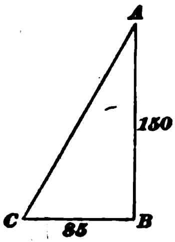


FIG. 40

SOLUTION.—In Fig. 40, AB represents the steeple 150 ft. high; C, a stake 85 ft. from the foot of the steeple; and AC, the rope. Here we have a triangle right-angled at B, of which AC is the hypotenuse. The square of $AC = 85^2 + 150^2 = 7{,}225 + 22{,}500 = 29{,}725$. Therefore,

$$AC = \sqrt{29{,}725} = 172.4 \text{ ft., nearly.} \quad \text{Ans.}$$

EXAMPLE 4.—Referring to Fig. 16, it is required to find the length of the post AB and that of the member BC.

SOLUTION.—Draw BK parallel to ED. Then, $BK = ED =$ **16 ft.** and $CK = CD - DK = CD - EB = 15 - 12 = 3$ ft. The right triangles AEB and BCK give

$$AB = \sqrt{\overline{AE}^2 + \overline{EB}^2} = \sqrt{16^2 + 12^2} = \sqrt{400} = 20 \text{ ft. Ans.}$$

$$BC = \sqrt{\overline{BK}^2 + \overline{CK}^2} = \sqrt{16^2 + 3^2} + \sqrt{265} = 16.279 \text{ ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. If the two sides about the right angle in a right triangle are 52 and 39 feet long, how long is the hypotenuse? Ans. 65 ft.

2. A ladder 65 feet long reaches to the top of a house when its foot is 25 feet from the house; how high is the house, supposing the ground to be level? Ans. 60 ft.

3. The shortest distance from a point to a line is 25 inches; the distances from this point to the extremities of the line are 54 inches and 40 inches, respectively; what is the length of the line?
Ans. 79.08 in.

4. Show that the diagonal of a square is equal to the side multiplied by $\sqrt{2}$.

REGULAR POLYGONS

57. A **regular** polygon is a polygon that has equal sides and equal angles, that is, it is equilateral and equiangular.

58. A circle can be circumscribed about any regular polygon.

Take any three vertexes of the regular polygon $ABCDE$, Fig. 41, as the vertexes A, B, C, and pass a circle through them. Let O be the center of this circle. Join O to A, B, C, D, and E. The polygon being equiangular, the angle ABC = angle BCD. The angles OCB and OBC, being opposite equal sides OC and OB of the triangle OBC, are equal. Hence,

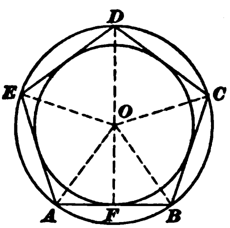


FIG. 41

$$ABC - OBC = BCD - QCB$$

or, $$ABO = OCD$$

The polygon being equilateral, the sides AB and CD are equal. Hence, the triangles AOB and OCD, having two sides and included angle of one equal to two sides and included angle of the other equal, are equal. Therefore, $OD = OA$, and a circle passing through A, *B*, and *C* must pass through *D*. In like manner, it can be shown that the circle passes through *E*.

59. A circle can be inscribed in any regular polygon.

In Fig. 41, *O A*, *O B*, *O C*, *O D*, and *O E*, being radii of the circumscribed circle, are equal and divide the polygon into equal isosceles triangles that have a common vertex *O*. The altitudes of these equal triangles are equal, hence the perpendicular distances, as *O F*, from *O* to each of the sides are the same. Therefore, a circle drawn with *O* as center and a radius equal to *O F* will be inscribed in the regular polygon.

60. The **center** of a regular polygon is the common center of the circumscribed and the inscribed circle.

61. The **radius** of a regular polygon is the radius of the circumscribed circle, as *O A*, Fig. 41.

62. The **apothem** of a regular polygon is the radius of the inscribed circle, as *O F*, Fig. 41.

63. The **angle at the center** of a regular polygon is the angle included by the radii drawn to the extremities of any side.

64. The angle at the center of any regular polygon is equal to four right angles, or 360°, divided by the number of the sides.

65. If n is the number of sides of a regular polygon, the sum of its interior angles is $2(n-2)$ right angles (see *Geometry*, Part 1), or, $90° \times 2(n-2) = 180° \times (n-2)$, and, since all the angles are equal, each angle is equal to $\frac{180° \times (n-2)}{n} = 180° - \frac{360°}{n}$. Since this value depends only on the number of sides, all regular polygons of the same number of sides have the same angles.

66. Regular polygons of the same number of sides are similar; their perimeters are to each other as any two homologous lines, and their areas are to each other as the squares of any two homologous lines.

67. The area of a regular polygon is equal to one-half the product of the perimeter and the apothem.

Let l be the side MN of a regular polygon, Fig. 42, n the number of sides, $p(= nl)$ the perimeter, $a(= OF)$ the apothem, and A the area. As A is equal to the sum of n triangles, each equal to MON, we have,

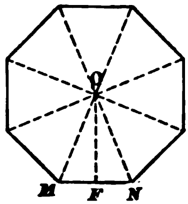


FIG. 42

$$A = (\tfrac{1}{2} MN \times OF) \times n = \tfrac{1}{2} la \times n = \tfrac{1}{2} nl \times a,$$

or,

$$A = \tfrac{1}{2} pa$$

EXAMPLE.—Find the area of a regular pentagon whose side is 25 feet and apothem is 17.2 feet.

SOLUTION.—The figure is a pentagon, hence it has five sides. The perimeter is 5×25 and the area is $\frac{5 \times 25 \times 17.2}{2} = 1{,}075$ sq. ft. Ans.

68. The areas of regular polygons each of whose sides is equal to 1 are given in the following table:

TABLE I

AREAS OF REGULAR POLYGONS

Name	Number of Sides	Area When Side = 1	Name	Number of Sides	Area When Side = 1
Triangle .	3	.4330	Octagon .	8	4.8284
Square . .	4	1.0000	Nonagon .	9	6.1818
Pentagon .	5	1.7205	Decagon .	10	7.6942
Hexagon[1] .	6	2.5981	Undecagon	11	9.3656
Heptagon .	7	3.6339	Dodecagon	12	11.1960

From the principle of Art. **55**, the following rule is derived:

Rule.—*To find the area of any regular polygon, square the length of a side and multiply by the area of the similar polygon whose side is equal to the unit of length.*

Let A = area; l = length of side of required polygon; a = area of similar polygon whose side is 1; then, by Art. **55**,

$$A : a = l^2 : 1^2$$

whence,

$$A = a l^2$$

EXAMPLE.—The side of a regular octagon is 3 inches, find its area.

SOLUTION.—From the table, the area of a regular octagon whose side is 1 in. is 4.8284 sq. in. Hence, the area of the octagon whose side is 3 in. is $4.8284 \times 3^2 = 43.456$ sq. in. Ans.

69. If the vertexes of a regular inscribed polygon are joined to the middle points of the arcs subtended by the sides of the polygon, the joining lines form a regular inscribed polygon of double the number of sides. Thus, the octagon *A F B G*, etc., Fig. 43, is formed by joining the middle points of the arcs subtended by the sides of the square *A B C D*.

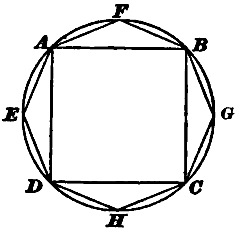


Fig. 43

70. If tangents are drawn at the middle points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed. Thus, in Fig. 44, the octagon *E F G H*, etc., is formed by drawing tangents at the middle points of the arcs between adjacent points of contact of the sides of the circumscribed square *A B C D*.

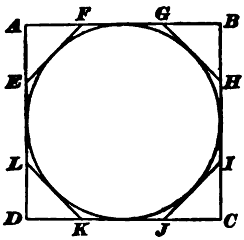


Fig. 44

CIRCULAR MEASUREMENTS

THE CIRCLE

LENGTH OF ANY ARC

71. If any two circles are taken, and two regular polygons of the same number of sides are inscribed in them, the perimeters of these polygons are to each other as the radii of the circles (Art. **66**). This relation holds whatever the number of sides of the polygon. Now, it is evident that, as this number increases, the perimeters of the two polygons approach the circumferences of their respective circles. We may, therefore, consider these circumferences as extreme cases of the perimeters of regular polygons, in which the number of sides is increased indefinitely; whence we conclude that the circumferences, also, are to each other as their radii.

If c and c' are the circumferences of any two circles, and r and r' their respective radii, we may write,

$$c : c' = r : r'$$

whence,
$$c : r = c' : r'$$

or,
$$\frac{c}{r} = \frac{c'}{r'}$$

Dividing both numbers by 2, and denoting the diameters by d and d',

$$\frac{c}{2r} = \frac{c'}{2r'}$$

that is,
$$\frac{c}{d} = \frac{c'}{d'}$$

As c and c' are any two circumferences, it is seen that the ratio obtained by dividing any circumference by its diameter is the same for all circumferences. This ratio is usually denoted by the Greek letter π (pronounced *pi*). We have, therefore, for any circle,

$$\frac{c}{d} = \pi$$

whence, $$c = \pi d = 2 \pi r$$

72. The quantity π can be determined by elementary geometrical methods, which may be found in treatises on geometry; but these methods are very laborious. A much better method is afforded by the theory of series, which is treated in works on trigonometry and the differential calculus. It is found that π cannot be expressed as an exact fraction, either decimal or vulgar. Its value can, however, be calculated to any desired degree of approximation. The following value is approximate to fifteen decimal places:

$$\pi = 3.141592653589793+$$

For nearly all practical purposes, 3.1416 is a sufficiently close value. This value is used very generally, and will be used in this Course, unless otherwise stated. The student should commit it to memory. A value that is often used in rough calculations is $\frac{22}{7}$; it can be used when no more than three significant figures are required in the result.

73. The length of an arc, when the number of degrees in the arc and the radius of the circle are given, may be found as follows:

The length of the arc is evidently the same part of the length of the circumference ($2 \pi r$) as the number of degrees in the arc is of the number of degrees in the whole circumference, or 360°. Thus, if n is the number of degrees in the arc, and l is its length, we shall have,

$$\frac{2 \pi r}{l} = \frac{360}{n}$$

whence, $$l = \frac{\pi r n}{180}$$

In applying this formula, minutes and seconds should be expressed as fractions of a degree.

EXAMPLE 1.—Find the length of a rope that will go around a wheel or drum 7.5 feet in diameter.

SOLUTION.—The required length is equal to the length c of the circumference of the wheel or drum. Here $d = 7.5$ ft., and, taking $\pi = 3.1416$, we have, by formula of Art. **71**,

$$c = 3.1416 \times 7.5 = 23.562 \text{ ft. Ans.}$$

Using $\frac{22}{7}$ for π, the result, to three significant figures, is

$$c = \frac{22}{7} \times 7.5 = 23.6 \text{ ft. Ans.}$$

EXAMPLE 2.—Find the diameter of a circular race track 1 mile in length.

SOLUTION.—Here c is given (= 1 mi. = 5,280 ft.) and the quantity required is d. From the formula $c = \pi d$, we get

$$d = \frac{c}{\pi} = \frac{5,280}{3.1416} = 1,680.7 \text{ ft. Ans.}$$

EXAMPLE 3.—What is the length of a railroad circular curve having a radius of 1,540 feet and subtending an angle at the center equal to 26° 35'?

SOLUTION.—To apply formula of Art. **73**, we have $r = 1,540$ ft., $n = 26\frac{35}{60}° = 26.583°$, nearly. Therefore,

$$l = \frac{3.1416 \times 1,540 \times 26.583}{180} = 714.50 \text{ ft. Ans.}$$

74. When only the chord AB, Fig. 45, of an arc and the height, or "rise," CD of the segment are known, the following approximate method gives good results. AC, the chord of half the arc, has the value

$$AC = \sqrt{\overline{AD}^2 + \overline{CD}^2} = \sqrt{\left(\frac{AB}{2}\right)^2 + \overline{CD}^2}$$

Then, to find the length of the arc:

Rule.—*From eight times the chord of half the arc, subtract the chord of the whole arc and divide the remainder by 3.*

That is,

$$\text{arc } ACB = \frac{8 \times AC - AB}{3}$$

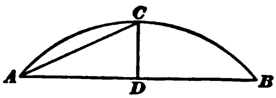


FIG. 45

Let c = chord of whole arc;
h = height of segment;
l = length of arc.

Then, $$AC = \sqrt{\frac{c^2}{4} + h^2} = \frac{1}{2}\sqrt{c^2 + 4h^2}$$

and $$l = \frac{4\sqrt{c^2 + 4h^2} - c}{3}$$

This formula gives the length of an arc less than one-sixth of the circumference correct to four figures, and it gives the length of an arc less than one-third of the circumference correct to three figures.

EXAMPLE.—Find the length of the arc $A\,C\,B$, Fig. 46.

SOLUTION.—In this example, $c = 72$, $h = 8$. Therefore,

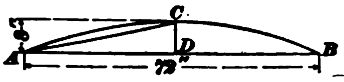


FIG. 46

$$l = \frac{4\sqrt{72^2 + 4 \times 8^2} - 72}{3} = 74.34 \text{ in.} \quad \text{Ans.}$$

75. For very flat arcs, that is, when $\frac{h}{c}$ is very small (say not greater than .1), the following approximate formula may be used, the notation being the same as in the preceding article:

$$l = c + \frac{8\,h^2}{3\,c}$$

EXAMPLE 1.—Find the length of the arc $A\,B$, Fig. 46.

SOLUTION.—

$$l = 72 + \frac{8 \times 8^2}{3 \times 72} = 72 + 2.37 = 74.37. \quad \text{Ans.}$$

This is not a very close approximation, because the ratio $\frac{c}{h}\left(= \frac{8}{72} = \frac{1}{9}\right)$ is not very small; however, the approximate value thus found would be close enough for most practical purposes.

EXAMPLE 2.—The chord of a railroad curve is 675 feet long, and the rise (or, "middle ordinate," as the rise is called in railroad work) is 40 feet; what is the length of the curve?

SOLUTION.—Here $c = 675$, $h = 40$, and therefore

$$l = 675 + \frac{8 \times 40^2}{3 \times 675} = 675 + 6.32 = 681.32 \text{ ft.} \quad \text{Ans.}$$

76. Circular Measure of an Angle.—The following equation follows from the formula of Art. **73**:

$$\frac{l}{r} = \frac{\pi n}{180} = \frac{\pi}{180} \times n$$

If we assume the radius to be 1, then

$$l = \frac{\pi}{180} \times n \qquad (1)$$

This equation gives the length of the arc that the angle subtends on a circle whose radius is equal to unity. The length of such arc is called the **circular measure** of the angle, and the angle is often referred to by stating that measure. Thus, an angle of 1.34, circular measure, means an angle that subtends an arc of length 1.34 on a circle whose radius is 1. An angle expressed in circular measure is also said to be expressed **in radians.**

If in equation 1 we make $n = 180°$, we obtain, for the circular measure of 180°, $l = \pi$, that is, 180° is equivalent to π radians. Likewise, 90° is equivalent to $\frac{\pi}{2}$ radians, etc.

EXAMPLES FOR PRACTICE

1. Find the distance around the outside of a waterwheel whose outside diameter is 22 feet 8 inches. Ans. 71.21 ft.

2. The wheel of a carriage is observed to turn 375 times in going from a certain place to another; the diameter of the wheel is 3.5 feet; what is the distance between the two places? Ans. 4,123.4 ft.

3. A circular column measures 45.5 inches around the outside; what is its diameter? Ans. 14.483 in.

4. A belt covers an arc of 50° on a pulley whose diameter is 5 feet; what length of the belt is in contact with the pulley? Ans. 2.1817 ft.

5. How long will it take a train to move over a curve subtending an angle of 100°, the radius of the curve being 1,800 feet, and the train going at the rate of 20 miles an hour? Ans. 1.79 min.

6. The length of arc of a circle is equal to the radius; find the number of degrees in the arc. Ans. 57.3° = 57° 18′, nearly

7. The chord of a railroad curve is 600 feet long and the middle ordinate is 80 feet; what is the length of the curve? Ans. 628 ft.

AREAS BOUNDED BY CIRCULAR ARCS

77. The area of a circle is equal to one-half the product of its circumference and radius (Art. **67**). This at once follows by considering the circle as an extreme case of a regular polygon.

Let A = area of circle;
c = circumference of circle;
r = radius of circle.

Then, $$A = \frac{1}{2} c r$$

or, since $c = 2\pi r$,

$$A = \frac{1}{2} 2\pi r \times r$$

or, simplifying,

$$A = \pi r^2 = 3.1416 r^2 \qquad (1)$$

Writing $\frac{d}{2}$ for r, we obtain for the area in terms of the diameter,

$$A = \frac{\pi d^2}{4} = .7854 d^2 \qquad (2)$$

These formulas serve likewise to find r or d when A is given. Since $2\pi r = c$, we have

$$r = \frac{c}{2\pi}, \text{ and } \pi r^2 = \pi\left(\frac{c}{2\pi}\right)^2 = \frac{c^2}{4\pi}$$

that is, $$A = \frac{c^2}{4\pi} \qquad (3)$$

This formula gives the area of a circle when its circumference is known.

EXAMPLE 1.—The steam pressure on a piston is 75 pounds per square inch, and the diameter of the piston is 15 inches; what is the pressure on the whole surface of the piston?

SOLUTION.—The required pressure is evidently seventy-five times the number of square inches in the surface of the piston, or seventy-five times the area A of the piston. Here $d = 15$ in., and formula **2** gives

$$A = .7854 \times 15^2$$

whence the total pressure is

$$75 \times .7854 \times 15^2 = 13{,}254 \text{ lb. Ans.}$$

EXAMPLE 2.—The distance around a circular park is 2.75 miles; what is the area of the park, in acres?

SOLUTION.—Here c is given equal to 2.75 mi. = (2.75×80) ch. Therefore, the area of the park, in square chains, is (formula **3**)

$$\frac{(2.75 \times 80)^2}{4 \times 3.1416}$$

The area, in acres, is one-tenth of this, or

$$\frac{1}{10} \times \frac{(2.75 \times 80)^2}{4 \times 3.1416} = \frac{220^2}{125.664} = 385.15 \text{ A. Ans.}$$

EXAMPLE 3.—What must be the diameter of a circular sewer pipe that its cross-section may be 12.75 square feet?

SOLUTION.—Solving formula **2** for d,

$$d = \sqrt{\frac{A}{.7854}} = \sqrt{\frac{12.75}{.7854}} = 4.03 \text{ ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. The cable of a suspension bridge measures 40 inches around its circumference; find: (*a*) the diameter d of the cable; (*b*) the area A of the cross-section.

Ans. $\begin{cases} (a)\ d = 12.732 \text{ in.} \\ (b)\ A = 127.32 \text{ sq. in.} \end{cases}$

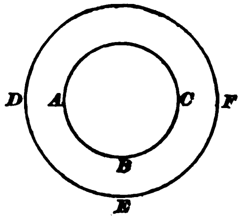


FIG. 47

2. Find a formula for the area A of the space enclosed between two circles ABC and DEF, Fig. 47, the diameter of the outer circle being D, and that of the inner circle d.

Ans. $\begin{cases} A = \frac{\pi}{4}(D^2 - d^2) \\ A = \frac{\pi}{4}(D + d)(D - d) \end{cases}$

3. What must be the inner diameter of a circular chimney, that its inner cross-section may be 14 square feet? Ans. 4.2221 ft.

4. The diameter of a circular airway of a mine is 10 feet; find: (*a*) the circumference c; (*b*) the area A of the cross-section.

Ans. $\begin{cases} (a)\ c = 31.416 \text{ ft.} \\ (b)\ A = 78.54 \text{ sq. ft.} \end{cases}$

---

78. A sector is the same part of a circle as its arc is of the circumference.

Let A = area of circle;
A' = area of sector;
n = number of degrees in arc of sector.

Then, $$A' : A = n : 360$$

whence, $$A' = \frac{nA}{360} = \frac{\pi r^2 n}{360}$$

EXAMPLE.—The angle of a sector of a circle is 75°; the diameter of the circle is 12 inches; what is the area of the sector?

SOLUTION.—The area A of the circle is $12^2 \times .7854$ sq. in. Then the area of the sector is

$$\frac{nA}{360} = \frac{75 \times 12^2 \times .7854}{360} = 23.562 \text{ sq. in. Ans.}$$

79. The area of a sector is equal to one-half the product of its base by the radius of the circle.

$$A' = \tfrac{1}{2} r l$$

If l is the length of the arc, or **base**, of a sector, we have (Art. **73**),

$$l = \frac{\pi r n}{180}.$$

whence,
$$n = \frac{180\, l}{\pi r}$$

This value of n substituted in formula of Art. **78** gives

$$A' = \frac{\pi r^2}{360} \times \frac{180\, l}{\pi r}.$$

or, reducing,
$$A' = \tfrac{1}{2} r l$$

EXAMPLE.—If the radius of an arc is 5 feet and the length of the arc is 4 feet, what is the area of the sector?

SOLUTION.—By formula of Art. **79**,

$$A' = \frac{l r}{2} = \frac{4 \times 5}{2} = 10 \text{ sq. ft. Ans.}$$

80. The area of a segment, as ADB, Fig. 48, is evidently equal to the area of the sector $AOBD$ minus the area of the triangle AOB.

EXAMPLE 1.—The diameter of a circle is 10 inches, and the chord of the arc of a segment is 7 inches; what is the area of the segment?

SOLUTION.—In Fig. 48, let AB = 7 in. and the diameter = 10 in. Then, OB = 5 in., and CB = 3.5 in. Hence, $OC = \sqrt{5^2 - 3.5^2}$ = 3.57 in., and CD = 5 − 3.57 = 1.43 in. Then, by formula of Art. **74**, arc $ADB = \frac{4\sqrt{7^2 + 4 \times 1.43^2} - 7}{3}$ = 7.75 in. Hence, area of sector $AOBD = \frac{1}{2} \times 5 \times 7.75$ = 19.38 sq. in. The area of the triangle $AOB = \frac{1}{2} \times 3.57 \times 7$ = 12.50 sq. in. Therefore, the area of the segment is 19.38 − 12.50 = 6.88 sq. in. Ans.

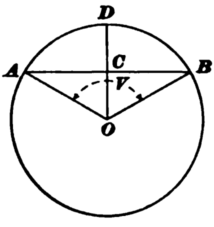


FIG. 48

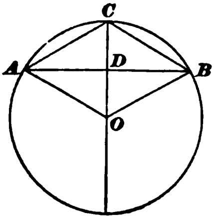


FIG. 49

EXAMPLE 2.—The chord of the arc of a segment is 79 inches and the height of the segment is 20 inches; find the area of the segment.

SOLUTION.—Let $ACBE$, Fig. 49, be the circle; let AB = 79 in. and CD = 20 in. Then, $AD = \frac{1}{2} \times 79$ in. = 39.5 in. By Art. **28**,

$$CD : AD = AD : DE$$

or,
$$20 : 39.5 = 39.5 : DE$$

whence,
$$DE = 78.01$$

Hence, the diameter = 20 + 78.01 = 98.01 in., and the radius = 49. Then the arc $ACB = \frac{4\sqrt{79^2 + 4 \times 20^2} - 79}{3} = 91.7$ in. Hence, the area of sector $AOBC = 91.7 \times \frac{1}{2} \times 49 = 2{,}246.65$ sq. in. The area of the triangle $AOB = \frac{1}{2} \times 79 \times 29 = 1{,}145.5$ sq. in. Therefore, the area of the segment = 2,246.65 − 1,145.50 = 1,101.15 sq. in. Ans.

THE ELLIPSE

81. An **ellipse** is a plane figure bounded by a curved line such that the sum of the distances of any point on that line from two fixed points within is always equal to the length of the line passing through the fixed points and terminating at both ends in the curved line.

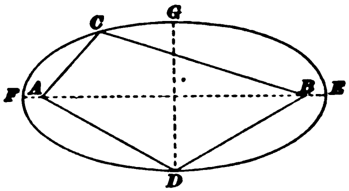


FIG. 50

In Fig. 50, the fixed points are A and B, and if C and D are any two points on the curve, $AC + CB = AD + DB = FE$. The two fixed points are the **foci**. The line FE through the foci is the **transverse, or major, axis.** The line GD, which is the perpendicular bisector of FE, is the **conjugate**, or **minor, axis.** The foci may be located from G or D as a center by striking arcs with a radius equal to one-half FE.

82. There is no simple and exact method of finding the periphery (perimeter) of an ellipse. The following formula gives values very nearly exact:

Let C = periphery;
a = half the major axis;
b = half the minor axis;
$D = \frac{a - b}{a + b}$.

Then, $$C = \pi(a + b)\frac{64 - 3D^4}{64 - 16D^2}$$

EXAMPLE.—What is the periphery of an ellipse whose axes are 10 inches and 4 inches long?

SOLUTION.—In this example, $a = 5$, $b = 2$, $D = \frac{5-2}{5+2} = \frac{3}{7}$.

Then, $C = 3.1416(5+2)\frac{64 - 3(\frac{3}{7})^4}{64 - 16(\frac{3}{7})^2} = 23.013$

Therefore, the periphery is 23.013 in. Ans.

83. The area of an ellipse is equal to the product of its two semiaxes multiplied by π.

Let a = half the major axis;
b = half the minor axis;
A = area.

Then, $A = \pi a b = 3.1416 a b$

EXAMPLE.—What is the area of an ellipse whose axes are 10 inches and 6 inches?

SOLUTION.—Here, $a = \frac{1}{2} \times 10 = 5$, $b = \frac{1}{2} \times 6 = 3$.
Then, $A = 3.1416 \times 5 \times 3 = 47.124$
Therefore, the area is 47.12 sq. in. Ans.

EXAMPLES FOR PRACTICE

1. The number of degrees in the angle formed by drawing radii from the center of a circle to the extremities of an arc of the circle is 84; the diameter of the circle is 17 inches; what is the area of the sector? Ans. 52.96 sq. in.

2. Given the chord of the arc of a segment equal to 24 inches, and the height of the segment equal to 6.5 inches, find: (*a*) the diameter of the circle; (*b*) the area of the segment. Ans. { (*a*) 28.7 in. (*b*) 109.5 sq. in.

3. (*a*) What is the perimeter of an ellipse whose axes are 15 inches and 9 inches? (*b*) What is the area? Ans. { (*a*) 38.29 in. (*b*) 106.03 sq. in.

4. The base of a sector is 24 inches and the diameter of the circle is 54 inches; what is the area of the sector? Ans. 324 sq. in.

THE MENSURATION OF SOLIDS

84. A **solid**, or body, has three dimensions: length. breadth, and thickness.

85. The **entire area** of a solid is the area of the whole outside of the solid.

The **convex area** of a solid having one or two flat ends is the same as the entire surface, except that the areas of the ends or bases are not included.

86. The **volume** of a solid is expressed by the number of times that it will contain another volume, called the unit of volume. Instead of the word *volume*, the expression **cubical contents** is frequently used.

THE PRISM AND CYLINDER

87. A **prism** is a solid whose ends are equal polygons in parallel planes, and whose sides are parallelograms.

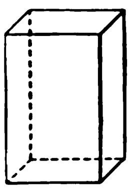


FIG. 51

88. A **parallelopipedon**, Fig. 51, is a prism whose bases (ends) are parallelograms.

89. A **cube**, Fig. 52, is a parallelopipedon whose faces and ends are squares.

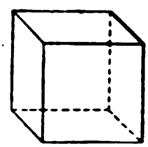


FIG. 52

90. The cube whose edges are equal to the unit of length is taken as the unit of volume when finding the volume of a solid.

Thus, if the unit of length is 1 inch, the unit of volume will be the cube each of whose edges measures 1 inch, or 1 cubic inch; and the number of cubic inches the solid contains will be its volume. If the unit of length is 1 foot, the unit of volume will be 1 cubic foot, etc. Cubic inch, cubic foot, and cubic yard are abbreviated to cu. in., cu. ft., and cu. yd., respectively.

91. Prisms take their names from their bases. Thus, a *triangular prism* is one whose bases are triangles; a *pentagonal prism* is one whose bases are pentagons, etc.

92. A **cylinder**, Fig. 53, is a round body of uniform diameter with circles for its ends.

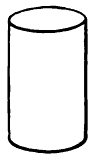


FIG. 53

93. A **right prism, or right cylinder,** is one whose center line (axis) is perpendicular to its bases.

94. The **altitude** of a prism or cylinder is the perpendicular distance between its two ends.

95. To find the convex area of any right prism, or right cylinder:

Rule.—*Multiply the perimeter of the base by the altitude.*

Let p = perimeter of base;
h = altitude;
c = convex area.

Then, $$c = p\,h$$

EXAMPLE 1.—What is the convex area of a right prism whose base is a square, one side of which is 9 inches, and whose altitude is 16 inches?

SOLUTION.— $9 \times 4 = 36$ in., the perimeter of the base. Applying formula of Art. **95**,

$c = 36 \times 16 = 576$ sq. in., the convex area. Ans.

To find the entire area, add the areas of the two ends to the convex area.

EXAMPLE 2.—What is the entire area of the parallelopipedon mentioned in the last question?

SOLUTION.—The area of one end is 9^2 = 81 sq. in. 81 × 2 = 162 sq. in., is the area of both ends. 576 + 162 = 738 sq. in., the entire area of the parallelopipedon. Ans.

EXAMPLE 3.—What is the entire area of a right cylinder whose base is 16 inches in diameter, and whose altitude is 24 inches?

SOLUTION.— 16 × 3.1416 = 50.27 in., or the perimeter (circumference) of the base. 50.27 × 24 = 1,206.48 sq. in., the convex area.

16^2 × .7854 × 2 = 402.12 sq. in., the area of the ends.

1,206.48 + 402.12 = 1,608.6 sq. in., the entire area. Ans.

96. To find the volume of a prism, or cylinder:

Rule.—*The volume of any prism or cylinder is equal to the area of the base multiplied by the altitude.*

Let A = area of base;
h = altitude;
V = volume.

Then, $$V = A h$$

If the given prism is a cube, the three dimensions are all equal, and the volume equals the cube of one of the edges. Hence, if the volume is given, the length of an edge is found by extracting the cube root.

If the volume and the area of the base are given, the altitude is $h = \frac{V}{A}$. If the cylinder or prism is hollow, the volume is equal to the area of the ring or base multiplied by the altitude.

EXAMPLE 1.—What is the volume of a rectangular prism whose base is 6 inches by 4 inches, and whose altitude is 12 inches?

SOLUTION.—The base of a rectangular prism is a rectangle; hence, 6 × 4 = 24 sq. in., the area of the base. Applying formula of Art. **96**,

V = 24 × 12 = 288 cu. in., or the volume. Ans.

EXAMPLE 2.—What is the volume of a cube whose edge is 9 inches?

SOLUTION.— 9^3 = 9 × 9 × 9 = 729 cu. in., the volume. Ans.

EXAMPLE 3.—What is the volume of a cylinder whose base is 7 inches in diameter, and whose altitude is 11 inches?

SOLUTION.— 7^2 × .7854 = 38.48 sq. in., the area of the base. Applying formula of Art. **96**,

V = 38.48 × 11 = 423.28 cu. in., the volume. Ans.

THE PYRAMID AND CONE

97. A **pyramid**, Fig. 54, is a solid whose base is a polygon, and whose sides are triangles uniting at a common point, called the **vertex.** If the base is a regular polygon, and the sides have the same inclination to the base, the pyramid is a **regular pyramid.**

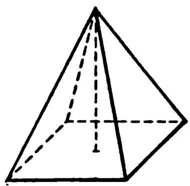


FIG. 54

98. A **cone**, Fig. 55, is a solid whose base is a circle, and whose convex surface tapers uniformly to a point called the **vertex.**

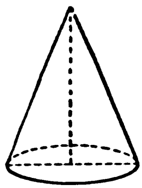


FIG. 55

99. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the base.

100. The **slant height** of a regular pyramid is a line drawn from the vertex perpendicular to one of the sides of the base. The slant height of a cone is a straight line drawn from the vertex to the circumference of the base, and lying on the surface of the cone.

101. To find the convex area of a regular pyramid or a cone:

Rule.—*The convex area of a regular pyramid or of a cone is equal to the perimeter of the base multiplied by one-half the slant height.*

Let p = perimeter;
s = slant height;
c = convex area.

Then, $$c = \frac{ps}{2}$$

EXAMPLE 1.—What is the convex area of a regular pentagonal pyramid, if each side of the base measures 6 inches and the slant height measures 14 inches?

SOLUTION.—The base of the pentagonal pyramid is a pentagon, and consequently it has five sides. $6 \times 5 = 30$ in., or the perimeter of the base. Applying formula of Art. **101**,

$$c = \frac{ps}{2} = \frac{30 \times 14}{2} = 210 \text{ sq. in., the convex area. Ans.}$$

EXAMPLE 2.—What is the entire area of a cone whose altitude is 15 inches, and whose base is 16 inches in diameter?

SOLUTION.—The slant height of the cone is the hypotenuse of a right triangle whose legs are the radius of the base and altitude of the cone, respectively. Therefore, the slant height is equal to $\sqrt{15^2 + 8^2} = 17$ in. (Art. **56**). The perimeter of the base is $16 \times 3.1416 = 50.2656$ in. Applying formula of Art. **101**,

$$c = \frac{50.2656 \times 17}{2} = 427.26 \text{ sq. in.}$$

The area of the base is $16^2 \times .7854 = 201.06$ sq. in. The entire area is, therefore, $427.26 + 201.06 = 628.32$ sq. in. Ans.

102. To find the volume of any pyramid or cone:

Rule.—*The volume of any pyramid or cone equals the area of the base multiplied by one-third of the altitude.*

Let A = area of base;
h = altitude;
V = volume.

Then,
$$V = \frac{Ah}{3}$$

EXAMPLE 1.—What is the volume of a triangular pyramid, each edge of whose base measures 6 inches, and whose altitude is 8 inches?

SOLUTION.—The base is an equilateral triangle; hence, applying the rule of Art. **68**, the area is $6^2 \times .433 = 15.59$ sq. in. Applying formula of Art. **102**,

$$V = \frac{Ah}{3} = \frac{15.59 \times 8}{3} = 41.57 \text{ cu. in. Ans.}$$

EXAMPLE 2.—What is the volume of a cone whose altitude is 18 inches, and whose base is 14 inches in diameter?

SOLUTION.— $14^2 \times .7854 = 153.94$ sq. in., the area of the base. Applying formula of Art. **102**,

$$V = \frac{Ah}{3} = \frac{153.94 \times 18}{3} = 923.64 \text{ cu. in., the volume. Ans.}$$

103. It has been stated that the volume of a cone or a pyramid is equal to one-third the product of the area of the base multiplied by the altitude. Similarly, the volume of any solid whose base is a plane figure and which tapers to a point like a cone or a pyramid is equal to one-third of the product of its base and altitude.

EXAMPLE.—Find the volume of an elliptical cone, whose base is an ellipse with diameters 8 inches and 6 inches, and the altitude is 7.5 inches.

SOLUTION.—The area of the ellipse at the base is $3.1416 \times 4 \times 3$. The volume is equal to one-third the product of the area of the base and altitude; that is,

$$V = \frac{1}{3} \times 3.1416 \times 4 \times 3 \times 7.5 = 94.248$$

Hence, the volume is 94.248 cu. in. Ans.

EXAMPLES FOR PRACTICE

1. Find the volume of a triangular pyramid of which the altitude is 4 inches and the base is an equilateral triangle having each side 3 inches long. Ans. 5.2 cu. in.

2. Find the weight of a steel bar 16 feet long and 2 inches in diameter, the weight of steel being taken as .28 pound per cubic inch. Ans. 168.89 lb.

3. What is the entire area of a hexagonal prism 12 inches long, each side of the base being 1 inch long? Ans. 77.196 sq. in.

4. (*a*) Find the convex area of a cone whose altitude is 12 inches, and the circumference of whose base is 31.416 inches. (*b*) Find the volume of the cone. Ans. { (*a*) 204.2 sq. in. (*b*) 314.16 cu. in.

THE FRUSTUM OF A PYRAMID OR A CONE

104. If a pyramid is cut by a plane parallel to the base, as in Fig. 56, so as to form two parts, the lower part is called a **frustum** of the pyramid.

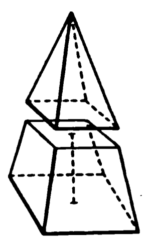


FIG. 56

105. If a cone is cut in a similar manner, as in Fig. 57, the lower part is called a **frustum of the cone**.

106. The upper end of a frustum of a pyramid or cone is called the **upper base**, and the lower end the **lower base**. The **altitude** of a frustum is the perpendicular distance between the bases.

107. To find the convex area of a frustum of a regular pyramid or of a cone:

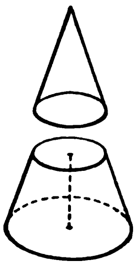


FIG. 57

Rule.—*The convex area of a frustum of a regular pyramid or of a cone equals one-half the sum of the perimeters of its bases multiplied by the slant height of the frustum.*

Let p = perimeter of lower base;
p' = perimeter of upper base;
s = slant height;
c = convex area.

Then, $$c = \left(\frac{p + p'}{2}\right)s$$

EXAMPLE 1.—Given the frustum of a triangular pyramid in which each side of the lower base measures 10 inches, each side of the upper base measures 6 inches, and whose slant height is 9 inches; find the convex area.

SOLUTION.— 10 in. × 3 = 30 in., the perimeter of the lower base. 6 in. × 3 = 18 in., the perimeter of the upper base. Applying formula of Art. **107**,

$c = \left(\frac{p + p'}{2}\right)s = \frac{30 + 18}{2} \times 9 = 216$ sq. in., the convex area. Ans.

EXAMPLE 2.—If the diameters of the two bases of a frustum of a cone are 12 inches and 8 inches, respectively, and the slant height is 12 inches, what is the entire area of the frustum?

SOLUTION.— $\frac{(12 \times 3.1416) + (8 \times 3.1416)}{2} \times 12 = 376.99$ sq. in., the convex area.

$8^2 \times .7854 = 50.27$ sq. in.
$12^2 \times .7854 = 113.1$ sq. in.

113.1 + 50.27 = 163.37 sq. in., the area of the two ends. 376.99 + 163.37 = 540.36 sq. in., the entire area of the frustum. Ans.

108. To find the volume of the frustum of a pyramid or a cone:

Rule.—*Add the areas of the upper base, the lower base, and the square root of the product of the areas of the two bases; multiply this sum by one-third of the altitude.*

Let A = area of lower base;
a = area of upper base;
h = altitude;
V = volume.

Then, $$V = (A + a + \sqrt{A\,a})\frac{h}{3}$$

EXAMPLE 1.—Given a frustum of a hexagonal pyramid in which each edge of the lower base measures 8 inches, and each edge of the upper base measures 5 inches, and whose altitude is 14 inches, what is its volume?

SOLUTION.—A hexagonal pyramid is one whose base is a regular hexagon, as shown in Fig. 58. Hence, applying formula of Art. **68**,

$$A = 8^2 \times 2.5981 = 166.28 \text{ sq. in.}$$

In a similar way, the area of the upper base is found to be 64.95 sq. in. Then, applying formula of Art. **108**,

$$V = (166.28 + 64.95 + \sqrt{166.28 \times 64.95})\frac{14}{3}$$

$$= 335.15 \times \frac{14}{3} = 1{,}564.03 \text{ cu. in., the volume. Ans.}$$

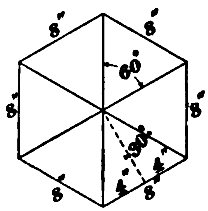


FIG. 58

EXAMPLE 2.—What is the volume of a frustum of a cone whose upper base is 8 inches in diameter, whose lower base is 12 inches in diameter, and whose altitude is 15 inches?

SOLUTION.—The area of the upper base is $8^2 \times .7854 = 50.27$ sq. in. The area of the lower base is $12^2 \times .7854 = 113.1$ sq. in., nearly. The square root of their product is $\sqrt{50.27 \times 113.1} = 75.4$.

Then, $$V = (50.27 + 113.1 + 75.4)\frac{15}{3}$$

$$= 238.77 \times \frac{15}{3} = 1{,}193.85 \text{ cu. in., the volume. Ans.}$$

THE WEDGE

109. A **wedge**, as here considered, is a solid whose base is a rectangle, two of whose opposite faces are parallel triangles, and two are parallelograms whose intersection is called the **edge** of the wedge. A wedge may therefore be defined as a triangular prism having one rectangular face, called the **base**. In Fig. 59, $ABCD$ is the base and EF the edge of the wedge.

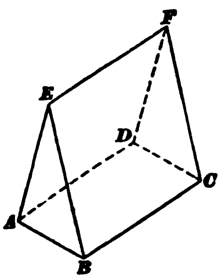


FIG. 59

110. The **altitude** of a wedge is the perpendicular distance between the base and the opposite edge.

111. To find the volume of a wedge:

Rule.—*The volume of any wedge is equal to the area of the base multiplied by one-half the altitude.*

Let A = area of base;
h = altitude;
V = volume.

Then,
$$V = \frac{A h}{2}$$

Example.—What is the volume of a wedge whose base is a rectangle 6 feet long and 4 feet wide, and whose altitude is 10 feet?

Solution.—The area of the base is $4 \times 6 = 24$ sq. ft. Applying formula of Art. **111**,

$$V = \frac{24 \times 10}{2} = 120 \text{ cu. ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. Steel weighs .28 pound per cubic inch; find the weight of a steel wedge whose base is a rectangle 3 inches by 1½ inches and whose altitude is 8 inches. Ans. 5.04 lb.

2. Find the volume of the frustum of a square pyramid of which the larger base is 15 inches square, the smaller base, 14 inches square, and the altitude, 3 inches. Ans. 631 cu. in.

3. A round tank is 8 feet in diameter at the top (inside) and 10 feet at the bottom; if the tank is 12 feet deep, how many gallons will it hold, there being 231 cubic inches in a gallon? Ans. 5,734.2 gal.

4. (*a*) What is the convex area of the frustum of a square pyramid whose altitude is 16 inches, one side of whose lower base is 28 inches long, and of the upper base 10 inches? (*b*) What is the volume of the frustum.
Ans. { (*a*) 1,395.18 sq. in.; (*b*) 6,208 cu. in. }

THE SPHERE

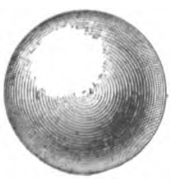


Fig. 60

112. A **sphere**, Fig. 60, is a solid bounded by a uniformly curved surface every point of which is equally distant from a point within, called the **center**.

The word **ball** is commonly used instead of sphere.

113. To find the area of the surface of a sphere:

Rule.—*The area of the surface of a sphere equals the square of the diameter multiplied by π.*

Let S = surface;
d = diameter.

Then, $$S = \pi d^2$$

EXAMPLE.—What is the area of the surface of a sphere whose diameter is 14 inches?

SOLUTION.—Applying formula of Art. **113**, $S = 3.1416 \times 14^2 = 3.1416 \times 14 \times 14 = 615.75$ sq. in., the area. Ans.

114. To find the volume of a sphere:

Rule.—*The volume of a sphere equals the cube of the diameter multiplied by $\frac{\pi}{6}$.*

Let V = volume;
d = diameter.

Then, $$V = \frac{\pi}{6}d^3 = .5236d^3$$

EXAMPLE.—What is the weight of a lead cannon ball 12 inches in diameter, a cubic inch of lead weighing .41 pound?

SOLUTION.—Applying formula of Art. **114**, $V = .5236 \times 12 \times 12 \times 12 = 904.78$ cu. in., the volume of the ball.

$904.78 \times .41 = 370.96$ lb. Ans.

The volume of a spherical shell, or hollow sphere, is equal to the difference in volume between two spheres having, respectively, the outer and the inner diameter of the shell.

115. To find the diameter of a sphere of known volume:

Rule.—*Divide the volume by .5236 and extract the cube root of the quotient. The result is the diameter.*

$$d = \sqrt[3]{\frac{V}{.5236}} = 1.2407\sqrt[3]{V}$$

EXAMPLE.—The volume of a sphere is 96.1 cubic inches; what is its diameter?

SOLUTION.—Applying formula of Art. **115**,

$$d = \sqrt[3]{\frac{V}{.5236}} = \sqrt[3]{\frac{96.1}{.5236}} = 1.2407\sqrt[3]{96.1} = 5.68 \text{ in. Ans.}$$

116. If any solid is cut into two parts by a plane, the surface of either part exposed by the removal of the other part is called a **plane section** of the solid.

Plane sections are divided into three classes: longitudinal sections, cross-sections, and right sections. A **longitudinal section** is any plane section taken lengthwise through the solid. Any other plane section is called a **cross-section.** If the surface exposed by taking a plane section of a solid is perpendicular to the center line of the solid, the section is called a **right section.** The surface exposed by any longitudinal section of a cylinder is a rectangle. The surface exposed by a right section of a cube is a square; of a cylinder or a cone, a circle. An oblique cross-section of a cylinder is an ellipse.

THE CYLINDRICAL RING

117. A **cylindrical ring** is a solid that may be generated by a circle revolving about an external axis in its plane.

118. To find the convex area of a cylindrical ring:

Rule.—*Multiply the circumference of an imaginary cross-section on the line A B, Fig. 61, by the length of the center line D.*

EXAMPLE.—A piece of round iron rod is bent into circular form to make a ring for a chain; if the outside diameter of the ring is 12 inches and the inside diameter is 8 inches, what is its convex area?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters, $\frac{12+8}{2} = 10$, and $10 \times 3.1416 = 31.416$ in., the length of the center line. The radius of the inside circle is 4 in., of the outside circle 6 in.; therefore, the diameter of the cross-section on the line *A B* is 2 in. Then, $2 \times 3.1416 = 6.2832$ in., and $6.2832 \times 31.416 = 197.4$ sq. in., or the convex area. Ans.

119. To find the volume of a cylindrical ring:

Rule.—*The volume will be the same as that of a cylinder whose altitude equals the length of the dotted center line D,*

Fig. 61, and whose base is the same as a cross-section of the ring on the line A B, drawn from the center O. Hence, to find the volume of a cylindrical ring, multiply the area of an imaginary cross-section on a line A B, by the length of the center line D.

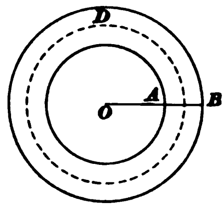


FIG. 61

EXAMPLE.—What is the volume of a cylindrical ring whose outside diameter is 12 inches, and whose inside diameter is 8 inches?

SOLUTION.—The diameter of the center circle equals one-half the sum of the inside and outside diameters, $\frac{12+8}{2} = 10$. $10 \times 3.1416 = 31.416$ in., the length of the center line. The radius of the outside circle is 6 in., of the inside circle, 4 in.; therefore, the diameter of the cross-section on the line $A B$ is 2 in. Then, $2^2 \times .7854 = 3.1416$ sq. in., the area of the imaginary cross-section; and $3.1416 \times 31.416 = 98.7$ cu. in., the volume. Ans.

EXAMPLES FOR PRACTICE

1. (*a*) What is the area of the surface of a sphere 30 inches in diameter? (*b*) What is the volume of the sphere?

Ans. { (*a*) 2,827.44 sq. in.; (*b*) 14,137.2 cu. in. }

2. (*a*) What is the convex area of a cylindrical ring, the outside diameter of the ring being 10 inches and the inside diameter 7½ inches? (*b*) What is the volume of the ring?

Ans. { (*a*) 107.95 sq. in.; (*b*) 33.734 cu. in. }

3. The volume of a sphere is 606.132 cubic inches; what is the convex area of a cone whose slant height is 10 inches, and the diameter of whose base is the same as the diameter of the sphere?

Ans. 164.934 sq. in.

THE PRISMOID

120. A **prismoid** is a solid whose two bases are any polygons in parallel planes, and whose lateral faces may be divided into triangles and trapezoids by lines joining the vertexes of one base with those of the other. Thus, the solid shown in Fig. 62 is a prismoid; its bases are the pentagon $A B C D E$ and the quadrilateral $F G H I$, which lie in parallel planes; and its faces are the triangle GBC and the trapezoids $GCDH$, $HDEI$, $IEAF$, and $FABG$.

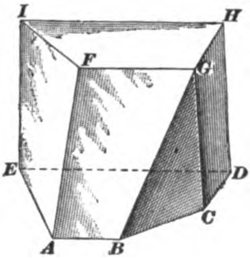


FIG. 62

121. The **altitude** of a prismoid is the perpendicular distance between the bases or parallel faces.

122. The parallel faces or bases of a prismoid are commonly called its **end sections.**

A prismoid is also defined as a solid having two parallel end faces, and composed of any combination of prisms, wedges, and pyramids, whose common altitude is the perpendicular distance between the parallel faces.

123. The **middle section** of a prismoid is the polygon formed by a plane, parallel to the bases, and cutting the prismoid at equal distances from the two bases or end sections. Thus, polygon $PQRS$ is the middle section of the prismoid shown in Fig. 63.

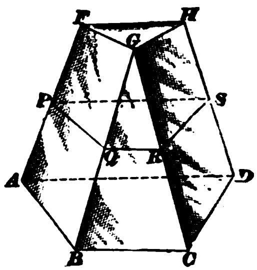


FIG. 63

124. Any dimension of the middle section of a prismoid may be taken equal to one-half the sum of the corresponding dimensions of the two end sections or bases. Thus, in Fig. 63, $PQ = \frac{1}{2}(AB + FG)$, $QR = \frac{1}{2}BC$, $RS = \frac{1}{2}(GH + CD)$, and $SP = \frac{1}{2}(HF + DA)$.

125. The area of the middle section of a prismoid may be measured directly, or calculated from its dimensions as determined from the dimensions of the end sections. It is not, in general, equal to one-half the sum of the areas of the bases.

The area of the middle section of a prism is the same as the area of either base; the area of the middle section of a wedge is equal to one-half the area of the base; the area of the middle section of a pyramid is equal to one-fourth the area of the base.

126. To find the volume of a prismoid:

Rule.—*Multiply the sum of the areas of the two end sections plus four times the area of the middle section by one-sixth the altitude.*

Let A = area of one base or end section;
A' = area of opposite base or end section;
M = area of middle section;
h = altitude;
V = volume of prismoid.

Then,
$$V = \frac{h}{6}(A + A' + 4\,M)$$

This formula for finding the volume of a prismoid is known as the **prismoidal formula.** It is theoretically exact for determining the volumes of those solids to which it applies.

The derivation of this formula is as follows:

A prismoid can always be divided into elementary parts that will be prisms, wedges, and pyramids. From formula of Art. **96,** the volume of a prism is $V = A\,h$; from formula of Art. **111,** the volume of a wedge is $V = \frac{A\,h}{2}$; and from formula of Art. **102,** the volume of a pyramid is $V = \frac{A\,h}{3}$. If these expressions are reduced to a common denominator, there will result,

For a prism, $V = \frac{6\,A\,h}{6}$ (1)

For a wedge, $V = \frac{3\,A\,h}{6}$ (2)

For a pyramid, $V = \frac{2\,A\,h}{6}$ (3)

Since any prism is of uniform cross-section throughout its length, every section will have the same area A, and equation (1) may be written

$$V = \frac{6\,A\,h}{6} = \frac{h}{6}(A + A' + 4\,M)$$

For a wedge, evidently $A' = 0$, and $M = \frac{1}{2}\,A$. Hence, equation (2) may be written

$$V = \frac{3\,A\,h}{6} = \frac{h}{6}(A + 0 + 2\,A) = \frac{h}{6}(A + A' + 4\,M)$$

For a pyramid, $A' = 0$, and $M = \frac{1}{4}\,A$. Hence, equation (3) may be written

$$V = \frac{2\,A\,h}{6} = \frac{h}{6}(A + 0 + A) = \frac{h}{6}(A + A' + 4\,M)$$

Each of these formulas is the same as the formula given in this article; which shows that the latter formula applies correctly to the volume of a prism, pyramid, or wedge, and since it applies to each, it applies also to their sum, or the volume of a prismoid.

EXAMPLE.—Find the volume of the prismoid shown in Fig. 64, whose altitude is 14 inches.

SOLUTION.—Let PQR be the middle section. Then,

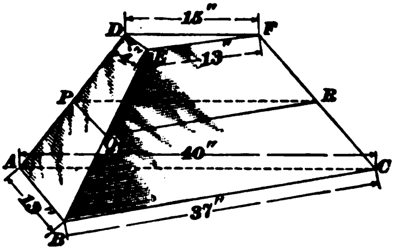


FIG. 64

$PQ = \frac{1}{2}(AB + DE) = \frac{1}{2}(13 + 4) = 8.5$ in.

$QR = \frac{1}{2}(BC + EF) = \frac{1}{2}(37 + 13) = 25$ in.

$RP = \frac{1}{2}(AC + DF) = \frac{1}{2}(40 + 15) = 27.5$ in.

The areas of the triangles are calculated by formula of Art. **47**, which gives the area of $ABC = 240$ sq. in., area of $DEF = 24$ sq. in., and area of $PQR = 105.2$ sq. in., nearly. Hence,

$$V = \frac{14}{6} \times (240 + 24 + 4 \times 105.2) = 1{,}597.9 \text{ cu. in., nearly. Ans.}$$

127. A familiar example of a prismoid is a railway cutting where the roadway is a horizontal plane, the side slopes are inclined planes, and the original surface of the ground is more or less inclined and irregular.

For calculating the volume of cuts and fills the prismoidal formula, though theoretically exact, gives results that are only approximate, on account of the inequalities of the surface of the ground. The nearer to each other the cross-sections are taken, the more accurate will be the result.

EXAMPLE 1.—Find, by the prismoidal formula, the volume of the frustum of a square pyramid of which the larger base is 2.5 feet square, the smaller base is 1 foot square, and the altitude is 16 feet.

SOLUTION.—The area of the larger base is $2.5 \times 2.5 = 6.25$ sq. ft.; the area of the smaller base is $1 \times 1 = 1$ sq. ft. The middle section is a square whose side is one-half the sum of the side of the upper and lower base; that is, $\frac{1}{2} \times (2.5 + 1) = 1.75$ ft. The area of the middle section is $1.75^2 = 3.0625$ sq. ft. Applying formula of Art. **126**, the volume of the frustum is

$$\frac{1}{6} \times 16 \times (6.25 + 1 + 4 \times 3.0625) = 52 \text{ cu. ft. Ans.}$$

EXAMPLE 2.—In a railway cutting 200 feet long, the following are the areas, in square feet, of the cross-sections taken every 50 feet, namely: 2,700, 2,619, 2,556, 2,484, 2,610. What is its volume?

SOLUTION.—The volume between the first and the third cross-section is, by formula of Art. **126**,

$$V = \frac{100}{6}(2{,}700 + 2{,}556 + 4 \times 2{,}619) = 262{,}200 \text{ cu. ft.}$$

The volume between the third and the fifth section is

$$V = \frac{100}{6}(2{,}556 + 2{,}610 + 4 \times 2{,}484) = 251{,}700 \text{ cu. ft.}$$

The volume of the cutting is the sum of the volumes of the two prismoids, which is 513,900 cu. ft. = 19,033 cu. yd. Ans.

128. Average End Areas.—In practice, the volume of cuts and fills is often calculated by what is known as the **method by average end areas**, or simply as the **end area method**. By this method, the volume of the solid is found by multiplying one-half the sum of the two end areas by the distance between the two sections. Thus, let

A = area of one cross-section;
A' = area of next cross-section;
h = perpendicular distance between sections;
V = volume.

Then,
$$V = \frac{h}{2}(A + A')$$

Results obtained by this formula are approximate and slightly larger than those given by the prismoidal formula. On account of its simplicity, the average end area formula is much used in practical earth-work calculations. The inequalities of the surface of the ground make it impossible to find the exact volume of a cut or fill, however accurate may be the formula applied.

EXAMPLE.—The areas of two cross-sections of a fill 50 feet apart are 2,700 and 2,619 square feet respectively; find the volume of the section, in cubic yards.

SOLUTION.—In this case, A = 2,700; A' = 2,619; and h = 50; then

$$V = \frac{50}{2}(2{,}700 + 2{,}619) = 132{,}975$$

Hence, the volume is 132,975 cu. ft. = 4,925 cu. yd. Ans.

EXAMPLES FOR PRACTICE

1. Find the volume of a right prismoid whose bases are rectangles that measure 10 inches by 8 inches and 8 inches by 6 inches, and whose height is 40 inches. Ans. 2,533.3 cu. in.

2. A railway cutting is 800 feet in length; the areas, in square yards, of cross-sections taken every 100 feet are: 237, 220, 204, 187, 171, 186, 204, 210, 220. Find the number of cubic yards in the cutting: (*a*) by the prismoidal formula; (*b*) by average end areas.

Ans. { (*a*) 53,633 cu. yd.
(*b*) 53,683 cu. yd.

3. Find, by the prismoidal formula, the volume of a frustum of a hexagonal pyramid, each side of the lower base being 12 inches; of the upper base, 8 inches; and the altitude being 12 inches.

Ans. 3,159.3 cu. in.

4. Find, by the prismoidal formula, the volume of a wedge whose base is a rectangle 15 feet in length and 9 feet in width, and whose altitude is 12 feet. Ans. 810 cu. ft.

PLANE TRIGONOMETRY

(PART 1)

THE TRIGONOMETRIC FUNCTIONS

DEFINITIONS

1. Trigonometric Functions and Trigonometry Defined.—Let A, Fig. 1, be any acute angle; AM and AN, its sides; BC, a perpendicular drawn to the side AN from any point on the side AM; and $B'C'$, a perpendicular drawn to the side AM from any point on the side AN. In the right triangle ABC, one of the vertexes of which is the vertex of the angle A, the hypotenuse AB will be referred to as *the hypotenuse;* the perpendicular BC, opposite the vertex of the angle A, as *the side opposite;* and the leg AC, containing the vertex of the angle A, as *the side adjacent.* Likewise, in the right triangle $AB'C'$, the hypotenuse is AB'; the side opposite is $B'C'$; and the side adjacent, or the leg containing the vertex of the angle A, is AC'. It should be borne in mind that these terms are used in connection with, or with reference to, the angle A.

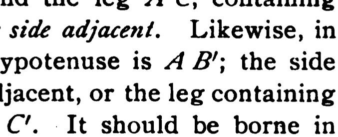


FIG. 1

The two right triangles ABC and $AB'C'$, having the acute angle A in common, are similar. Therefore,

$$\frac{AB}{AC} = \frac{AB'}{AC'}, \quad \frac{BC}{AB} = \frac{B'C'}{AB'}, \quad \frac{BC}{AC} = \frac{B'C'}{AC'}$$

It will be observed that, from whichever side the perpendicular is drawn, and whatever the point from which it is drawn, the ratio of the hypotenuse to the side adjacent remains unchanged, or is **constant.** The same is true of the ratio of the side opposite to the side adjacent, and, in general, of the ratio of any two of the three lines—hypotenuse, side adjacent, and side opposite. Evidently, these ratios are different for different angles. Thus, if A is 45°, both acute angles B and B' are also 45°; the triangles ABC and $AB'C'$ are isosceles; and therefore

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = 1$$

If A is greater than 45°, BC is greater than AC, and the ratio $\frac{BC}{AC}$, having its numerator greater than its denominator, is greater than 1.

Confining ourselves to the ratio $\frac{BC}{AC}$ of the side opposite to the side adjacent, it is seen that the value of this ratio depends on the magnitude of the angle, and may, therefore, be used for the determination of the angle. Thus, it has just been shown that when the angle is 45° the ratio is equal to 1; hence, if in the solution of a problem it is found that the two legs of a right triangle are equal, or that their ratio is 1, it can be at once concluded that each of the acute angles is 45°.

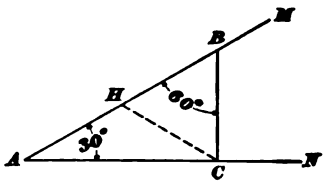


FIG. 2

Consider now an angle A, Fig. 2, of 30°. The right triangle ABC having been constructed, BC is the side opposite and AC the side adjacent. If H is the middle point of the hypotenuse, the line HC is equal to AH, or $\frac{AB}{2}$; for, if a semicircle is described on AB as a diameter, with HA as a radius, that semicircle must pass through C, since the angle ACB is a right angle. Now, HC being equal to HB, the angle HCB is equal to B, or 60°; and, as the sum of the three angles of the triangle BHC is 180°, the angle BHC must be 60°. The triangle HBC being equiangular, it is also equilateral, and therefore $BC = BH = \frac{AB}{2}$, and the ratio of the side opposite to the hypotenuse is $\frac{BC}{AB} = \frac{\frac{1}{2}AB}{AB} = \frac{1}{2}$. Suppose, now, that in dealing with a right triangle the hypotenuse is found, by measurement, to be 1,500 feet and one of the sides 750 feet. Since the ratio of 750 to 1,500 is $\frac{1}{2}$, we at once conclude that the angle opposite the 750-foot side is 30°, and the other angle of the triangle, 60°.

These illustrations give a general idea of the practical value and use of the ratios under consideration. These ratios are determined for each angle, by methods that will be again referred to further on, and collected together in a table, from which the angle corresponding to any given ratio can be determined. Thus, if in a certain angle the ratio of the opposite side to the hypotenuse is $\frac{1}{2}$, this ratio is looked for in the table, where it is found as that belonging to 30°. In this manner, the value of the angle is determined from the ratio in question, that ratio being obtained from the measured lengths of certain lines.

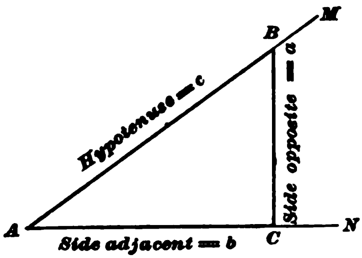


Fig. 3

2. The ratios considered in the preceding article are called **trigonometric functions** of the angle A. In the triangle ABC, Fig. 3, two ratios are obtained by dividing any of three sides by each of the other two. Hence, there are six trigonometric functions of the angle A. This is true of any angle, since A is here used to represent any angle whatever. These functions have very important and useful properties, which make them exceedingly valuable for the solution of geometrical problems by computation.

3. Trigonometry is that branch of mathematics that treats of the properties of trigonometric functions and of their application to the solution of triangles.

4. The Sine and the Tangent.—Two of the most important of the trigonometric functions are the ratio of the side opposite to the hypotenuse, and that of the side opposite to the side adjacent; that is, $\frac{a}{c}$ and $\frac{a}{b}$, Fig. 3. They are called, respectively, the **sine** of A and the **tangent** of A. The words *sine* and *tangent* are abbreviated to *sin* and *tan*, respectively, and the expressions *sin A*, *tan A*, are for brevity read *sine A*, *tangent A*, instead of *sine of A*, and *tangent of A*. We have, then,

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad (1)$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b} \qquad (2)$$

If these formulas are fixed in the mind, little difficulty will be experienced in remembering the others that will be given. It should be noticed that the side opposite is the numerator in both ratios. The occurrence of the letter *a* in both the words *adjacent* and *tangent* will help one to remember which of the two fractions represents the tangent and which the sine.

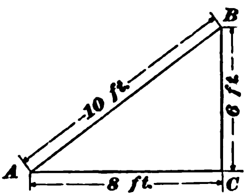


FIG. 4

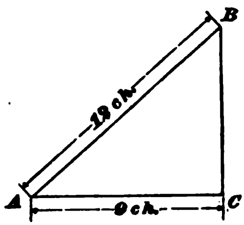


FIG. 5

EXAMPLE 1.—In the right triangle $A\,B\,C$, Fig. 4, the lengths of the sides are shown; find the sine and the tangent of A.

SOLUTION.—In this case, the hypotenuse $A\,B = 10$; the side adjacent, $A\,C = 8$; side opposite, $B\,C = 6$. These values in formulas 1 and 2 give

$$\sin A = \frac{6}{10} = .6. \quad \text{Ans.}$$

$$\tan A = \frac{6}{8} = .75. \quad \text{Ans.}$$

EXAMPLE 2.—In the right triangle $A\,B\,C$, Fig. 5, the hypotenuse is 12 chains, and the side $A\,C$ is 9 chains; find: (*a*) the sine and the tangent of A; (*b*) the sine and the tangent of B.

SOLUTION.—(*a*) For the angle A, we have

hypotenuse $AB = 12$

side adjacent, $AC = 9$

side opposite, $BC = \sqrt{\overline{AB}^2 - \overline{AC}^2} = \sqrt{12^2 - 9^2} = 7.9372$

Substituting in formulas **1** and **2**,

$$\sin A = \frac{BC}{AB} = \frac{7.9372}{12} = .66143. \quad \text{Ans.}$$

$$\tan A = \frac{BC}{AC} = \frac{7.9372}{9} = .88191. \quad \text{Ans.}$$

(*b*) For angle B, we have

hypotenuse $BA = 12$

side opposite, $AC = 9$

side adjacent, $BC = 7.9372$

Therefore,

$$\sin B = \frac{AC}{AB} = \frac{9}{12} = .75. \quad \text{Ans.}$$

$$\tan B = \frac{AC}{BC} = \frac{9}{7.9372} = 1.1339. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. In a right triangle ABC (make a sketch of this triangle), A and B are the two acute angles; the hypotenuse = 40 feet; side opposite B = 15 feet; find: (*a*) sin A and tan A; (*b*) sin B and tan B.

Ans. $\begin{cases} (a) \sin A = .92703, \tan A = 2.47207 \\ (b) \sin B = .37500, \tan B = .40452 \end{cases}$

2. From a point on one side of an angle M, a perpendicular is drawn on the other side; it is found that this perpendicular is 12.5 inches long, and that it meets the other side at a distance of 7.75 inches from the vertex; find the sine and the tangent of the angle M. (Make a sketch of this triangle.)

Ans. $\begin{cases} \sin M = .84988 \\ \tan M = 1.61290 \end{cases}$

3. From a point on one side of an angle A distant 10 inches from the vertex, a perpendicular is drawn on the other side; the distance from the vertex to the foot of the perpendicular is 6 inches; find sin A and tan A.

Ans. $\begin{cases} \sin A = .80000 \\ \tan A = 1.33333 \end{cases}$

4. The two acute angles of a right triangle are P and Q; the side opposite P is 150 feet, and that opposite Q is 225 feet; find: (*a*) sin P and tan P; (*b*) sin Q and tan Q.

Ans. $\begin{cases} (a) \sin P = .55469, \tan P = .66667 \\ (b) \sin Q = .83204, \tan Q = 1.50000 \end{cases}$

5. The Cosine and Cotangent.—The **cosine** and **cotangent** of an angle are, respectively, the sine and the tangent of the complement of the angle. The words *cosine* and *cotangent* are abbreviated to *cos* and *cot*, respectively, and the expressions *cos A*, *cot A* are read *cosine A*, *cotangent A*. Denoting any angle by A, its complement is $90° - A$; therefore, according to the definitions just given,

$$\cos A = \sin (90° - A) \qquad (1)$$
$$\cot A = \tan (90° - A) \qquad (2)$$

Since the complement of $90° - A$ is A, it also follows that

$$\cos (90° - A) = \sin A \qquad (3)$$
$$\cot (90° - A) = \tan A \qquad (4)$$

With reference to the angle B, Fig. 3, BC is the side adjacent and AC the side opposite. Therefore, by formulas **1** and **2**, Art. **4**,

$$\sin B = \frac{b}{c}, \ \tan B = \frac{b}{a}$$

and therefore, since A is the complement of B,

$$\cos A = \sin B = \frac{b}{c}$$

$$\cot A = \tan B = \frac{b}{a}$$

or, again referring to the angle A, which is the angle under consideration,

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad (5)$$

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} \qquad (6)$$

The student will, after some practice, become familiar with these formulas. Whenever he forgets them, he should refer to the definitions of the cosine and cotangent, which will at once enable him to write down the formulas, provided that he remembers those for the sine and the tangent.

6. The Secant and Cosecant.—The **secant** of an angle is the reciprocal of the cosine of the angle; that is, 1 divided by the cosine.

The word *secant* is abbreviated to *sec.* According to the definition, we have

$$\sec A = \frac{1}{\cos A} \qquad (1)$$

It follows that

$$\cos A = \frac{1}{\sec A} \qquad (2)$$

7. The **cosecant** of an angle is the secant of the complement of the angle. The abbreviations *cosec* and *csc* are used for *cosecant.* According to the definition, we have

$$\csc A = \sec\,(90° - A) \qquad (1)$$

Since A is the complement of $90° - A$, we have also

$$\csc\,(90° - A) = \sec A \qquad (2)$$

By means of formula **1**, Art. **6**, this relation may be written

$$\csc A = \sec\,(90° - A) = \frac{1}{\cos\,(90° - A)}$$

or, since $\cos\,(90° - A) = \sin A$ (formula **3**, Art. **5**),

$$\csc A = \frac{1}{\sin A} \qquad (3)$$

Therefore, the cosecant of an angle may also be defined as the reciprocal of the sine. Notice very particularly that

secant = reciprocal of *cosine*
cosecant = reciprocal of *sine*

From formula **3** above follows

$$\sin A = \frac{1}{\csc A} \qquad (4)$$

8. Cofunctions and Complementary Functions. The functions cosine, cotangent, and cosecant are sometimes called **cofunctions** of the angle considered; while the sine, tangent, and secant are called **fundamental functions.** As has been explained, the cofunctions of an angle are the corresponding fundamental functions of the complement of the angle. Thus, the cosine of A is the sine of $90° - A$; the cotangent of A is the tangent of $90° - A$; etc.

A **fundamental function** and its corresponding cofunction are called **complementary functions** of each other. The sine, for example, is the complementary function of the cosine; and the cosine is the complementary function of the sine.

EXAMPLE 1.—Find: (*a*) the cosine of the angle A, Fig. 5; (*b*) the cotangent; (*c*) the secant; (*d*) the cosecant.

SOLUTION.—(*a*) The cosine of A is equal to the sine of B, or

$$\frac{AC}{AB} = \frac{9}{12} = .75. \quad \text{Ans.}$$

(*b*) The cotangent of A is equal to the tangent of B, or (see example 2, Art. 4)

$$\frac{AC}{BC} = \frac{9}{7.9372} = 1.1339. \quad \text{Ans.}$$

(*c*) The secant of A is 1 divided by cos A, or

$$1 \div \frac{9}{12} = \frac{12}{9} = 1.33333. \quad \text{Ans.}$$

(*d*) The cosecant of A is 1 divided by sin A, or

$$1 \div \frac{BC}{AB} = \frac{AB}{BC} = \frac{12}{7.9372} = 1.51187. \quad \text{Ans.}$$

EXAMPLE 2.—Find the functions of 30°.

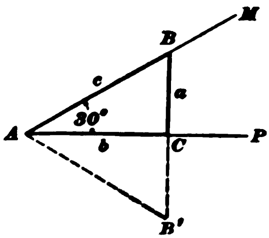


FIG. 6

SOLUTION.—Let the angle MAP, Fig. 6, be 30°. Draw BC perpendicular to AP, produce it to B', making $CB' = CB$, and draw AB'. The triangle BAB' thus formed is isosceles, and angle $CAB' = CAB = 30°$. Therefore, $BAB' = 30° + 30° = 60°$. Also, angle $B = 90° - 30° = 60°$; and angle B' = angle $B = 60°$. As the three angles of ABB' are equal, the sides are also equal, and $c = BB' = 2a$. Now, the figure gives,

$$b = \sqrt{c^2 - a^2} = \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}$$

Bearing these values in mind, we have

$$\sin 30° = \frac{a}{c} = \frac{a}{2a} = \frac{1}{2}. \quad \text{Ans.}$$

$$\tan 30° = \frac{a}{b} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \quad \text{Ans.}$$

$$\cos 30° = \frac{b}{c} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}. \quad \text{Ans.}$$

$$\cot 30° = \frac{b}{a} = \frac{a\sqrt{3}}{a} = \sqrt{3}. \text{ Ans.}$$

$$\sec 30° = \frac{1}{\cos 30°} = 1 \div \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}. \text{ Ans.}$$

$$\csc 30° = \frac{1}{\sin 30°} = 1 \div \frac{1}{2} = 2. \text{ Ans.}$$

NOTE.—It is only in a few cases that the values of the trigonometric functions of an angle can be derived by elementary principles, as above. The general method for determining the functions of any angle is comparatively complicated, and is beyond the scope of this work. The trigonometric functions of any angle can be obtained from a table, as will be presently explained.

EXAMPLES FOR PRACTICE

1. The acute angles of a right triangle are B and C; the side opposite B is 1,200 feet; and that opposite C is 1,500 feet; find the fundamental functions of B, and from them the cofunctions of C.

$$\text{Ans.}\begin{cases}\sin B = .62471, \tan B = .8, \sec B = 1.2806\\ \cos C = .62471, \cot C = .8, \csc C = 1.2806\end{cases}$$

2. From example 2, Art. 8, derive the functions of 60° (= 90° − 30°).

$$\text{Ans.}\begin{cases}\sin 60° = \frac{\sqrt{3}}{2}, \tan 60° = \sqrt{3}, \cos 60° = \frac{1}{2}\\ \cot 60° = \frac{\sqrt{3}}{3}, \sec 60° = 2, \csc 60° = \frac{2}{3}\sqrt{3}\end{cases}$$

3. Given $\sin A = \frac{2}{3}$ and $\cos B = \frac{4}{5}$, find csc A and sec B.

$$\text{Ans.}\begin{cases}\csc A = 1.5\\ \sec B = 1.25\end{cases}$$

4. Find the trigonometric functions of 45°. (Notice that here the side opposite is equal to the side adjacent. Denote the hypotenuse by c, and express the other two sides in terms of c.)

$$\text{Ans.}\begin{cases}\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}\\ \tan 45° = \cot 45° = 1\\ \sec 45° = \csc 45° = \sqrt{2}\end{cases}$$

---

9. The Versed Sine and Coversed Sine.—The **versed sine** (*vers*) of an angle is 1 minus the cosine; and the **coversed sine** (*covers*) is 1 minus the sine.

$$\text{vers } A = 1 - \cos A \qquad (1)$$
$$\text{covers } A = 1 - \sin A \qquad (2)$$

These two functions are not much used, except in railroad work.

10. Summing Up.—The foregoing definitions are summed up in the table given below, which contains the expressions for the functions of the angle A, Fig. 7, in terms of the hypotenuse c, the side opposite, a, and the side adjacent, b.

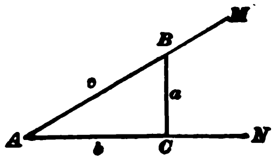


FIG. 7

TABLE I

Function	sin	tan	cos	cot	sec	csc	vers	covers
Value . . .	$\frac{a}{c}$	$\frac{a}{b}$	$\frac{b}{c}$	$\frac{b}{a}$	$\frac{c}{b}$	$\frac{c}{a}$	$1-\frac{b}{c}$	$1-\frac{a}{c}$

The ratios $\frac{c}{b}$ and $\frac{c}{a}$ for the secant and cosecant are obtained from the formulas $\sec A = 1 \div \cos A = 1 \div \frac{b}{c} = \frac{c}{b}$, $\csc A = 1 \div \sin A = 1 \div \frac{a}{c} = \frac{c}{a}$.

11. Representation of the Trigonometric Functions by Lines.—Let A, Fig. 8, be any angle. From its

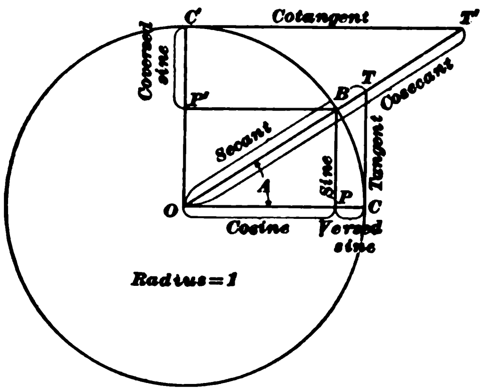


FIG. 8

vertex O, describe a circle of radius 1; or, otherwise, describe any circle and take its radius as unity. This circle intersects the sides of the angle at B and C. Draw the tangent CT, meeting OB produced at T; the radius OC' perpendicular to OC; the lines BP and BP' perpendicular to OC and OC', respectively; and the tangent $C'T'$, meeting OB produced at T'.

Since the angle A is measured by the arc CB, the trigonometric functions of the angle are said to be likewise the trigonometric functions of the arc. It is, for instance, immaterial whether we say that 1 is the tangent of an angle of 45° or of an arc of 45°.

In the figure constructed as just explained, the trigonometric functions of the angle A, or of the arc CB, may be represented by lines, as marked. For, in the right triangle OPB, in which BP, OP, and OB are, respectively, the side opposite, the side adjacent, and the hypotenuse, we have

$$\sin A = \frac{BP}{OB}, \cos A = \frac{OP}{OB}$$

or, since $OB = 1$,

$$\sin A = \frac{BP}{1} = BP, \cos A = \frac{OP}{1} = OP$$

In the triangle OCT, in which CT and OC are, respectively, the side opposite and the side adjacent, and OT is the hypotenuse,

$$\tan A = \frac{CT}{OC} = \frac{CT}{1} = CT$$

$$\sec A = \frac{OT}{OC} = \frac{OT}{1} = OT$$

By the same reasoning, it can be shown that $C'T'$ and OT' are, respectively, the tangent and the secant of the angle $C'OT'$, or the cotangent and the cosecant of A, since $C'OT'$ is the complement of A.

Let the student verify that, according to the definitions of the versed sine and coversed sine, these functions are represented by PC and $P'C'$, respectively.

RELATIONS AMONG THE FUNCTIONS OF AN ANGLE

12. Method of Marking a Triangle.—The triangle ABC, Fig. 7, has the angles marked by the capital letters A, B, and C and the sides opposite these angles marked by the small letters, a, b, and c, respectively. This method of marking a triangle is very useful and convenient, as it points out at once the relative position of the sides and the angles. In a right triangle, the right angle is usually designated by C. In the figures that follow, when only the angles are marked, the sides opposite are taken as marked by the small letters corresponding to the capital letters that mark the angles.

13. Relation Between Tangent and Cotangent.—In Fig. 7,

$$\tan A = \frac{a}{b}, \cot A = \frac{b}{a}$$

Multiplying these equations together gives

$$\tan A \times \cot A = \frac{a}{b} \times \frac{b}{a} = 1$$

whence,

$$\cot A = \frac{1}{\tan A}$$

$$\tan A = \frac{1}{\cot A}$$

That is, the tangent and cotangent are each the reciprocal of the other. This is a very important relation, and should be committed to memory, together with those given in the two articles following.

14. Tangent and Cotangent in Terms of Sine and Cosine.—In Fig. 7,

$$\sin A = \frac{a}{c}, \cos A = \frac{b}{c}$$

Dividing these equations member by member gives

$$\frac{\sin A}{\cos A} = \frac{a}{c} \div \frac{b}{c} = \frac{a}{b}$$

that is, since $\frac{a}{b} = \tan A$,

the top with the given number of degrees, in the subdivision of that column headed by the name of the given function, and horizontally opposite the number in the left-hand column (marked ′) that expresses the number of odd minutes in the angle. When the function considered is a sine or a cosine, it is taken from the table headed Natural Sines and Cosines; when a tangent or cotangent, from the table headed Natural Tangents and Cotangents.

EXAMPLE.—Find the natural functions of an angle of 37° 23′.

SOLUTION.—On page 30 of the table headed Natural Sines and Cosines, the double column headed 37° is found. Looking in the left-hand minute column for 23 (number of odd minutes in the given angle), and glancing along the horizontal row to the right of 23, the number .60714 is found in the single column marked Sine under 37°; and the number .79459 is found in the column marked Cosine. Therefore,

$$\sin 37° 23' = .60714. \text{ Ans.}$$
$$\cos 37° 23' = .79459. \text{ Ans.}$$

The tangent and cotangent are taken in a similar manner from the table headed Natural Tangents and Cotangents, page 39. The results are:

$$\tan 37° 23' = .76410. \text{ Ans.}$$
$$\cot 37° 23' = 1.30873. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Verify the following values:

(*a*) $\sin 39° 55' = .64167$; $\cos 39° 55' = .76698$; $\tan 39° 55' = .83662$; $\cot 39° 55' = 1.19528$.

(*b*) $\tan 16° 32' = .29685$; $\cos 16° 32' = .95865$; $\sec 16° 32' = 1.04313$; $\csc 16° 32' = 3.51407$.

(*c*) $\cot 43° 2' = 1.07112$; $\csc 43° 2' = 1.46537$; $\tan 43° 2' = .93360$; $\cos 43° 2' = .73096$.

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19. To Find the Natural Functions of an Angle Greater Than 45° and Containing No Odd Seconds. The required function is found in the double column marked at the bottom with the given number of degrees, in the subdivision of that column having at the bottom the name of the given function, and horizontally opposite the number in the right-hand column (marked ′) that expresses the odd minutes in the angle. It will be observed that the number of degrees at the bottom of the pages decrease as the pages increase, and that the number of minutes in the right-hand column increase from bottom to top.

EXAMPLE.—Find the functions of 53° 43′.

SOLUTION.—The double column marked 53° at the bottom is found on page 30 of Natural Sines and Cosines. Looking along the horizontal row determined by the number 43 in the right-hand minute column, the number .80610 is found in the single column marked Sine at the bottom, and the number .59178 in the single column marked Cosine at the bottom, these two columns forming the double column marked 53° at the bottom. Therefore,

$$\sin 53° 43' = .80610. \quad \text{Ans.}$$
$$\cos 53° 43' = .59178. \quad \text{Ans.}$$

The tangent and cotangent are similarly taken from page 39 of Natural Tangents and Cotangents. The results are:

$$\tan 53° 43' = 1.36217. \quad \text{Ans.}$$
$$\cot 53° 43' = .73413. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Verify the following values:

(*a*) sin 67° 45′ = .92554; cos 67° 45′ = .37865; tan 67° 45′ = 2.44433; cot 67° 45′ = .40911.

(*b*) cot 74° 3′ = .28580; csc 74° 3′ = 1.04004; sin 74° 3′ = .96150.

(*c*) cos 48° 9′ = .66718; cot 48° 9′ = .89567; csc 48° 9′ = 1.34248.

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20. To Find the Natural Functions of an Angle Containing Odd Seconds.—The method of solving this problem by means of the table is founded on the following principle, which applies within the limits of approximation with which the table is constructed:

If several angles are taken within an interval not greater than 1′; that is, so that the difference between the greatest and the smallest shall not exceed 1′, the ratio of the difference between any two of these angles to the difference between any other two is the same as the ratio obtained by dividing the difference between the values of any trigonometric function for the first pair of angles, by the difference between the values of the same function for the second pair of angles. For instance, if the angles 43° 46′ 32″, 43° 46′ 34″, 43° 46′ 40″, and 43° 47′ are taken between 43° 46′ and 43° 47′, then

$$\frac{43^\circ\ 47' - 43^\circ\ 46'\ 40''}{43^\circ\ 46'\ 34'' - 43^\circ\ 46'\ 32''} = \frac{\sin 43^\circ\ 47' - \sin 43^\circ\ 46'\ 40''}{\sin 43^\circ\ 46'\ 34'' - \sin 43^\circ\ 46'\ 32''}$$

In general, if A, B, C, D are any angles within an interval of $1'$, then

$$\frac{A-B}{C-D} = \frac{\sin A - \sin B}{\sin C - \sin D} = \frac{\cos A - \cos B}{\cos C - \cos D}$$
$$= \frac{\tan A - \tan B}{\tan C - \tan D} = \frac{\cot A - \cot B}{\cot C - \cot D}$$

Similarly,

$$\frac{A-B}{B-C} = \frac{\sin A - \sin B}{\sin B - \sin C} = \frac{\cos A - \cos B}{\cos B - \cos C}, \text{ etc.}$$

Let A be the number of degrees and minutes in any angle, and s the number of odd seconds. Then the angle, which will be represented by $A + s''$, lies between A and $A + 1'$ or between A and $A + 60''$. For instance, if the angle is $25^\circ\ 15'\ 37''$, it lies between $25^\circ\ 15'$, which is represented by A, and $25^\circ\ 16'$, which is $25^\circ\ 15' + 1'$, or $A + 1'$, or $A + 60''$. In this case s represents $37''$. From the principle stated above we have,

$$\frac{(A + 60'') - A}{(A + s'') - A} = \frac{\sin (A + 60'') - \sin A}{\sin (A + s'') - \sin A}$$

or,

$$\frac{60}{s} = \frac{\sin (A + 1') - \sin A}{\sin (A + s'') - \sin A}$$

whence, solving this equation for $\sin (A + s'')$,

$$\sin (A + s'') = \sin A + [\sin (A + 1') - \sin A]\frac{s}{60} \qquad (1)$$

Similarly,

$$\tan (A + s'') = \tan A + [\tan (A + 1') - \tan A]\frac{s}{60} \qquad (2)$$

For the cosine, we have

$$\cos (A + s'') = \cos A + [\cos (A + 1') - \cos A]\frac{s}{60}$$

but, since the cosine of an angle decreases as the angle increases, $\cos A$ is greater than $\cos (A + 1')$, and therefore it is better to write the formula thus,

$$\cos (A + s'') = \cos A - [\cos A - \cos (A + 1')]\frac{s}{60} \qquad (3)$$

Similarly,

$$\cot (A + s'') = \cot A - [\cot A - \cot (A + 1')]\frac{s}{60} \qquad (4)$$

The functions of A and $A + 1'$ can be readily taken from the table, as explained in the preceding articles, and from them the functions of $A + s''$ are determined by the formulas just given, or by the following rule, which states in words what the formulas express in symbols:

Rule.—*Find, in the table, the sine, cosine, tangent, or cotangent corresponding to the degrees and minutes in the angle.*

For the seconds, find the difference between this value and the value of the sine, cosine, tangent, or cotangent of an angle 1 minute greater; multiply this difference by a fraction whose numerator is the number of seconds in the given angle and whose denominator is 60.

If the sine or tangent is sought, add this correction to the value first found; if the cosine or cotangent is sought, subtract the correction.

Example.—Find: (a) the sine of 56° 43′ 17″; (b) the cosine; (c) the tangent; and (d) the cotangent.

Solution.—(a) Here A = 56° 43′, s = 17, $A + 1'$ = 56° 44′.

$$\begin{aligned} \sin(A + 1') &= \sin 56° 44' = .83613 \\ \sin A &= \sin 56° 43' = .83597 \\ \text{Difference} &= .00016 \\ &\times \tfrac{17}{60} \\ &\overline{.00005}, \text{ nearly} \end{aligned}$$

Adding this product to sin A, we have

$$\sin 56° 43' 17'' = .83597 + .00005 = .83602. \quad \text{Ans.}$$

(b)

$$\begin{aligned} \cos A &= \cos 56° 43' = .54878 \\ \cos(A + 1') &= \cos 56° 44' = .54854 \\ \text{Difference} &= .00024 \\ &\times \tfrac{17}{60} \\ &\overline{.00007}, \text{ nearly} \end{aligned}$$

Subtracting this product from cos A, we have

$$\cos 56° 43' 17'' = .54878 - .00007 = .54871. \quad \text{Ans.}$$

(c)

$$\begin{aligned} \tan(A + 1') &= \tan 56° 44' = 1.52429 \\ \tan A &= \tan 56° 43' = 1.52332 \\ \text{Difference} &= .00097 \\ &\times \tfrac{17}{60} \\ &\overline{.00027}, \text{ nearly} \end{aligned}$$

Adding this product to tan A, we have

$$\tan 56° \, 43' \, 17'' = 1.52332 + .00027 = 1.52359. \quad \text{Ans.}$$

(*d*)

$$\cot A = \cot 56° \, 43' = .65646$$
$$\cot (A + 1') = \cot 56° \, 44' = .65604$$
$$\text{Difference} = .00042$$
$$\times \frac{17}{60}$$
$$.00012, \text{ nearly}$$

Subtracting this product from cot A, we have

$$\cot 56° \, 43' \, 17'' = .65646 - .00012 = .65634. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Verify the following values:

(*a*) sin 18° 54′ 45″ = .32412; tan 18° 54′ 45″ = .34262.
(*b*) cos 34° 17′ 18″ = .82621; cot 34° 17′ 18″ = 1.46659.
(*c*) sin 72° 26′ 20″ = .95340; cot 72° 26′ 20″ = .31647.
(*d*) cos 65° 6′ 9″ = .42100; tan 65° 6′ 9″ = 2.15457.
(*e*) sin 80° 0′ 3″ = .98481; cot 80° 0′ 3″ = .17631.
(*f*) tan 14° 14′ 14″ = .25373; cos 14° 14′ 14″ = .96928.

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21. To Find the Angle Corresponding to a Given Function, When the Function Is in the Table.—This case does not present any difficulty. Having found the given function in the table, the degrees in the angle are taken from the top or the bottom, and the minutes from the left- or the right-hand column, according as the name of the function is at the top or at the bottom of the page.

EXAMPLE 1.—The sine of an angle is .47486; what is the angle?

SOLUTION.—Glancing down the columns marked Sine in the table of Natural Sines and Cosines, .47486 is found (on page 28) in the column headed 28°. The number of minutes, 21, is found in the left-hand minute column, horizontally opposite .47486. Therefore, .47486 = sin 28° 21′. Ans.

EXAMPLE 2.—Find the angle whose cosine is .27032.

SOLUTION.—Looking in the columns marked Cosine at the top of the page, the given cosine is not found; hence, the angle is greater than 45°. Consequently, looking in the columns marked Cosine at the bottom of the page, .27032 is found (on page 26) in the double column marked 74° at the bottom, and in the horizontal row beginning with 19 in the right-hand minute column. Therefore, the angle whose cosine is .27032 is 74° 19′; or, .27032 = cos 74° 19′. Ans.

EXAMPLE 3.—Find the angle whose tangent is 2.15925.

SOLUTION.—On searching the table of Natural Tangents, the given tangent is found to belong to an angle greater than 45°, so that it must be looked for in the column marked Tangent at the bottom. It is found in the column having 65° at the bottom and opposite 9′ in the right-hand minute column. Therefore, 2.15925 = tan 65° 9′. Ans.

EXAMPLE 4.—Find the angle whose cotangent is .43412.

SOLUTION.—From the table of Natural Cotangents, it is found that this value is less than the cotangent of 45°, so it must be found in the column marked Cotangent at the bottom. Looking there, it is found in the column having 66° at the bottom, and opposite 32′, in the right-hand column of minutes. Therefore, the angle whose cotangent is .43412 is 66° 32′, or .43412 = cot 66° 32′. Ans.

EXAMPLES FOR PRACTICE

1. Find the angle whose sine is .47486. Ans. 28° 21′
2. Find the angle whose cosine is .74353. Ans. 41° 58′
3. Find the angle whose tangent is 2.06247. Ans. 64° 8′
4. Find the angle whose cotangent is 1.20665. Ans. 39° 39′
5. Find the angle whose sine is .76903. Ans. 50° 16′
6. Find the angle whose tangent is 9.93101. Ans. 84° 15′

22. To Find the Angle Corresponding to a Given Function, When the Function Is Not in the Table. Since the table includes the functions of all angles containing no odd seconds, a function not found in the table must correspond to an angle having odd seconds. Let the odd seconds that are to be determined be denoted by s, and the degrees and minutes by A, as in Art. **20**. Now, two consecutive functions including the given function can always be found in the table; that is, two consecutive functions of which one is greater and the other less than the given function. The required angle must, therefore, lie between the two angles corresponding to these two consecutive functions, and its number of degrees and minutes, A, is the number of degrees and minutes in the smaller of the two angles. The larger angle is $A + 1'$, or $A + 60''$, while the required angle is $A + s''$. Having determined A, it only remains to determine the number of odd seconds, or s. This is done by means of the following formulas, obtained by solving for s the formulas found in Art. **20**.

If the given function is a sine or tangent,

$$s = \frac{\sin(A + s'') - \sin A}{\sin(A + 1') - \sin A} \times 60 \qquad (1)$$

$$s = \frac{\tan(A + s'') - \tan A}{\tan(A + 1') - \tan A} \times 60 \qquad (2)$$

If the given function is a cosine or cotangent,

$$s = \frac{\cos A - \cos(A + s'')}{\cos A - \cos(A + 1')} \times 60 \qquad (3)$$

$$s = \frac{\cot A - \cot(A + s'')}{\cot A - \cot(A + 1')} \times 60 \qquad (4)$$

Observe that, although $A + s''$ is not known, its sine, cosine, etc., as the case may be, is known, or given. Thus, if the problem is to find the angle whose cotangent is .97888, we have $\cot(A + s'') = .97888$.

The foregoing formulas lead to the following general rule for finding the angle corresponding to a given function:

Rule.—*Find the difference of the two numbers in the table between which the given function lies, and use that difference as the denominator of a fraction.*

Find the difference between the function belonging to the smaller angle and the given function, and use that difference as the numerator of the fraction mentioned above. Multiply this fraction by 60. The result will be the number of seconds to be added to the smaller angle in order to obtain the required angle.

EXAMPLE 1.—Find the angle whose sine is .57698.

SOLUTION.—Looking in the table of Natural Sines, in the columns marked Sine, it is found that the given sine lies between .57691 ($= \sin 35° 14'$) and .57715($= \sin 35° 15'$). The difference between them is $.57715 - .57691 = .00024$. The difference between the sine of the smaller angle, or .57691, and the given sine, or .57698, is $.57698 - .57691 = .00007$. Then, $\frac{.00007}{.00024} \times 60 = \frac{7}{24} \times 60 = 18''$, nearly, and the required angle is $35° 14' 18''$; or, $.57698 = \sin 35° 14' 18''$. Ans.

NOTE.—In practice, only the significant figures of the differences forming the terms of the function are used, the decimal point being dispensed with. Thus, $.57715 - .57691 = 24$, it being understood that this means 24 units of the fifth decimal order, or .00024.

EXAMPLE 2.—Find the angle whose cosine is .27052.

SOLUTION.—Looking in the table of Cosines, the given cosine is found to belong to a greater angle than 45° and therefore it must be looked for in the columns marked Cosine at the bottom of the page. It is found between the numbers .27060(= cos 74° 18′) and .27032(= cos 74° 19′). The difference between the two numbers is .27060 − .27032 = 28 units of the fifth order. The cosine of the smaller angle, or 74° 18′, is .27060, and the difference between this and the given cosine is .27060 − .27052 = 8 units of the fifth order. Hence, $\frac{8}{28} \times 60 = 17''$; and, therefore, .27052 = cos 74° 18′ 17″. Ans.

EXAMPLE 3.—Find the angle whose tangent is 2.15841.

SOLUTION.— 2.15841 falls between 2.15760(= tan 65° 08′) and 2.15925 (= tan 65° 9′). The difference between these numbers is 2.15925 − 2.15760 = 165 units of the fifth order; 2.15841 − 2.15760 = 81 units of the fifth order. Hence, $\frac{81}{165} \times 60 = 30''$, nearly, and therefore 2.15841 = tan 65° 8′ 30″. Ans.

EXAMPLE 4.—Find the angle whose cotangent is 1.26342.

SOLUTION.— 1.26342 falls between 1.26395(= cot 38° 21′) and 1.26319 (= cot 38° 22′). The difference between these numbers is 1.26395 − 1.26319 = .00076. Also, 1.26395 − 1.26342 = .00053. $\frac{53}{76} \times 60 = 42''$, and therefore 1.26342 = cot 38° 21′ 42″. Ans.

EXAMPLES FOR PRACTICE

1. Find: (*a*) the sine of 48° 17′; (*b*) the cosine; (*c*) the tangent.

Ans. (*a*) .74644; (*b*) .66545; (*c*) 1.12172

2. Find: (*a*) the sine of 13° 11′ 6″; (*b*) the cosine; (*c*) the tangent.

Ans. (*a*) .22810; (*b*) .97364; (*c*) .23427

3. Find: (*a*) the sine of 72° 0′ 2″; (*b*) the cosine; (*c*) the tangent.

Ans. (*a*) .95106; (*b*) .30901; (*c*) 3.07778

4. (*a*) Of what angle is .26489 the sine? (*b*) Of what angle is it the cosine?

Ans. (*a*) 15° 21′ 37″; (*b*) 74° 38′ 23″

5. (*a*) Of what angle is .688 the sine? (*b*) Of what angle is it the cosine? (*c*) Of what angle is it the tangent?

Ans. (*a*) 43° 28′ 20″; (*b*) 46° 31′ 40″; (*c*) 34° 31′ 40″

TABLE OF LOGARITHMIC FUNCTIONS

23. The student is already familiar with the use of the table of logarithms of numbers. As stated in Art. **17**, a **table of logarithmic functions** is a table containing the logarithms of the natural functions, these logarithms being, for convenience, called **logarithmic functions.** Thus, the logarithm of the sine of an angle is referred to as **the logarithmic sine** of the angle.

The connection between the tables can be seen from the following:

From table of natural functions, cot 44° . . . . . = 1.03553
From table of logarithms, log 1.03553 . . . . . = .01516
From table of logarithmic functions, log cot 44° = .01516

Few tables give the logarithmic secants and cosecants. These logarithmic functions may be obtained from the relations,

$$\sec A = \frac{1}{\cos A}, \csc A = \frac{1}{\sin A}$$

which give,

$$\log \sec A = -\log \cos A, \log \csc A = -\log \sin A$$

That is, instead of adding the logarithmic secant or cosecant, the logarithmic cosine or sine, respectively, may be subtracted. Likewise, instead of subtracting the logarithmic secant, the logarithmic cosine may be added, and instead of subtracting the logarithmic cosecant, the logarithmic sine may be added.

24. Description of the Table.—The table of logarithmic functions contains for every minute the logarithms, to five decimal places, of the trigonometric sines, cosines, tangents, and cotangents of angles from 0° to 90°. From 0° to 45°, the degrees are placed at the top of the page and the minutes in the column headed ′ on the left. From 45° to 90°, the degrees are at the bottom of the page, the minutes in the last whole column at the right, and the name of the trigonometric function is placed at the bottom of the column.

This arrangement is similar to that in the table of natural functions. It will be observed that the numbers of degrees at the top of the pages increase in the order of the pages from 0° to 44°, while those at the bottom decrease from 89° to 44°.

The general description of the table will be better understood by referring to one of its pages. Take, for instance, the page marked 11° at the top and 78° at the bottom. The first column on the left (marked ′) contains the natural numbers from 1 to 60. These numbers represent minutes. Horizontally opposite to these numbers, and in the columns marked at the top log sin, log tan, etc., are printed the logarithmic functions, each function being in the same horizontal line as the number of minutes by which the corresponding angle exceeds 11°. Thus, the logarithmic tangent of 11° 39′, which is $\bar{1}.31425$, is found in the column marked log tan at the top, and in the same horizontal line as the number 39 in the left-hand column. Similarly, the number $\bar{1}.99072$, being in the column marked at the top log cos, and in the same horizontal line as 48 in the left-hand column, is the logarithmic cosine of 11° 48′. In some tables, several mantissas are printed under and to the right of the same characteristic, and are understood to belong with that characteristic. Thus, in the logarithm just considered, only the mantissa .99072 is printed, the characteristic being the same as the first one found above that mantissa.

The last column but one (marked ′ at the bottom) contains the natural numbers from 1 to 60, increasing from bottom to top. It will be observed that any angle determined by the number of degrees at the bottom (78 in this case) and any number of minutes in the right-hand minute column, is the complement of the angle determined by the number of degrees at the top (11 in this case) and the number of minutes in the left-hand minute column, horizontally opposite the number of minutes in the right-hand minute column. Thus, the number 18 in the right-hand minute column is horizontally opposite the number 42 in the left-hand column, and we have, 78° 18′ + 11° 42′ = 90°. Therefore, since the fundamental functions of an angle are equal to the cofunctions of its complement,

$$\sin 11° \, 42' = \cos 78° \, 18'$$

$$\cot 11° \, 42' = \tan 78° \, 18', \text{ etc.}$$

and $$\log \sin 11° \, 42' = \log \cos 78° \, 18', \text{ etc.}$$

For this reason, the notation log tan is written at the bottom of the column headed log cot, to indicate that the logarithms in this column are the logarithmic tangents of angles whose number of degrees is the number (78 in this case) at the bottom of the page, and whose number of minutes is opposite those logarithms in the right-hand minute column. Similarly, the columns marked log sin, log tan, and log cos at the top are marked, respectively, log cos, log cot, and log sin at the bottom.

25. After the column marked log sin there is a column marked d. This column contains the differences, expressed in units of the fifth decimal order, between the consecutive logarithmic sines given in the sine column. Thus, referring to the page headed 11°, the first number in the d-column following the sine column is 65; it will be observed that this number is opposite the space between the logarithmic sines $\bar{1}.28125$ and $\bar{1}.28060$, and is the difference, in units of the fifth decimal order, or expressed in hundred thousandths, between these two logarithmic sines. These differences are called **tabular differences.** Similar differences are printed in the column marked d after the cosine column, and in the column marked c. d. between the tangent and the cotangent column. The notation c. d. means *common difference*, as the differences between the successive logarithmic tangents are the same as those between the corresponding cotangents. although obtained by reversing the order in which the functions are subtracted; that is to say, $\log \tan A - \log \tan B = \log \cot B - \log \cot A$.

The tabular differences for the cosines are not given in the first ten pages, both for want of space and because they are so small that they can be readily determined by mental subtraction.

The use of the tabular differences, the use and contents of the column marked p. p. in all pages but the first three, and the peculiarities and applications of these first three pages of the table will be explained further on.

26. To Find the Logarithmic Functions of an Angle Having No Odd Seconds.

Rule.—*For an angle less than 45°, look for the degrees at the top of the page and for the minutes in the column (marked ′) at the left of the page on which the number of degrees is found. Then look across the page along the horizontal row containing the given number of minutes, into the column headed by the name of the function whose logarithm is required. The desired logarithm is found in this row and column.*

For an angle between 45° and 90°, find the degrees at the bottom of the page and the minutes in the column (marked ′) at the right of the page. Then look across the page, along the horizontal row containing the given number of minutes, into the column marked at the bottom with the name of the function whose logarithm is to be found. The row and column thus determined contain the desired logarithm.

EXAMPLE 1.—Find the logarithmic sine and the logarithmic tangent of 15° 24′.

SOLUTION.—On the page marked 15° at the top, in the column headed log sin, and in the same horizontal row with 24, the number $\bar{1}.42416$ is found; and in the column headed log tan, the number $\bar{1}.44004$ is found. Hence,

$$\log \sin 15° \, 24' = \bar{1}.42416. \quad \text{Ans.}$$
$$\log \tan 15° \, 24' = \bar{1}.44004. \quad \text{Ans.}$$

EXAMPLE 2.—Find the logarithmic tangent and cosine of 73° 10′.

SOLUTION.—As 73 is greater than 45, it is found at the bottom of the page. Looking for the number of minutes (10′) in the right-hand minute column, and following the horizontal row determined by this number into the column marked log tan at the bottom, the number .51920 is found. Likewise, the number $\bar{1}.46178$ is found in the column marked log cos at the bottom, and horizontally opposite the number 10 in the right-hand minute column. Therefore,

$$\log \tan 73° \, 10' = .51920. \quad \text{Ans.}$$
$$\log \cos 73° \, 10' = \bar{1}.46178. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find: (*a*) the logarithmic cosine of 36° 58′; (*b*) the logarithmic tangent.

Ans. { (*a*) $\bar{1}.90254$; (*b*) $\bar{1}.87659$ }

2. Find: (*a*) the logarithmic tangent of 23° 39′; (*b*) the logarithmic cotangent.

Ans. { (*a*) $\bar{1}.64140$; (*b*) .35860 }

3. Find: (*a*) the logarithmic sine of 79° 45′; (*b*) the logarithmic cosine.

Ans. { (*a*) $\bar{1}.99301$; (*b*) $\bar{1}.25028$ }

4. Find: (*a*) the logarithmic tangent of 46° 59′; (*b*) the logarithmic cotangent.

Ans. { (*a*) .03009 ; (*b*) $\bar{1}.96991$ }

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27. To Find the Logarithmic Functions of an Angle Containing an Odd Number of Seconds.—Let the number of degrees and minutes in an angle any of whose logarithmic functions is required be denoted by A, and the number of odd seconds by s. Thus, if the angle is 37° 43′ 19″, A will equal 37° 43′, and s will equal 19″; also, $A + 1'$, or $A + 60''$, will equal 37° 43′ + 1′, or 37° 44′. (See Art. **20.**) Since the table gives the logarithmic functions of any angle containing no odd seconds, the logarithmic functions of A and $A + 1'$ may be readily found, as explained in the last article. Let these logarithmic functions be denoted by l and l', respectively, and the required logarithmic function by L. In the general theory of logarithms, treated in advanced works on mathematics, it is shown that if two consecutive angles (as 37° 43′ and 37° 44′) are taken from the table, the difference between any logarithmic function of the greater and the same logarithmic function of the smaller angle is to the difference between the same logarithmic function of any intermediate angle (as 37° 43′ 19″) and the same function of the smaller angle, as the difference between the greater and the smaller angle is to the difference between the intermediate and the smaller angle. If the notation $F(A)$, read *function of A*, is employed to denote any logarithmic function of an angle A, we have, writing $A + 60''$ instead of $A + 1'$,

$$\frac{F(A+60'')-F(A)}{F(A+s)-F(A)}=\frac{(A+60'')-A}{(A+s)-A}=\frac{60}{s}$$

that is,
$$\frac{l'-l}{L-l}=\frac{60}{s}$$

whence,
$$L-l \doteq (l'-l)\frac{s}{60}$$

and
$$L=l+(l'-l)\frac{s}{60}$$

The difference between l' and l, being the difference between two consecutive logarithmic functions, may be taken from the column of tabular differences in the table. (See Art. **25.**) Denoting the tabular difference $l'-l$ by D, the preceding equation becomes

$$L=l+D\times\frac{s}{60}$$

It should be observed that, since the sine and the tangent increase with the angle, while the cosine and cotangent decrease as the angle increases, $l'-l$ is positive or negative according as the functions considered are fundamental functions (sine, tangent) or cofunctions (cosine, cotangent). In the latter case, D in the formula should be treated as negative; that is, the product $D\times\frac{s}{60}$ should be subtracted from l.

It should also be borne in mind that the tabular difference D is expressed in units of the fifth order of decimals, or hundred thousandths: Thus, if the number of seconds s is 15, and the tabular difference is 36, the quantity to be added to l is $.00036\times\frac{15}{60}=.00009$.

If $l=\bar{1}.59812$, the work is arranged as follows:

$$\begin{aligned} l &= \bar{1}.59812\\ D\times\frac{s}{60} &= \qquad\;\; 9\\ \hline L &= \bar{1}.59821\end{aligned}$$

When, as in this case, the product $D\times\frac{s}{60}$ is small, it can readily be added or subtracted mentally. Only the significant figures of D (those given in the d-column) are used, it being understood that the result expresses units of the fifth order of decimals. Thus, instead of writing $D = .00036$, and $D \times \frac{s}{60} = .00036 \times \frac{s}{60}$, the following abbreviated notation is used: $D = 36$; $D \times \frac{s}{60} = 36 \times \frac{s}{60}$, the latter product expressing decimal units of the fifth order, or hundred thousandths.

The foregoing formula indicates the process by which the logarithmic functions of an angle containing odd seconds are obtained. It may be stated in words as follows:

Rule.—*Drop the seconds, and find the logarithmic function of the remaining angle. Find the tabular difference between this logarithmic function and the same function of the angle next higher in the table. Multiply this tabular difference by the number of seconds in the angle and divide the product by 60. Add this result to or subtract it from the logarithm found, according as the logarithm to be determined is that of a fundamental function or that of a cofunction. The result thus obtained is the required logarithmic function.*

EXAMPLE 1.—Find: (*a*) the logarithmic sine of 15° 40′ 32″; (*b*) the logarithmic cosine.

SOLUTION.—(*a*) Dropping the seconds, 15° 40′ is obtained, whose logarithmic sine, found as in Art. **25**, is $\bar{1}.43143$; that is, $l = \bar{1}.43143$. Opposite the space between this logarithm and the following, and in the column marked d, is found the tabular difference 45($= D$). Applying the formula given in Art. **27**,

$$L = \bar{1}.43143 + .00045 \times \tfrac{32}{60}$$

$$l = \bar{1}.43143$$

$$D \times \frac{s}{60} = 45 \times \frac{32}{60} = \quad 24$$

$$L = \bar{1}.43167$$

that is, log sin 15° 40′ 32″ = $\bar{1}.43167$. Ans.

In practice, it is not necessary to write all the figures of l before adding the correction $D \times \frac{s}{60}$. Having found the value of l in the table, one places and keeps the finger on that value and calculates the correction $D \times \frac{s}{70}$. In the majority of cases, this correction can be added mentally to l. Thus, in the example just explained, the correction is 24, which, being mentally added to the number 43 formed by the last two figures of l, gives 67 as the last two figures of L. The other figures of L are the same as those of l.

(*b*) The logarithmic cosine of 15° 40′ is $\bar{1}.98356(=l)$. Horizontally opposite the space between this logarithm and the following, the tabular difference $4(=D)$ is found in the column marked d on the right of the cosine column. As the function under consideration is a cofunction, the correction $D\times\frac{s}{60}$ must be subtracted for l. We have, then,

$$l = \bar{1}.98356$$

$$D\times\frac{s}{60} = 4\times\frac{32}{60} = \quad 2, \text{ to the nearest unit}$$

$$L = \bar{1}.98354$$

Therefore, $\log\cos 15°\ 40'\ 32'' = \bar{1}.98354$. Ans.

In practice, the correction 2 would be subtracted mentally, without previously writing the value of l.

Example 2.—Find the logarithmic tangent of 63° 39′ 27″.

Solution.—Dropping the seconds, and referring to the page marked 63° at the bottom, the logarithmic tangent of 63° 39′ is found to be $.30512(=l)$. Since in this case the angles increase from bottom to top, the tabular difference to be used is that horizontally opposite the space between the logarithm just taken and the one immediately above it in the column (that is, .30543). This difference is 31, printed in the column marked c. d. on the left of the cotangent column. We have, therefore,

$$l = .30512$$

$$\frac{s}{60}\times D = \frac{27}{60}\times 31 = \quad 14, \text{ to the nearest unit}$$

$$L = .30526$$

Therefore, $\log\tan 63°\ 39'\ 27'' = .30526$. Ans.

Example 3.—Find the logarithmic cotangent of 54° 8′ 9″.

Solution.—Dropping the seconds, the value of l is found to be $\bar{1}.85913$. The tabular difference in the c. d. column and horizontally opposite the space between this logarithm and the one immediately above it is 26. As the cotangent is a cofunction, the correction $\frac{s}{60}\times D$ is to be subtracted from l. Then,

$$l = \bar{1}.85913$$

$$\frac{s}{60}\times D = \frac{9}{60}\times 26 = \quad 4$$

$$L = \bar{1}.85909$$

Therefore, $\log\cot 54°\ 8'\ 9'' = \bar{1}.85909$. Ans.

EXAMPLES FOR PRACTICE

1. Find the logarithmic sine, tangent, and cosine of 33° 21′ 46″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.74032\\ \log\tan = \bar{1}.81852\\ \log\cos = \bar{1}.92179\end{cases}$$

2. Find the logarithmic sine and cotangent of 23° 3′ 17″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.59286\\ \log\cot = .37100\end{cases}$$

3. Find the logarithmic tangent and cosine of 49° 12′ 12″.

$$\text{Ans.}\begin{cases}\log\tan = .06395\\ \log\cos = \bar{1}.81516\end{cases}$$

4. Find the logarithmic sine, tangent, and cosine of 72° 52′ 49″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.98031\\ \log\tan = .51143\\ \log\cos = \bar{1}.46889\end{cases}$$

5. Find the logarithmic sine and cotangent of 81° 38′ 28″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.99536\\ \log\cot = \bar{1}.16712\end{cases}$$

6. Find the logarithmic tangent and cosine of 65° 0′ 47″.

$$\text{Ans.}\begin{cases}\log\tan = .33159\\ \log\cos = \bar{1}.62574\end{cases}$$

7. Find the logarithmic secant and cosecant of 59° 0′ 9″.

$$\text{Ans.}\begin{cases}\log\sec = .28819\\ \log\csc = .06692\end{cases}$$

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28. Use of the Column of Proportional Parts.—The method described in the preceding article can be applied to any table of logarithmic functions. Some tables, however, among them the table furnished with this Course, contain a column giving the products of the tabular differences by the fractions $\frac{6}{60}$, $\frac{7}{60}$, $\frac{8}{60}$, $\frac{9}{60}$, $\frac{10}{60}$, $\frac{20}{60}$, $\frac{30}{60}$, $\frac{40}{60}$, and $\frac{50}{60}$. These products are called **proportional parts**, and are given in the right-hand column (marked p. p. at the top) of each page, beginning with 3°. The tabular differences are here printed in heavy figures. Under each tabular difference are given the products of it by $\frac{6}{60}$, $\frac{7}{60}$, etc., the number of sixtieths being printed horizontally opposite the product, on the left of a vertical line. Thus, referring to the right-hand column of the page marked 13° at the top, the numbers 54, 53, 52, printed in heavy type, are tabular differences. The number 27, directly under 54, and horizontally opposite the number 30 on the left of the vertical line, is the product of 54 by $\frac{30}{60}$. Likewise, 17.3, found under 52, and horizontally opposite 20, is the product of 52 by $\frac{20}{60}$. The proportional parts for 1, 2, 3, 4, 5 are obtained from those for 10, 20, 30, etc., by moving the decimal point one place to the left. Thus, the proportional part for 20, under the tabular difference 52, is 17.3, as just explained. The proportional part for 2, that is, the product of 52 by $\frac{2}{60}$, is 1.73.

In the first three pages of the logarithmic table, no proportional parts are given, the use of these pages being different from that of the others. In pages 45, 46, and 47, not all the tabular differences are given in the p. p. column, owing to want of space; but the proportional part for any tabular difference is easily obtained by means of the proportional parts for digits given at the bottom of the p. p. column. Referring, for example, to page 45, the tabular difference 215, which is found in the c. d. column, does not appear in the p. p. column. If we wish to find the product of 215 by $\frac{30}{60}$, we look in the p. p. column for the tabular difference next lower than 215, which is 212. Horizontally opposite 30, and under 212, we find 106; that is, $212 \times \frac{30}{60} = 106$. As $215 = 212 + 3$, we must add to the product just found (106), the product of $3 \times \frac{30}{60}$. This is taken from the column headed 3 near the bottom of the p. p. column: there we find 1.5 horizontally opposite 30; that is, $3 \times \frac{30}{60} = 1.5$. Therefore, $215 \times \frac{30}{60} = 106 + 1.5 = 107.5$. The addition of these two products can usually be effected mentally.

The correction $D \times \frac{s}{60}$ to be applied to l in order to find L (formula of Art. 27) is found from the table of proportional parts as follows:

Rule.—*Having found the tabular difference D, look for this difference in the column of proportional parts. If this difference is found in that column and the number of seconds is a digit greater than 5 or a digit followed by a cipher, look for it on the left of the vertical line under D; the correction is then found horizontally opposite this number, and directly under D. If the* *number of seconds is a digit less than 6, add a cipher, find the proportional part corresponding to the resulting number, and move the decimal point one place to the left. If the number of seconds consists of two significant digits (as 39), find the correction for the first digit followed by a cipher, and that for the second digit, and add the two corrections. (Thus, if the number of seconds is 43, the correction is found by adding the corrections for 40 and 3.)*

If the tabular difference D is not found in the p. p. column (which may happen only on pages 45 to 47), take, as just explained, the proportional part corresponding to the next lower tabular difference found in the p. p. column; then, from the digit columns found at the bottom of the p. p. column, find the proportional part corresponding to the difference between D and the tabular difference just used. Add the two proportional parts thus found.

EXAMPLE 1.—Find: (*a*) the logarithmic tangent of 22° 17′ 8″; (*b*) the logarithmic cosine.

SOLUTION.—(*a*) Dropping the seconds, we find log tan 22° 17′ = $\bar{1}.61256 (= l)$; $D = 36$. Turning to the column of proportional parts, 36 is found in heavy type near the top of the page. Following the horizontal row that begins with 8 (number of seconds) at the left of the vertical line under 36, we find in that row, and directly under 36, the correction 4.8, which may be called 5, as there are no other numbers to be combined with it. Therefore,

$$l = \bar{1}.61256$$

$$\frac{s}{60} \times D = \text{p. p.} = \quad 5$$

$$L = \bar{1}.61261$$

That is, log tan 22° 17′ 8″ = $\bar{1}.61261$. Ans.

(*b*) l = log cos 22° 17′ = $\bar{1}.96629$; $D = 5$. Looking for the column headed 5 among the proportional parts, the correction .7 (or say 1) is found directly under 5 and horizontally opposite 8. Therefore,

$$l = \bar{1}.96629$$

$$\frac{s}{60} \times D = \text{p. p.} = \quad 1$$

$$L = \bar{1}.96628$$

That is, log cos 22° 17′ 8″ = $\bar{1}.96628$. Ans.

EXAMPLE 2.—Find the logarithmic sine of 3° 18′ 9″.

SOLUTION.— l = sin 3° 18′ = $\bar{2}.76015$; $D = 219$. The difference 219 is not found in the p. p. column; the tabular difference in the p. p. column next lower is 216. Under 216, and horizontally opposite 9, is found 32.4. The difference between 219 and 216 is 3. Looking for 3 in the digit columns at the bottom of the p. p. column, .5 is found under 3, and horizontally opposite 9. Therefore, $219 \times \frac{9}{10} = 32.4 + .5 = 33$, nearly.

$$\begin{aligned} l &= \bar{2}.76015 \\ 219 \times \frac{9}{60} &= \quad\quad 33 \\ L &= \bar{2}.76048 \end{aligned}$$

That is, $\log 3° 18' 9'' = \bar{2}.76048$. Ans.

Example 3.—Find: (*a*) the logarithmic tangent of 53° 47′ 04″; (*b*) the logarithmic cosine.

Solution.—(*a*) $l = \log \tan 53° 47' = .13529$; $D = 26$; the proportional part for 40, under D, that is, under 26, is 17.3; the proportional part for 4 is $\frac{17.3}{10}$, or 2, nearly.

$$\begin{aligned} l &= .13529 \\ 26 \times \frac{4}{60} &= \quad\quad 2 \\ L &= .13531 \end{aligned}$$

That is, $\log \tan 53° 47' 4'' = .13531$. Ans.

(*b*) $l = \log \cos 53° 47' = \bar{1}.77147$; $D = 17$. The number horizontally opposite 40, in the column headed 17 among the proportional parts, is 11.3; the proportional part for 4 is, therefore, $\frac{11.3}{10} = 1$, nearly.

$$\begin{aligned} l &= \bar{1}.77147 \\ 17 \times \frac{4}{60} &= \quad\quad 1 \\ L &= \bar{1}.77146 \end{aligned}$$

That is, $\log \cos 53° 47' 4'' = \bar{1}.77146$. Ans.

Example 4.—To find the logarithmic cotangent of 72° 35′ 47″.

Solution.— $l = \log \cot 72° 35' = \bar{1}.49652$; $D = 45$. Looking among the proportional parts for the column headed 45, the correction for 40 is found to be 30, and that for 7 is found to be 5.3. Therefore,

$$\begin{aligned} l &= \bar{1}.49652 \\ \text{p. p. for } 40 = 30.0 & \\ \text{p. p. for } 7 = \underline{\ 5.3} & \\ \text{p. p. for } 47 = \quad\quad & \quad\quad 35 \\ L &= \bar{1}.49617 \end{aligned}$$

That is, $\log \cot 72° 35' 47'' = \bar{1}.49617$. Ans.

In practice, it would not be necessary to write down the corrections 30 and 5.3, which would be added mentally. The same remark applies to all similar cases.

EXAMPLES FOR PRACTICE

1. Find the logarithmic sine and cotangent of 9° 39′ 17″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.22456\\ \log\cot = .76924\end{cases}$$

2. Find the logarithmic sine, tangent, and cosine of 39° 8′ 52″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.80025\\ \log\tan = \bar{1}.91065\\ \log\cos = \bar{1}.88959\end{cases}$$

3. Find the logarithmic cotangent and cosecant of 80° 3′ 46″.

$$\text{Ans.}\begin{cases}\log\cot = \bar{1}.24352\\ \log\csc = .00657\end{cases}$$

4. Find the logarithmic sine, secant, and tangent of 49° 0′ 54″.

$$\text{Ans.}\begin{cases}\log\sin = \bar{1}.87788\\ \log\sec = .18319\\ \log\tan = .06107\end{cases}$$

5. Find the logarithmic tangent and cosine of 4° 2′ 4″.

$$\text{Ans.}\begin{cases}\log\tan = \bar{2}.84838\\ \log\cos = \bar{1}.99892\end{cases}$$

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29. To Find the Angle Corresponding to Any Logarithmic Function When the Given Function Is Found in the Table.—In this case, the angle, which contains no odd seconds, is found as follows:

Rule.—*Find the given logarithm in the column marked by the name of the function whose logarithm is given. Then, if the name of the given function is at the top of the column, the number of degrees in the angle is that at the top of the page, and the number of minutes is horizontally opposite the logarithm, in the left-hand minute column. If the name of the function is at the foot of the column, the number of degrees in the angle is that at the foot of the page, and the number of minutes is in the right-hand minute column, horizontally opposite the given logarithm.*

In searching the table for a given logarithm, it should be borne in mind that the logarithmic sines and tangents increase, and the cosines and cotangents decrease, from 0° to 90°. Therefore, in the columns marked log sin and log tan at the top, the logarithms increase, and in the columns headed log cos and log cot the logarithms decrease, from the first to the last page. The sines and tangents continue to increase, and the cosines and cotangents to decrease, from the last page to the first, in the columns marked with the names of these functions, respectively, at the bottom. Thus, the last page contains, in the column headed log sin, the logarithmic sines of the angles between 44° and 45°. The sines are continued in the column marked log sin at the bottom, which contains the logarithmic sines of the angles between 45° and 46°; the preceding page contains the sines of angles between 46° and 47°, etc. Here the logarithmic sines increase from bottom to top, and in the inverse order of the pages.

When looking for a given logarithmic sine, open the table at random. Glance at both of the sine columns, that is, the column marked log sin at the top and the column marked log sin at the bottom, and compare the logarithms in them with the given logarithm. If the given logarithm is less than those found in the column marked log sin at the top, said given logarithm must be in that column, but in a preceding page. If the given logarithm is greater than those in the column marked log sin at the bottom, said given logarithm must be in that column, but in a preceding page. If neither of these is the case, the given logarithm must be in a subsequent page. Turn a few pages forwards or backwards, as the case may be, and repeat the operation. The comparison of the two columns, however, is not usually necessary after the first three figures of the given logarithm have been found in one of them, as that logarithm is then found in that column, and can be readily seen among the logarithms beginning with those three figures.

Proceed exactly in the same manner when the given function is a cosine; that is, treat the cosine as though it were a sine; but, having found the given logarithm, treat it as that of a cosine and take the angle accordingly.

As the tangents of angles less than 45° are less than 1, their logarithmic tangents have negative characteristics, and as the tangents of angles greater than 45° are greater than 1, their logarithmic tangents have positive characteristics. Therefore, a logarithmic tangent should be looked for in the column marked log tan at the top or at the bottom, according as its characteristic is negative or positive. For a logarithmic cotangent, the rule should be reversed.

EXAMPLE 1.—Find the angle whose logarithmic sine is $\bar{1}.57669$.

SOLUTION.—Opening the table at random, say at the page marked 36° at the top, it is at once seen that the logarithms in the column marked log sin at the top are greater than the given logarithm. This logarithm must, therefore, be in that column, but in a preceding page. Turning the pages backwards, a few at a time, the given logarithm is found on page 64, among those logarithms whose first three figures are $\bar{1}.57$. As the name of the function is at the head of the column, the number of degrees (22) is taken from the top of the page, and that of minutes (10) from the left-hand minute column. Therefore, the angle whose logarithmic sine is $\bar{1}.57669$ is 22° 10′, or $\bar{1}.57669 = \log \sin 22° 10'$.

Suppose that the table had first been opened at page 56. Since the given logarithm is greater than those in the column marked log sin at the top and less than those in the column marked log sin at the bottom (or log cos at the top), the given logarithm is to be found in a subsequent page. Suppose also that, turning the pages forwards, a few at a time, we come to page 63, and find the first three figures ($\bar{1}.57$) of the given logarithm in the column marked log sin at the top. Then, without consulting the other column, we follow the former column to the bottom, and into the next page, where we find the given logarithm, and take the corresponding angle as before.

EXAMPLE 2.—To find the angle whose logarithmic sine is $\bar{1}.89810$.

SOLUTION.—Open the table at random, say at page 73. Since the given logarithm is greater than those in the column marked log sin at the top, and less than those in the column marked log sin at the bottom, it must be found in a subsequent page. Suppose that we turn next to page 85. We see at once that the given logarithm is greater than those in the column headed log sin, and also than those in the column marked log sin at the bottom. Therefore, it must be in the latter column in some preceding page. Turning the pages backwards, we find the first three figures ($\bar{1}.89$) of the given logarithm on page 79, and among the logarithms to which these three figures are common, we find $\bar{1}.89810$. As this is a logarithmic sine, and the name sine is at the bottom of the column, the degrees in the corresponding angle are taken from the bottom of the page, and the minutes from the right-hand minute column. Therefore, 52° 16′ is the angle whose logarithmic sine is $\bar{1}.89810$; that is, $\bar{1}.89810 = \log \sin 52° 16'$. Ans.

EXAMPLE 3.—Find the angle whose logarithmic cosine is $\bar{1}.86924$.

SOLUTION.—Treating this as though it were a logarithmic sine, it is found, as explained above, on page 84, in the column marked log sin at the bottom. Since the name cosine is at the top of the column, the required angle is 42° 16′. That is, $\bar{1}.86924$ = log cos 42° 16′. Ans.

Example 4.—Find the angle whose logarithmic cotangent is .15639.

Solution.—As the characteristic is positive, the logarithm should be looked for in the column marked log cot at the top. After looking in a few pages, the first three figures (0.15) of the logarithm are found on page 76, and among them is found the given logarithm. The name of the function being at the head of the column, the degrees in the angle are taken from the top of the page, and the minutes from the left-hand minute column. Therefore, .15639 = log cot 34° 54′. Ans.

EXAMPLES FOR PRACTICE

1. Find the angle whose logarithmic sine is $\bar{1}.57885$. Ans. 22° 17′
2. Find the angle whose logarithmic sine is $\bar{1}.66731$. Ans. 27° 42′
3. Find the angle whose logarithmic sine is $\bar{2}.93740$. Ans. 4° 58′
4. Find the angle whose logarithmic sine is $\bar{1}.98345$. Ans. 74° 17′
5. Find the angle whose logarithmic cosine is $\bar{1}.92086$. Ans. 33° 33′
6. Find the angle whose logarithmic cosine is $\bar{1}.57232$. Ans. 68° 4′
7. Find the angle whose logarithmic cosine is $\bar{1}.84949$. Ans. 45° 0′
8. Find the angle whose logarithmic tangent is $\bar{1}.97649$. Ans. 43° 27′
9. Find the angle whose logarithmic cotangent is $\bar{2}.89274$. Ans. 85° 32′
10. Find the angle whose logarithmic tangent is .67377. Ans. 78° 2′
11. Find the angle whose logarithmic cotangent is .35517. Ans. 23° 49′
12. Find the angle whose logarithmic tangent is 1.28060. Ans. 87° 0′

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30. To Find the Angle Corresponding to a Given Logarithmic Function When the Function Is Not in the Table.—*Without the Use of Proportional Parts.*—From the formula given in Art. 27, the following may be obtained:

$$s = \frac{(L - l) \times 60}{D} = \frac{(L - l) \times 60}{l' - l}$$

Therefore, if the function L is given and it is found to lie between the consecutive logarithms l and l', the corresponding angle $A + s$ is that corresponding to l increased by the number of seconds determined by the formula just given. It will be remembered (see Art. **27**) that l and l' are, respectively, the logarithmic functions of two angles (A and $A + 1'$) differing by one minute. If the function is a fundamental function (sine or tangent) l' is greater than l; and since L lies between l and l', L is also greater than l; therefore, both $L - l$ and $l' - l$ are positive. If the function is a cofunction, l is greater than l', and also greater than L; therefore, both $L - l$ and $l' - l$ are negative, and $\frac{L - l}{l' - l}$ is positive. In such case, however, it is better to write this fraction in the form $\frac{l - L}{l - l'}$.

From the formula and the explanations just given, the following rule is derived for finding the angle corresponding to any given logarithmic function:

Rule.—*Find in the table the two consecutive logarithmic functions between which the given function lies. The degrees and minutes in the smaller of the angles corresponding to these two functions are the degrees and minutes in the required angle.*

Find the difference between the given function and that of the smaller angle; multiply that difference by 60, and divide the product by the tabular difference between the two functions in the table. The result will be the number of odd seconds in the required angle.

As the tabular difference is expressed in units of the fifth decimal order, the difference $L - l$ should be likewise expressed. Thus, if $L = \bar{1}.25198$, and $l = \bar{1}.25168$, the difference $L - l$ will be called 30.

Example 1.—Find the angle whose logarithmic sine is $\bar{1}.47867$ ($= L$).

Solution.—The first three figures of the given logarithm are always found in the table, and this makes it easy to determine the functions between which the given logarithm lies. Searching the sine columns of the table, it is found that $\bar{1}.47867$ lies between $\bar{1}.47854$($= l$) and

$\bar{1}.47894(= l')$ on page 59. The smaller of the two angles corresponding to these two logarithms is $17° 31'(= A)$. Now, $L - l = 13$, $l' - l$ (tabular difference taken from table) $= 40$. Therefore,

$$s = \frac{13 \times 60}{40} = 19.5'', \text{ or, say, } 20''$$

and $\quad A + s = 17° 31' + 20'' = 17° 31' 20''$

that is, $\quad \bar{1}.47867 = \log \sin 17° 31' 20''$. Ans.

EXAMPLE 2.—Find the angle whose logarithmic tangent is .27743 $(= L)$.

SOLUTION.—As the characteristic is positive, the logarithms between which L lies should be looked for in the column marked log tan at the bottom. These two logarithms are $.27738(= l)$ and $.27769(= l')$. The smaller angle corresponds to .27738, and is $62° 10'(= A)$. Also,

$$L - l = 5, \; l' - l(= D) = 31$$

$$A + s = A + \frac{5 \times 60}{31} = 62° 10' + 10'', \text{ nearly, } = 62° 10' 10''$$

that is, $\quad .27743 = \log \tan 62° 10' 10''$. Ans.

EXAMPLE 3.—Find the angle whose logarithmic cotangent is $\bar{1}.85899(= L)$.

SOLUTION.— L is found to lie between $\bar{1}.85887(= l')$ and $\bar{1}.85913$ $(= l)$. It will be noticed that here l is the greater, and l' the smaller of the two logarithms. Angle corresponding to $l = 54° 8'(= A)$.

$$l = \bar{1}.85913$$
$$L = \bar{1}.85899$$
$$l - L = 14; \; l - l' = 26$$

$$A + s = 54° 8' + \frac{14 \times 60}{26} = 54° 8' 32'', \text{ nearly}$$

that is, $\quad \bar{1}.85899 = \log \cot 54° 8' 32''$. Ans.

EXAMPLES FOR PRACTICE

1. Find the angle whose logarithmic sine is $\bar{1}.45566$.
Ans. 16° 35′ 27″

2. Find the angle whose logarithmic tangent is $\bar{1}.33471$.
Ans. 12° 11′ 44″

3. Find the angle whose logarithmic sine is $\bar{1}.89798$.
Ans. 52° 14′ 42″

4. Find the angle whose logarithmic cosine is $\bar{1}.67412$.
Ans. 61° 49′ 23″

5. Find the angle whose logarithmic cosine is $\bar{1}.92386$.
Ans. 32° 56′ 45″

6. Find the angle whose logarithmic cotangent is .54139.

Ans. 16° 2′ 20″

7. Find the angle whose logarithmic tangent is $\bar{1}.86712$.

Ans. 36° 22′ 7″

8. Find the angle whose logarithmic cosine is $\bar{1}.99785$.

Ans. 5° 42′ 0″

9. Find the angle whose logarithmic cotangent is $\bar{1}.12345$.

Ans. 82° 25′ 52″

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31. *With the Use of Proportional Parts.*—Having found the degrees and minutes in the angle as in the preceding case, the number s of odd seconds may be conveniently found from the column of proportional parts. In order to facilitate the explanations that follow, the proportional parts corresponding to the tabular difference 105 are here copied from page 48 of the table. It will, therefore, be assumed that the value of D is 105, and, for what is said below, the student should refer to these proportional parts. Such being the case, the formula given at the beginning of the preceding article may be written,

	105
6	10.5
7	12.3
8	14.0
9	15.8
10	17.5
20	35.0
30	52.5
40	70.0
50	87.5

$$s = \frac{(L - l) \times 60}{105}$$

The value $L - l$, which is the difference between the given logarithm and the logarithm of the degrees and minutes (A) in the required angle, is readily determined, as already explained. It is only necessary to repeat that, if the function is a cofunction, $l - L$ should be used instead of $L - l$. Since the numbers on the right of the vertical line are the products of $\frac{105}{60}$ by the numbers on the left, it follows that the numbers on the left are the products of those on the right by $\frac{60}{105}$. Thus, $52.5 = \frac{105}{60} \times 30$, and $30 = 52.5 \times \frac{60}{105} = \frac{52.5 \times 60}{105}$. Therefore, if $L - l$ is found among the numbers directly under 105, the value of s is the number on the left of the vertical column horizontally opposite $L - l$. For example, if $L - l = 35$, then $s = 20''$. If $L - l = 16$, then

$s = 9''$, the number 9 being opposite 15.8, which, to the nearest unit, may be called 16.

It will be remembered that the proportional parts opposite 10, 20, 30, 40, 50, when divided by 10 (that is, when the period is moved one place to the left), give the products of $\frac{105}{60}$ by 1, 2, 3, 4, and 5. From those parts we may, therefore, find by inspection the products of $\frac{105}{60}$ by all the digits from 1 to 9; and, in what follows, we shall proceed as if the products 1.75, 3.50, 5.25, 7.00, 8.75 of $\frac{105}{60}$ by 1, 2, 3, 4, and 5 were actually printed in the table opposite those digits; that is, it will be assumed that the proportional parts run in this order: 1.75, 3.50, 5.25, 7.00, 8.75, 10.5, 12.3, 14.0, etc., up to 87.5, the corresponding numbers on the left being, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50. The proportional parts 1.75, 3.50, 5.25, 7.00, 8.75 will be referred to as proportional parts found in the table, corresponding to 1, 2, 3, 4, and 5 seconds, respectively.

This being understood, the number s of odd seconds in the angle is determined as follows:

Rule.—*Find l, l', $L - l$, and $l' - l$ (= tabular difference, or D), as before. Look for the tabular difference D in the column of proportional parts. Look for $L - l$ in the column of proportional parts directly under D. If $L - l$ is found there, the number horizontally opposite it on the left of the vertical line is the required number of seconds s. If $L - l$ is not found under D, take the proportional part next lower, which call p. Find the difference between $L - l$ and p, and look among the proportional parts under D for this difference, or the part nearest to it, whether higher or lower. Call this part p'. Add the numbers horizontally opposite p and p' on the left of the vertical line. The result will be the required number of seconds s.*

EXAMPLE 1.—Find the angle whose logarithmic tangent is $\bar{1}.42822 (= L)$.

SOLUTION.— $l = \bar{1}.42805$, $A = 15° 0'$, $L - l = 17$, $D = 51$. Looking in the column marked p. p. for 51, the number 17(= $L - l$) is found under it, horizontally opposite the number 20 on the left of the vertical column. Therefore, $s = 20''$, and

$$\bar{1}.42822 = \log \tan 15° \; 0' \; 20''. \quad \text{Ans.}$$

Example 2.—Find the angle whose logarithmic cosine is $\bar{1}.52783(=L)$.

Solution.— $l=\bar{1}.52811$, $A=70°\ 17'$, $l-L=28$, $D=36$. The proportional part under 36 next lower than 28 is 24; $28-24=4$; the proportional part nearest 4 is 4.2; the number horizontally opposite 24 is 40; and the number horizontally opposite 4.2 is 7; hence, $s=40+7=47''$, and therefore

$$\bar{1}.52783=\log\cos 70°\ 17'\ 47''. \quad \text{Ans.}$$

Example 3.—Find the angle whose logarithmic sine is $\bar{1}.66191(L)$.

Solution.— $l=\bar{1}.66173$; $A=27°\ 19'$; $L-l=18$; $D=24$. Looking in the p. p. column for 24, the proportional part next lower than 18 is $16(=p)$, horizontally opposite which is 40. $18-p=18-16=2$. This difference is found among the proportional parts in the table (since it is the same as 20 with the decimal point moved one place to the left), and corresponds to $5''\left(=\frac{50}{10}\right)$. Therefore, $s=40+5=45''$, and

$$\bar{1}.66191=\log\sin 27°\ 19'\ 45''. \quad \text{Ans.}$$

Example 4.—Find the angle whose logarithmic cotangent is $\bar{1}.00375(=L)$.

Solution.— $l=\bar{1}.00427$; $A=84°\ 14'$; $l-L=52$; $D=126$. The proportional part under 126 next lower than 52 is $42(=p)$, which corresponds to $20''$; $52-42=10$. The proportional part nearest to 10 is $10.50\left(=\frac{105.0}{10}\right)$, which corresponds to $5''\left(=\frac{50}{10}\right)$. Therefore, $s=20''+5''=25''$, and

$$\bar{1}.00375=\log\cot 84°\ 14'\ 25''. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Find the angle whose logarithmic sine is $\bar{1}.78988$. Ans. 38° 3′ 20″

2. Find the angle whose logarithmic tangent is $\bar{1}.78540$.
Ans. 31° 23′ 15″

3. Find the angle whose logarithmic sine is $\bar{1}.77777$.
Ans. 36° 49′ 56″

4. Find the angle whose logarithmic cosine is $\bar{1}.87341$.
Ans. 41° 39′ 21″

5. Find the angle whose logarithmic cotangent is .31789.
Ans. 25° 41′ 9″

6. Find the angle whose logarithmic cosine is $\bar{1}.34567$.
Ans. 77° 11′ 38″

7. Find the angle whose logarithmic cotangent is $\bar{1}.00381$.

Ans. 84° 14′ 22″

8. Find the angle whose logarithmic tangent is 1.00300.

Ans. 84° 19′ 42″

9. Find the angle whose logarithmic sine is $\bar{2}.99001$.

Ans. 5° 36′ 30″

---

32. Tabular Logarithms Increased by 10.—In printing a table of logarithms of the trigonometric functions, the characteristics cannot be omitted, since they cannot be ascertained by inspection. To avoid printing the bars over the negative characteristics, the latter are in many tables increased by 10. The logarithms as then printed are **tabular logarithms**, and are denoted by *L* sin, *L* cos, etc. Thus,

$$L \sin 27° = \log \sin 27° + 10 = \bar{1}.65705 + 10 = 9.65705$$
$$L \cos 27° = \log \cos 27° + 10 = \bar{1}.94988 + 10 = 9.94988$$

GENERAL PRINCIPLE OF INTERPOLATION

33. It has been explained in some of the preceding articles how to determine the natural or the logarithmic functions of any angle containing an odd number of seconds, and therefore, not found in the table; also, how to find the angle corresponding to a given function, when that function is not in the table but lies between two values given in the table. The operation by which such intermediate values are determined from a table is called **interpolation.** The values that are actually given in the table are called **tabular values.** For example, in the table of logarithmic functions already described are found all angles that lie between 0° and 90° and contain no odd seconds, and also the logarithmic sines, cosines, etc. of such angles; those are all tabular values. Angles containing odd seconds are not in the table, nor are their logarithmic functions. Both these angles and their functions are intermediate values, and it is in connection with them that interpolation is used.

34. The general principle of interpolation, to be explained presently, is of the utmost importance, and of great value to the engineer, whose work requires the frequent use of tables of various kinds. That principle, although only approximately true, applies to nearly all tables with which the engineer has to deal, and the student should endeavor to make himself thoroughly familiar with it.

Let a table be constructed on the general type shown on the margin, the left-hand column containing values of a quantity X, and the right-hand column corresponding values of some quantity whose values depend on the values of X. Thus, the values of X may be the natural numbers 1, 2, 3, 4, etc., and the corresponding values of F may be the logarithms or the square roots of those numbers; or the values of X may be angles, and those of F may be sines, cosines, etc., either natural or logarithmic. So far as the principle of interpolation is concerned, it is immaterial what kind of quantity is represented by X, and what kind of quantity is tabulated under F. It should be stated, however, that the principle applies only to tables in which the differences between consecutive values of X and the differences between the corresponding values of F do not vary very rapidly.

X	F
—	—
—	—
—	—
x_1	f_1
x	f
x_2	f_2
—	—

Let x_1 and x_2, as shown in the above general form, be two consecutive values of X given in the table, and f_1 and f_2 the corresponding values of F. Let x be a value of X lying between x_1 and x_2, and f the corresponding value of F. Neither x nor f is in the table, but one of them is given, and the problem is to find the other by interpolation. For instance, if the table is one of natural tangents in which the angles increase by whole minutes, x_1 and x_2 may be, respectively, 31° 42′ and 31° 43′, and f_1 and f_2 their corresponding tangents; while x may be any angle between 31° 42′ and 31° 43′, and f its tangent. Either x may be given to find f; or f may be given to find x.

The quantity by which the tabular value x_1 must be algebraically increased in order to obtain x will be called the **increment** of x_1, and denoted by $i(x_1)$, read *increment of* x_1 (mathematicians use the notation Δx_1, read *delta* x_1). We have, then,

$$x = x_1 + i(x_1) \qquad (1)$$

Using a similar notation for f_1,

$$f = f_1 + i(f_1) \qquad (2)$$

If x is given, $i(x_1)$ may be assumed as given, since $i(x_1) = x - x_1$. Then $i(f_1)$ is determined by interpolation, as explained below, and f is found from formula **2**. Similarly, if f is given, $i(f_1)$ is likewise given, and x is found by interpolation.

The difference, as $x_2 - x_1$, of two consecutive values of X, will be called the **interval** of X; and that between two consecutive values of F, the interval of F. The notation $I(x_1)$, read *interval of* x_1, will be used to denote the interval $x_2 - x_1$. Similarly, $I(f_1)$ will denote the interval $f_2 - f_1$.

The principle of interpolation is this: *The increments $i(x_1)$ and $i(f_1)$ are to each other as the corresponding intervals $I(x_1)$ and $I(f_1)$; or, algebraically,*

$$\frac{i(x_1)}{i(f_1)} = \frac{I(x_1)}{I(f_1)} \qquad (3)$$

This formula is very easily remembered on account of its symmetry. The following, derived from it, serve, respectively, to find $i\,(f_1)$ when x is given, and $i\,(x_1)$ when f is given:

$$i(f_1) = I(f_1) \times \frac{i(x_1)}{I(x_1)} \qquad (4)$$

$$i(x_1) = I(x_1) \times \frac{i(f_1)}{I(f_1)} \qquad (5)$$

The last two formulas may be stated in the form of a general principle, as follows: *Either increment is equal to the corresponding interval multiplied by the ratio of the other increment to the other interval.* It is easy to remember what the numerator of this ratio is, by noticing that the ratio is always less than 1, and that, since the increment is always less than the interval, the former must be the numerator and the latter the denominator. It should be noted that $i(x_1)$, $i(f_1)$, $I(x_1)$, and $I(f_1)$ may be expressed in any convenient units, it being understood that $i(f_1)$, as determined from formula **4**, is in the same units as $I(f_1)$; and that $i(x_1)$, as determined from formula **5**, is in the same units as $I(x_1)$. Thus, if the values of f_2 and f_1 in the table are, respectively, 4.3476 and 4.3463, then, $I(f_1) = f_2 - f_1 = .0013$, or, if one ten-thousandth is taken as the unit, we may write $I(f_1) = 13$. The value of $i(f_1)$, determined from formula **4**, must be understood to express ten-thousandths. For instance, if $\frac{i(x_1)}{I(x_1)} = .3$, then, $i(f_1) = 13 \times .3 = 3.9$ (ten-thousandths) $= 4$ (ten-thousandths), nearly.

The value of f is then found thus,

$$\begin{aligned} f_1 &= 4.3463 \\ i(f_1) &= \underline{\qquad 4} \\ f &= 4.3467 \end{aligned}$$

Usually, the correction $i(f_1)$ can be added to f_1 mentally, in order to find f.

EXAMPLE 1.—Find the logarithm of 57,846 by means of a five-place table giving the logarithms of numbers consisting of four figures.

SOLUTION.—Only the mantissas will be considered, since the characteristics are determined by inspection. The given number lies between 57,840($=x_1$) and 57,850($= x_2$), whose logarithms are, respectively, .76223($= f_1$) and .76230($= f_2$). We have, therefore, expressing $f_2 - f_1$, or $I(f_1)$, in units of the fifth order,

$$\begin{aligned} x &= 57846 & f_2 &= .76230 \\ x_1 &= \underline{57840} & f_1 &= \underline{.76223} \\ i(x_1) &= \quad 6 & I(f_1) &= \quad 7 \end{aligned}$$

$$I(x_1) = x_2 - x_1 = 10$$

Then (formula 4),

$$i(f_1) = 7 \times \frac{6}{10} = 4.2 = 4, \text{ nearly}$$

and

$$f = \begin{cases} f_1 \\ + i(f_1) \end{cases} = \begin{cases} .76223 \\ +4 \end{cases} = .76227. \quad \text{Ans.}$$

EXAMPLE 2.—Find, by means of a five-place table, the number the mantissa of whose logarithm is .47693.

SOLUTION.—Here $f(= .47693)$ lies between the tabular values $.47683(= f_1)$ and $.47698(= f_2)$, which are, respectively, the logarithms of $29{,}980(= x_1)$ and $29{,}990(= x_2)$. We have, then,

$$\begin{aligned} f_2 &= .47698 \\ f &= .47693 \\ f_1 &= .47683 \\ I(f_1) = f_2 - f_1 &= 15 \\ i(f_1) = f - f_1 &= 10 \end{aligned} \qquad \begin{aligned} x_2 &= 29{,}990 \\ x_1 &= 29{,}980 \\ I(x_1) &= 10 \end{aligned}$$

Then (formula 5),

$$i(x_1) = 10 \times \frac{10}{15} = 7, \text{ nearly}$$

and $$x = x_1 + i(x_1) = 29{,}980 + 7 = 29{,}987. \quad \text{Ans.}$$

This gives the significant figures of the number. The decimal point should be placed according to the characteristic of the given logarithm.

EXAMPLE 3.—Find the angle whose natural tangent is $.56781(= f)$ by means of a table giving the natural tangents of angles varying by minutes.

SOLUTION.—Here f is found to lie between $.56769(= \tan 29° 35' = f_1)$ and $.56808(= \tan 29° 36' = f_2)$. Expressing $x_2 - x_1$, or $I(x_1)$, in seconds, we have

$$\begin{aligned} x_2 &= 29° 36' \\ x_1 &= 29° 35' \\ I(x_1) &= 60'' \end{aligned} \qquad \begin{aligned} f_2 &= .56808 \\ f &= .56781 \\ f_1 &= .56769 \\ I(f_1) &= 39 \\ i(f_1) &= 12 \end{aligned}$$

Then (formula 5),

$$i(x_1) = 60'' \times \frac{12}{39} = 18'', \text{ nearly}$$

and $$x = x_1 + i(x_1) = 29° 35' 18''. \quad \text{Ans.}$$

EXAMPLE 4.—In Searles' field book is given a table of lengths of arcs for different degrees of curvature. Part of it is as follows (lengths in feet):

Degree of Curve $(=X)$	Length of Arc for One Station $(=F)$
10° 10′	100.131
10° 20′	100.136
10° 30′	100.140

Find the length of the arc between two stations for a 10° 26′ curve.

SOLUTION.—Here we have, $x = 10° 26'$, which lies between $10° 20'$ $(= x_1)$ and $10° 30 (= x_2)$. Expressing $I(x_1)$ and $i(x_1)$ in minutes, and $I(f_1)$ and $i(f_1)$ in thousandths, we have $I(x_1) = 10$, $i(x_1) = 6$, $I(f_1) = 140 - 136 = 4$.

Therefore (formula 4),

$$i(f_1) = 4 \times \frac{6}{10} = 2, \text{ nearly}$$

and $$f = f_1 + i(f_1) = \begin{Bmatrix} 100.136 \\ +2 \end{Bmatrix} = 100.138. \quad \text{Ans.}$$

In all simple cases like this the operations can be performed mentally and very rapidly.

EXAMPLES FOR PRACTICE

1. From the following table, find, by interpolation, the cube root of 347.3 and that of 349.7.

Number	Cube Root
347	7.0271
348	7.0338
349	7.0406
350	7.0473

Ans. $\begin{cases} 7.0291 \\ 7.0453 \end{cases}$

2. Find, from the following table, the diameter of a circle whose circumference is 63.57318.

Diameter	Circumference
20.1	63.14601
20.2	63.46017
20.3	63.77433

Ans. 20.236

SOLUTION OF RIGHT TRIANGLES

35. Fundamental Equations.—Let ABC, Fig. 9, be a right triangle, in which A, B, and C are the angles and a, b, and c are the lengths of the sides, c being the hypotenuse. Since A and B are complementary angles, we have

$$\sin A = \cos B \qquad \tan A = \cot B$$
$$\cos A = \sin B \qquad \cot A = \tan B$$

Also, from the definitions of the trigonometric functions, $\sin A = \frac{a}{c}$, $\tan A = \frac{a}{b}$, $\cos B = \sin A = \frac{a}{c}$, $\cot A = \frac{b}{a}$; whence, expressing the value of a from each of these equations,

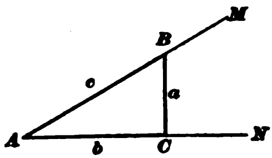

Fig. 9

$$a = c \sin A \qquad (1)$$
$$a = b \tan A \qquad (2)$$
$$a = c \cos B \qquad (3)$$
$$a = b \cot B \qquad (4)$$

From formulas **1** and **3**, the following values are found for c:

$$c = \frac{a}{\sin A} = a \csc A \qquad (5)$$

$$c = \frac{a}{\cos B} = a \sec B \qquad (6)$$

Finally, from geometry,

$$c^2 = a^2 + b^2 \qquad (7)$$

Of the trigonometric formulas just given, it is only necessary to commit to memory formulas **1** and **2**, as the others are immediate consequences of these. These two formulas may be stated in words thus:

Either leg of a right triangle is equal to the hypotenuse multiplied by the sine, or to the other leg multiplied by the tangent, of the opposite angle.

It should be observed that, since a is either leg whose opposite angle is A, and adjacent angle B, the letters a and b may be interchanged in the preceding formulas, provided that A and B are likewise interchanged. Thus, by interchanging a and b, A and B in formulas **1** and **5**, we obtain,

$$b = c \sin B, \quad c = \frac{b}{\sin B} = b \csc B$$

36. Solution of a Right Triangle.—In general, when some of the parts of a triangle are given, the process of determining the others is called **solving the triangle,** or the **solution of the triangle.** The latter expression is applied also to the triangle determined in accordance with the given data.

In order to solve a right triangle, two parts, one at least of which should be a side, must be known in addition to the right angle. The two parts may be either (1) one side and one of the acute angles, or (2) two sides.

37. Case I.—*Given a Side and an Acute Angle.* The other acute angle is found from the relation $A + B = 90°$, and the other two sides by means of formulas **1** to **7**, Art. **35**, as illustrated by the following examples:

EXAMPLE 1.—In Fig. 10, the length of the hypotenuse $A\,B$ of the right triangle $A\,C\,B$, right-angled at C, is 24 feet, and the angle A is 29° 31′; find the sides $A\,C$ and $B\,C$, and the angle B.

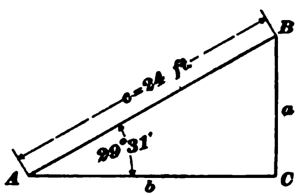

FIG. 10

NOTE.—When working examples of this kind, make a sketch and mark the known parts, as shown in the figure.

SOLUTION WITHOUT LOGARITHMS.— $B = 90° - A = 90° - 29° 31' = 60° 29'$. By formula **3**, Art. **35**, interchanging a and b, and A and B,

$b = c \cos A = 24 \cos 29° 31' = 24 \times .87021 = 20.89$ ft., nearly.

By formula **1**, Art **35**,

$a = 24 \sin 29° 31' = 24 \times .49268 = 11.82$ ft., nearly.

Ans. $\begin{cases} B = 60° 29' \\ A\,C = 20.89 \text{ ft.} \\ B\,C = 11.82 \text{ ft.} \end{cases}$

SOLUTION BY LOGARITHMS.—By formulas **3** and **1**, Art. **35**,

$$b = 24 \cos 29° 31' \qquad (1)$$
$$a = 24 \sin 29° 31' \qquad (2)$$

LOGARITHMS FOR (1)		LOGARITHMS FOR (2)	
$\log 24$	$= 1.38021$	$\log 24$	$= 1.38021$
$\log \cos 29° 31'$	$= \bar{1}.93963$	$\log \sin 29° 31'$	$= \bar{1}.69256$
$\log b$	$= 1.31984$	$\log a$	$= 1.07277$
b	$= 20.89$	a	$= 11.82$

In working examples of this kind, the two logarithmic functions should be taken from the table at the same time. It saves time and space to arrange the operations as follows:

$$\log a = 1.07277;\ a = 11.82$$
$$\log \sin 29° 31' = \bar{1}.69256$$
$$\log 24 = 1.38021$$
$$\log \cos 29° 31' = \bar{1}.93963$$
$$\log b = 1.31984;\ b = 20.89. \quad \text{Ans.}$$

The logarithm of 24 is written first, and then the logarithms of the sine and cosine, one over, the other under, log 24, the addition being performed upwards in one case and downwards in the other.

EXAMPLE 2.—One leg of a right triangle $A\,C\,B$, Fig. 11, is 37 feet 7 inches long; the angle opposite is 25° 33′ 7″; what are the lengths of the hypotenuse and the side adjacent, and what is the other angle?

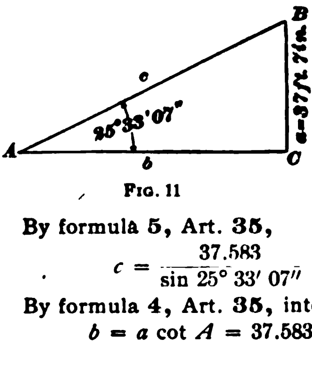

FIG. 11

SOLUTION WITHOUT LOGARITHMS. $B = 90° - 25° 33' 07'' = 64° 26' 53''$. Reducing 37 ft. 7 in. to ft., we have, $a = 37.583$ ft., nearly.

By formula **5**, Art. **35**,

$$c = \frac{37.583}{\sin 25° 33' 07''} = \frac{37.583}{.43133} = 87.133 \text{ ft., nearly.}$$

By formula **4**, Art. **35**, interchanging a and b, and A and B,

$$b = a \cot A = 37.583 \times 2.09166 = 78.611 \text{ ft., nearly.}$$

$$\text{Ans.} \begin{cases} B = 64° 26' 53'' \\ A\,C = 78.611 \\ A\,B = 87.133 \text{ ft.} \end{cases}$$

SOLUTION BY LOGARITHMS.—As before,

$$c = \frac{37.583}{\sin 25° 33' 7''}$$

Also,

$$b = 37.583 \cot 25° 33' 7''$$

$$\log b = 1.89548;\ b = A\,C = 78.611 \text{ ft.}$$
$$\log \cot 25° 33' 7'' = .32049$$
$$\log 37.583 = 1.57499$$
$$\log \sin 25° 33' 7'' = \bar{1}.63481$$
$$\log c = 1.94018;\ c = A\,B = 87.132 \text{ ft.} \quad \text{Ans.}$$

It is to be noted that the value of $A B$ given by logarithms is different in the fifth figure from the result given by natural functions. This is due to the fact that in using five-place tables the results can be depended on to be correct to only four figures, and to have a very close approximation to the fifth figure.

NOTE.—In the majority of cases, the solution by logarithms is far more expeditious than the solution by natural functions. The student is strongly advised to form the habit of solving all trigonometric problems by means of logarithms and the logarithmic functions, whenever these functions can be used.

38. Case II.—*Given Two Sides.* If the given sides are the two legs a and b, A is found from formula **2**, Art. **35**, and B from the relation $A + B = 90°$. To find c, formula **7**, Art. **35**, may be used; but, unless a and b are convenient numbers to square, it is preferable to determine c by formula **5**, Art. **35**, after having determined A.

If the given sides are the hypotenuse c and one leg, say a, the

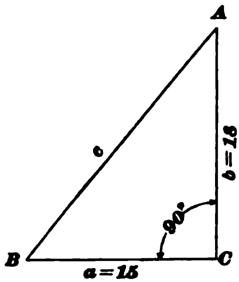

FIG. 12

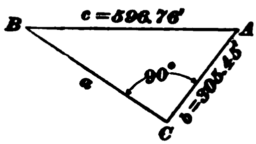

FIG. 13

angle A is found by formula **1**, Art. **35**, B from the relation $A + B = 90°$, and b from either formula **4**, or formula **7**, Art. **35**. The latter gives

$$b = \sqrt{c^2 - a^2}$$

Unless c and a are convenient numbers to square, the quantity under the radical should be replaced by the product $(c + a)\ (c - a)$, and then

$$\log b = \tfrac{1}{2}\ [\log\ (c + a) + \log\ (c - a)]$$

from which b can be readily determined.

EXAMPLE 1.—Given a and b as shown in Fig. 12, to find A, B, and c.

SOLUTION.—Formula **2**, Art. **35**,

$$\tan A = \frac{a}{b} = \frac{15}{18} = \frac{5}{6} = .83333$$

$$A = 39°\ 48'\ 20''$$

$$B = 90° - 39°\ 48'\ 20'' = 50°\ 11'\ 40''$$

Formula **5**, Art. **35**,

$$c = \frac{15}{\sin A} = \frac{15}{\sin 39° 48' 20''}$$

$$\log 15 = 1.17609$$
$$\log \sin 39° 48' 20'' = \bar{1}.80630$$
$$\log c = 1.36979;\ c = 23.431$$

Otherwise,

$$c = \sqrt{15^2 + 18^2} = \sqrt{(3 \times 5)^2 + (3 \times 6)^2} = 3\sqrt{5^2 + 6^2} = 3\sqrt{61} = 23.431.$$

Ans.

EXAMPLE 2.—The hypotenuse c and the leg b having the values shown in Fig. 13, find the acute angles and the leg a.

SOLUTION.—By formula **3**, Art. **35**, interchanging a and b, A and B,

$$\cos A = \frac{b}{c} = \frac{305.45}{596.76}$$

$$\log 305.45 = 2.48494 \qquad 90° = 89° 59' 60''$$
$$\log 596.76 = 2.77580 \qquad A = 59° 12' 46''$$
$$\log \cos A = \bar{1}.70914;\ A = 59° 12' 46'' \qquad B = 30° 47' 14''$$

Formula **2**, Art. **35**,

$$a = 305.45 \tan 59° 12' 47''$$
$$\log 305.45 = 2.48494$$
$$\log \tan 59° 12' 46'' = .22489$$
$$\log a = 2.70983;\ a = 512.66$$

Otherwise, $a = \sqrt{c^2 - b^2} = \sqrt{(c + b)(c - b)}$

$$c + b = 902.21 \qquad \log (c + b) = 2.95531$$
$$c = 596.76 \qquad \log (c - b) = 2.46435$$
$$b = 305.45 \qquad 2)5.41966$$
$$c - b = 291.31 \qquad \log a = 2.70983;\ a = 512.66$$

Ans. $\begin{cases} A = 59° 12' 47'' \\ B = 30° 47' 13'' \\ a = 512.66 \text{ ft.} \end{cases}$

EXAMPLES FOR PRACTICE

1. In a right triangle ACB, right-angled at C (let the student make a sketch), the hypotenuse AB = 40 inches and angle A = 28° 14′ 14″; solve the triangle.

Ans. $\begin{cases} \text{Angle } B = 61° 45' 46'' \\ AC = 35.239 \text{ in.} \\ BC = 18.925 \text{ in.} \end{cases}$

2. In a right triangle ACB, right-angled at C, the side BC = 10 feet 4 inches; if angle A = 26° 59′ 6″, what are the other parts?

Ans. $\begin{cases} \text{Angle } B = 63° 0' 54'' \\ AB = 22 \text{ ft. } 9\frac{1}{4} \text{ in., nearly} \\ AC = 20 \text{ ft. } 3\frac{1}{4} \text{ in., nearly} \end{cases}$

PLANE TRIGONOMETRY

(PART 2)

LOGARITHMIC FUNCTIONS OF SMALL ANGLES

1. Angles less than 3° are of comparatively rare occurrence in practice. When, however, they do occur, and they contain odd seconds, their logarithmic sines, tangents, and cotangents cannot be accurately determined by the general formulas and rules given in *Plane Trigonometry*, Part 1. These functions are found from a special table, which covers the first three pages of the general table of logarithmic functions furnished with this Course. These pages differ from the others in several respects, namely:

(*a*) The column of seconds on the left, marked ″ at the top, gives the total number of seconds in all angles between 0° and 3°, at intervals of 1 minute. Thus, on page 43, the number 6,360 in the column of seconds is horizontally opposite 46 in the minute column, and is, therefore, the total number of seconds in 1° 46′.

(*b*) The column headed S T, between the sine and the tangent column, contains the values of log tan A − log A'', and log sin A − log A'' for all values of A between 0° and 3°, varying from minute to minute; A'' is the total number of seconds in the angle A. The first four figures of these differences are common to the tangent and the sine and are printed near the head of the column; the other two figures are printed under S for the sine and under T for the tangent. The two figures corresponding to any angle are horizontally opposite the total number of seconds in the angle, this total number of seconds being given in the left-hand column. Thus, for 1° 45′ (= 6,300″), the value of S, or of log sin 1° 45′ − log 6,300, is $\bar{6}.68551$; and the value of T, or of log tan 1° 45′ − log 6,300, is $\bar{6}.68571$.

(*c*) Next to the cotangent column, there is a column marked C, containing the values of $-T$. The first four figures of these values are common to all angles between 0° and 3°, and are printed but once; the other two are printed horizontally opposite the number of seconds in the corresponding angles. Thus, for 1° 51′ (= 6,660″), the value of C is 5.31427. The values of S, T, and C will here be referred to as **corrections.**

2. To Find the Logarithmic Sine or Tangent of an Angle Between 0° and 3°.—If there are no odd seconds in the angle, the logarithm may be at once taken from the table, as in *Plane Trigonometry*, Part 1. Here it will be assumed that the angle contains a number of odd seconds. Let the angle be denoted by A, and the total number of seconds in it by A''; that is, let A'' be the angle reduced to seconds. (See Art. **1.**)

Rule.—*Open the table at the page headed by the number of degrees in the given angle. Look in the minute column for the number of minutes nearest (whether greater or less) to the number of odd minutes and seconds in the given angle. (Thus, if the given angle is 2° 36′ 40″, look for 2° 37′; if the given angle is 2° 36′ 21″, look for 2° 36′.) Take from the column headed* S T *the correction horizontally opposite the number of minutes found as just described, using the correction under* S *for the sine, and that under* T *for the tangent. Look in the column of seconds at the left of the page for the number horizontally opposite the number of minutes in the given angle, and to it add the number of odd seconds in that angle. The result will be the total number of seconds (A'') in the given angle. Find the logarithm of this number of seconds from the table of logarithms of numbers. Add to this logarithm the correction found as above. The result will be the required logarithmic sine or tangent, according to the correction used.*

EXAMPLE 1.—To find the logarithmic sine of 1° 3′ 45″ (= A).

SOLUTION.—Opening the table at page 43 (headed 1°), we look for 4′ in the minute column, since 3′ 45″ is nearer to 4′ than to 3′. Horizontally opposite 4, and in the column headed S T, the sine correction $\bar{6}.68555$ (= S) is found. We now look in the minute column for the number of minutes (3) in the given angle; horizontally opposite it in the left-hand column is the number 3,780, number of seconds in 1° 3′; adding 45″, we obtain 3,825 (= A'') for the total number of seconds in the given angle.

$$\log A'' = \log 3{,}825 = 3.58263$$
$$S = \bar{6}.68555$$
$$\log \sin A = \bar{2}.26818$$

that is, $\log \sin 1° 3' 45'' = \bar{2}.26818$. Ans.

EXAMPLE 2.—To find the logarithmic tangent of 2° 36′ 17″.

SOLUTION.—On page 44, the correction for the tangent, opposite 36′, is $\bar{6}.68587$ (= T). Number of seconds opposite 36′ in the left-hand column, 9,360; $A'' = 9{,}360 + 17 = 9{,}377$.

$$\log 9{,}377 = 3.97206$$
$$T = \bar{6}.68587$$
$$\log \tan 2° 36' 17'' = \bar{2}.65793. \text{ Ans.}$$

3. To Find the Logarithmic Cotangent of an Angle Between 0° and 3°.

Rule.—*Find C, A″, and log A″ exactly as in the last article, C being taken from the correction column next to the cotangent column. Subtract log A″ from C. The result will be the required logarithmic cotangent.*

EXAMPLE.—To find the logarithmic cotangent of 1° 52′ 37″.

SOLUTION.—On page 43, the correction under C, and horizontally opposite 53′, is 5.31427; $A'' = 6{,}720 + 37 = 6{,}757$.

$$C = 5.31427$$
$$\log A'' = \log 6{,}757 = 3.82975$$
$$C - \log A'' = 1.48452$$

that is, $\log \cot 1° 52' 37'' = 1.48452$. Ans.

4. To Find the Logarithmic Tangent, Cosine, or Cotangent of an Angle Between 87° and 90°.—These functions also are to be taken from the first three pages of the table of logarithmic functions. The simplest way to proceed is to subtract the angle from 90° and look for the corresponding complementary function as explained in Arts. **2** and **3**. Thus, log cos 88° 55′ 38″ is obtained by looking for log sin (90° − 88° 55′ 38″) = log sin 1° 4′ 22″.

EXAMPLES FOR PRACTICE

1. Find the logarithmic sine of 1° 6′ 19″. Ans. $\bar{2}.28532$
2. Find the logarithmic sine of 0° 2′ 41″. Ans. $\bar{4}.89240$
3. Find the logarithmic tangent of 2° 56′ 57″. Ans. $\bar{2}.71196$
4. Find the logarithmic cotangent of 1° 30′ 18″. Ans. 1.58049
5. Find the logarithmic cosine of 88° 50′ 49″. Ans. $\bar{2}.30370$
6. Find the logarithmic tangent of 89° 3′ 9″. Ans. 1.78151
7. Find the logarithmic cotangent of 88° 0′ 25″. Ans. $\bar{2}.54157$

---

5. To Find the Angle Corresponding to a Given Logarithmic Function, When the Function Lies Between Two of the Functions in the First Three Pages of the Table.—I. *Sine and Tangent.*—As explained in Art. **1**, log sin $A = S + \log A''$; therefore,

$$\log A'' = \log \sin A - S \qquad (1)$$

Likewise, when log tan A is given,

$$\log A'' = \log \tan A - T \qquad (2)$$

From these formulas is derived the following

Rule.—*Find in the table the logarithm nearest to the given one. Take the correction horizontally opposite this logarithm, and subtract it from the given logarithm. The result will be the logarithm of the total number of seconds* (A'') *in the given angle. Find the number corresponding to this logarithm, and reduce it to degrees, minutes, and seconds.*

It is here assumed that the given function lies between two functions in the column marked log sin or log tan, as the case may be, at the top. If the names of the functions are at the bottom, the sine should be treated as in *Plane*

Trigonometry, Part 1; the tangent should be treated as if it were a cotangent, according to the directions to be given presently, and when the angle corresponding to that cotangent is found, it should be subtracted from 90°.

II. *Cotangent.*—Since log cot $A = C - \log A''$ (Art. **3**), we have

$$\log A'' = C - \log \cot A \qquad \textbf{(3)}$$

From this formula is derived the following

Rule.—*Find in the table the logarithmic function nearest the given cotangent. Take from the* C *column the correction horizontally opposite the logarithm just found, and from it subtract the given logarithmic cotangent. The result will be the logarithm of the total number of seconds in the angle.*

Here, as before, it is assumed that the given cotangent lies between two of those marked log cot at the top. If it lies between two logarithms in the column marked log cot at the bottom, it should be treated as if it were a tangent, and having found the angle corresponding to this tangent, it should be subtracted from 90° to obtain the required angle.

III. *Cosine.*

Rule.—*If the given cosine lies between two of those in the column headed log cos, apply the general rule given in Plane Trigonometry, Part 1. If it lies between two of the logarithms in the column marked log cos at the bottom, treat it as if it were a sine, find the angle corresponding to that sine as above, and subtract the result from 90°.*

EXAMPLE 1.—To find the angle whose logarithmic tangent is $\bar{2}.32803$.

SOLUTION.—The logarithmic tangent nearest to $\bar{2}.32803$ is $\bar{2}.32711$, found in the column headed log tan on page 43. The T correction horizontally opposite $\bar{2}.32711$ is $\bar{6}.68564$.

$$\begin{aligned} \log \tan A &= \bar{2}.32803 \\ T &= \underline{\bar{6}.68564} \\ \log A'' &= 3.64239 \end{aligned}$$

From the table of logarithms of numbers,

$$A'' = 4{,}389'' = 1°\ 13'\ 9''. \quad \text{Ans.}$$

EXAMPLE 2.—To find the angle whose logarithmic cotangent is 2.49567.

SOLUTION.—The nearest logarithmic cotangent found in the table is 2.49488. The number opposite this logarithm in the C column is 5.31442.

$$\begin{aligned} C &= 5.31442 \\ \log \cot A &= \underline{2.49567} \\ \log A'' &= 2.81875; \\ A'' = 659'' &= 0° \ 10' \ 59''. \quad \text{Ans.} \end{aligned}$$

NOTE.—Angles are here given to the nearest whole second.

EXAMPLE 3.—To find the angle whose logarithmic cosine is $\bar{2}.63723$.

SOLUTION.—The nearest logarithm, $\bar{2}.63678$, is found on page 44, in the column headed log sin. The given function is, therefore, to be treated as if it were a logarithmic sine, and the angle A, corresponding to this sine is to be subtracted from 90° to obtain the required angle A. The correction horizontally opposite $\bar{2}.63678$, in the S column, is $\bar{6}.68544$.

$$\begin{aligned} \log \sin A_1 &= \bar{2}.63723 \\ S &= \underline{\bar{6}.68544} \\ \log A_1'' &= 3.95179; \\ A_1 = 8{,}949'' &= 2° \ 29' \ 9'' \\ A = 90° - 2° \ 29' \ 9'' &= 87° \ 30' \ 51''. \quad \text{Ans.} \end{aligned}$$

EXAMPLES FOR PRACTICE

Verify the following values:

(a)	$\bar{2}.17645 = \log \sin 0° \ 51' \ 37''$	(e)	$\bar{2}.48790 = \log \cot 88° \ 14' \ 19''$
(b)	$\bar{3}.94316 = \log \sin 0° \ 30' \ 10''$	(f)	$2.47608 = \log \cot 0° \ 11' \ 29''$
(c)	$\bar{2}.65783 = \log \cos 87° \ 23' \ 36''$	(g)	$1.31009 = \log \tan 87° \ 11' \ 48''$
(d)	$\bar{2}.58349 = \log \tan 2° \ 11' \ 41''$	(h)	$\bar{3}.95377 = \log \cos 89° \ 29' \ 6''$

---

6. Use of the Column of Seconds for Obtaining the Angle Corresponding to a Given Function.—In order to avoid confusing the student by too many rules, the reduction of A'' to degrees, minutes, and seconds was, in the preceding articles, effected by the ordinary rules of arithmetic, without any reference to the table. The following is a more expeditious method:

Let the given function lie between the functions of two consecutive angles, A_1 and $A_1 + 1'$. Then, the degrees and minutes in the required angle are those in A_1, and may be at once written down. The number in the column of seconds on the left, horizontally opposite the number of minutes in A_1, gives the total number of seconds in A_1. Denoting that number by A_1'' and the number of odd seconds in the required angle by s, we have

$$s = A'' - A_1''$$

EXAMPLE.—To find the angle whose logarithmic tangent is $\bar{2}.30217$.

SOLUTION.—The given function lies between $\bar{2}.29629$ and $\bar{2}.30263$. The angle corresponding to the first of these two functions is 1° 8′ ($= A_1$); $A_1'' = 4{,}080''$.

$$\begin{aligned} \log \tan A &= \bar{2}.30217 \\ T &= \bar{6}.68563 \\ \log A'' &= 3.61654;\ A'' = 4{,}136 \\ s &= A'' - A_1'' = 4{,}136 - 4{,}080 = 56'' \\ A &= A_1 + s = 1°\ 8'\ 56''.\ \text{Ans.} \end{aligned}$$

The subtraction $A'' - A_1''$ can usually be effected mentally.

EXAMPLES FOR PRACTICE

Apply the method just described to the Examples for Practice given after Art. 4.

GENERAL TRIGONOMETRIC FORMULAS

ANGLES AND THEIR TRIGONOMETRIC FUNCTIONS

7. Angle of Any Magnitude.—In trigonometry, an angle is considered as being generated by a straight line turning about one of its ends, which is the vertex of the angle. In this motion, any point in the turning line describes a circular arc, whose number of degrees is the measure of the angle. The turning line is called the **generating line.** The position that this line occupies before it begins to turn, and from which arcs are measured, is called the **initial line,** or the **initial position** of the generating line; and the position it occupies after turning through a certain angle

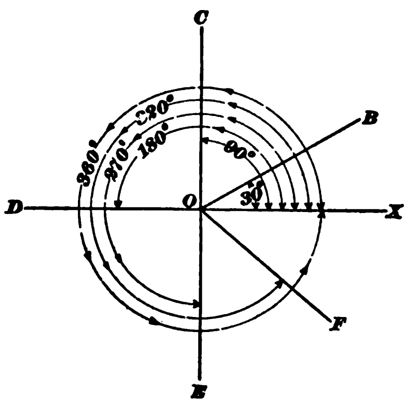

FIG. 1

is called the **final position.** In Fig. 1, for example, the initial position of the generating line is OX. The turning is supposed to take place about the point O and in a direction opposite to that in which the hands of a clock move. When the line turns from the position OX to the final positions OB, OC, OD, OE, OF, it generates angles of 30°, 90°, 180°, 270°, 320°, respectively, as indicated on the figure. If the line makes a complete turn, so that its final position coincides with its initial position OX, the angle generated is 360°.

8. Positive and Negative Angles.—When an angle is described by a line turning in a direction contrary to that in which the hands of a watch move, the angle is considered **positive**; if described in the opposite direction, it is considered **negative**. Referring to Fig. 2, the angle XOB, whose supplement is A, may be regarded as having been described in any of the following manners:

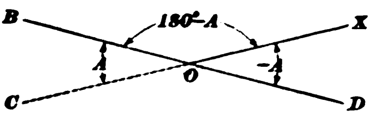

FIG. 2

(a) By turning the generating line about O from the position OX in the positive direction through $(180 - A)$ degrees to the position OB.

(b) By turning the generating line about O in a positive direction through an angle of 180°, when it will be in the position OC, and then turning it back from OC in the negative direction through the angle $-A$ (negative, because turned in the negative direction) into the position OB.

(c) By turning the generating line about O in the negative direction through the angle $-A$, into the position OD, and then turning it back in the positive direction through 180° into the position OB.

It is to be noticed that, however the angle $(180° - A)$ may be regarded as described, the resulting angle XOB is the same.

9. Quadrants.—Let OX, Fig. 3, be the initial position of the generating line, and OM_1, OM_2, OM_3, OM_4 final positions, determining, respectively, the angles A_1, A_2, A_3, A_4, all measured from OX upwards and toward the left. Producing XO and drawing through O a perpendicular YY' to OX, the plane of the figure is divided into four right angles, called **quadrants.** Taking them in order, following the direction in which positive angles are reckoned, they are distinguished as follows: XOY is the **first quadrant;** YOX', the **second quadrant;** $X'OY'$, the **third quadrant;** and $Y'OX$, the **fourth quadrant.**

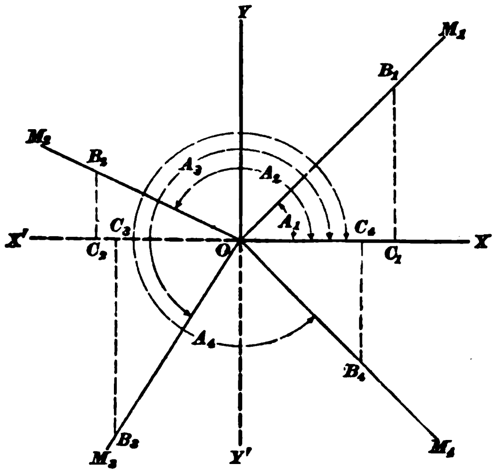

FIG. 3

10. Trigonometric Functions of Any Angle.—In the definitions given in *Plane Trigonometry*, Part 1, only acute angles were considered. Referring to Fig. 3, in which B_1C_1 is perpendicular to OX, the trigonometric functions of the acute angle A_1 were defined by the following equations:

$$\sin A_1 = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{B_1C_1}{OB_1} \qquad \tan A_1 = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{B_1C_1}{OC_1}$$

$$\cos A_1 = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{OC_1}{OB_1} \qquad \cot A_1 = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{OC_1}{B_1C_1}$$

$$\sec A_1 = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{OB_1}{OC_1} \qquad \csc A_1 = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{OB_1}{B_1C_1}$$

These formulas serve as the definitions of the trigonometric functions of any angle; that is, the **sine** of any angle is the ratio of the side opposite to the hypotenuse; the tangent is the ratio of the side opposite to the side adjacent, etc. But, in order that these definitions may be correct, it is necessary to apply to them some algebraic rules relating to signs.

In Fig. 3, the hypotenuse used for the determination of the functions of A_1 is any portion OB_1 of the side OM_1, which is the final position of the generating line. From B_1, a perpendicular B_1C_1 is drawn on the initial line OX, thus determining the right triangle OB_1C_1. The length of the perpendicular B_1C_1, which is the side opposite the vertex of the angle, is the distance of B_1 above the initial line OX, and the length of the adjacent side OC_1 is the distance of the point B_1 to the right of the vertex, measured along the initial line; or, what is the same thing, OC_1 is the distance of B_1C_1 from the vertex, measured toward the right.

Consider now the angle XOM_2, or A_2, in which the final position OM_2 of the generating line lies in the second quadrant. As before, the hypotenuse to be used in the definitions of the trigonometric functions of A_2 is any portion OB_2 of the side OM_2, which is the final position of the generating line. As before, also, a perpendicular from B_2 is drawn on the initial line OX; but, in this case, the perpendicular falls on OX produced. In the right triangle OB_2C_2, the perpendicular B_2C_2 is the side opposite the vertex of the angle A_2, and OC_2 is the side adjacent. It should be noted very particularly that the terms *side opposite* and *side adjacent* are used to describe the positions of the legs of the right triangle with reference to the *vertex* of the angle considered, not to the angle itself. Thus, B_2C_2 is not opposite the angle A_2, but opposite the vertex O of that angle. The length of the side opposite, B_2C_2, measures the distance of B_2 above the initial line; and the length of OC_2, or the side adjacent, measures the distance of the opposite side B_2C_2 to the left of the vertex; or, in the language of algebra, it may be said that $-OC_2$ is the distance of B_2C_2 to the *right* of O.

Having defined the cosine of any angle as the ratio of the side adjacent to the hypotenuse, and the side adjacent as the distance of the side opposite from the vertex, measured toward the right of the vertex, it is necessary, when the side opposite is to the left of the vertex, to consider its distance from the vertex, or the side adjacent, as negative. This is in accordance with the general principle of algebra, that, if distances counted in one direction are treated as positive, distances in the opposite direction must be treated as negative. In the triangle $O B_2 C_2$, therefore, $O C_2$ should be treated as negative, and therefore, the cosine of A_2 is $\frac{-O C_2}{O B_2}$.

Considering now the angle A_3, the hypotenuse is, as above, any portion $O B_3$ of the side $O M_3$, which is the final position of the generating line. From B_3, the perpendicular $B_3 C_3$ on the initial line (produced) is drawn, and thus a right triangle is determined, in which $B_3 C_3$ is the side opposite, and $O C_3$ the side adjacent. As previously explained, $O C_3$ should be treated as negative. The opposite side $B_3 C_3$, which is the distance of B_3 below the initial line, should also be treated as negative; for if distances above the initial line are treated as positive, those below the initial line must be treated as negative.

Finally, in the angle A_4, which terminates in the fourth quadrant, $O C_4$, the side adjacent, is positive, while $B_4 C_4$, the side opposite, is negative.

The foregoing explanations may be summed up as follows: The side opposite is positive or negative according as the hypotenuse is above or below the initial line. The side adjacent is positive or negative according as it extends toward the right or toward the left of the vertex. The hypotenuse is always positive.

11. Algebraic Signs of the Functions.—Referring again to Fig. 3, it will be observed that, for any angle, as A_1, terminating in the first quadrant, both the side adjacent and the side opposite, or $O C_1$ and $B_1 C_1$, are positive, and therefore all the functions are positive; for any angle, as A_2, terminating in the second quadrant, the side adjacent, or $O C_2$, is negative, and the side opposite, or $B_2 C_2$, is positive. Therefore,

$$\sin A_2 = \frac{+B_2C_2}{+OB_2}, \text{ positive} \qquad \tan A_2 = \frac{+B_2C_2}{-OC_2}, \text{ negative}$$

$$\cos A_2 = \frac{-OC_2}{+OB_2}, \text{ negative} \qquad \sec A_2 = \frac{+OB_2}{-OC_2}, \text{ negative}$$

$$\csc A_2 = \frac{+OB_2}{+B_2C_2}, \text{ positive}$$

The signs of the functions of angles terminating in the third and in the fourth quadrant are similarly determined. The results are tabulated below.

TABLE I

Function	Quadrant			
	First	Second	Third	Fourth
	Sign of Function			
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−
Cotangent	+	−	+	−
Secant	+	−	−	+
Cosecant	+	+	−	−

12. Trigonometric Functions of 0° and 90°.—In the right triangles ACB, Fig. 4, the hypotenuse AB may be taken to have any value whatever. It is evident that BC, the side opposite, decreases as the angle CAB decreases, and becomes zero when the angle becomes zero; and that BC coincides with the hypotenuse AB when the angle CAB is 90°. Again, AC, the adjacent side, increases as the angle decreases, and is equal to the hypotenuse AB when the angle CAB is 0°. Also, AC becomes zero when CAB is 90°. Now, from the definitions of the trigonometric functions,

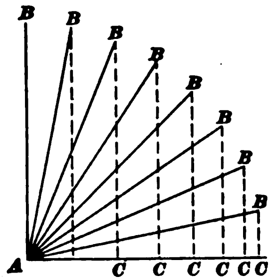

FIG. 4

$$\sin CAB = \frac{\text{side opposite}}{\text{hypotenuse}}, \text{ whence} \begin{cases} \sin 0° = \dfrac{0}{AB} = 0 \\ \sin 90° = \dfrac{AB}{AB} = 1 \end{cases}$$

$$\cos CAB = \frac{\text{side adjacent}}{\text{hypotenuse}}, \text{ whence} \begin{cases} \cos 0° = \dfrac{AB}{AB} = 1 \\ \cos 90° = \dfrac{0}{AB} = 0 \end{cases}$$

In like manner,

$$\tan 0° = \frac{0}{AC} = 0 \qquad \tan 90° = \frac{BC}{0} = \infty$$

$$\cot 0° = \frac{AC}{0} = \infty \qquad \cot 90° = \frac{0}{CB} = 0$$

NOTE.—The cotangent of CAB is equal to $\frac{CA}{CB}$. Now, as the angle decreases, the side CB becomes less and less, and it is evident that, as the denominator of a fraction becomes less and less, the numerator remaining the same, the value of the fraction increases. As the denominator decreases indefinitely, the value of the fraction increases indefinitely, and when the value of the fraction exceeds any known quantity, however great, it is said to be infinite. The sign ∞ is used to express an infinite number.

13. Functions of (180° − A).—Let XOM, Fig. 5, be any angle, and A ($= MOX'$) its supplement. Draw OM' making with OX an angle equal to A, as shown. Take any part OB of OM for the hypotenuse, and draw BC perpendicular to OX produced; draw BB' parallel to OX, and $B'C'$ perpendicular to OX. Then, $BC = B'C'$; $OB = OB'$; $OC = -OC'$ (Art. **10**); and, by the definitions of the functions,

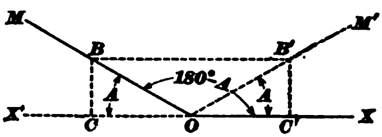

FIG. 5

$$\sin XOM = \frac{BC}{OB} = \frac{B'C'}{OB'} = \sin A$$

$$\cos XOM = \frac{OC}{OB} = \frac{-OC'}{OB} = -\cos A$$

that is, $\sin(180° - A) = \sin A$ (1)

$\cos(180° - A) = -\cos A$ (2)

Similarly, $\tan(180° - A) = -\tan A$ (3)

$$\cot(180° - A) = -\cot A \quad (4)$$

These relations are especially useful for finding the logarithmic functions of angles greater than 90°, since these functions are arithmetically equal to those of the supplements of the angles; that is, when signs are disregarded, any function of an angle and that of its supplement are equal. For example, $\sin 105° = \sin(180° - 105°) = \sin 75°$; $\cos 105° = -\cos(180° - 105°) = -\cos 75°$.

14. Functions of $(90° + A)$.—By formula **1** of Art. **13**,
$\sin(90° + A) = \sin[180° - (90 + A)] = \sin(90° - A)$
or, since $\sin(90° - A) = \cos A$,

$$\sin(90° + A) = \cos A \quad (1)$$

The following formulas may be derived in a similar manner:

$$\tan(90° + A) = -\cot A \quad (2)$$

$$\cos(90° + A) = -\sin A \quad (3)$$

$$\cot(90° + A) = -\tan A \quad (4)$$

15. Functions of Negative Angles.—The complement of an angle is the algebraic difference between the angle and 90°. If the angle is greater than 90°, its complement is negative. Thus, the complement of 95° is $90° - 95° = -5°$. The cofunctions of an angle are the corresponding fundamental functions of its complement, whether that complement be positive or negative. Thus, $\cos 85° = \sin(90° - 85°) = \sin 5°$; $\cos 95° = \sin(90° - 95°) = \sin(-5°)$. Similarly, $\sin 95° = \cos(90° - 95°) = \cos(-5°)$. It is, therefore, necessary to know how to determine the functions of negative angles.

If $90° + A$ is any angle, its complement is $90° - (90° + A) = -A$; and, therefore,

$$\cos(90° + A) = \sin(-A), \quad \cot(90° + A) = \tan(-A)$$
$$\sin(90° + A) = \cos(-A), \quad \tan(90° + A) = \cot(-A)$$

whence, replacing the values of $\cos(90° + A)$, $\cot(90° + A)$, etc. from the preceding article,

$$\sin(-A) = -\sin A \qquad (1)$$

$$\tan(-A) = -\tan A \qquad (2)$$

$$\cos(-A) = \cos A \qquad (3)$$

$$\cot(-A) = -\cot A \qquad (4)$$

ADDITION OF ANGLES

16. To Express the Sine or Cosine of the Sum or Difference of Two Angles in Terms of the Sine and Cosine of the Angles.—The following formulas are fundamental; being of frequent occurrence, they are very important, and should be committed to memory:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (1)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad (2)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad (4)$$

NOTE.—The derivation of these formulas is given in the Appendix at the end of this Section, under the Roman numeral I. That Appendix contains this and a few other demonstrations that are comparatively laborious and may be found irksome by some. They are not essential to the understanding of the formulas, and the student is not required to learn them. He is, however, advised to peruse them carefully, as they are good exercises in the handling and transforming of both algebraic and trigonometric expressions.

These formulas are not used, as they seem to imply, to determine the sine or the cosine of the sum or difference of two angles, when the sine and cosine of those angles are given. They can be used for this purpose, but there would be no advantage in so doing. Their main value consists in their application to transforming complicated trigonometric expressions into simpler ones. The student will often have occasion to employ them in this manner. In order that he may have an idea of this application of the formulas, two examples are given here.

EXAMPLE 1.—To determine the angle A from the relation

$$\frac{\sin(A + 28°)}{\sin A} = .95$$

SOLUTION.—Applying formula **1**, we have

$$\frac{\sin (A + 28°)}{\sin A} = \frac{\sin A \cos 28° + \cos A \sin 28°}{\sin A}$$

$$= \frac{\sin A \cos 28°}{\sin A} + \frac{\cos A \sin 28°}{\sin A} = \cos 28° + \cot A \sin 28°$$

replacing $\frac{\cos A}{\sin A}$ by its equal $\cot A$ (see *Plane Trigonometry*, Part 1). Substituting this value of the quotient $\frac{\sin (A + 28°)}{\sin A}$ in the given equation, we have,

$$\cos 28° + \cot A \sin 28° = .95$$

whence $$\cot A = \frac{.95 - \cos 28°}{\sin 28°} = \frac{.95 - .88295}{.46947} = .14282$$

and, therefore, $A = 81° 52' 19''$. Ans.

EXAMPLE 2.—To transform the expression $\tan A + \tan B$ into the expression $\frac{\sin (A + B)}{\cos A \cos B}$.

NOTE.—Transformations of this kind are very often useful, when logarithms are employed. Thus, if $\tan A + \tan B$ were to be multiplied by 39.578, it would be necessary first to find the natural tangent of A, then that of B, add the two together, take the logarithm of the sum thus obtained, and add this logarithm to that of 39.578. If, however, the expression $\frac{\sin (A + B)}{\cos A \cos B}$ is used, the logarithms of $\sin (A + B)$, $\cos A$, $\cos B$ can be taken from the table, and the operation performed without having recourse to natural functions, which are often inconvenient.

SOLUTION.—We have (*Plane Trigonometry*, Part 1),

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$$

According to formula **1**, the numerator of this last fraction is equal to $\sin (A + B)$. Therefore,

$$\tan A + \tan B = \frac{\sin (A + B)}{\cos A \cos B}. \quad \text{Ans.}$$

17. Sine and Cosine of 2 A and of $\frac{1}{2}$ A.—From the formulas for the sine and cosine of the sum of two angles, the following are deduced:

$$\sin 2A = 2 \sin A \cos A \qquad (1)$$

$$\cos 2A = \cos^2 A - \sin^2 A \qquad (2)$$

$$\cos 2A = 1 - 2 \sin^2 A \qquad (3)$$

$$\cos 2A = 2 \cos^2 A - 1 \qquad (4)$$

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A \qquad (5)$$

$$\cos A = \cos^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A \qquad (6)$$

$$\cos A = 1 - 2 \sin^2 \tfrac{1}{2} A \qquad (7)$$

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 \qquad (8)$$

As in the case of formulas **1** to **4**, Art. **16**, these formulas are used mainly for the purposes of transformation. They are very simply derived as follows:

When B is made equal to A, formula **1**, Art. **16**, becomes

$$\sin (A + A) = \sin A \cos A + \cos A \sin A$$

that is, $\sin 2 A = 2 \sin A \cos A$

Similarly, formula **2**, Art. **16**, becomes

$$\cos (A + A) = \cos A \cos A - \sin A \sin A$$

that is, $\cos 2 A = \cos^2 A - \sin^2 A$

Formula **3** follows from this, by writing $1 - \sin^2 A$ instead of $\cos^2 A$ (since $\sin^2 A + \cos^2 A = 1$); and formula **4**, by writing $1 - \cos^2 A$ instead of $\sin^2 A$.

Formulas **1** to **4** give the sine and cosine of twice any angle in terms of the sine and cosine of the angle. If the angle is denoted by $\tfrac{1}{2} A$, twice the angle will be A, and formulas **1** to **4** take the forms of formulas **5** to **8**.

OBLIQUE TRIANGLES

FUNDAMENTAL PRINCIPLES

NOTE.—For the general method of marking and naming the sides and angles of a triangle, see *Plane Trigonometry*, Part 1.

18. Principle of Sines.—*In any triangle, the sides are proportional to the sines of the opposite angles.* That is,

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}, \quad \frac{b}{c} = \frac{\sin B}{\sin C}$$

Let $A B C$, Fig. 6, be any triangle and p the perpendicular from C on the opposite side. Then, in (a), the right triangles $A C D$ and $B C D$ give, respectively,

$$p = b \sin A, \; p = a \sin B$$

whence, putting the two values of p equal to each other,

$$a \sin B = b \sin A$$

and, therefore, dividing by $b \sin B$,

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

In (b), the right triangles ACD and BCD give, respectively,
$$p = b \sin A, p = a \sin CBD$$
whence,
$$a \sin CBD = b \sin A$$

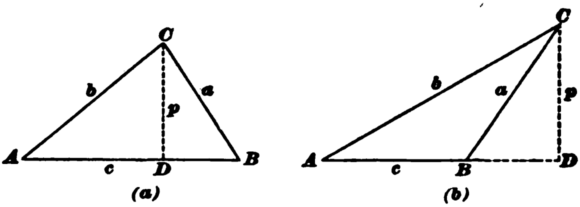

FIG. 6

But, as $CBD = 180° - B$, we may write $\sin B$ instead of $\sin CBD$ (Art. **13**), and, therefore,

$$a \sin B = b \sin A$$

whence, as before,

$$\frac{a}{b} = \frac{\sin A}{\sin B} \qquad (1)$$

By drawing a perpendicular from B on AC, and reasoning in the same manner, it may be shown that

$$\frac{a}{c} = \frac{\sin A}{\sin C} \qquad (2)$$

Similarly,
$$\frac{b}{c} = \frac{\sin B}{\sin C}$$

By transforming equation (1), we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

and by a similar transformation of equation (2),

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

We have, therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The principle of sines may, then, be stated in this form: *In every triangle, the quotient obtained by dividing the length of any side by the sine of the opposite angle is the same, whatever the side taken.*

This quotient is called the **modulus** of the triangle, and will here be denoted by M. The modulus can be found when any of the sides and the opposite angle are known.

The principle of sines is one of the most important in trigonometry, and both forms in which it is stated in this article should be committed to memory.

19. The Cosine Principle.—*In any triangle, the square of one side is equal to the sum of the squares of the other two sides minus twice the product of these two sides and the cosine of their included angle.* That is (Fig. 6),

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$
$$b^2 = a^2 + c^2 - 2\,a\,c\cos B$$
$$c^2 = a^2 + b^2 - 2\,a\,b\cos C$$

These formulas are derived in Appendix II.

20. Principle of Tangents.—*The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* That is (Fig. 6),

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

The derivation of this formula is given in Appendix III. The student should have no difficulty in committing the formula to memory, as its symmetry makes it very easy to remember.

SOLUTION OF OBLIQUE TRIANGLES

21. The solution of oblique triangles is treated under four cases:

Case I: Given Two Sides and the Included Angle. Let a, b, and C, Fig. 6, be given and A, B, and c be required. Of the two methods given below, the first is preferable in most cases.

First Method.—From the formula in Art. **20**, the following is readily derived:

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B) \qquad (1)$$

Now, since $A + B + C = 180°$, we have also,

$$A + B = 180° - C; \text{ and } \tfrac{1}{2}(A + B) = \tfrac{1}{2}(180° - C) = 90° - \tfrac{1}{2}C$$

Therefore, $\frac{1}{2}C$ is the complement of $\frac{1}{2}(A + B)$, and hence, $\tan\frac{1}{2}(A + B) = \cot\frac{1}{2}C$. Substituting this value in equation (1), the following formula is derived:

$$\tan\tfrac{1}{2}(A - B) = \frac{a - b}{a + b}\cot\tfrac{1}{2}C \qquad (1)$$

If the student remembers the formula in Art. **20**, or the principle of tangents, he will have no difficulty in remembering this formula, which is derived from the formula in Art. **20**, by simply writing $\cot\frac{1}{2}C$ instead of $\tan\frac{1}{2}(A + B)$.

From this formula $\frac{1}{2}(A - B)$ can be found. Let this value of $\frac{1}{2}(A - B)$ be denoted by D. We have also, as explained above, $\frac{1}{2}(A + B) = \frac{1}{2}(180° - C) = 90° - \frac{1}{2}C$.

$$\tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2}C \qquad (2)$$

$$\tfrac{1}{2}(A - B) = D \qquad (3)$$

Adding equations (2) and (3) gives

$$A = (90° - \tfrac{1}{2}C) + D$$

Subtracting equation (3) from (2) gives

$$B = (90° - \tfrac{1}{2}C) - D$$

Knowing A and B, the side c may be found from the relation (Art. **18**),

$$\frac{c}{\sin C} = \frac{a}{\sin A}, \text{ which gives } c = \frac{a \sin C}{\sin A}$$

It is, however, more convenient to find c from the following formula, the derivation of which is given in Appendix IV:

$$c = \frac{(a - b)\cos\frac{1}{2}C}{\sin\frac{1}{2}(A - B)} \qquad (2)$$

It will be noticed that, for calculating $\tan\frac{1}{2}(A - B)$, the logarithms of $(a - b)$ and $\cot\frac{1}{2}C$ have to be found. The logarithm of $\cos\frac{1}{2}C$ may be taken out of the table at the same time as that of $\cot\frac{1}{2}C$. Also, when the angle $\frac{1}{2}(A - B)$ is taken from the table, its logarithmic sine should be taken at the same time. This greatly simplifies the application of formula **2**.

Second Method.—The third side c can be found directly from the formula in Art. **19**, which gives

$$c = \sqrt{a^2 + b^2 - 2\,a\,b \cos C}$$

Then, by the principle of sines,

$$\sin A = \frac{a \sin C}{c},\ \sin B = \frac{b \sin C}{c}$$

This method is of value when the only required part is the side c, especially if a and b are convenient numbers to square.

EXAMPLE 1.—In a triangle, $a = 17$ feet, $b = 12$ feet, and the included angle $C = 59° 23'$. To find the other parts of the triangle.

SOLUTION.—Here $\frac{1}{2}\,C = 29° 41' 30''$; $a + b = 17 + 12 = 29$, and $a - b = 17 - 12 = 5$. Then, by the first method,

$$\tan \tfrac{1}{2}\,(A - B) = \frac{5}{29} \times \cot 29° 41' 30''$$

log 5 =	.69897	log 5 =	.69897
log 29 =	1.46240	log cos 29° 41′ 30″ =	$\bar{1}$.93887
	$\bar{1}$.23657		.63784
log cot 29° 41′ 30″ =	.24397	log sin D =	$\bar{1}$.46154
log tan $\frac{1}{2}(A - B)$ =	$\bar{1}$.48054	log c =	1.17630
$D = \frac{1}{2}(A - B)$ =	16° 49′ 25″;	c =	15.007. Ans.

$A = (90° - 29° 41' 30'') + 16° 49' 25'' = 77° 7' 55''$. Ans.
$B = (90° - 29° 41' 30'') - 16° 49' 25'' = 43° 29' 5''$. Ans.

EXAMPLE 2.—Given $a = 10$, $b = 15$, and $C = 60°$; to find c.

SOLUTION.—By the second method,

$$c = \sqrt{10^2 + 15^2 - 2 \times 10 \times 15 \cos 60°}$$
$$= \sqrt{325 - 300 \times .5} = \sqrt{175} \doteq 13.229 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Given $a = 37.46$ feet, $b = 59.17$ feet, and $C = 69° 13'$; find A, B, and c.
Ans. $\begin{cases} A = 37° 21' 30'' \\ B = 73° 25' 30'' \\ c = 57.72 \text{ ft.} \end{cases}$

2. Two sides of a triangle are, respectively, 687.64 and 319.58 feet long, and their included angle is 47° 15′ 8″; find the other two angles and the third side.
Ans. $\begin{cases} \text{Angles, } 106° 14' 56'' \text{ and } 26° 29' 56'' \\ \text{Third side} = 525.97 \end{cases}$

3. Given $c = 4$ chains, $a = 6$ chains, and $B = 45° 18'$; find b.
Ans. $b = 4.271$ ch.

4. Given $b = 43.16$ chains, $c = 51.29$ chains, and $A = 35° 8' 10''$; find B, C, and a.

Ans. $\begin{cases} B = 57° 13' 20'' \\ C = 87° 38' 30'' \\ a = 29.544 \text{ ch.} \end{cases}$

---

22. Case II: Given a Side and Two Angles.—Let c, A, and B be known, to find a, b, and C. The angle $C = 180° - A - B$. By the principle of sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C}, \text{ whence } a = \frac{c}{\sin C} \sin A$$

Similarly,

$$b = \frac{c}{\sin C} \sin B$$

Since $\frac{c}{\sin C}$ is the modulus of the triangle (Art. 18), these formulas may be thus stated: *Any side of a triangle is equal to the modulus of the triangle multiplied by the sine of the angle opposite that side.*

EXAMPLE.—Given $a = 98.48$, $B = 60° 45'$, and $C = 39° 15'$; to find b, c, and A.

SOLUTION.— $A = 180° - (60° 45' + 39° 15') = 80°$. Ans.

$$M = \frac{98.48}{\sin 80°}; \; b = \frac{98.48}{\sin 80°} \sin 60° 45'; \; c = \frac{98.48}{\sin 80°} \sin 39° 15'$$

$\log 98.48 = 1.99335$
$\log \sin 80° = \bar{1}.99335$
$\log M = 2.00000$

$\log b = 1.94076$; $b = 87.248$. Ans.
$\log \sin 60° 45' = \bar{1}.94076$
$\log M = 2.00000$
$\log \sin 39° 15' = \bar{1}.80120$
$\log c = 1.80120$; $c = 63.27$. Ans.

NOTE.—Attention is called to the convenient way in which the work is here arranged. Having determined log M, this logarithm is copied, and then one of the logarithms to be added to it is written above it, the other under it, the addition being performed upwards in one case, and downwards in the other.

EXAMPLES FOR PRACTICE

1. Given $a = 45.39$ feet, $B = 38° 12'$, and $C = 11° 11' 34''$; find A, b, and c.

Ans. $\begin{cases} A = 130° 36' 26'' \\ b = 36.973 \text{ ft.} \\ c = 11.605 \text{ ft.} \end{cases}$

2. Given $c = 101.11$ chains, $C = 55° 55' 55''$, and $A = 10° 10' 10''$; find B, a, and b.

Ans. $\begin{cases} B = 113° 53' 55'' \\ a = 21.551 \text{ ch.} \\ b = 111.59 \text{ ch.} \end{cases}$

23. Case III: Given Three Sides.—Let a, b, and c be given, to find A, B, and C.

First Method.—The angles can be found directly from the cosine formulas (Art. **19**), which, being solved for cos A, cos B, and cos C, respectively, give

$$\left.\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2\,bc}\\ \cos B &= \frac{a^2 + c^2 - b^2}{2\,ac}\\ \cos C &= \frac{a^2 + b^2 - c^2}{2\,ab}\end{aligned}\right\} \quad (1)$$

These formulas are to be used when the numbers a, b, c are convenient to square; otherwise, they are too cumbersome, and those given below for the functions of half the angles should be employed. It is necessary to apply the formulas in determining only two of the angles, as the third follows from the relation $A + B + C = 180°$. As a check, however, the formulas should be applied to the third angle also.

It should be borne in mind that, if the cosine of an angle is found to be negative, this implies that the angle is obtuse (Art. **13**). In such case, the cosine is treated as positive, and the corresponding angle taken from the table is subtracted from 180° to obtain the required angle. Thus, if $\cos A = -.97030$, we look for the angle whose cosine is $+.97030$, which is 14°. Then, $A = 180° - 14° = 166°$.

EXAMPLE.—Given $a = 4$ inches, $b = 5$ inches, and $c = 7$ inches; to find A, B, and C.

SOLUTION.— $\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{5^2 + 7^2 - 4^2}{2 \times 5 \times 7} = \frac{58}{70} = .82857,$

and, therefore, $A = 34°\ 2'\ 53''$. Ans.

$$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac} = \frac{4^2 + 7^2 - 5^2}{2 \times 4 \times 7} = \frac{40}{56} = .71429$$

and, therefore, $B = 44°\ 24'\ 54''$. Ans.

$$C = 180° - A - B = 101°\ 32'\ 13''. \text{ Ans.}$$

As a check, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab} = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5} = -\frac{8}{40} = -.20000$$

The angle whose cosine is .20000 is 78° 27′ 47″. Therefore, $C = 180° - 78° 27' 47'' = 101° 32' 13''$.

Second Method.—As said before, this method is to be applied when the operations required by formula **1** involve too much labor, which happens when the lengths of the given sides consist of three or more significant figures—the usual case. If the sum of the sides is denoted by $2s$, or half their sum by s, the angles A, B, C may be found by the following formulas, which are derived in Appendix V:

$$\left.\begin{aligned}\tan \tfrac{1}{2} A &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\ \tan \tfrac{1}{2} B &= \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\\ \tan \tfrac{1}{2} C &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}\cos \tfrac{1}{2} A &= \sqrt{\frac{s(s-a)}{bc}}\\ \cos \tfrac{1}{2} B &= \sqrt{\frac{s(s-b)}{ac}}\\ \cos \tfrac{1}{2} C &= \sqrt{\frac{s(s-c)}{ab}}\end{aligned}\right\} \qquad (3)$$

For angles differing but little from 90° (say between 85° and 90°), use the cosine formulas **3**; in all other cases, the tangent formulas **2**.

We have also,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad (4)$$

with similar formulas for $\sin \frac{1}{2} B$ and $\sin \frac{1}{2} C$. These formulas are of value for deriving the tangent formulas **2**, as well as for deriving an expression for the area of a triangle when the sides are given. They may also be used instead of the tangent formulas **2** for the determination of the angles, but the latter are preferable.

EXAMPLE.—In the triangle ABC, $a = 567$ feet, $b = 736$ feet, and $c = 264$ feet; to find the angles A, B, and C.

SOLUTION.—The tangent formulas will be used.

To find A

$a = 567$	$\log(s-c) = 2.71559$
$b = 736$	$\log(s-b) = 1.67669$
$c = 264$	4.39228
$2s = 1{,}567$	$\log s = 2.89404$
$s = 783.5$	$\log(s-a) = 2.33546$
$s-a = 216.5$	5.22950
$s-b = 47.5$	
$s-c = 519.5$	$2)\bar{1}.16278$
	$\log\tan\frac{1}{2}A = \bar{1}.58139$

$\frac{1}{2}A = 20° 52' 38''$, $A = 41° 45' 16''$. Ans.

To find B

$\log(s-a) = 2.33546$
$\log(s-c) = 2.71559$
5.05105
$\log s = 2.89404$
$\log(s-b) = 1.67669$
4.57073
$2)0.48032$
$\log\tan\frac{1}{2}B = 0.24016$
$\frac{1}{2}B = 60° 5' 29''$; $B = 120° 10' 58''$ Ans.

To find C

$\log(s-a) = 2.33546$
$\log(s-b) = 1.67669$
4.01215
$\log s = 2.89404$
$\log(s-c) = 2.71559$
5.60963
$2)\bar{2}.40252$
$\log\tan\frac{1}{2}C = \bar{1}.20126$
$\frac{1}{2}C = 9° 1' 54''$; $C = 18° 3' 48''$ Ans.

To check, add the angles:

41°	45′	16″
120	10	58
18	3	48
180°	00′	2″

The triangle closes within 2 sec. This error is due to the use of five-place tables, and to the fact that the angle in each case was taken out to the nearest second.

EXAMPLES FOR PRACTICE

1. Given $a = 1$ mile, $b = 2$ miles, and $c = 1.5$ miles; find A, B, and C. (Use first method.)
Ans. $\begin{cases} A = 28° 57' 17'' \\ B = 104° 28' 39'' \\ C = 46° 34' 4'' \end{cases}$

2. Given $a = 50$ chains, $b = 30$ chains, and $c = 45$ chains; find A, B, and C. (Use first method.)
Ans. $\begin{cases} A = 80° 56' 36'' \\ B = 36° 20' 7'' \\ C = 62° 43' 17'' \end{cases}$

3. Given a = 63.47 feet, b = 89.36 feet, and c = 109.83 feet; find A, B, and C. (Use second method.)

Ans. $\begin{cases} A = 35° \ 18' \ 10'' \\ B = 54° \ 27' \ 2'' \\ C = 90° \ 14' \ 50'' \end{cases}$

4. Given a = 2,354 feet, b = 3,115 feet, and c = 836.6 feet; find A, B, and C. (Use second method.)

Ans. $\begin{cases} A = 21° \ 7' \ 24'' \\ B = 151° \ 31' \ 8'' \\ C = 7° \ 21' \ 30'' \end{cases}$

---

24. Case IV: Given Two Sides and the Angle Opposite One of Them.—In the triangle ABC, let a, b, and A be given, to find B, C, and c. The angle B or C is found by means of the principle of sines; thus,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ whence } \sin B = \frac{b \sin A}{a}$$

Then, $$C = 180° - A - B, \text{ and } c = \frac{a}{\sin A} \sin C$$

When the data are given as above, without any further restrictions, there may be two triangles that will answer the given conditions; and the problem is said to have two solutions. For here the angle B is determined from its sine, and as every sine corresponds to two supplementary angles, either of these angles may be taken. Thus, if sin B is found to be .64746, the corresponding angle may be either 40° 21′ or 180° − 40° 21′ = 139° 39′, since these angles both have the same sine (Art. **13**).

The same result is obtained from geometrical considerations. On any line AX, Fig. 7, construct an angle equal to the given angle A, and on its side AC take AC equal to one of the given sides b. From C as a center, with a radius equal to the side a, describe an arc. This arc will generally cut AX at two points, B_1 and B_2, and either of the triangles ACB_1 or ACB_2 will answer the conditions of the problem, for they both contain the given sides b and a, and the angle A opposite a.

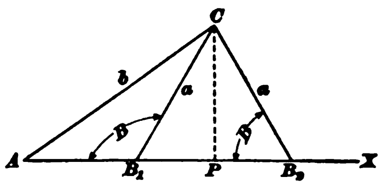

FIG. 7

The problem will have but one solution in the following cases:

6. The connecting-rod AB, Fig. 16, of an engine is 9 feet 3 inches, and the crank-arm CB is $10\frac{1}{4}$ inches; the figure shows the crank after

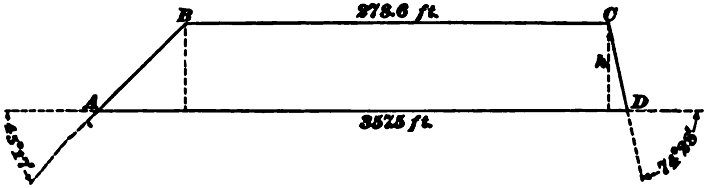

FIG. 15

it has performed one-eighth of a revolution, starting from the position CB'. Find: (*a*) the inclination M of the connecting-rod to the axis

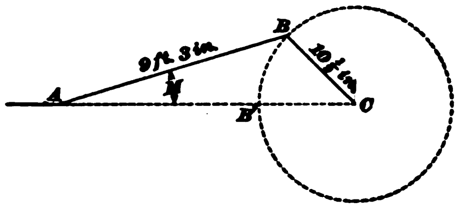

FIG. 16

of the piston rod, which is in line with CA; (*b*) the distance AC of the joint A from the center of the crank-circle.

Ans. $\begin{cases} (a) \ M = 3° 50' 7'' \\ (b) \ AC = 9 \text{ ft. } 10\frac{1}{4} \text{ in., nearly} \end{cases}$

AREAS

LAND MEASURE

25. In surveying the public lands of the United States and Canada, all linear measurements are made with the surveyors' chain, also known as Gunter's chain, from the name of the inventor. This chain is 66 feet in length and contains 100 links, each 7.92 inches long. In private surveys, the foot is commonly taken as the unit of linear measure, and small land areas are expressed in square feet.

Land areas of considerable extent in the countries mentioned are generally expressed in acres. Fractional parts of an acre, which formerly were expressed in roods, square rods or perches, and square links, are now expressed decimally by nearly all surveyors. Thus, 40.35 acres is written instead of 40 acres, 1 rood, and 16 square rods.

Tables of linear and square measure are given in *Arithmetic*, and to those tables the student is referred for detailed information regarding the subject. The following table gives the relative values of the units of area used in land surveying in the countries referred to above. As already stated, the square foot and acre are now the units most commonly employed.

TABLE OF LAND MEASURE

1 square yard (sq. yd.) .	=	9	square feet (sq. ft.)
1 square rod* (sq. rd) .	=	30¼	square yards = 272¼ square feet
1 square chain (sq. ch.) .	=	16	square rods = 4,356 square feet
1 acre (A.)	=	10	square chains = 43,560 square feet
1 rood (R.)	=	40	square rods = 10,890 square feet
1 acre	=	4	roods = 160 square rods
1 square mile (sq. mi.)	=	640	acres = 6,400 square chains
1 township (Tp.)	=	36	square miles = 23,040 acres (app.)

*Sometimes called a perch or pole, and designated by the abbreviation P.

As will be observed, there are 10 square chains in an acre. In order, therefore, to reduce to acres any number of square chains, it is sufficient to move the decimal point one place toward the left, which is equivalent to dividing by 10. It must also be borne in mind that, since there are 100 links in 1 chain, links are usually expressed decimally as hundredths of a chain. Thus, 6.72 chains is written instead of 6 chains 72 links.

EXAMPLE 1.—A rectangular piece of land is 1,060 feet in length by 820 feet in breadth; what is its area: (*a*) in acres and decimals? (*b*) in acres, roods, and perches?

SOLUTION.— (*a*) 1,060 × 820 = 869,200 sq. ft.; 869,200 ÷ 43,560 = 19.954 A. Ans.

(*b*) .954 A. = .954 × 4 = 3.816 R.; .816 R. is equal to .816 × 40 = 32.64 P. Hence, the area is 19 A. 3 R. 32.64 P. Ans.

EXAMPLE 2.—A rectangular piece of land is 12 chains and 6 links (12.06 chains) in length by 8 chains and 55 links (8.55 chains) in breadth; what is its area: (*a*) in acres and decimals? (*b*) in acres, roods, and perches?

SOLUTION.— (*a*) 12.06 × 8.55 = 103.11 sq. ch.; 103.11 ÷ 10 = 10.311 A. Ans.

(*b*) .311 A. = .311 × 4 = 1.244 R.; .244 R. is equal to .244 × 40 = 9.76 P. Hence, the area is 10 A. 1 R. 9.76 P. Ans.

EXAMPLES FOR PRACTICE

1. A rectangular piece of land is 1,190 feet in length by 700 feet in breadth; what is its area: (*a*) in acres and decimals? (*b*) in acres, roods, and perches?

Ans. { (*a*) 19.123 A.
(*b*) 19 A. 0 R. 19.7 P.

2. A rectangular piece of land is 525 feet long by 250 feet wide; what is its area: (*a*) in acres and decimals? (*b*) in acres, roods, and perches?

Ans. { (*a*) 3.013 A.
(*b*) 3 A. 0 R. 2.08 P.

3. A rectangular piece of land is 15 chains and 65 links in length by 8 chains and 16 links in breadth; what is its area: (*a*) in acres and decimals? (*b*) in acres, roods, and perches?

Ans. { (*a*) 12.77 A.
(*b*) 12 A. 3 R. 3.2 P.

AREAS OF POLYGONS

THE TRIANGLE

NOTE.—In all that follows, the area of any figure under consideration will be designated by S, unless otherwise stated.

26. Given the Base and Altitude.—Any of the sides of a triangle may be taken as the base, the altitude being the length of the perpendicular drawn on the base from the vertex of the opposite angle. In Fig. 17, b is taken as the base, and the perpendicular BH, denoted by h, is the altitude.

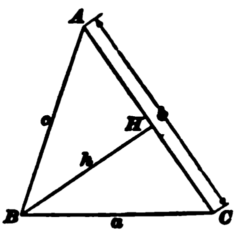

FIG. 17

It was shown in *Geometry*, Part 2, that the area of a triangle, when the base b and altitude h are known, is given by the formula

$$S = \tfrac{1}{2} b h$$

27. Given Two Sides and the Included Angle.—Let b, c, and A, Fig. 17, be given. In the right triangle ABH, we have $h = c \sin A$. The substitution of this value of h in the formula in Art. 26 gives

$$S = \tfrac{1}{2} b c \sin A$$

In words, *the area of a triangle is equal to one-half the product of any two sides and the sine of their included angle.*

EXAMPLE.—Two of the sides of a triangular field are 39.47 and 59.23 chains, respectively, and their included angle is 65° 10′ 40″. To find the contents of the field, in acres.

SOLUTION.—By the formula, S (square chains) $= \tfrac{1}{2} \times 39.47 \times 59.23 \sin 65° 10' 40'' = 1{,}060.9$ sq. ch.; whence, dividing by 10 (Art. 25),

S (acres) = 106.09 A. Ans.

28. Given One Side and Two Angles.—The other angle may be at once found by subtracting the sum of the two given angles from 180°. It may, therefore, be assumed that the three angles are known. Let b, Fig. 17, be the given side. From Art. 22, the value of c is equal to the modulus of the triangle multiplied by sin C, or,

$$c = \frac{b}{\sin B} \sin C$$

Substituting this value in the formula in Art. **27**, we obtain

$$S = \frac{b^2 \sin A \sin C}{2 \sin B}$$

29. The formula in Art. **28** is convenient when logarithmic functions are employed. For the use of natural functions, the following is preferable:

In the right triangles ABH and CBH, Fig. 17, we have,

$$AH = h \cot A, \; CH = h \cot C$$

whence, adding these two equations,

$$AH + CH = h \cot A + h \cot C$$

that is, $$b = h(\cot A + \cot C)$$

and, therefore, $$h = \frac{b}{\cot A + \cot C} \qquad (1)$$

This formula is useful and should be committed to memory. It may be stated in words thus: *The altitude of a triangle is equal to the base divided by the sum of the cotangents of the adjacent angles.*

By substituting, in the formula in Art. **26**, the value of h given in formula **1**, we obtain

$$S = \frac{b^2}{2(\cot A + \cot C)} \qquad (2)$$

In words, *the area of a triangle is equal to the square of any side divided by twice the sum of the cotangents of the angles adjacent to that side.*

Example.—One side of a triangular field is 127.64 chains, and the adjacent angles are 46° 15′ and 60° 41′. To find the area.

Solution by Logarithmic Functions.—Here, $b = 127.64$, $A = 46° 15'$, $C = 60° 41'$, and $B = 180° - 46° 15' - 60° 41' = 73° 4'$. Formula of Art. **28**,

$$S = \frac{127.64^2 \sin 46° 15' \sin 60° 41'}{2 \sin 73° 4'}$$
$$= 5{,}363.4 \text{ sq. ch.} = 536.34 \text{ A. Ans.}$$

Solution by Natural Functions.—By formula **2**,

$$S = \frac{127.64^2}{2(\cot 46° 15' + \cot 60° 41')} = \frac{127.64^2}{2(.95729 + .56156)}$$
$$= \frac{127.64^2}{3.0377} = 5{,}363.4 \text{ sq. ch.} = 536.34 \text{ A. Ans.}$$

Note.—Even if natural functions are used, the division is advantageously performed by means of logarithms.

EXAMPLES FOR PRACTICE

1. Two sides of a triangular field are 3,760 and 2,757 feet, respectively, and their included angle is 54° 13′ 13″. What is the area of the field, in acres? Ans. S = 96.534 A.

2. One side of a triangle is 96.34 chains; the opposite angle is 49° 10′, and one of the adjacent angles, 69° 45′ 30″. What is the area of the triangle, in acres? Ans. S = 503.69 A.

3. One side of a triangle is 8.93 inches, and the adjacent angles are 34° 16′ and 17° 37′ 18″. What is the area of the triangle? Ans. S = 8.638 sq. in.

4. Two sides of a triangle are 17 and 25 feet, respectively, and the included angle is 76° 13′. What is the area of the triangle? Ans. S = 206.38 sq. ft.

---

30. Given the Three Sides.—Let a, b, and c, Fig. 17, be given, and denote $\frac{1}{2}(a + b + c)$ by s. The area S of the triangle is given by the following formula, which is derived in Appendix VI:

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

EXAMPLE.—The sides of a triangular tract are 1,634.6 (= a, say), 978.28 (= b, say), and 2,176.4 (= c, say) feet, respectively; to find the area, in acres.

SOLUTION.—The work may be conveniently arranged as shown below. The numbers in marks of parenthesis indicate the order in which the several quantities are set down. In (6), s is placed above a, b, c in order to facilitate the subtractions. The differences $s - a$, $s - b$, $s - c$ are written, as the subtractions are performed, horizontally opposite a, b, and c, respectively.

(6) s = 2,394.64		
(1) a = 1,634.60	(7) $s - a$ =	760.04
(2) b = 978.28	(8) $s - b$ =	1,416.36
(3) c = 2,176.40	(9) $s - c$ =	218.24
(4) 2 s = 4,789.28		
(5) s = 2,394.64		

(10) log s = 3.37924
(11) log $(s - a)$ = 2.88083
(12) log $(s - b)$ = 3.15117
(13) log $(s - c)$ = 2.33893
2)11.75017
log S = 5.87509; S = 750,050 sq. ft. = 17.22 A. Ans.

EXAMPLES FOR PRACTICE

1. Find the area of a triangular tract whose sides are 54.36, 73.19, and 101.76 chains, respectively. Ans. $S = 192.26$ A.

2. Find the area of a triangular plate whose sides are 17.12, 12.75, and 8.95 inches, respectively. Ans. $S = 55.646$ sq. in.

THE TRAPEZOID

31. Notation.—In Fig. 18, the bases, or parallel sides, of the trapezoid $ABCD$ are denoted by b_1 and b_2; the altitude, by h; and the sides AD and BC, by a and c, respectively. The angles will be designated by the letters A, B, C, D at the vertexes. The line DB' is drawn through D parallel to CB, thus forming a parallelogram in which $B'B = DC = b_2$, and $DB' = CB = c$. Also, angle $DB'A = B$, and $AB' = AB - B'B = b_1 - b_2$. For some purposes, it is convenient to represent this difference by a single letter d, as shown in the figure.

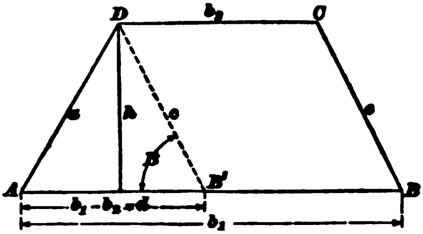

FIG. 18

32. Given the Bases and the Altitude.—As shown in *Geometry*, Part 2, the area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases; that is,

$$S = \frac{1}{2}(b_1 + b_2)h$$

33. Given the Bases and the Angles Adjacent to One of Them.—Let b_1, b_2, A, and B, Fig. 18, be given. In the triangle ADB' we have (formula **1**, Art. **29**),

$$h = \frac{b_1 - b_2}{\cot A + \cot B}$$

If this value of h is substituted in the formula of Art. **32**, the result is,

$$S = \frac{(b_1 - b_2)(b_1 + b_2)}{2(\cot A + \cot B)} \qquad (1)$$

As the product of the sum of two quantities by their difference is equal to the difference between the squares of the quantities, $(b_1 - b_2)(b_1 + b_2)$ is equal to $b_1^2 - b_2^2$; and, therefore, formula **1** may also be written:

$$S = \frac{b_1^2 - b_2^2}{2(\cot A + \cot B)} \qquad (2)$$

For the use of logarithmic functions, formula **1** may be transformed into the following (see Appendix VII):

$$S = \frac{(b_1 - b_2)(b_1 + b_2) \sin A \sin B}{2 \sin (A + B)} \qquad (3)$$

In the application of these formulas, the student should bear in mind that the cotangent of an angle greater than 90° is negative, and numerically equal to the cotangent of the supplement of the angle; also, that the sine of an angle greater than 90° is equal to the sine of its supplement. Thus, $\cot 125° = -\cot(180° - 125°) = -\cot 75° = -.26795$; and $\sin 125° = \sin 75° = .96593$.

EXAMPLE 1.—The two bases of a trapezoid are 350 and 137 chains, respectively; the angles adjacent to the longer base are 75° 10′ and 63° 54′. What is the area of the trapezoid?

SOLUTION BY NATURAL FUNCTIONS.—Let $350 = b_1$, $137 = b_2$, $A = 75° 10'$, $B = 63° 54'$. As b_1 and b_2 are not convenient numbers to square, formula **1**, which is better adapted to logarithmic work, will be used.

$$S = \frac{(350 - 137)(350 + 137)}{2(\cot 75° 10' + \cot 63° 54')} = \frac{213 \times 487}{2(.26483 + .48989)} = 68{,}721 \text{ sq. ch.}$$
$$= 6{,}872.1 \text{ A. Ans.}$$

SOLUTION BY LOGARITHMIC FUNCTIONS.—By formula **3**,

$$S = \frac{(350 - 137)(350 + 137) \sin 75° 10' \sin 63° 54'}{2 \sin 139° 4'}$$

or, replacing $\sin 139° 4'$ by $\sin(180° - 139° 4') = \sin 40° 56'$,

$$S = \frac{213 \times 487 \sin 75° 10' \sin 63° 54'}{2 \sin 40° 56'} = 68{,}721 \text{ sq. ch.} = 6{,}872.1 \text{ A. Ans.}$$

EXAMPLE 2.—The bases of a trapezoid are 100 and 70 feet, the angles adjacent to the shorter base being 52° 47′ and 143° 14′. What is the area of the trapezoid?

SOLUTION.—Since the bases are parallel, the two angles adjacent to each of the non-parallel sides are supplementary. Thus, in Fig. 18, $A + D = 180°$, $B + C = 180°$; and, therefore, $A = 180° - D$; $B = 180° - C$. Let $52° 47' = D$, $143° 14' = C$. Then,

$$A = 180^\circ - 52^\circ\ 47' = 127^\circ\ 13'$$
$$B = 180^\circ - 143^\circ\ 14' = 36^\circ\ 46'$$
$$\cot A = -\cot (180^\circ - 127^\circ\ 13') = -\cot 52^\circ 47' = -.75950$$
$$\cot B = \cot 36^\circ\ 46' = 1.33835$$

Formula 2,

$$S = \frac{100^2 - 70^2}{2(+1.33835 - .7595)} = \frac{5,100}{1.1577} = 4,405.3 \text{ sq. ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. The bases of a trapezoidal tract are 78.63 and 54.71 chains, respectively; the angles adjacent to the longer base are 55° 18′ and 62° 53′. Find the area, in acres. Ans. $S = 132.4$ A.

2. Find the number of square feet in a trapezoidal cross-section of a canal 40 feet wide at the bottom, 65 feet wide at the top, and whose non-parallel sides are inclined to the horizontal at an angle of 50°. (The dimensions across the top and bottom are measured horizontally.) Ans. $S = 782.09$ sq. ft.

3. The two bases of a trapezoid are 10.25 and 18.76 inches, respectively; one of the angles adjacent to the shorter base is 76° 45′ 10″, and the angle diagonally opposite is 66° 8′ 9″; find the area of the trapezoid. (Use logarithmic functions.) Ans. $S = 596.4$ sq. in.

---

34. Given the Four Sides.—If the difference between the two bases added to the sum of the non-parallel sides is denoted by $2s$; that is, if the expression $\frac{1}{2}(a + c + d)$, Fig. 18, is denoted by s, the area of the trapezoid is given by the following formula (see Appendix VIII):

$$S = \frac{b_1 + b_2}{d}\sqrt{s(s-a)(s-c)(s-d)}$$

EXAMPLE FOR PRACTICE

The bases of a trapezoidal field are 136.43 and 210.18 chains, respectively; one of the non-parallel sides is 96.73 chains, and the other 164.37 chains. Find the area of the tract, in acres.

Ans. $S = 864.97$ A.

THE REGULAR POLYGON

35. Given the Number of Sides and the Radius. Let MN, Fig. 19, be one of the sides of a regular polygon of n sides; O, the center, and r the radius, of the circumscribed circle (called also the center and radius, respectively, of the polygon); and A, the angle at the center subtended by a side of the polygon. The length of the side MN will be denoted by l.

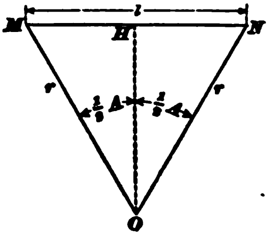

FIG. 19

Let n and r be given, to find the area S of the polygon and the length l of each of its sides. From *Geometry*, Part 2, the angle MON, or A, is found by dividing 360° by the number of sides in the polygon; that is,

$$A = \frac{360°}{n}$$

The area of the triangle MON is (Art. **27**) $\frac{1}{2} OM \times ON \sin MON$, or $\frac{1}{2} r \times r \sin A = \frac{1}{2} r^2 \sin A = \frac{1}{2} r^2 \sin \frac{360°}{n}$. Since the polygon consists of n triangles equal to MON, its area S is equal to n times the area of MON; that is,

$$S = n \times \tfrac{1}{2} r^2 \sin \frac{360°}{n}$$

or

$$S = \tfrac{1}{2} n r^2 \sin \frac{360°}{n} \qquad (1)$$

In the right triangle MOH, we have,

$$MH = r \sin \frac{A}{2}$$

or, since MH is one-half of MN, or of l,

$$\frac{l}{2} = r \sin \frac{A}{2}$$

whence, multiplying by 2,

$$l = 2r \sin \frac{A}{2}$$

Finally, $\frac{A}{2} = \frac{1}{2} \frac{360°}{n} = \frac{180°}{n}$. By the substitution of this value in the expression for l just found, we get, finally,

$$l = 2r \sin \frac{180^\circ}{n} \qquad (2)$$

36. When the Number of Sides and Their Common Length Are Given.—Let n and l, Fig. 19, be given, to find the radius r and the area S. The radius is found by solving formula **2**, Art. **35**, for r, which gives,

$$r = \frac{l}{2 \sin \frac{180^\circ}{n}} \qquad (1)$$

In the triangle MOH, we have,

$$OH = MH \cot \tfrac{1}{2}A = \frac{MN}{2} \cot \tfrac{1}{2}A$$

The area of MON is $\frac{1}{2} MN \times OH$. Writing instead of OH the value just found,

$$\tfrac{1}{2} MN \times \frac{MN}{2} \cot \tfrac{1}{2}A = \frac{MN^2}{4} \cot \tfrac{1}{2}A$$

$$= \frac{l^2}{4} \cot \tfrac{1}{2}A = \frac{l^2}{4} \cot \frac{180^\circ}{n}$$

Multiplying this by n, we obtain, for the area of the polygon,

$$S = \frac{nl^2}{4} \cot \frac{180^\circ}{n} \qquad (2)$$

EXAMPLE 1.—Find the area, and also the length of the side, of a regular decagon inscribed in a 15-inch circle.

SOLUTION.—In practice, it is usual to refer to a circle by its diameter, and so a 15-in. circle is a circle whose diameter is 15 in. We have, therefore, $r = \frac{15}{2} = 7.5$, $n = 10$, $\frac{360^\circ}{n} = \frac{360^\circ}{10} = 36^\circ$, $\frac{180^\circ}{n} = 18^\circ$, and formulas **1** and **2**, Art. **35**, give

$S = \frac{1}{2} \times 10 \times 7.5^2 \sin 36^\circ = 165.32$ sq. in. Ans.
$l = 2 \times 7.5 \sin 18^\circ = 4.635$ in. Ans.

EXAMPLE 2.—Each of the sides of an octagonal park is 150 feet; what is the area of the park, in acres?

SOLUTION.—Here $l = 150$ ft., $n = 8$, $\frac{180^\circ}{n} = \frac{180^\circ}{8} = 22\frac{1}{2}^\circ = 22^\circ\ 30'$, and formula **2**, Art. **36**, gives,

$S = \frac{1}{4} \times 8 \times 150^2 \cot 22° 30' = 2 \times 22{,}500 \cot 22° 30' = (45{,}000 \cot 22° 30')$ sq. ft. $= \frac{45{,}000 \cot 22° 30'}{43{,}560}$ A. = 2.494 A. Ans.

EXAMPLES FOR PRACTICE

1. Find the side and area of an equilateral triangle inscribed in a 20-inch circle.

Ans. $\begin{cases} l = 17.321 \text{ in.} \\ S = 129.9 \text{ sq. in.} \end{cases}$

2. What must be the length of the side and the radius of a regular pentagon, that its area may be 46.97 square feet?

Ans. $\begin{cases} l = 5.225 \text{ ft.} \\ r = 4.445 \text{ ft.} \end{cases}$

3. An eight-sided drive is to be built around a circular park 1,500 feet in diameter, the drive to be 15 feet wide, with its outer corners on the circumference of the park. Find: (*a*) the length of each of the sides of the outer boundary of the drive; (*b*) the length of each of the sides of the inner boundary; (*c*) the cost of paving the drive with asphalt, at $2.25 per square yard; (*d*) the difference between the exact area of the drive and the approximate area found by assuming the polygonal boundaries to coincide with the circumferences of their respective circumscribed circles.

Ans. (*a*) 574.02 ft.
(*b*) 561.60 ft.
(*c*) $17,025
(*d*) 844 sq. yd.

OTHER POLYGONS

37. The area of any polygon can be determined by dividing the polygon into triangles, and measuring in each triangle whatever parts are necessary for the determination of its area. The parts to be measured depend on special conditions and on the instruments used. The polygon may be divided into triangles either by diagonals or by lines drawn from a convenient interior point to the different vertexes. Illustrations of these methods of division will be given in connection with surveying. When the area is to be determined from a plat, the base and altitude of each triangle are usually the most convenient parts to measure.

AREAS BOUNDED BY IRREGULAR OUTLINES

AREA INCLUDED BETWEEN A STRAIGHT LINE AND AN IRREGULAR CURVE

38. By Selected Ordinates.—Let it be required to determine the area between the curve *D C* and the straight line *A B*, Fig. 20. A very convenient method is to draw perpendiculars on *A B* from the points of the curve at which its direction changes appreciably, and to consider the portion of the curve between two consecutive perpendiculars to be a straight line. The

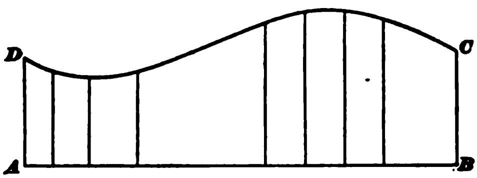

FIG. 20

figure is then treated as if divided into a number of trapezoids, whose areas can be computed by the rules of geometry. The perpendiculars are called **ordinates.** Both the lengths of the ordinates and the distances between every two consecutive ordinates should be measured. The area of any of the (approximate) trapezoids into which the figure is thus divided is equal to one-half the sum of the two ordinates enclosing it multiplied by the distance between them. It should be understood that both this rule and those given further on relating to the same subject are only approximate. Since the bounding curve is irregular, that is, does not follow any mathematical law, no exact formula can be found for the area.

EXAMPLE.—Referring to Fig. 20, suppose that, beginning at the left of the figure, the successive ordinates measure 15, 13, 12, 13.5, 20, 21.5, 22, 20, and 16 feet, respectively, and that the successive distances between the offsets, from left to right, measure 7.5, 10, 15, 41, 10.5, 11.5, 11.5, and 21 feet, respectively; what is the area of the surface?

SOLUTION.—The area of the figure is approximately equal to the sum of the areas of the trapezoids into which it is divided, and the area of each trapezoid is equal to one-half the sum of its parallel sides multiplied by the perpendicular distance between them. Therefore, the area of the figure is equal to

$$\frac{15+13}{2}\times 7.5+\frac{13+12}{2}\times 10+\frac{12+13.5}{2}\times 15+\frac{13.5+20}{2}\times 41$$
$$+\frac{20+21.5}{2}\times 10.5+\frac{21.5+22}{2}\times 11.5+\frac{22+20}{2}\times 11.5+\frac{20+16}{2}\times 21$$
$$= 2{,}195.5 \text{ sq. ft.} \quad \text{Ans.}$$

39. Trapezoidal Rule: Sigma Notation.—In order to facilitate the calculations, the ordinates are often measured at regular intervals along the straight line, as shown in Fig. 21. The area $ABCD$ included between the straight line and the irregular boundary can then be more easily calculated by what is commonly known as the **trapezoidal rule.** This

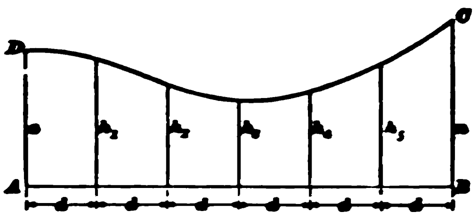

Fig. 21

is merely a rule for calculating the combined area of a series of trapezoids that have the same altitude, the areas being combined for convenience of calculation. The result given by this rule is closer the smaller the distance between the ordinates. The rule is as follows:

Rule.—*Add together one-half the two end ordinates and all the intermediate ordinates, and multiply the sum by the common distance between the ordinates.*

Let a = first ordinate;
n = last ordinate;
h_1, h_2, h_3 = intermediate ordinates;
d = common distance between ordinates;
S = area of surface.

Then, $S = [\frac{1}{2}(a+n) + h_1 + h_2 + h_3 + \ldots]d$

This expression may be put in a simpler form by using the **sigma notation,** which is as follows: As will be noticed, all the intermediate ordinates are denoted by h, different subscripts being used to indicate different values of h. We may, therefore, write the value of S thus,

$$S = [\tfrac{1}{2}(a + n) + \text{sum of all values of } h]d$$

Instead of the phrase *sum of all values of h*, the expression Σh, read *sigma h*, is used. The symbol Σ is the Greek letter sigma, corresponding to English S, and is very commonly used, as here, to indicate the addition of several quantities of the same character, denoted by a single symbol; hence, the name **sign of summation**, which also is often given to that letter.

By using the sigma notation, the value of S may be written

$$S = \left(\frac{a + n}{2} + \Sigma h\right)d$$

EXAMPLE.—If the ordinates from the straight line AB to the curved boundary DC, Fig. 21, are 19, 18, 14, 12, 13, 17, and 23 links, respectively, and are at equal distances of 50 links, what is the area included between the curved boundary and the straight line?

SOLUTION.—Area $ABCD = \left(\frac{19 + 23}{2} + 18 + 14 + 12 + 13 + 17\right) \times 50 = 4{,}750$ sq. li. Ans.

40. Simpson's Rule.—The foregoing rule assumes that all the small figures into which the area is divided are perfect trapezoids, which assumption always involves more or less error, since the irregular boundary is in nearly all cases an irregular curve. When the offsets are taken at

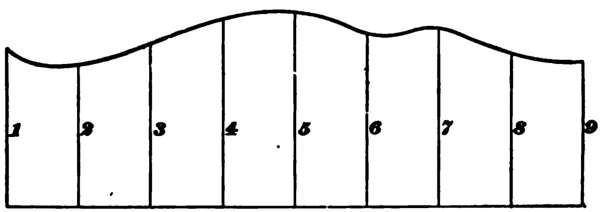

FIG. 22

regular intervals, the following rule, known as **Simpson's one-third rule**, gives a closer approximation. In applying this rule, the base line must be divided into an even number of equal parts; the ordinates measured at the points of division are numbered consecutively, as shown in Fig. 22.

Rule.—*Divide the base line into an even number of equal parts, and at the points of division erect ordinates terminating in the curve. Number the ordinates 1, 2, 3, etc., from left to right, including those at the ends of the base. Add together the end ordinates, four times the sum of all intermediate even-numbered ordinates, and twice the sum of all intermediate odd-numbered ordinates; multiply the total sum by one-third the common distance between adjacent ordinates.*

This rule has been used extensively; it can be expressed by a formula as follows:

Let h_e = any intermediate even-numbered ordinate;
h_o = any intermediate odd-numbered ordinate;
and let all other quantities be represented by the same letters as in the preceding article. Then,

$$S = (a + n + 4\,\Sigma h_e + 2\,\Sigma h_o)\frac{d}{3}$$

The notation will be readily understood by reference to Fig. 23. The expression $4\,\Sigma h_e$ means four times the sum of all the ordinates h_e, or, in other words, four times the sum of all the even-numbered ordinates.

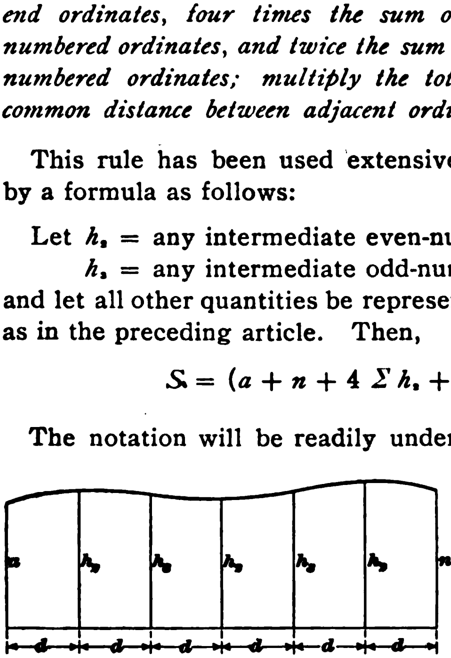

FIG. 23

EXAMPLE.—What is the area $ABCD$, Fig. 21, by Simpson's rule, using the same values as in the example in Art. **39**?

SOLUTION.— $S = [19 + 23 + 4(18 + 12 + 17) + 2(14 + 13)] \times \frac{50}{3}$ = 4,733 sq. li. Ans.

EXAMPLES FOR PRACTICE

1. A figure included between a straight base line, a curve, and two perpendiculars to the base at the ends has nine ordinates, including the two end perpendiculars, whose lengths are 43, 48, 39, 50, 41, 32, 37, 31, and 22 feet, respectively; the common distance between the ordinates is 60 feet. Find the area: (*a*) by the trapezoidal rule; (*b*) by Simpson's rule.

Ans. { (*a*) 18,630 sq. ft.
{ (*b*) 18,860 sq. ft.

2. In order to determine the area included between an irregular boundary, a straight base line, and two perpendiculars to the base at the ends, eight ordinates, including the two end perpendiculars, are measured from the straight line to the boundary. The ordinates are found to measure 16, 18, 12, 13, 15, 17, 19, and 20.5 feet, and the successive distances between them are found to measure 7.8, 10, 15, 20, 12, 40, and 5 feet, respectively. What is the area of the surface?

Ans. 1,760.9 sq. ft.

3. A surface lying between a straight base line and a curve is limited by two perpendiculars to the base line at the ends; the base line is divided into eight parts 50 feet each, and at the points of division ordinates are measured. The lengths of the successive ordinates, including the two end perpendiculars, are 10, 25, 38, 49, 58, 65, 70, 73, and 74 feet, respectively. Find the area of the surface: (*a*) by the trapezoidal rule; (*b*) by Simpson's rule.

Ans. { (*a*) 21,000 sq. ft.; (*b*) 21,067 sq. ft.

AREA BOUNDED BY AN IRREGULAR CURVE

41. By Ordinates.—Suppose that it is required to find the area enclosed by the heavy irregular curve shown in Fig. 24. A broken line *AEFMGHIA* is drawn around

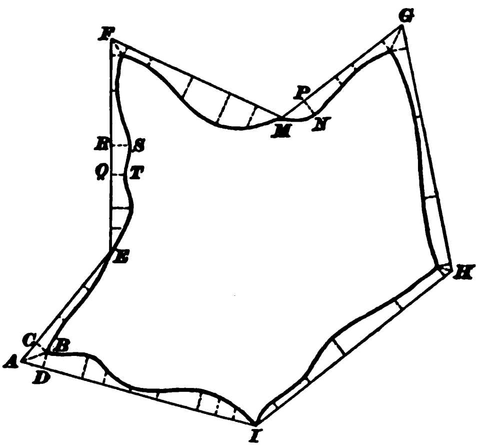

FIG. 24

the curved boundary line, and as close to it as convenient. Ordinates to the straight lines thus drawn are measured from the points where the direction of the curved boundary changes materially, as shown. The area of the polygon

AEFMGHIA is calculated by one of the methods explained in preceding articles, and from it is subtracted the sum of the areas included between the curved boundary and the broken line, calculated as in Art. **39.**

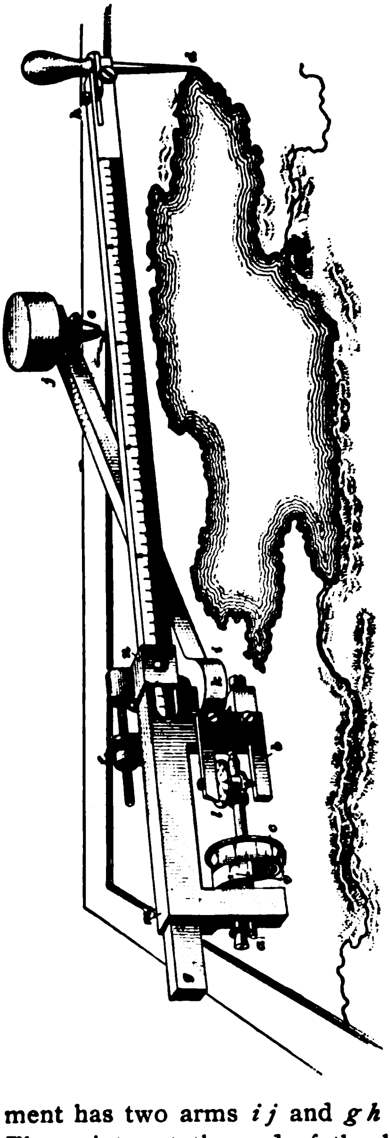

Fig. 25

At such corners as *A*, the triangles *ABC* and *ABD* are computed from the measured bases *AC* and *AD* and the altitudes *BC* and *BD*. All the quadrilaterals, as *QRST*, are treated as trapezoids; and such three-sided figures as *MPN*, as triangles. The process is so simple that it does not require any further explanation.

42. By the Planimeter.—The most convenient way to find the area of a plane surface having an irregular boundary is by the **planimeter.** There are several forms of planimeters; the one most commonly used is the **polar planimeter** (see Fig. 25). As will be seen from the illustration, this instrument has two arms *ij* and *gh* connected by a hinge joint. The point *e* at the end of the bar *ij* is called the **anchor**

point; it remains stationary while the point *d*, called the **pointer** or **tracer**, at the end of the bar *g h* is moved over the outline of the figure whose area is to be determined. The movement of the pointer *d* causes the wheel *c* on the opposite end of the bar to roll on the paper; this wheel is called the **measuring wheel** or **counter wheel.** The graduated bar *g h* can be adjusted by sliding it in or out through the socket *m* in the top of the frame. This bar is clamped by means of a clamp screw, a part of which is shown back of the small movable socket *n*, and is set at the exact length required by means of the thumbscrew *f*. The bar *i j* is of fixed length; it is pivoted at *k*, the junction of the two bars. The measuring wheel *c* is mounted on the main axis *a b*, which is parallel with the bar *g h*. The complete revolutions of the wheel *c* are read on the disk *l*, and the fractional parts of revolutions are read on the wheel *c* and the vernier *v*, the tenths and hundredths being read on the wheel itself, and the thousandths on the vernier.

To use the planimeter, the anchor point *e* is fixed on the paper or drawing board, preferably outside the figure to be measured, the pointer *d* is placed on some point in the periphery of the figure, and a reading of the wheel *c* is taken. The point *d* is then moved carefully around the periphery of the figure, in a clockwise direction, or from left to right, to the point of beginning. A second reading of the wheel *c* is then taken, and the difference between the two readings is the number of revolutions of the wheel. If the wheel is set to read zero, the number of revolutions is given directly by the second reading.

If the anchor point is outside the area to be measured, the distance traversed by the wheel, or the product of the number of revolutions by the circumference of the wheel, in inches, multiplied by the length of the bar *n h*, in inches, is the area, in square inches, bounded by the path of the pointer *d*.

If the anchor point is inside the area, the product just referred to must be added to the area of the **zero circle**, whose radius is equal to $\sqrt{p^2 + q^2 + 2\,p\,r}$, p being the length of the arm $n\,h$; r, the distance from the center of the wheel c to the center of the joint k; and q, the length of the bar $k\,j$. The bar $g\,h$ is generally set at such a length that ten times the number of revolutions of the wheel c is the area measured. This area is the actual area of the figure measured, and the area *represented* by the figure is determined from the scale of the plat. The area given by the planimeter, in square inches, must be multiplied by the square of the scale of the plat, in order to get the area sought. Thus, if the plat has been drawn to a scale of 50 feet to an inch, each square inch of the plat is equivalent to $50 \times 50 = 2{,}500$ square feet of area.

Suppose that the area bounded by the irregular line in Fig. 25, as measured by the planimeter, is 2.535 square inches, and that the scale of the plat is 100 feet to an inch; then the area represented by a square inch of the plat is $100 \times 100 = 10{,}000$ square feet, and the area represented by the closed figure is $10{,}000 \times 2.535 = 25{,}350$ square feet.

Full directions for using the planimeter are usually furnished by the maker.

APPENDIX: DERIVATION OF FORMULAS

I—FORMULAS 1 TO 4 OF ART. 16

Let $R\,O\,Q$, Fig. 26, be any angle A, and $Q\,O\,S$ any angle B. Then, $A + B = R\,O\,S$. From any point P on $O\,S$, draw $P\,N$ and $P\,M$, perpendicular, respectively, to $O\,R$ and $O\,Q$. Draw $M\,K$ parallel to $O\,R$ and therefore perpendicular to $P\,N$; also, $M\,L$ perpendicular to $O\,R$. The angles $M\,P\,K$ and $R\,O\,Q$, having their sides perpendicular each to each, are equal. Now,

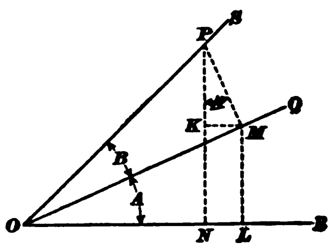

FIG. 26

$$\sin(A+B) = \frac{NP}{OP} = \frac{NK + KP}{OP} = \frac{ML}{OP} + \frac{KP}{OP} = \frac{OM \sin A}{OP} + \frac{PM \cos A}{OP}$$

(triangles MLO and PMK) $= \sin A \dfrac{OM}{OP} + \cos A \dfrac{PM}{OP} = \sin A \cos B + \cos A \sin B$ (triangle OPM)

This is formula **1.**

Also,

$$\cos (A - B) = \sin [90° - (A - B)] = \sin [(90° - A) + B]$$

or, by formula **1**,

$$\cos (A - B) = \sin (90° - A) \cos B + \cos (90° - A) \sin B$$
$$= \cos A \cos B + \sin A \sin B$$

which is formula **4.**

Formula **3** follows from this; for

$$\sin (A - B) = \cos [90° - (A - B)] = \cos [(90° + B) - A]$$
$$= \cos (90° + B) \cos A + \sin (90° + B) \sin A$$

or, because $\cos (90° + B) = - \sin B$, and $\sin (90° + B) = \cos B$ (Art. **14**),
$\sin (A - B) = - \sin B \cos A + \cos B \sin A = \sin A \cos B - \cos A \sin B.$

Finally, applying this formula,

$$\cos (A + B) = \sin [90° - (A + B)] = \sin [(90° - A) - B]$$
$$= \sin (90° - A) \cos B - \cos (90° - A) \sin B$$
$$= \cos A \cos B - \sin A \sin B$$

which is formula **2.**

II—FORMULAS OF ART. 19

Referring to Fig. 6 (a) and (b), Art. **18**,

$$a^2 = p^2 + BD^2 \qquad (1)$$

In (a), $BD = c - AD$, whence $\overline{BD}^2 = c^2 - 2c \times AD + \overline{AD}^2$.

In (b), $BD = AD - c$, whence $\overline{BD}^2 = \overline{AD}^2 - 2c \times AD + c^2$.

Substituting this value of BD in equation (1),

$$a^2 = p^2 + \overline{AD}^2 + c^2 - 2c \times AD \qquad (2)$$

But $p^2 + \overline{AD}^2 = b^2$, and $AD = b \cos A$; therefore,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

When the angle opposite the side is obtuse, as B in Fig. 6 (b), the same reasoning leads to the relation,

$$b^2 = a^2 + c^2 + 2ac \times \cos CBD$$

the second member of which becomes $a^2 + c^2 - 2ac \cos B$, when $\cos CBD$ is replaced by its equal $- \cos B$ (Art. **13**).

III—FORMULAS OF ART. 20

Let ABC, Fig. 27, be any triangle. As usual, the angles of the triangle will be denoted by A, B, C, and the opposite sides by a, b, c, respectively; that is, angle $CAB = A$, $BC = a$, etc. Produce AC to A', making $CA' = BC = a$. Draw BA', and AP perpendicular to it, meeting BC at Q.

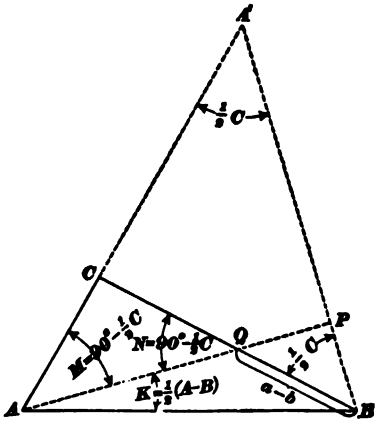

FIG. 27

Since $BC = CA'$, the triangle BCA' is isosceles, and, therefore, the angles $CA'B$ and CBA' are equal. The sum of these two angles, or twice either of them, is equal to the external angle BCA, or C, and therefore each of these two angles is equal to $\frac{1}{2}C$. In the right triangle APA', the angle M, being the complement of A', is equal to $90° - \frac{1}{2}C$.

We have also

$$K = A - M = A - (90° - \tfrac{1}{2}C),$$

or, since $C = 180° - (A+B) = 180° - A - B$,

$$K = A - [90° - \tfrac{1}{2}(180° - A - B)] = \tfrac{1}{2}(A - B)$$

The angle N being external to the triangle AQB, we have

$$N = K + B = \tfrac{1}{2}(A - B) + B = \tfrac{1}{2}(A + B)$$
$$= \tfrac{1}{2}(180° - C) = 90° - \tfrac{1}{2}C = M$$

Therefore, the triangle AQC is isosceles, and $QC = AC = b$; and, consequently, $BQ = a - b$.

The right triangle ABP gives,

$$\tan \tfrac{1}{2}(A - B) = \frac{BP}{AP}$$

or, writing the values of BP and AP from the triangles BQP and APA',

$$\tan \tfrac{1}{2}(A - B) = \frac{BQ \cos \frac{1}{2}C}{AA' \sin \frac{1}{2}C} = \frac{BQ}{AA'} \cot \tfrac{1}{2}C$$

that is,
$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \tfrac{1}{2}C \qquad (1)$$

Now, $\frac{1}{2}C = \frac{1}{2}[180° - (A+B)] = 90° - \frac{1}{2}(A+B)$, and therefore, $\cot \frac{1}{2}C = \tan \frac{1}{2}(A+B)$. By substituting this value in equation (1), and transforming, the formula in Art. **20** is obtained.

IV—FORMULA 2 OF ART. 21

This formula is derived from Fig. 27 as follows: In the triangle BPQ,

$$BP = BQ \cos \tfrac{1}{2} C = (a - b) \cos \tfrac{1}{2} C \qquad (1)$$

and, in the triangle ABP,

$$c\,(= AB) = \frac{BP}{\sin \frac{1}{2}(A - B)}$$

which becomes formula 2 when BP is replaced by its value (1).

V—FORMULAS 2 TO 4 OF ART. 23

We have (formula 8, Art. 17),

$$2 \cos^2 \tfrac{1}{2} A = 1 + \cos A$$

or, substituting the value of $\cos A$ from formula 1, Art. 23,

$$2 \cos^2 \tfrac{1}{2} A = 1 + \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc} = \frac{(b + c)^2 - a^2}{2\,bc}$$

or, remembering that the difference between the squares of two numbers is equal to their sum multiplied by their difference,

$$2 \cos^2 \tfrac{1}{2} A = \frac{(b + c + a)(b + c - a)}{2\,bc} \qquad (1)$$

Now, since $a + b + c = 2s$, we have, subtracting $2a$ from both members, $b + c - a = 2s - 2a = 2(s - a)$. Likewise, $a + b - c = 2(s - c)$, and $a + c - b = 2(s - b)$. Substituting these values in equation (1),

$$2 \cos^2 \tfrac{1}{2} A = \frac{2s \times 2(s - a)}{2\,bc} = \frac{2s(s - a)}{bc}$$

whence,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}} \qquad (2)$$

which is formula 3, Art. 23.

Likewise (formula 7, Art. 17),

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc}$$

$$= \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc} = \frac{a^2 - (b^2 - 2\,bc + c^2)}{2\,bc} = \frac{a^2 - (b - c)^2}{2\,bc}$$

$$= \frac{(a + b - c)(a - b + c)}{2\,bc} = \frac{2(s - c) \times 2(s - b)}{2\,bc} = \frac{2(s - b)(s - c)}{bc}$$

whence,

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}} \qquad (3)$$

which is formula 4, Art. 23.

Formula 2 is obtained by dividing equation (3) by equation (2).

VI—FORMULA OF ART. 30

Formulas 3 and 4 of Art. 23 are:

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{b c}} \qquad (1)$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{b c}} \qquad (2)$$

Also (formula 5, Art. 17),

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A \qquad (3)$$

Substituting in equation (3) the values of $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$ from equations (1) and (2),

$$\sin A = 2\sqrt{\frac{(s-b)(s-c)}{b c}}\sqrt{\frac{s(s-a)}{b c}} = 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2 c^2}}$$

$$= 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{b c}.$$

Substituting this value in formula of Art. 27,

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

VII—FORMULA 3 OF ART. 33

We have, since $\cot = \frac{\cos}{\sin}$,

$$\frac{1}{\cot A + \cot B} = \frac{1}{\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B}} = \frac{\sin A \sin B}{\sin B \cos A + \cos B \sin A}$$

$$= \frac{\sin A \sin B}{\sin (A + B)}$$

By substituting this value in formula 1, we obtain

$$S = \frac{(b_1 - b_2)(b_1 + b_2) \sin A \sin B}{2 \sin (A + B)}$$

VIII—FORMULA OF ART. 34

Let the area of the triangle $A D B'$, Fig. 18, be denoted by T, and that of the parallelogram $B C D B'$ by P. Then,

$$S = P + T \qquad (1)$$

Now, $P = b_2 h$, $T = \frac{1}{2} d h$

Dividing the first of these equations by the second,

$$\frac{P}{T} = \frac{b_2}{\frac{1}{2} d} = \frac{2 b_2}{d} = \frac{2 b_2}{b_1 - b_2}$$

whence, $P = \frac{2 b_2}{b_1 - b_2} T$

Substituting this value of P in equation (1),

$$S = \frac{2b_2}{b_1 - b_2}T + T = \left(\frac{2b_2}{b_1 - b_2} + 1\right)T = \frac{b_1 + b_2}{b_1 - b_2}T = \frac{b_1 + b_2}{d}T \qquad (2)$$

Let $\frac{1}{2}(a + c + d) = s$. Then (formula of Art. 30),

$$T = \sqrt{s(s-a)(s-c)(s-d)}$$

and, substituting this value in equation (2),

$$S = \frac{b_1 + b_2}{d}\sqrt{s(s-a)(s-c)(s-d)}$$

TABLE OF TRIGONOMETRIC FORMULAS

The principal formulas occurring in the text, and others that can be readily derived from these, are tabulated in the following pages for convenient reference. As these formulas, which include those for the solution of triangles, are here systematically classified and arranged, the student will find this table useful in the solution of all kinds of problems requiring the application of trigonometry. He is advised to refer to it often, so as to become familiar with its contents and use.

FORMULAS DEFINING THE TRIGONOMETRIC FUNCTIONS

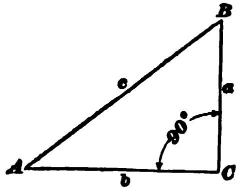

FIG. 28

1. $\sin A = \frac{a}{c}$
2. $\tan A = \frac{a}{b}$
3. $\cos A = \sin (90° - A) = \frac{b}{c}$
4. $\cot A = \tan (90° - A) = \frac{b}{a}$
5. $\sec A = \frac{c}{b}$
6. $\csc A = \sec (90° - A) = \frac{c}{a}$
7. $\text{vers } A = 1 - \cos A = 1 - \frac{b}{c}$
8. $\text{covers } A = \text{vers } (90° - A) = 1 - \sin A = 1 - \frac{a}{c}$

FUNCTIONS OF 0° AND 90°

9. $\sin 0° = 0$
10. $\tan 0° = 0$
11. $\cos 0° = 1$
12. $\cot 0° = \infty$
13. $\sec 0° = 1$
14. $\csc 0° = \infty$
15. $\sin 90° = 1$
16. $\tan 90° = \infty$
17. $\cos 90° = 0$
18. $\cot 90° = 0$
19. $\sec 90° = \infty$
20. $\csc 90° = 1$

FUNCTIONS OF NEGATIVE ANGLES

21. $\sin (-A) = -\sin A$
22. $\tan (-A) = -\tan A$
23. $\cos (-A) = \cos A$
24. $\cot (-A) = -\cot A$
25. $\sec (-A) = \sec A$
26. $\csc (-A) = -\csc A$

FUNCTIONS OF 90° + A

27. $\sin(90° + A) = \cos A$
28. $\tan(90° + A) = -\cot A$
29. $\cos(90° + A) = -\sin A$
30. $\cot(90° + A) = -\tan A$
31. $\sec(90° + A) = -\csc A$
32. $\csc(90° + A) = \sec A$

FUNCTIONS OF 180° − A AND OF 180° + A

33. $\sin(180° - A) = \sin A$
34. $\tan(180° - A) = -\tan A$
35. $\cos(180° - A) = -\cos A$
36. $\cot(180° - A) = -\cot A$
37. $\sec(180° - A) = -\sec A$
38. $\csc(180° - A) = \csc A$
39. $\sin(180° + A) = -\sin A$
40. $\tan(180° + A) = \tan A$
41. $\cos(180° + A) = -\cos A$
42. $\cot(180° + A) = \cot A$
43. $\sec(180° + A) = -\sec A$
44. $\csc(180° + A) = -\csc A$

FUNCTIONS OF 360° − A AND OF 360° + A

45. $\sin(360° - A) = -\sin A$
46. $\tan(360° - A) = -\tan A$
47. $\cos(360° - A) = \cos A$
48. $\cot(360° - A) = -\cot A$
49. $\sec(360° - A) = \sec A$
50. $\csc(360° - A) = -\csc A$
51. $\sin(360° + A) = \sin A$
52. $\tan(360° + A) = \tan A$
53. $\cos(360° + A) = \cos A$
54. $\cot(360° + A) = \cot A$
55. $\sec(360° + A) = \sec A$
56. $\csc(360° + A) = \csc A$

FUNCTIONS OF (A + B) AND OF (A − B)

57. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
58. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
59. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
60. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
61. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
62. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

FUNCTIONS OF $2A$ AND OF $\frac{1}{2}A$

63. $\sin 2A = 2 \sin A \cos A$
64. $\cos 2A = \cos^2 A - \sin^2 A$
65. $\cos 2A = 2 \cos^2 A - 1$
66. $\cos 2A = 1 - 2 \sin^2 A$
67. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$
68. $\sin \frac{1}{2} A = \sqrt{\dfrac{1 - \cos A}{2}}$
69. $\cos \frac{1}{2} A = \sqrt{\dfrac{1 + \cos A}{2}}$
70. $\tan \frac{1}{2} A = \sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$
71. $\tan \frac{1}{2} A = \dfrac{1 - \cos A}{\sin A}$

SUMS AND DIFFERENCES OF FUNCTIONS

72. $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$
73. $\sin A - \sin B = 2 \sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)$
74. $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$
75. $\cos A - \cos B = 2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(B - A)$
76. $\tan A + \tan B = \dfrac{\sin (A + B)}{\cos A \cos B}$
77. $\tan A - \tan B = \dfrac{\sin (A - B)}{\cos A \cos B}$
78. $\sin^2 A - \sin^2 B = \sin (A + B) \sin (A - B)$
79. $\cos^2 A - \cos^2 B = \sin (A + B) \sin (B - A)$
80. $\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B)$

CHAIN SURVEYING

CHAINING

DEFINITIONS AND CLASSIFICATION

1. Definitions.—**Surveying** is that branch of civil engineering which treats of the principles and methods employed for determining the relative positions of points on the earth's surface. By the earth's "surface" is here meant all that part of the earth that can be explored; the term includes, therefore, the bottoms of seas and rivers, and the interior of mines, as well as the more accessible portions.

2. A treatise on surveying comprises not only directions for making measurements on the earth's surface, but also full descriptions of the instruments employed, as well as the methods whereby the results of the measurements are made to serve special purposes, such as determining the area or making a map of a tract of land.

3. **To survey, or make a survey of,** a part of the earth's surface, is to make the measurements that are necessary for determining the positions of those of its points, or of the objects it contains, that it is desired to locate. The process of making such measurements is also referred to as a *survey*. The points of a survey may be located with reference to one another only, or with reference to some external points or objects also.

4. Divisions of the Subject.—Surveying is divided into three general branches, namely:

1. *Chain surveying*, in which no other measuring instrument is employed than a chain or tape for measuring distances.

2. *Angular surveying*, in which angle-measuring instruments are employed in connection with distance-measuring instruments.

3. *Leveling*, which treats of the determination of elevations, or vertical distances.

5. Surveying is further subdivided into a great many special branches, which receive their names either from the instruments employed or from the ends to be attained. Thus, *compass surveying* treats of the methods of surveying with a compass; *farm surveying* treats of the methods used in surveying farms; etc.

THE CHAIN, TAPE, AND ACCESSORIES

6. **Instruments Used for Linear Measurement.** The instruments used most commonly for measuring distances are the *engineers' chain*, the *surveyors' chain*, and the *steel tape*. A tape composed of linen and having fine brass threads woven into it longitudinally, called a **metallic tape,** is commonly used for short measurements. Marking pins and range poles, to be described presently, are used in connection with the chain, especially in measuring long lines.

7. **The Engineers' Chain.**—This chain is 100 feet long and is composed of 100 links of steel or iron wire, each two adjacent links being connected by small rings. Some chains have two and some three rings between the adjacent links. In the former class, the length of a link, including a ring at each end, is 1 foot. The best chains are made of No. 12 steel wire and have all joints in the links and rings brazed to prevent their springing apart when the chain is under tension. At the end of every tenth link is attached a brass tag, or **tally mark,** and the distances of 10, 20, 30, and 40 links from either end of the chain are distinguished, respectively, by one, two, three, and four points on the tag, separated by

notches or by slits in the tag. The center of the chain, 50 links from either end, is marked by a plain oval tag. The handles are usually of brass, and attach by means of swivels having nuts and threads for adjusting the length of the chain. Each handle forms part of the end link, the length of the chain being measured to the extreme ends or outer edges of the handles. In Fig. 1 is shown an engineers' chain folded and tied so as to be convenient for carrying.

The engineers' chain is used chiefly in railroad surveying, but it is also used to some extent in city surveying and in other kinds of surveying where the foot is the unit of measurement. For measuring over very rough ground, a chain 50 feet long, called a **half chain**, is sometimes used; it has two handles, but is otherwise exactly the same as one-half of a chain 100 feet long.

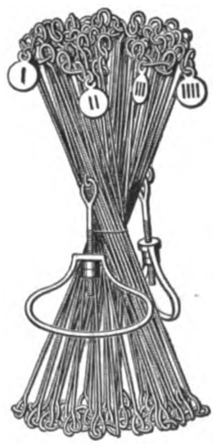

FIG. 1

8. The Surveyors' Chain.—The surveyors' chain, often called **Gunter's chain**, from the name of its inventor, is the same as the engineers' chain in every respect, except that its length is 66 feet, or 4 rods, instead of 100 feet. Like the engineers' chain, it is divided into 100 links, and consequently the length of each link is .66 foot, or 7.92 inches.

This chain is mainly used in land surveying, where the acre is the unit of area. It is very convenient for this purpose, as areas expressed in square chains can be expressed in acres by simply moving the decimal point one place to the left, there being 10 square chains in 1 acre. It is also well for the student to remember that there are 80 chains in 1 statute mile.

The surveyors' chain is used in all United States land surveys, and whenever the word *chain* occurs in a legal document, it is understood to mean a surveyors' chain, or 66 feet.

9. The Steel Tape.—Steel tapes are now used extensively in surveying and are largely superseding both the engineers' and the surveyors' chain. They are steel ribbons varying from about $\frac{1}{8}$ to $\frac{1}{2}$ inch in width and graduated in various ways, according to the purposes for which they are

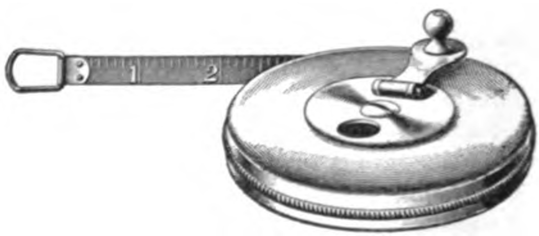

FIG. 2

used. They can be obtained in any length from 1 yard to 1,000 feet and graduated to order. For city surveying, and for many other purposes, a tape 50 feet long is generally preferred. This tape is usually from $\frac{1}{4}$ to $\frac{3}{8}$ inch wide, graduated in feet, tenths, and hundredths, and is enclosed in a hard-leather case having a folding handle for winding up the tape. Such a tape is shown in Fig. 2.

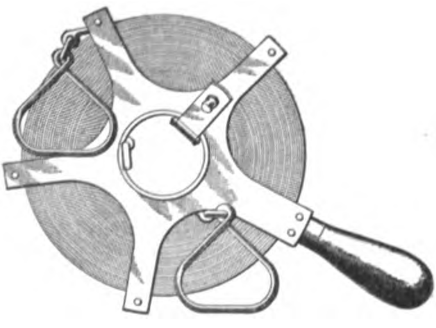

FIG. 3

For some purposes, tapes 300 or 500 feet long and even of greater length are used. They are of small cross-section and are graduated in different ways, but are not uncommonly graduated every 10 feet, with the first and last 10 feet graduated to feet and the first and last foot to tenths. They usually have detachable handles and are wound on some form of reel. A tape of this kind wound on a metal reel is shown in Fig. 3.

In some tapes, the handle forms part of the end division or graduation, the length of the tape counting from the outside of the handle. In others, the graduations begin on the inside of the handle, where the tape is attached (see Fig. 2); and in others the graduations begin on the tape itself, a short distance from the handle. When using a tape, the surveyor should ascertain where the graduations begin, as otherwise he may make serious errors.

10. Marking Pins.—These are used for marking and recording chain lengths, and consist of slender rods pointed at one end and bent into a ring at the other. They are usually made of No. 6 steel wire, and measure about 14 inches in length. When used in grassy or weedy ground, pieces of red or white cloth should be tied to the rings, so that the positions of the pins may be easily found. A set of marking pins should preferably consist of eleven pins. The use of this number of pins greatly facilitates the reckoning of distances, as will be presently explained.

11. Range Poles.—This name is given to poles or rods held or placed at points to which measurements are taken, for the purpose of defining the direction of lines on the ground. Usually, a piece of white cloth is tied to each pole so that the pole may be easily seen. For this reason, range poles are often called **flagpoles**, or simply **flags**, and the men holding them, **flagmen**. Wooden poles are generally 10 feet long, made with a pointed iron shoe on their lower end. They are graduated in feet and the foot-spaces are painted alternately red and white. Metal poles, or rods, are about 6 feet long. They are much more slender than wooden poles, and are specially suited for marking points to which the line of sight of an angle-measuring instrument is to be directed. If it is desired to sight past the pole, the flagman should stand to one side of the line of sight. Otherwise, he should stand facing the observer and directly behind the pole, which should be held lightly with the fingers of both hands, and as nearly vertical as possible.

THE PROCESS OF CHAINING

12. Definition.—**Chaining** is the process of measuring a distance on the ground, with either a chain or a tape; whichever instrument is used, the process is substantially the same.

13. Survey Distances Are Horizontal.—All surveying distances, except those used in leveling, are understood to be horizontal; they are determined either by measuring horizontally or by reducing inclined distances to horizontal distances. Thus, in Fig. 4, by the distance from *A* to *B* is meant the distance *A B′*, counted along the horizontal line *A X*, from *A* to *B′*, the latter point being the foot of a

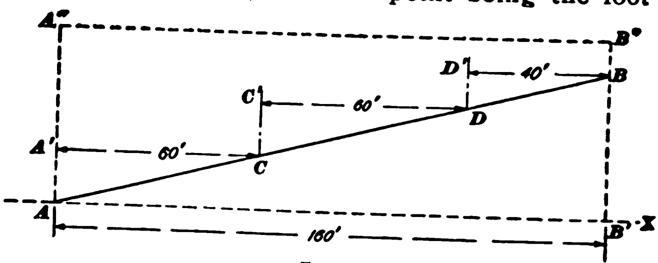

FIG. 4

perpendicular (a vertical line in this case) from *B* to *A X*. The line *A B′*, or any other horizontal line, as *A″ B″*, between two verticals through *A* and *B*, is called the **horizontal projection** of *A B*.

The distance *A B′* is obtained by a series of horizontal measurements *A′ C*, *C′ D*, and *D′ B*, the points *A′*, *C′*, and *D′* being vertically above *A*, *C*, and *D*, respectively. The sum of these horizontal measurements *A′ C*, *C′ D*, and *D′ B* is equal to *A B′*, which, as stated above, is referred to as the distance between *A* and *B*. If the distance *B B′*, which is the difference in elevation between *A* and *B*, is known, the inclined distance *A B* may be measured, and the distance *A B′* calculated from the relation $AB' = \sqrt{\overline{AB}^2 - \overline{BB'}^2}$.

The lines platted on a map are the horizontal projections of the real lines in the field, and the map itself represents the horizontal projection of the tract surveyed. It should be borne in mind that, when a line is referred to, its horizontal projection is meant; that, by the distance between two points is meant the horizontal distance between two verticals through those points; and that by the angle between two lines is meant the angle between the horizontal projections of the lines.

14. Fixing the Direction of a Line.—The point from which a line is to be measured being occupied by the surveyor, a flagpole is stuck or held at the other end, or at some intermediate point, in order that the chainmen, by sighting to the pole, may make the measurements in the proper direction; that is, exactly along the line.

15. Chaining on Level Ground.—Two men are necessary for all chaining, and are designated **head chainman** and **rear chainman**, from their position in reference to the direction in which the measurement is being taken. If a chain is to be used, it should be thrown out by the rear chainman, preferably in a direction opposite to that in which chaining is to be done, so that he may observe and straighten out any kinks as the chain is drawn past him by the head chainman, who drags the chain by one of the handles in the direction of the line, while the rear chainman, holding the other handle, remains at the beginning of the line. In undoing the chain, the two handles should be grasped firmly in one hand, and with the other hand the chain should be thrown out with force so that as far as possible the links will be free from each other. When the head chainman starts, he carries ten of the marking pins, leaving the eleventh with the rear chainman. When the chain has been drawn out almost to its full length, the rear chainman calls, "Chain." The head chainman then stops and tightens up the chain, the end of which he holds against a pin to be stuck in the ground at the end of a chain length. He should hold the pin as nearly vertical as possible and tight against the end of the chain, keeping his body to one side so as not to obstruct the view of the rear chainman in sighting to the range pole. He should move the pin, and with it his end of the chain to right or left, as indicated by word or motion from the rear chainman, who holds his end of the chain at the point or pin from which the measurement is being made. The rear chainman gives the correct line for the pin by sighting along the line determined by the point he occupies and the range pole, and directing the head chainman to move the pin held by the latter until it is cut by the former's line of sight. When, the chain being taut and *straight* (not caught on any intervening obstruction), the head chainman's pin is in line, the rear chainman calls, "Stick," to which the head chainman replies, "Stuck," when the pin has been properly placed.

The rear chainman pulls out each pin after the measurement from it has been taken. When ten pins have been set, the head chainman calls, "Tally." He then receives from the rear chainman ten pins with which to start again. Each tally should be recorded, either by writing or by putting a pebble in the pocket, or in some other convenient manner. When the point to which measurement is being made is reached, the head chainman walks on with the chain until the rear end is at the last pin. He then returns and notes the number of links from the last pin to the point. This, added to the number of chains, will give the total distance chained. In running a line from which measurements are to be taken locating objects near the line, the end of each chain length should be marked by wooden stakes or otherwise.

16. Method of Reckoning.—The distance measured to any point is obtained by first adding the number of tallies recorded, and then counting the number of pins held by the rear chainman, each of which represents a chain length, except the last one, if the last measurement is not a whole chain. It is here assumed that the only pin remaining in the ground is the one marking the point to which the last measurement was made. Suppose, for example, that four tallies have been recorded, that the rear chainman has 7 pins, and that the last measurement was 47 links. The four tallies represent $4 \times 10 = 40$ chains; six of the pins held by the rear chainman represent 1 chain each, and the other, 47 links, or .47 chain. Therefore, the length of line is $40 + 6 + .47 = 46.47$ chains.

17. Chaining Down Hill.—In chaining down hill, the head end of the chain is raised as nearly as possible to the same level as the rear end. The head chainman transfers the measurement from the elevated end of the chain by holding a range pole vertically, or by dropping from the end of the chain a marking pin with the heavy end down, or one or more pebbles of sufficient size to make a mark where they fall. This transference is done more accurately by means of a plumb-bob, the string of which is held against the end of the chain. The chain should be drawn as taut as possible, in order to avoid the sagging of its center part, the effect of sagging being to make the recorded measurements too long.

If the slope of the ground is so great that the head chainman cannot hold his end of the chain level with the rear end, the chain is *broken;* that is, a part only of it is used in making the measurement. The head chainman should break chain at such a point that the part used will be as long as he can conveniently hold level with the rear end.

The process of chaining down hill is illustrated in Fig. 4, where *B A* is the line to be measured. The head chainman draws out the chain to its full length, in the direction *B A*. Finding that he cannot hold his end level with *B*, he returns to a point that he can hold level, say the fortieth link from the rear end of the chain. When this measurement is transferred to the ground at *D*, the rear chainman, dropping his end of the chain, holds at *D* the point where the chain was broken, and the head chainman transfers the measurement from the end of the chain to the ground at *C*. This process is repeated until the total distance between *B* and *A* has been measured.

In very steep slopes, it may be necessary to break chain several times in measuring a chain length. If pins are used for marking intermediate points between two full chain lengths, they should be given or thrown to the head chainman as the rear chainman advances from those points. The rear chainman should keep only those pins that mark full chain lengths.

18. Chaining Up Hill.—In chaining up hill, the process is reversed; the rear chainman holds his end of the chain directly over the point last occupied and marked by the head chainman, and raises it so that it will be level with the end held by the head chainman. The rear chainman may keep his end of the chain over the proper point by using either a plumb-line or a range pole. An experienced chainman can estimate the proper position by the eye, with sufficient accuracy. The head chainman should draw the chain taut and mark the end of the measurement as soon as the rear chainman, having put him in line, calls out, "Stick." This process is repeated as often as necessary. Care should be taken not to mistake the point where the chain is broken, taking, for instance, the fortieth for the sixtieth link.

As it is much easier to chain down than up hill, it is often best, especially where the slope is very steep, to set a point in line at the top of the hill and then chain backwards from it.

19. Folding the Chain.—The chain may be folded either from one end or from the center. To fold it from the center, take the two middle links together in the left hand, grasp the third pair of links from the middle with the right hand and fold the second and the third pair of links across the middle links and nearly parallel to them; then grasp the fifth pair and fold the fourth and the fifth pair across and nearly parallel to those already folded, and so on until the chain is entirely folded. It should then be secured by a cord or strap around the centers of the links, as shown in Fig. 1. The chain may be folded from one end in a similar manner.

20. Correction for Erroneous Length of Chain. The length of a chain or tape is altered by changes in temperature, and by wear and distortion. The variations due to temperature are very small, and need not be considered except in very accurate work, as will be explained in another

Section of this Course. The alterations due to wear and distortion are sometimes considerable, and should be allowed for. An error of 2 or 3 inches in the length of a chain is not uncommon, especially when the chain has been roughly used. This shows the necessity of handling the chain very carefully. If the chain is caught, as in a stump or stone, it should not be disengaged by pulling from the end, as this may stretch the rings and bend the links. Although, in measuring, the chain should be held taut, it should not be pulled with so much force as to cause stretching of the rings.

21. The length of the chain should be tested often. This is done either by comparing the chain with a chain or tape of standard length, or by stretching it between two points whose exact distance is known. It is advisable to have two such points marked permanently on an office floor, smooth pavement, curb, or some other convenient place.

The length of the chain can be adjusted to the standard length by means of swivels in the end links.

If, after a line has been measured, the length of the chain (or tape) is found to be in error, the true length of the line can be easily determined by means of the following formula:

$$L_o = L \pm eL$$

in which L_o = true length of line;

L = length of line as actually measured;

e = error in length of one unit of measure.

If, for instance, the length of a line is measured in feet, and the measurements are made with a 50-foot tape that is found to be .1 foot too long, the error is $\frac{.1}{50}$, or .002 foot in 1 foot. In this case, $e = .002$. If the measurement is recorded in chains, and the chain is found to be .1 link too long, the error is .1 link, or .001 chain per chain, and $e = .001$.

It should be understood that the correction eL expresses the same kind of units as e. If, for instance, e is 1.5 inches per chain, and the length of the line is expressed in chains, its true length is L chains $\pm$ 1.5 L inches.

In order to avoid this mixture of units, it is better to express e in the same kind of units as those in which the line is measured. Thus, an error of 1.5 inches per chain is equivalent to $\frac{1.5}{7.92}$ links per chain, since there are 7.92 inches in 1 link; and, since there are 100 links in a chain, an error of $\frac{1.5}{7.92}$ link is equivalent to $\frac{1.5}{7.92 \times 100}$ chain. This last value is the one that should be used for e, and then the correction eL will represent chains. If the recorded length of the line is, say, 24 chains, the correction is

$$eL = \frac{1.5 \times 24}{7.92 \times 100} = .045 \text{ chain} = 4.5 \text{ links}$$

If the chain is too long, the distance measured with it will be recorded as too short, and the correction eL should be added; while if the chain is too short, the distance measured will be recorded as too long, and the correction eL should be subtracted.

EXAMPLE 1.—The length of a line, measured with a 100-foot chain, was found to be 1,048 feet. It was afterwards found that the chain was .19 foot too long. What was the true length of the line?

SOLUTION.—If the error is .19 ft. in 100 ft., it is $\frac{1}{100}$ of .19 = .0019 ft., or, say, .002 ft. per ft. Then, e = .002, L = 1,048, and, therefore,

$$L_e = 1{,}048 + .002 \times 1{,}048 = 1{,}050 \text{ ft., nearly. Ans.}$$

The error is added, because, the chain being too long, the recorded length of the line was too small.

EXAMPLE 2.—The length of a line was recorded as 19.89 Gunter's chains. The length of the chain having been found to be 1¾ inches too short, it is required to determine the true length of the line.

SOLUTION.—An error of 1.75 in. is equivalent to an error of $\frac{1.75}{7.92 \times 100}$ ch. Making $e = \frac{1.75}{7.92 \times 100}$ and L = 19.89 in the formula, and noticing that the correction eL is to be subtracted, we obtain,

$$L_e = 19.89 - \frac{1.75 \times 19.89}{7.92 \times 100} = 19.85 \text{ ch. Ans.}$$

EXAMPLES FOR PRACTICE

1. The length of a line as measured with an engineers' chain is 1,946 feet; it is found that the chain is 2.5 inches too short; what is the true length of the line? Ans. 1,942 ft., nearly

2. A line is measured with a Gunter's chain and found to be 8.94 chains long; if the chain is .5 link too long, what is the true length of the line? Ans. 8.98 ch., nearly

3. A certain line is measured with a 100-foot tape that is .2 foot too long; if the recorded length is 647 feet, what is the true length of the line? Ans. 648.3 ft.

4. The length of a line as measured with a Gunter's chain is 3.36 chains; if it is known that the chain is 2 inches too short, what is the true length of the line? Ans. 3.35 ch.

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22. Precision.—In chain surveying, an error of 1 in 500 is generally permissible, and should not be exceeded; that is, two measurements of the same line should not give results differing by more than 1 foot for every 500 feet measured. If, however, the chaining is done carefully, and the ground is not rough, the error need not exceed 1 in 800 or 1,000.

23. Approximate Methods of Determining Distances.—It is often desirable to know approximately the distance between two points. By a little practice a person can obtain a long stride that is very close to 3 feet in length; or, better still, a pace $2\frac{1}{2}$ feet long. By counting the paces between two points and multiplying their number by 3 or $2\frac{1}{2}$, as the case may be, he can determine the distance quite closely.

Where it is possible to drive over the line, a convenient method is to tie a piece of cloth to a spoke of a wheel near the tire, and, having measured the circumference of the wheel, count the number of revolutions made by the wheel in passing over the line. The approximate distance is obtained by multiplying the number of revolutions of the wheel by the length of its circumference.

FIELD PROBLEMS

24. To Prolong a Line.—In Fig. 5, *A B* is a line that it is desired to prolong beyond its extremity *B*. Having marked *A* and *B* by range poles stuck in the ground vertically at these points, the surveyor walks some distance back of *A* and places himself at a point *P* in line with *A* and *B*, by sighting to the pole at *A*, and stepping to the right or to the left until the pole at *B* is covered by that at *A*. He then directs a pole to be held beyond *B* and motions to the flagman until the latter has the pole in such position that it is

FIG. 5

covered by the poles at *A* and *B*. Let *Q* be the point thus determined. A pole being stuck at *Q*, the surveyor moves to *A*, and sighting along *B Q*, lines in the flagman at a point *R* beyond *Q*. The process may be repeated and the line prolonged as far as necessary. The distances *A B*, *B Q*, *Q R*, etc. should be as long as they can be conveniently made.

25. To Run a Line Over a Hill When the Ends of the Line Are Invisible From Each Other.—The points *A* and *B*, Fig. 6, are supposed to be on opposite sides of a hill, and to be invisible from each other. It is desired to run a line between them, or to locate some intermediate points.

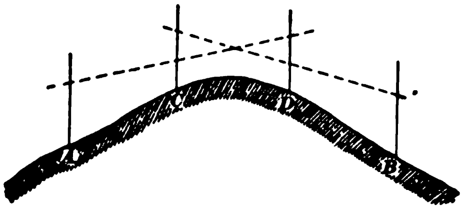

FIG. 6

Having set two poles at *A* and *B*, two flagmen with poles station themselves at *C* and *D*, approximately in line with *A* and *B*, and in such positions that the poles at *B* and *D* are visible from *C*, and those at *C* and *A* are visible from *D*. The flagman at *C* lines in the flagman at *D* between *C* and *B*, and then the flagman at *D* lines in that at *C* between *D* and *A*. Then the flagman at *C* again lines in that at *D*, and so on, until *C* is in line between *D* and *A* at the same time that *D* is in line between *C* and *B*. The points *C* and *D* will then be in line with *A* and *B*.

26. To Erect a Perpendicular to a Line at a Given Point.—Let it be required to erect a perpendicular to the line *A B* at the point *B*, Fig. 7. A triangle whose sides are in the proportion of 3, 4, and 5 is a right triangle, the longest side being the hypotenuse; for $5^2 = 4^2 + 3^2$. The following method is based on this principle: Lay off on the line *B A*, Fig. 7, a distance of 30 feet (or links). Fix one end of the chain at one of the extremities of *B C*, and the end of the nintieth link at the other extremity. Hold the end of the fiftieth link and draw the chain until both parts are taut. The point *D* where the end of the fiftieth link is held will then be a point in the perpendicular, and the direction of the latter will therefore be *B D*.

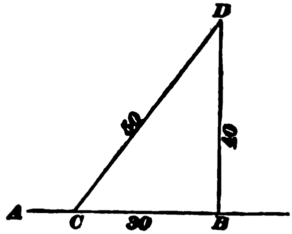

FIG. 7

The distance *B C* may be any other convenient multiple of 3. In general, if *B C* is denoted by $3a$, *B D* must be $4a$, and *C D* must be $5a$. Thus, *B C* may be made equal to 21 ($= 3 \times 7$) links; in which case *B D* must be $4 \times 7 = 28$, and *C D* must be $5 \times 7 = 35$, links. As $35 + 28 = 63$, one end of the chain must be fixed at one of the extremities of *B C*, the end of the sixty-third link at the other extremity, and the chain pulled from the end of the thirty-fifth link until both parts are taut. Let the student draw a sketch representing these conditions.

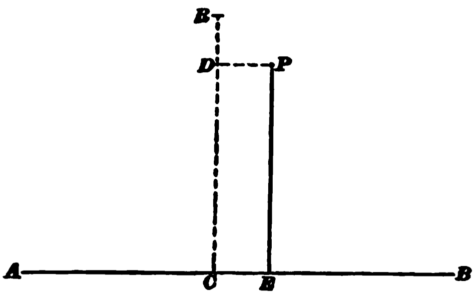

FIG. 8

27. To Drop a Perpendicular to a Line From a Point Without. In Fig. 8, let *P* be a point without the line *A B* from which it is desired to drop a perpendicular to the line. Estimate, by eye, the point at which the perpendicular will meet *A B*. Let *C* be the point thus estimated. Erect a perpendicular *C R* to *A B* at *C*, as explained in the preceding article. Prolong this perpendicular a short distance past *P*, and measure its distance PD from P. As CR is not far from P, the point D directly opposite P can be located by eye with sufficient accuracy. Measuring from C a distance CE equal to PD, the point E, where the perpendicular from P meets AB, is determined.

28. To Run a Line Parallel to Another.—Let AB, Fig. 9, be a line to which a parallel through a given point P

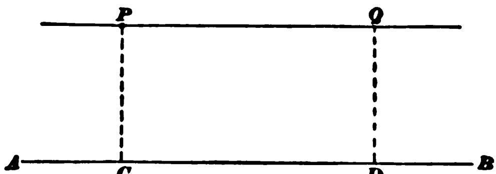

FIG. 9

is to be run. Drop a perpendicular from P to AB, as explained in Art. **27**. Let C be the point where this perpendicular meets AB. Measure CP. At any other point D in the line AB, erect a perpendicular to the latter line. Measure on this perpendicular a distance DQ, equal to CP. A line through P and Q will be the required parallel.

29. To Determine the Angle Between Two Lines. Let AD and AE, Fig. 10, be two lines on the ground. To determine the angle DAE, measure off from A on AD and AE equal distances AB and AC. Measure the distance BC. Then

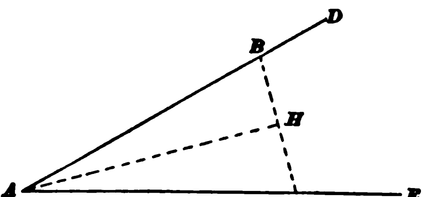

FIG. 10

the angle DAE is calculated from the relation

$$\sin \frac{1}{2} DAE = \frac{\frac{1}{2} BC}{AB}$$

The derivation of this equation is very simple. The triangle BAC being isosceles, the perpendicular AH on BC bisects both the angle DAE and the base BC; that is, $BAH = \frac{1}{2} DAE$, and $BH = \frac{1}{2} BC$. The right triangle BAH gives $\sin BAH = \frac{BH}{AB}$; that is, $\sin \frac{1}{2} DAE = \frac{\frac{1}{2} BC}{AB}$.

EXAMPLE.—If, in Fig. 10, AB and AC are each 100 feet and BC is 57.6 feet, what is the value of the angle DAE?

SOLUTION.—Substituting the values of BC and AB in the equation given above, we obtain,

$$\sin \frac{1}{2} DAE = \frac{\frac{1}{2} \times 57.6}{100} = .28800$$

whence $\frac{1}{2} DAE = 16° 44'$, nearly; and, therefore, $DAE = 16° 44' \times 2 = 33° 28'$. Ans.

EXAMPLES FOR PRACTICE

1. If, in Fig. 10, AB and AC are each 100 feet and BC is 64.8 feet, what is the value of the angle DAE? Ans. 37° 48'

2. If the distances AB and AC, Fig. 10, are each 120 feet and the distance BC is 96.7 feet, what is the value of the angle DAE? Ans. 47° 32'

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30. To Lay Out an Angle.—In Fig. 11, let it be

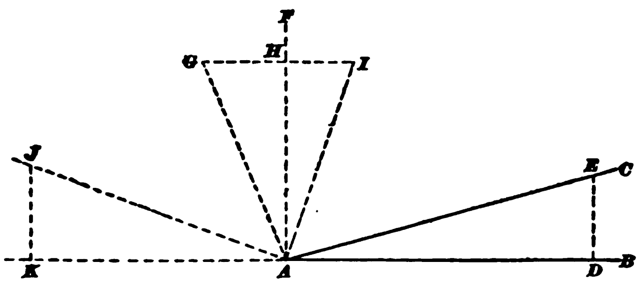

FIG. 11

required to lay out an angle BAC equal to X degrees, from the line AB. Measure any convenient distance AD on AB and from D erect a perpendicular DE equal to AD tan X. Then DAE is the required angle.

If the angle is between 45° and 135°, it is better to erect a perpendicular AF at A, and lay off from it, in the manner described above, an angle equal to the difference between the given angle and 90°.

Thus, if it is desired to lay off an angle BAI greater than 45°, but less than 90°, the angle HAI, which is the complement of BAI, is laid off from the perpendicular AF, by making HI equal to AH tan HAI. If it is desired to lay off an angle BAG that is greater than 90°, but less than 135°, the angle HAG, which is equal to $BAG - 90°$, is laid off from the perpendicular AF, by making HG equal to AH tan HAG.

If the angle is greater than 135°, its supplement is laid off from the prolongation AK of BA. The angle BAJ, for example, is laid off by erecting the perpendicular KJ and taking $KJ = AK \tan (180° - BAJ)$.

EXAMPLE 1.—If the angle BAG, Fig. 11, is 114° 45′, and the distance AH is 100 feet, what is the length of the perpendicular HG to be laid off from AF in constructing the angle BAG?

SOLUTION.— 114° 45′ − 90° = 24° 45′., and HG = 100 tan 24° 45′ = 46.10 ft. Ans.

EXAMPLE 2.—What is the length of the perpendicular KJ to be laid off from AK in constructing the angle BAJ = 152° 30′, the distance AK being 100 feet?

SOLUTION.—The supplement of 152° 30′ is 180° − 152° 30′ = 27° 30′. Therefore, KJ = 100 tan 27° 30′ = 52.1 ft. Ans.

EXAMPLES FOR PRACTICE

1. If the angle BAG, Fig. 11, is 123° 30′, and the distance AH is 80 feet, what is the length of the perpendicular HG to be laid off from AF in constructing the angle BAG? Ans. 53 ft.

2. If an angle BAJ, Fig. 11, is to be constructed equal to 156° 15′, what must be the length of the perpendicular JK, when AK is equal to 100 feet? Ans. 44 ft.

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31. To Prolong a Line Through an Obstacle.—In prolonging a line AB, Fig. 12, suppose that an obstacle K,

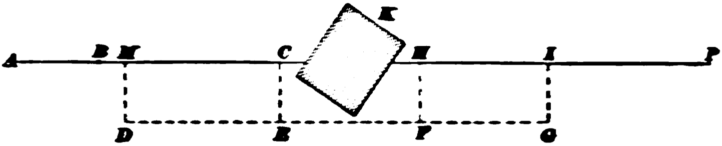

FIG. 12

as a building, makes it impossible to run and measure the produced line directly. The way to proceed under such circumstances is usually as follows:

The line AB is produced to a point C as near the obstacle as possible; a perpendicular CE is erected at C, and its length CE made long enough to clear the obstacle. At a point M, 30 or 40 feet from C, another perpendicular to AC is erected, and a distance MD equal to CE is measured on it. The points D and E determine a line parallel to AC. This parallel is prolonged beyond the obstacle, and at two points F and G, the former just beyond the obstacle and the latter about 30 or 40 feet from it, perpendiculars are erected, on which distances FH and GI are measured equal to CE and MD. The points H and I thus determined are in the prolongation of AB. The distance CH is equal to EF, which should be measured.

32. To Determine the Distance to an Inaccessible Point.—In Fig. 13, let it be required to determine the distance from the point B to an inaccessible point P. Measure BC in any convenient direction and run a line $A'D'$ parallel to BC. Measure AD, the distance between the points where the lines PB and PC intersect $A'D'$. Measure also AB. Then,

$$BP = \frac{AB \times BC}{AD - BC}$$

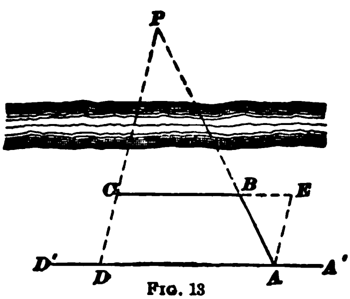

FIG. 13

The derivation of this equation is as follows: Draw AE parallel to DP, and intersecting CB produced at E. Then, the angles of the triangle BCP being equal to those of the triangle ABE, these two triangles are similar, and therefore $\frac{BP}{AB} = \frac{BC}{BE}$; whence,

$$BP = \frac{AB \times BC}{BE}$$

or, because $BE = AD - BC$,

$$BP = \frac{AB \times BC}{AD - BC}$$

EXAMPLE.—If in Fig. 13, BC = 100 feet, AB = 52.4 feet, and AD = 124.2 feet, what is the distance BP?

SOLUTION.—Substituting these values in the equation given above,

$$BP = \frac{52.4 \times 100}{124.2 - 100} = 216.5 \text{ ft. Ans.}$$

33. To Determine by a Random Line the Distance Between Two Points That Are Invisible From Each Other.—In Fig. 14 is represented a case that often occurs in practice. The extremities A and B of a line whose distance is to be determined are known and fixed, but they are not visible from each other on account of some obstruction to the line of sight. The distance AB is determined by means of an auxiliary line AD', called a **random line.** This line is run from one of the extremities (A in this case) of the given line, in such a manner that it will pass as near the other extremity as can be estimated. From this other extremity, a perpendicular is dropped on the random line,

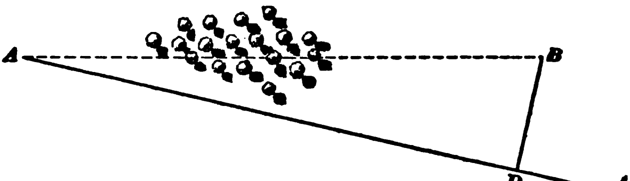

FIG. 14

and the required distance is computed by solving a right triangle. In the present case, the perpendicular BD from B to AD' is measured, the distance AD is noted, and the distance AB is given by the equation $AB = \sqrt{\overline{AD}^2 + \overline{BD}^2}$.

EXAMPLE.—If, in Fig. 14, the distance AD is 206.1 feet and the distance BD is 35.1 feet, what is the distance from A to B?

SOLUTION.—Here $AD = 206.1$, $BD = 35.1$, and, therefore, substituting in the equation given above,

$$AB = \sqrt{206.1^2 + 35.1^2} = 209.1 \text{ ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. If the distances CB, CD, and DA, Fig. 13, are, respectively, 75, 42, and 103 links, what is the distance CP? Ans. 1.13 ch.

2. Two points A and B are invisible from each other. A random line being run from A, it is found that its distance from B is 12.6 feet; if the distance from A to the foot of the perpendicular is 192.5 feet, what is the distance from A to B? Ans. 193 ft.

SURVEY OF A CLOSED FIELD

THE FIELD WORK

34. Preliminary Examination.—Before making the survey of a field, the surveyor should, if possible, walk around it and make in his notebook a sketch showing the boundary lines. He should mark all corners and study the field in a general way, so as to obtain a fairly approximate idea of its form and contents.

35. Measurements.—The sides of the field should first be measured. If they are straight lines, the measurements

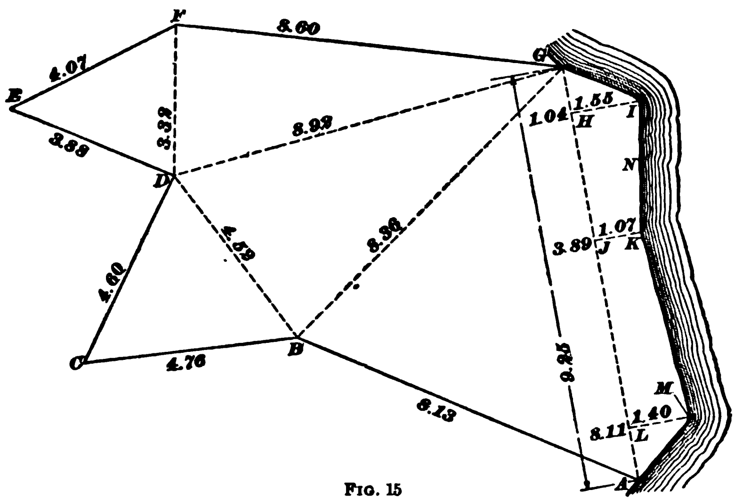

FIG. 15

are made as explained in Art. **15.** In the case of an irregular boundary line, such as *G N A*, Fig. 15, which is the edge of a stream, one or more straight lines are run as close to the irregular boundary as possible. From these auxiliary lines are taken perpendicular measurements, called **offsets,** to those points of the boundary where any considerable change in its direction occurs.

Thus, in Fig. 15, the straight line *G A* is run close to the shore line *G N A*. The distances *G H*, *G J*, and *G L* from *G* to the points of *G A* opposite those points of *G N A* to which offsets are to be taken, are noted, and the offsets *H I*, *J K*, *L M* measured, when the line *G A* is being run. It will be observed that all distances along a main or auxiliary line are counted from the beginning of that line. For instance, instead of recording the distances *G H*, *H J*, and *J L*, the distances *G H*, *G J*, and *G L* are recorded. The parts *G I*, *I K*, *K M*, and *M A* of the curved boundary being nearly straight, the portion of the field between the right line *G A* and the curved line *G N A* is divided, approximately, into the right triangles *G H I* and *A L M* and the trapezoids *H I J K* and *J K L M*.

36. It is often difficult to measure directly along a line, as when the latter is determined by a fence. In such cases, an offset of 3 or 4 feet is measured from the line at each extremity, and the distance between the ends of the offsets measured. The direction of such short offsets is determined by the eye with sufficient accuracy.

37. Dividing the Field Into Triangles.—In order to make a plat of the field and calculate its contents, the field is divided into triangles by means of diagonals, which are measured on the ground. The surveyor should use his own judgment as to the best and most convenient diagonals to measure. He should avoid using diagonals that make very acute or very obtuse angles. Thus, in Fig. 15, *D G* is a better diagonal to use than *B F*.

38. Tie-Lines.—Obstacles often make it impossible to measure directly the diagonals of a field. In this case the lengths of the diagonals may be calculated by means of the principles of similar triangles. The process is illustrated in Fig. 16, which represents a field whose diagonals *B D* and *B E* cross a pond, and cannot, therefore, be measured directly. If the sides *B A* and *E A* are produced to *F* and *G*, in such manner that AF and AG are proportional to AB and AE, and a line is run from F to G, a triangle FAG will be formed similar to BAE; for in these two triangles the angles at A are equal, and, as shown in geometry, two triangles having an equal angle included by proportional sides are similar. In order to make AF and AG proportional to AB and AE, fix the point F at some convenient distance from A. The distance AG that must be measured

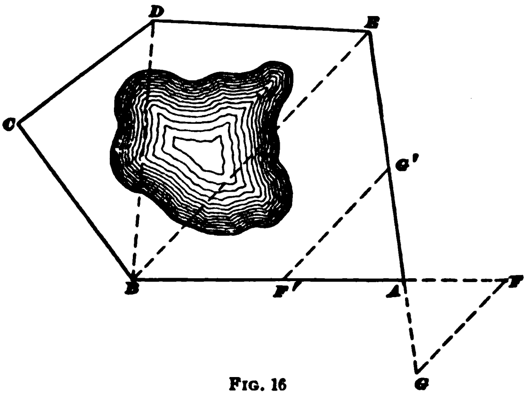

FIG. 16

along EA produced is determined from the proportion $AF:AG = AB:AE$; whence,

$$AG = \frac{AF \times AE}{AB}$$

Having fixed F and G, measure FG. The diagonal BE can then be calculated from the proportion $AF:FG = AB:BE$, which gives

$$BE = \frac{FG \times AB}{AF}$$

Such lines as FG are called **tie-lines**, because they tie the sides together. The diagonal BD can be calculated by means of tie-lines in a similar manner.

Instead of constructing a triangle such as AFG by producing two of the sides, a triangle, such as $AF'G'$, may be constructed on the inside of the tract in substantially the same manner as described above for the triangle AFG. The distance AF' is first measured; the point G' is then fixed by making AG' equal to $\frac{AF' \times AE}{AB}$. Finally, $F'G'$ is measured, and BE calculated from the relation

$$BE = \frac{F'G' \times AB}{AF'}$$

EXAMPLE.—In Fig. 16, let the lengths of the sides be as follows: AB = 320 feet, BC = 217 feet, CD = 196 feet, DE = 285 feet, and EA = 304 feet. It is required to calculate the length of the diagonal BE by means of a tie-line.

SOLUTION.—Let the line BA be prolonged 100 ft. beyond A; that is, make AF = 100 ft. Then AG must be equal to

$$\frac{AF \times AE}{AB} = \frac{100 \times 304}{320} = 95 \text{ ft.}$$

Let the length of FG, as found by measurement, be 125 ft.; then,

$$BE = \frac{FG \times AB}{AF} = \frac{125 \times 320}{100} = 400 \text{ ft. Ans.}$$

EXAMPLE FOR PRACTICE

In order to compute the length of BD, Fig. 16, by a tie-line, the side BC was prolonged a distance of 80 feet from C. (*a*) What distance must DC have been prolonged? (*b*) The length of the tie-line was found to be 98 feet; find the length of BD. Ans. { (*a*) 72 ft. (*b*) 266 ft.

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39. The Location of Objects.—It is often desirable to locate important objects, such as buildings, roads, fences, etc., with reference to the main lines of the survey. This is most easily done by perpendicular measurements, or offsets, from those lines. In the case of objects very near the main line, the points from which the offsets should be measured can be estimated by the eye; in the case of a distant object, its perpendicular distance from the main line is determined by the method explained in Art. **27**.

The points from which measurements are to be made should be determined as far as possible while the line is being run, and their distances from the next preceding corner should be noted and recorded in every case.

In Fig. 17, AB represents one side of a field from which certain objects are to be located. The point W (a spring, say) may be located by measuring the perpendicular distance *O W*, and noting the distance *A O*.

40. A point may also be located by measuring its distances from two points in the main line. Thus, in Fig. 17, *W* can be located by noting the distances *A L* and *A M*, and then measuring the distances *L W* and *M W*. In platting the survey, the point *W* can be readily located by laying off the distances *A L* and *A M* and from *L* and *M* as centers, with radii of .76 chain and .80 chain, respectively, drawing two intersecting arcs; their point of intersection is the position of the point *W* on the map.

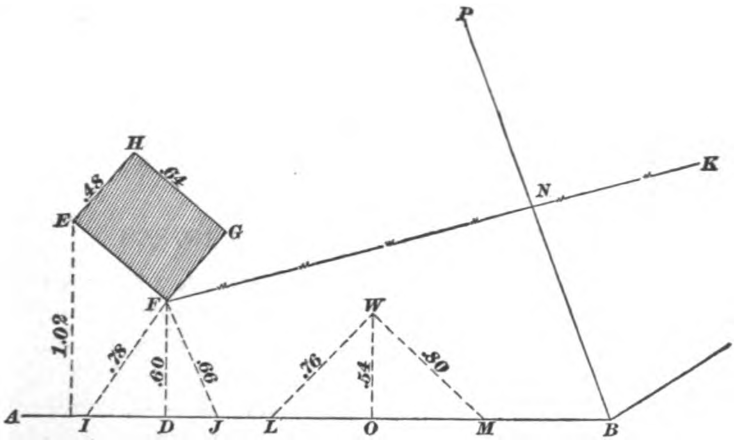

FIG. 17

41. A rectangular building *H G E F*, Fig. 17, can be located by measuring the perpendicular distances of two of its corners from the main line. The dimensions of the building should be measured and recorded. The building can also be located by measuring the distance of one corner from two points in the main line; one of the points being in a straight line with one side of the building. Thus, in Fig. 17, the point *I* is in a straight line with the side *G F*, and *J* is any convenient point on the main line. The position of the building is determined by noting the distances *A I* and *A J*, and measuring the distances *F I* and *F J*.

42. The direction of a straight line is determined by two points in the line. Therefore, a straight road or a fence is located by locating two of its points (in the case of a road, two points in its center line). Thus, in Fig. 17, the point *F*, which is the point of beginning of the straight fence *FK*, having been located as previously described, the direction of the fence may be determined by locating another point in it, as the point of intersection *N* of the fence with the diagonal *B P*.

43. Objects within a field may sometimes be more easily located from one of its diagonals than from one of the main lines of the survey. The fence just referred to is an example of this mode of location.

NOTE.—It should be understood that an object is located by only one of the methods explained above. For instance, the spring *W*, Fig. 17, is located either by the offset *O W* or by the two lines *L W* and *M W*. The two methods of location are shown in the figure simply for the purposes of illustration; but, in practice, one only is used.

KEEPING AND PLATTING NOTES

44. Keeping the Notes.—The **notes**, or record, of a chain survey are usually kept in an ordinary field book, commonly called a **transit book**, which is of a convenient size to slip into the pocket. The left-hand page of the notebook is ruled in six columns. In one of these columns are written the letters or numbers by which the corners or points between which measurements are taken are designated. Horizontally opposite these letters or numbers, in the next column, are recorded the distances of each point from the one immediately preceding it. As only two columns of this page are needed, the fourth and the fifth column are used for this purpose, so as to bring them nearer the right-hand page, and leave plenty of space to the left for descriptive remarks.

The right-hand page is ruled in blue with a red line down the center of the page. This page is used for sketches and remarks.

The line being run is commonly represented by the red center line. In case more room is needed for sketching, the line being run may be represented by a line drawn on one side of the center line of the page and parallel to it. On this page are noted also the date and location of the survey, the names and positions of the different members of the party, and any other remarks that the surveyor may deem necessary and useful. In sketching, it is better to face in the direction in which the line is being measured, and to represent the line as running from bottom to top in the notebook. For this reason, nearly all surveying notes read from the bottom to the top of the page.

The notes given below are for part of the field represented in Fig. 15. The survey is supposed to have begun at *A*,

41			Point	Distance Chains		Chain Survey.	John Jones' Field 42
						½ Mile South of Elmdale,	Pa.
						Wm Johnson, Surveyor, Henry	Fox, Head Chainman,
						June 18, 1905. Geo. Hillis, Gunter's	Rear Chainman. Chain was checked.
			D	4.60	Corner	Pile of Stones	
				3.89			
				3.56			
				1.92		.71 1.32	
			C	4.76	Corner		Marshall Road .50 Ch. wide
				4.18			.71
				3.50			.32 John Jones' Dwelling
				1.14		Center of Marshall Road	
			B	8.13	Corner		
			A			Fence Post at Edge of Stream	

from which the line *A B* was run, then *B C*, etc. The number 8.13, opposite *B*, denotes the distance from *B* to *A*; the number 4.76, opposite *C*, denotes the distance of *C* from *B*; and so on with the other points. The distances recorded between those opposite the corners, as those between *B* and *C*, are the distances of the points directly opposite in the sketch, from the immediately preceding corner. Thus, the distances 3.50 and 4.18, directly opposite the corners of the dwelling in the sketch, are the distances of these corners from *B*.

Notes should be full and plain, and should be kept as neatly as possible. The surveyor should keep his notes in such manner that they can be readily understood, not only by himself, but by any one having a knowledge of surveying. This is especially necessary when the notes are not to be platted by the same person that takes them. The pages of the notebook should be numbered.

45. Platting the Notes.—If a map of the field is desired, the notes of the survey are platted to some convenient scale, determined by the size of the field and the size of the map desired. A plat of the field represented in Fig. 15, Art. **35**, would be constructed in the following manner:

Any line of the plat may be drawn first, but it should be drawn in such a position that the map will come within the limits of the paper and be approximately in the center. This can be easily done by an inspection of the sketch and the notes. In the present case it will be assumed that the line *A B* is drawn first; its length is made equal to 8.13 chains to the scale chosen for the map. From *A* and *B* as centers, and with radii equivalent, according to scale, to 9.25 and 8.36 chains, respectively, two arcs are described, whose point of intersection locates the point *G* on the map. From *B* and *G* as centers and with radii of 4.52 and 8.92 chains, respectively, two arcs are described, whose point of intersection locates the point *D* on the map. The remaining corners are located in a similar manner. Joining them by straight lines, the straight boundaries of the field are obtained.

To plat the irregular boundary *G N A*, the distances *G H*, *G J*, and *G L* are laid off from *G*, after *G A* has been platted, and at the points *H*, *J*, and *L* are erected perpendiculars *H I*, *J K*, and *L M* equal to the recorded offsets 1.55, 1.07, and 1.40 chains, respectively. The irregular boundary *G N A* is platted by drawing a freehand line through the points *G*, *I*, *K*, *M*, and *A*, making the parts of the line between the points *G* and *I*, *I* and *K*, etc. nearly straight.

CALCULATING THE AREA

46. The area of a closed field is obtained by calculating the areas of the triangles and trapezoids into which the field has been divided.

As explained in *Geometry*, Part 2, and *Plane Trigonometry*, Part 2, the area of a triangle the lengths of whose sides are known, may be found by the formula

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

in which a, b, and c represent the three sides, and s, half their sum, or $\frac{a+b+c}{2}$.

It was also explained in *Geometry*, Part 2, that the area of a trapezoid is equal to the product of one-half the sum of the parallel sides and the perpendicular distance between them.

In calculating the area of the part of the field having an irregular boundary, the parts of the boundary lines between the points to which offsets are measured are considered to be straight lines.

EXAMPLE.—Required the area of the portion $ABGNA$ of the field represented in Fig. 15, the dimensions being as shown on the figure.

SOLUTION.—To find the area of the triangle ABG we have, $AB = 8.13$ ($= a$, say); $BG = 8.36$ ($= b$, say); $GA = 9.25$ ($= c$, say). Therefore, $s = \frac{8.13 + 8.36 + 9.25}{2} = 12.87$; $s - a = 4.74$; $s - b = 4.51$; $s - c = 3.62$.

Substituting in the formula, we have,

$$S = \sqrt{12.87 \times 4.74 \times 4.51 \times 3.62} = 31.56 \text{ sq. ch.}$$

The area of the triangle GHI is

$$\tfrac{1}{2}(1.04 \times 1.55) = .81 \text{ sq. ch.}$$

The distance between the parallel sides of the trapezoid $HIJK$ is equal to $3.89 - 1.04 = 2.85$ ch. The area of this trapezoid is, then,

$$\left(\frac{1.07 + 1.55}{2}\right) 2.85 = 3.73 \text{ sq. ch.}$$

The distance between the parallel sides of the trapezoid $JKML$ is equal to $8.11 - 3.89 = 4.22$ ch. The area of this trapezoid is, then,

$$\left(\frac{1.40 + 1.07}{2}\right) 4.22 = 5.21 \text{ sq. ch.}$$

In the triangle ALM, $AL = 9.25 - 8.11 = 1.14$ ch. The area of the triangle is, then,

$$\tfrac{1}{2}(1.14 \times 1.40) = .80 \text{ sq. ch.}$$

The area of the part $ABGNA$ is equal to the sum of the areas into which it is divided, or

$$31.56 + .81 + 3.73 + 5.21 + .80 = 42.11 \text{ sq. ch.} = 4.21 \text{ A.}$$

As already stated, square chains are reduced to acres by dividing by 10.

EXAMPLES FOR PRACTICE

1. Plat the field represented by Fig. 15 and calculate its area, in acres. Ans. 9.03 A.

2. Plat the field represented by Fig. 18 and calculate its area, in acres. Ans. 10.02 A.

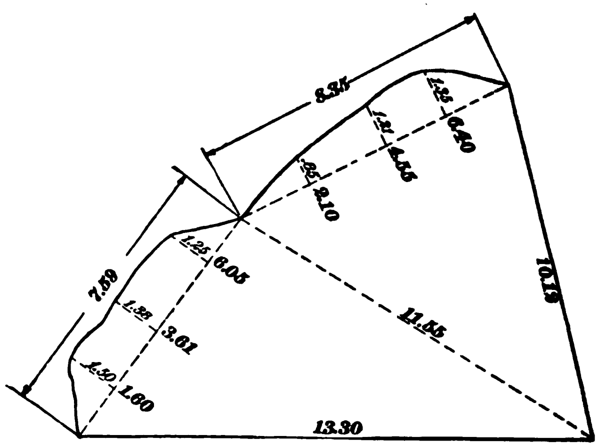

Fig. 18

COMPASS SURVEYING
(PART 1)

THE SURVEYORS' COMPASS

DESCRIPTION

1. The **compass** used in surveying consists essentially of a magnetic needle supported freely on a pivot at the center of a horizontal graduated circle. To this circle are attached

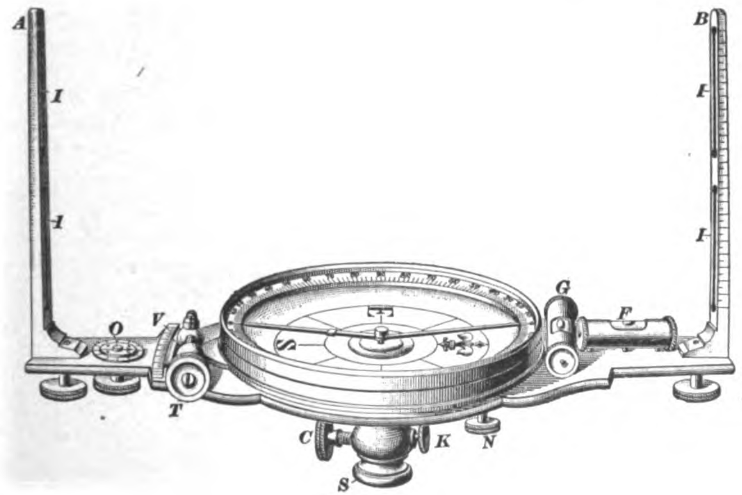

FIG. 1

a pair of sights. The needle and graduated circle are enclosed in a brass case having a glass cover, and the whole rests on a convenient support, with facilities for leveling. A compass is shown in Fig. 1.

2. The Magnetic Needle.—This is a slender bar of steel, usually from about 3 to 6 inches long, strongly magnetized and supported on a fine-pointed pivot on which it turns freely. By means of the small thumbscrew *N*, Fig. 1, it can be lifted from the pivot and held against the glass of the compass box, so that it will not wear the pivot by oscillating when the compass is being moved. In any given locality, the magnetic needle, when at rest, assumes a direction that is nearly constant and coincides, approximately, with a north-and-south line.

3. Magnetic Poles of the Earth.—The ends of the needle are attracted by two forces acting at two points on nearly diametrically opposite sides of the earth, called the **magnetic poles**, which are not very far from the geographical poles of the earth. The one near the geographical north pole is called the **north magnetic pole**; the other, the **south magnetic pole**. The line joining them is not a true north-and-south line, and for this reason the needle, which lies in the line of the magnetic poles, does not point exactly north and south. The end of the needle that points to the north magnetic pole is called the **north end**; the other, the **south end**.

4. Dip of the Needle.—At all points equidistant from the magnetic poles, the needle is horizontal, its two ends being equally attracted. At points nearer to one magnetic pole than to the other, the needle inclines toward the nearer pole and forms with the horizon an angle called the **dip of the needle**. In northern latitudes, the north end dips below the horizon, and, in order to keep the needle horizontal, a coil of platinum wire, movable along the needle, is wound around its south half. This coil serves both to preserve the horizontal position of the needle and to indicate which is the south, and which is the north, end.

5. The Sights.—Attached to two diametrically opposite ends of the compass plate are the **sights**, also called **sight vanes**, by means of which the compass is directed toward any object. They consist of upright bars of brass *A*, *B*,

Fig. 1, through each of which is a narrow vertical slit, with holes at its top and bottom, as shown at *I*. In some compasses, the narrow slit extends only through the upper half of one sight and the lower half of the other, the remaining half of each having a wider opening, in which the line of sight is defined by a vertical hair. Each sight is attached to the end of the compass plate by means of a screw having a milled head that projects below the plate. This permits the sights to be quickly detached when the compass is placed in its case. The sights of some compasses, instead of being detachable, have joints near the base, so as to fold down on the needle box.

6. The Graduated Needle Circle.—Also attached to the compass plate and within the circular box containing the needle, is a graduated circle having a diameter slightly greater than the length of the needle, and at the center of which is the pivot supporting the needle. This circle, sometimes called the **needle circle**, is usually graduated to half degrees and numbered at every tenth degree mark from 0° to 90° in each quadrant. The zero-degree marks are in line with the sights, as shown in Fig. 1. One of them, called the *north end of the compass*, is indicated either by the letter *N* or a fleur-de-lis; the other, called the *south end*, is indicated by the letter *S*. These names do not imply that the line determined by the sights, or passing through the zero marks, is a north-and-south line; they are used for the purpose of facilitating the reading of the instrument, as will be explained further on. The 90° divisions are marked *E* (east) and *W* (west), respectively; but the position of these letters is the reverse of the position of the cardinal points named by them; that is, *E* is on the left of north, and *W* on the right. This arrangement is also adopted for convenience in reading the instrument.

7. Plate Levels.—For the purpose of indicating when the compass is in a horizontal position, two small spirit levels are attached to the compass plate, as shown at *F* and *G*, Fig. 1. Each level consists of a glass tube curved slightly upwards. The greater part of the tube is filled with alcohol, the remaining space being occupied by a small bubble of air. The tube is mounted in a brass tube, or case, that is attached to the compass plate by screws. One of these tubes, *F*, is parallel to the line of sight, and the other, *G*, is at right angles to that line. When these levels are in proper adjustment, the compass can be brought to a perfectly horizontal position by so moving or tilting it as to bring the air bubbles to the centers of the tubes.

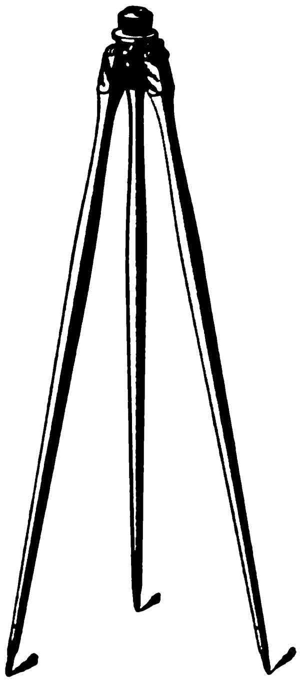

Fig. 2

8. Outkeeper.—On the main plate at *O*, Fig. 1, is shown a small dial plate having an index, or pointer, that is turned by means of a milled-headed thumb wheel, shown beneath the main plate. This device is called an **outkeeper**, and is used for keeping count in chaining. The dial is numbered from 0 to 16 and the index is turned one division for each chain or each 10 chains measured, as may be most convenient or advantageous. The outkeeper is not essential, and is not found on all compasses.

9. The Tripod.—The compass is often supported on a single standard, called a **Jacob's staff.** This is merely a straight pole about 4½ feet long, having on its lower end a pointed steel shoe, for thrusting the staff into the ground, and on its upper end brass mountings to fit the compass. A more perfect support, called a **tripod,** is sometimes employed. This consists (see Fig. 2) of three legs shod with steel and connecting by hinge joints to a metal **tripod head.**

10. Socket and Spindle.—Attached to the under side of the compass plate is a projecting piece, or **center** (see

Fig. 1), having in its lower end a vertical **socket** *S*, by means of which the compass is connected with its support. This socket is slightly conical in form and fits on a **spindle**, on which it turns freely, permitting the compass to be revolved in a horizontal plane. On the lower end of the spindle is a ball turned perfectly spherical and held in a spherical socket, the upper part of which screws on its lower part in such manner that its pressure on the ball can be regulated by tightening or loosening the screw. This ball-and-socket joint permits the compass to be moved or tilted in any direction in the operation of leveling it, the screw being at the same time sufficiently tight so that the plate will remain in position when leveled. The lower portion of the ball-and-socket joint is attached to the top of the Jacob's staff or tripod head, as the case may be. When it attaches to the top of the former, it is sometimes called the **staff head**. A spindle with ball-and-socket joint and staff head is shown in Fig. 3: *H* is the staff head, *B* is the ball-and-socket joint, and *S* is the spindle, which fits into the socket *S* of the compass shown in Fig. 1, and is secured by the spring catch *K*, which drops into the small groove *G* shown in the spindle of Fig. 3. The socket turns freely on the spindle, but can be clamped in any position by means of the clamp screw *C*, Fig. 1.

FIG. 3

ADJUSTMENTS OF THE COMPASS

11. Conditions of Perfect Adjustment.—Besides several conditions that are attended to by the instrument maker, the following are indispensable for accurate work:

1. The plane tangent to the level bubbles when at the centers of their respective tubes must be perpendicular to the vertical axis of the socket.

2. The two ends of the needle must be in the same vertical plane as the pivot.

3. The needle pivot must be in the center of the graduated circle.

A new compass made by a good manufacturer always satisfies these conditions, as the instrument is sold by the maker in perfect adjustment. Rough usage, however, a fall, or a hard blow may throw the compass out of order, and it is necessary that the surveyor should know how to test and readjust it.

12. To Adjust the Plate Levels.—Bring the bubbles to the centers of the level tubes by moving or rotating the plate carefully by means of the ball-and-socket joint; then revolve the compass horizontally through 180°; that is, turn it end for end. If the bubbles remain in the centers of the tubes, the levels are in adjustment. But if in turning the compass end for end, either bubble runs toward one end of the tube, lower that end and raise the opposite end sufficiently to bring the bubble half way back, by means of small screws that attach the ends of the tube to the plate; then bring the bubble to the center by moving the plate as before. Repeat the operation until both bubbles remain in the centers of the tubes in every position of the compass.

13. To Straighten the Needle.—Level the compass and turn it so that the north end of the needle points exactly toward or cuts some prominent graduation mark of the needle circle, and note the exact reading of the south end of the needle. In order to read either end of the needle accurately, the eye should be directly above a line in the prolongation of the opposite end of the needle. Then reverse the compass end for end and turn it so that the south end of the needle cuts the same graduation mark and note if the north end reads the same as the south end did before reversing. If it does not read the same, correct one-half the error by bending the needle carefully, and repeat the operation, using different graduation marks, until exact reversals are obtained.

14. To Center the Needle Pivot.—Having, if necessary, straightened the needle, turn the compass so that the north end of the needle will exactly cut some prominent graduation mark, and observe if the south end exactly cuts the opposite graduation mark. If it does not, find the position of the needle that shows the greatest difference in the readings of its opposite ends; then remove the needle from the pivot and bend the pivot carefully at right angles to this position an amount equal to one-half the error. Repeat the operation until the needle cuts accurately all opposite graduation marks.

SURVEYING WITH THE COMPASS

PRELIMINARY DEFINITIONS AND EXPLANATIONS

15. Object of Compass Measurements.—The compass is used primarily for determining the directions of lines. By means of it, the angle between any line and the direction of the needle can be measured directly. The angle between any two lines is found by a simple arithmetical operation from the angles that they make with the direction of the needle.

16. Angle Between a Line and the Needle.—Let *A B*, Fig. 4, be a straight line on the ground, *A H* a horizontal line through *A*, and *B H* a vertical line through *B*, intersecting *A H* at *H*. As explained in *Chain Surveying*, in all surveying operations (except leveling), the line *A B* is represented by its horizontal projection *A H*, and by the

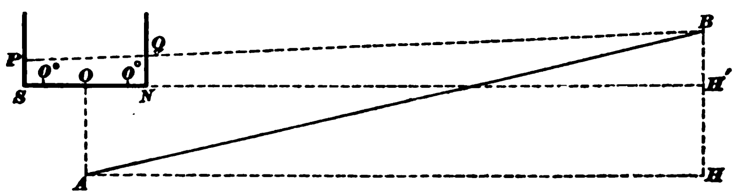

FIG. 4

distance from *A* to *B* is understood the length of the projection *A H*. So, too, by the angle between two lines is meant the angle between their horizontal projections; and by the angle between a line and the needle is meant the angle between the direction of the needle and the horizontal projection of the line.

Let the compass $PSNQ$ be set with its center O directly over A, and let the plate, after being leveled, be turned until the point B is seen by looking through the slits in the sights P and Q. It is evident that, with the compass in this position, the horizontal line SNH', passing through the zero marks and the feet of the sights is parallel to the horizontal projection AH of AB. Therefore, the angle between AB (that is, between AH) and the direction of the needle is the same as the angle made by the needle with the line passing through the 0° marks of the graduated circle. It will soon be explained how this angle is read from the circle.

17. Meridians.—A line having the direction of the magnetic needle is called a **magnetic meridian.** A north-and-south line, that is, a line directed toward the geographical poles of the earth, is called a **true meridian,** or simply a **meridian.** The term *meridian* is also often applied to any line used as a line of reference from which to locate the positions of points and the directions of other lines.

18. Bearing of a Line.—The angle that a line (that is, the horizontal projection of the line) makes with the magnetic meridian is called the **magnetic bearing** of the line. The **true bearing** of a line is the angle between the line and the true meridian. In compass surveying, all bearings are understood to be magnetic bearings, and the word *magnetic* is usually omitted. Since the magnetic meridian is indicated by the direction of the needle, the magnetic bearing of a line is the angle between the line and the needle.

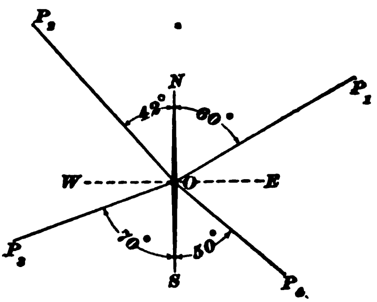

FIG. 5

Bearings are reckoned from 0° to 90°, and indicate the amount by which a line is east or west of north or south. In Fig. 5, NS represents the magnetic needle, N and S being, respectively, the north and the south end; OP_1, OP_2,

OP_3, and OP_4 are supposed to be lines of a survey. The line OP_1 makes with the north half of the needle an angle of 60°; as the line lies between the north point N and the east point E, its bearing is said to be 60° northeast, or 60° to the east of north. This is indicated by the notation N 60° E, read *north 60° east.* Similarly, the bearings of OP_2, OP_3, and OP_4 are, respectively, N 42° W (north 42° west), S 70° W (south 70° west), and S 50° E (south 50° east).

If a line points directly to the north, to the east, etc., it is said to be **due north, due east,** etc.

19. To Determine the Magnetic Bearing of a Line. Let OP, Fig. 6, be a line whose magnetic bearing is to be determined. The compass is set over the extremity O, so that its center shall be directly over that extremity. After the plate has been leveled, it is turned until the point P is in line with the slits in the sights, and therefore with the line SN passing through the zeros of the graduated circle. It will be noticed that the north end N of the compass is the forward end, or the one that is between the eye and the extremity P of the line, the observer's eye being supposed to be at the slit in the south sight and looking in the direction SN. Having directed his line of sight in this manner, the surveyor looks at the *north* end of the needle FG, which is distinguished from the south end by the coil s wound around the south half of the needle. The letters between which the north end of the needle lies indicate whether the bearing is northeast, northwest, southeast, or southwest—if the north end of the needle lies, as in Fig. 6, between N and E, the bearing is northeast; if between S and W, southwest; etc. The number of degrees in the bearing is indicated by the number of the graduated circle toward which the *north* end of the needle points. Degrees and half degrees are read directly on the

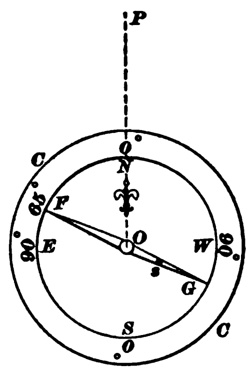

FIG. 6

circle, and quarter degrees are estimated by the eye. In Fig. 6 the north end of the needle points toward the 65° mark, or "reads" 65°; therefore, the bearing of the line is N 65° E. Sometimes the needle is read to the nearest 5 minutes; but the imperfections of this instrument make such refinement of little practical value for the purposes of surveying.

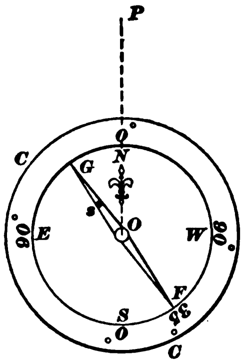

FIG. 7

In Fig. 7, where the bearing of OP is to be determined, the north end of the needle lies between the letters S and W, which indicates that OP is a southwest line; and, as that end reads 35°, the bearing is S 35° W.

Should not the compass circle be marked as here assumed, the bearing of a line can be very simply determined by inspection. Thus, in Fig. 6, OP, being nearer to the north end F of the needle than to the south end, is a north line; and, being on the right of F, it is east of north; the line is, therefore, a northeast line. In Fig. 7, OP is nearer to the south end G of the needle than to the north end; this indicates that OP is a south line; and, as it runs to the right of G, or to the west of south, it is a southwest line.

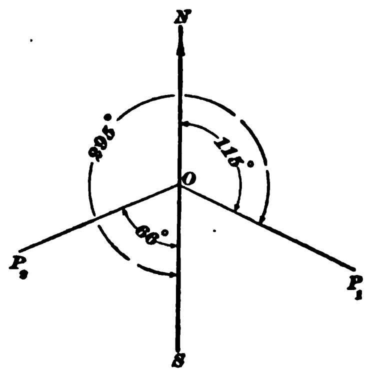

FIG. 8

20. Course.—A line whose length and bearing are measured is commonly called **a course.** The term *course* is sometimes used, especially in navigation, in the sense of direction or bearing, as when it is said that the course of a ship is N 50° E.

21. Azimuth.—The **azimuth** of a line is the angle that the line forms with the meridian. Azimuth may be either

true azimuth or **magnetic azimuth,** according as it is referred to the true or to the magnetic meridian. Azimuth differs from bearing in that it is always measured in the direction corresponding to the movement of the hands of a clock when lying face upwards, through any required angle up to 360°, starting from either the north or the south point. In Fig. 8, in which SN represents the meridian, and N and S the north and the south point, respectively, the azimuth of OP_1, counted from the north, is 115°; counted from the south, 295°. The azimuth of OP_2, counted from the south, is 66°; counted from the north, 180° + 66° = 246°.

FIELD WORK

22. Taking Bearings.—To take the bearing of a line, the compass is set directly over a point of the line—one extremity, if possible. When the compass is mounted on a tripod, this can be done accurately by means of a plumb-bob suspended from the center of the compass; or, approximately, by dropping a pebble from below the center. If the compass is mounted on a Jacob's staff, the staff should be planted in a vertical position directly on the line. Then, by means of the levels, the compass is brought to a perfectly level position. A flagman holds a flag perfectly vertical at some other point of the line—preferably the other extremity of it, if the flag can be seen distinctly. The surveyor then turns the compass plate horizontally, walking around the instrument so as to be constantly behind the south sight, and looks over the sights until they are approximately in line with the pole held by the flagman. He then looks through the slits—his eye being at the south slit, so that the north end of the compass is the forward end—and turns the plate until his line of sight cuts the flag. With the compass in this position, he reads the bearing according to the directions given in Art. **19.** When it is not possible to set the compass over a point of the line, as when the latter is determined by a fence or the foot of a wall, it is set a short distance from the line, and the flag is held opposite the other end, at the same distance from the line.

The line of sight should be directed as nearly to the bottom of the flag as possible, in order to avoid or diminish the error that may arise from any deviation of the flag from the vertical.

When reading a bearing, the eye should be directly over the needle, so as to avoid the error from **parallax**, or the apparent change in the position of the needle that occurs when the needle is looked at along a line not in the same vertical plane with it.

23. Foresights and Backsights.—A sight taken from a point of a line on another point, in the direction in which the line is being run or measured, is called a **foresight**; it is the sight taken to ascertain the bearing of the line, as explained in the last article. In order to check the bearing as thus determined, the compass is set at the point first sighted, or at a point between the point first sighted and the point first occupied by the compass; a sight, called a **backsight**, is then taken on the latter point, and the bearing read. If the two bearings are the reverse of each other, it may be assumed that no mistake or error has been made, and the work may be proceeded with. Thus, suppose that the bearing of OP, Fig. 9, has been found, by setting the compass at O and sighting to P, to be N 50° E.

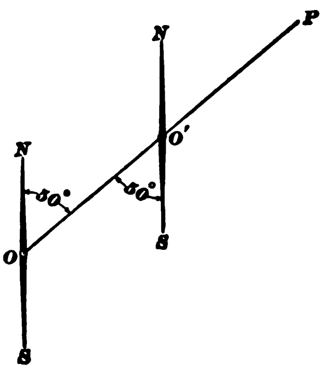

FIG. 9

A backsight may be obtained by setting the compass at some intermediate point, as O', and directing the line of sight to O. It is evident that, if the bearing of OP, or of OO', is N 50° E, that of $O'O$ must be S 50° W. Likewise, if the bearing of OO' were S 30° E, that of $O'O$ should be N 30° W. In every case, the bearing determined by a backsight must be obtainable from that determined by the corresponding foresight by simply interchanging the letters N and S, E and W.

24. Local Attraction.—The compass needle may be deflected from its natural direction by the attraction of any magnetic substance near it, such as iron ore, the rails of a railway, a gas pipe, the chain, a bunch of keys, etc. This disturbing influence, called **local attraction**, is very frequently met with, and the surveyor should take special pains to avoid the errors to which it may give rise.

When the bearing determined by a foresight does not agree with that determined by a backsight, the usual cause of the difference is local attraction. To determine where the disturbing influence lies, a backsight is taken from an intermediate point along the line, or foresights and backsights are taken on an auxiliary point outside of the line. Suppose that the foresight from *O*, Fig. 10, on *P* is N 85° 45′ E, and that the backsight from *P* to *O* is S 75° 30′ W. If the instrument has been read accurately, there must be local attraction either at *O* or at *P*, since otherwise the bearing of *P O* would be S 85° 45′ W. Selecting

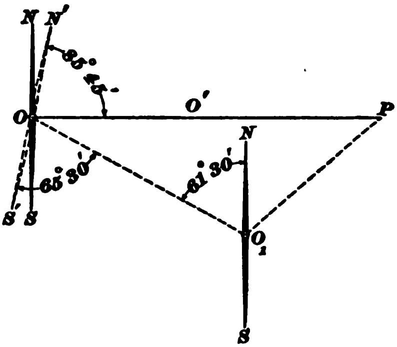

FIG. 10

some intermediate point, as *O′*, and taking a backsight on *O*, it will usually be found that the bearing thus determined agrees with one of the bearings determined from *O* and *P*. If the bearing of *O′ O* is S 85° 45′ W, which agrees with the bearing of *O O′*, the local attraction is at *P*, and the correct bearing of *O P* is N 85° 45′ E. If the bearing of *O′ O* is S 75° 30′ W, which agrees with the bearing of *P O*, the local attraction is at *O*, and the correct bearing of *O P* is N 75° 30′ E. The point *O′* should not be taken too near either *O* or *P*, as otherwise the needle might be disturbed by the local attraction existing at one of the latter points.

If it is not convenient to take an intermediate point on the line. or if there is reason to suspect that there is local attraction all along the line, foresights and backsights on an outside point, as O_1, may be taken from both O and P. It may be found that there is local attraction at both O and P, in which case the bearing must be corrected by determining the angle by which the needle is deflected by the disturbing influences. Thus, if the bearing of $O_1 O$ is N 61° 30′ W and that of $O O_1$ is, according to the reading of the compass, S 65° 30′ E, the latter bearing, which should be S 61° 30′ E, shows that the effect of local attraction is to deflect the needle toward the west of south, or toward the east of north, by an amount equal to 4°. This is shown in the figure,

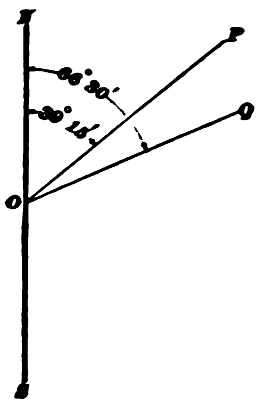

FIG. 11

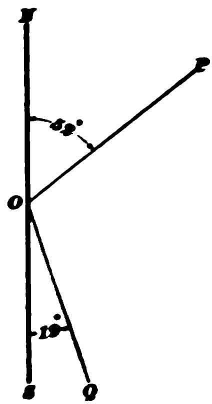

FIG. 12

where SN is the position that the needle would have if there were no local attraction, and $S'N'$ is the position that it assumes at O. The true bearing of OP is, therefore, N (85° 45′ + 4°) E = N 89° 45′ E. In all this it is assumed that there is no local attraction at O_1. This point must be tested for local attraction by taking an intermediate point between it and O, and proceeding as with the point O', Fig. 10.

When a bearing is to be corrected for local attraction, it is advisable to draw a diagram similar to Fig. 10, in order to avoid confusion and the mistakes that may arise therefrom.

25. To Determine the Angle Between Two Lines Whose Bearings Are Known.—When the bearings of two lines are of the same kind, that is, both N E, N W, S E, or S W, the angle between the lines is obviously equal to the difference between the bearings. Thus, if the bearing of a line OQ, Fig. 11, is N 66° 30′ E, and that of another OP is N 39° 15′ E, the angle between the lines is 66° 30′ − 39° 15′ = 27° 15′. In this, and in what follows, the bearings of the intersecting lines should be reckoned from their point of intersection. If, for instance, the bearing of PO has been found, by sighting from P to O, to be S 39° 15′ W, its bearing as determined from O, that is, by sighting from O along OP, is N 39° 15′ E.

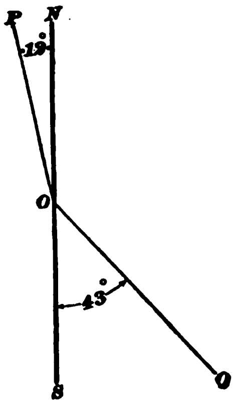

FIG. 13

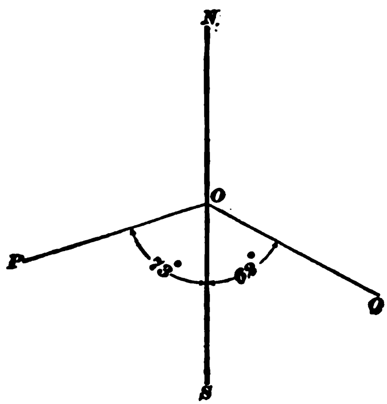

FIG. 14

In the majority of cases, the simplest and surest way to determine the angle between two lines is to make a rough sketch, in which a line is drawn to represent the meridian, and from any point of it two lines are drawn about in the positions that the given lines have with reference to the needle, as indicated by their bearings. Thus, in Fig. 11, NS represents the meridian, N being the north point; OP represents the line whose bearing is N 39° 15′ E, and OQ, the line whose bearing is N 66° 30′ E. Having marked the values of the angles NOP and NOQ on the figure, the angle POQ can at once be determined by inspection. Figs. 12, 13, and 14 are further illustrations of this method. In

Fig. 12, the bearing of OP is N 52° E, and that of OQ is S 19° E. The figure at once shows that POQ is equal to 180° − (52° + 19°) = 109°. Similarly, in Fig. 13, $POQ = PON + NOQ$ = 12° + (180° − 43°) = 149°; and in Fig. 14, $POQ = POS + SOQ$ = 73° + 62° = 135°.

26. From a Point on a Line of Known Bearing, to Run a Line Making a Given Angle With the Given Line.—The bearing of the required line is best determined by a diagram showing the angles that the given line makes with the needle and with the other line. Suppose, for instance, that the bearing of the given line is S 48° 45′ W, and that the required line is to make an angle of 160° 30′ with the given line, it being known from other conditions that the required line is to lie on the right of the given line.

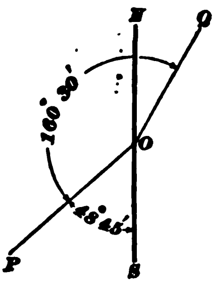

Fig. 15

Fig. 15 shows these conditions, NS being the meridian, OP the given line, and OQ the required line. Evidently, NOQ = 160° 30′ − PON = 160° 30′ − (180° − 48° 45′) = 29° 15′. Therefore, the bearing of OQ is N 29° 15′ E. Having determined that bearing, the compass is set at O, and the plate turned until the north end of the needle lies between the letters N and E, and reads 29° 15′. The line of sight through the slits (the north slit being ahead) has then the direction of the required line OQ, which may be marked by setting a stake or pole at any convenient point of it.

EXAMPLES FOR PRACTICE

1. What is the angle between two lines whose bearings are N 32° 15′ E and N 42° 30′ W, respectively? Ans. 74° 45′

2. What is the angle between two lines whose bearings are N 15° 30′ E and S 46° 45′ W, respectively? Ans. 148° 45′

3. The bearings of two lines are N 48° 15′ W and S 76° 30′ E; what is the angle between them? Ans. 151° 45′

4. The bearing of a line is N 47° 30′ W; what is the bearing of a line that lies to the left of and makes an angle of 138° 15′ with the given line? Ans. S 5° 45′ E

5. The bearing of a line is N 20° 30′ E; what is the bearing of a line that lies on the right of and makes an angle of 130° 45′ with the given line? Ans. S 28° 45′ E

6. The bearing of a line is S 40° 30′ W; what is the bearing of a line to the left of and making an angle of 60° 45′ with the given line? Ans. S 20° 15′ E

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27. To Survey a Closed Field.—In surveying a closed field bounded by straight lines, the corners are first marked. The bearings of the lines are measured by setting the compass at the different corners in succession, and sighting from each corner on the one immediately following it. The lengths of the lines are measured with the chain or tape as explained in *Chain Surveying*. Care should be taken to check every foresight by a backsight. In chaining, the head chainman may be lined in by the rear chainman, as in chain surveying, or by the compassman. The latter, having directed the sights in the direction of the line, looks through the slits when the head chainman is about to stick a pin in the ground, and motions to him to move the pin right or left, until it is covered by the line of sight.

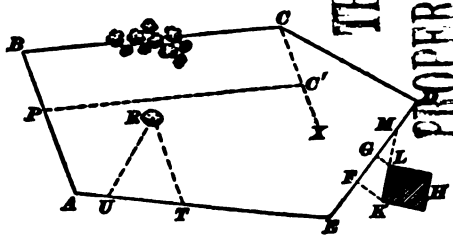


FIG. 16

28. It often happens that a corner is not visible from the one immediately preceding. This case is illustrated in Fig. 16, where the line of sight from *B* to *C* is obstructed by trees. The length and bearing of *A B* are supposed to have been measured. The length and bearing of *B C* can be determined in several ways, of which the following is one of the simplest.

Measure back from *B* any convenient distance *B P* along *B A*; set the compass at *C* and direct the line of sight along a line CX parallel to BA. In order to do this, it is only necessary to turn the compass plate until the needle reading is equal to the bearing of BA. It should be borne in mind that the bearing of BA is the reverse of that of AB. If, for instance, the bearing of AB is N 20° W, that of BA is S 20° E, and the reading of the compass at C must be S 20° E. Measure along CX a distance CC' equal to BP. Move the compass to C', take the bearing of $C'P$, and have the chainman measure the distance PC'. This distance is equal to BC, and the bearing of BC is the reverse of that of $C'P$. Thus, if the bearing of $C'P$ is S 70° W, that of BC is N 70° E.

29. Important objects near the line of survey are located in various ways, according to circumstances. When they are very close to the line being run, they are usually located by perpendicular offsets. Thus, the house H, Fig. 16, may be located by measuring and noting the perpendiculars GL and FK and the distances DG and DF. It may also be located by prolonging KL to its intersection M with DE and measuring and noting the distances DM, ML, LK, and the bearing of ML or MK. Such short perpendiculars as GL and FK are usually run by the eye.

A distant object, as the rock R, is located by measuring its distance from any convenient point, as T, on one of the boundary lines, and taking the bearing of TR; or by simply taking the bearings of lines, as TR and UR, directed to it from two points on the line of survey. The latter method is especially useful for the location of inaccessible points. In making the map of the field, T and U are located by their recorded distances from E. The intersection of two lines having, respectively, the bearings of TR and UR and drawn in the map through the points representing T and U, will determine the position of R.

30. Passing Obstacles.—In running and measuring a line of known bearing, it is sometimes impossible to measure its whole length directly, either because parts of the line run through inaccessible or not easily accessible ground, or because the line goes through objects, as trees or houses, that prevent direct measurement. An instance of this condition is illustrated in Fig. 17, where the line of survey AB runs through the buildings K and L. The same figure illustrates two of the most convenient methods of passing obstacles, one method being used for passing K and the other for passing L.

The method used in passing the building K is as follows: Having run the line to a point P as near K as possible, the compassman sets the instrument at P and turns off a line PQ in any convenient direction. He reads the bearing of PQ, which should preferably be a whole number of degrees. A stake is driven at a point Q of PQ such that a line through it and parallel to AB will clear the obstacle K. The instrument is then set at Q, a backsight on P is taken in order to check the bearing of PQ, and a line QQ' is run from Q parallel to, or having the same bearing as, AB, and long

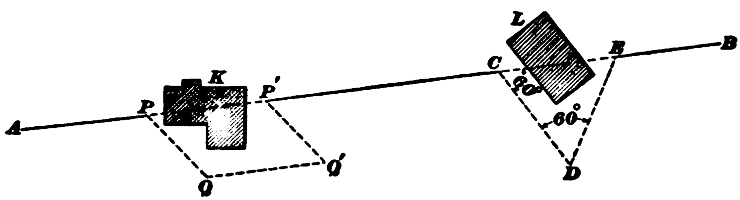


FIG. 17

enough to clear the obstacle. The distances PQ and QQ' are measured. The instrument is now set at Q', from which a line $Q'P'$ is run parallel to QP (that is, having the same bearing as QP, not as PQ). On this parallel, a distance $Q'P'$ equal to QP is measured. The point P' thus determined is in the line AB and the distance PP' is equal to QQ'. Setting the compass at P' and turning the plate until the reading of the needle is equal to the bearing of AP, the line of sight will be directed along AB and the work may be continued in the usual manner.

31. The obstacle L, Fig. 17, is passed by means of an equilateral triangle. The line having been run to a point C

near *L*, a line *CD* making an angle of 60° with *AB* is turned off (see Art. **26**). The distance *CD* is taken so that a line from *D*, at 60° with *CD*, will clear the obstacle. The proper position of *D* can be estimated by the eye. The distance *CD* having been measured, the compass is set at *D*, and a line *DE* equal to *CD* and at 60° with it is run. The point *E* thus determined is on *AB*, and the distance *CE* is equal to *CD*.

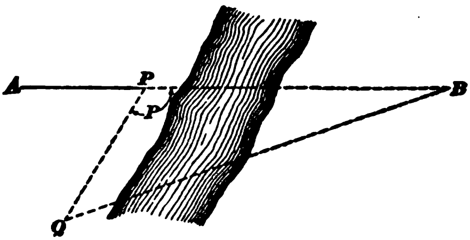


FIG. 18

In this, as in all compass work, the surveyor should test all bearings by backsights.

32. In Fig. 18 is represented a very common condition. Here the line of survey *AB* crosses a river, the distance across which cannot be measured directly, nor ascertained by either of the methods explained in the last two articles. The distance from *A* to a point *P* near the bank having been measured, the distance *PB* is determined as follows:

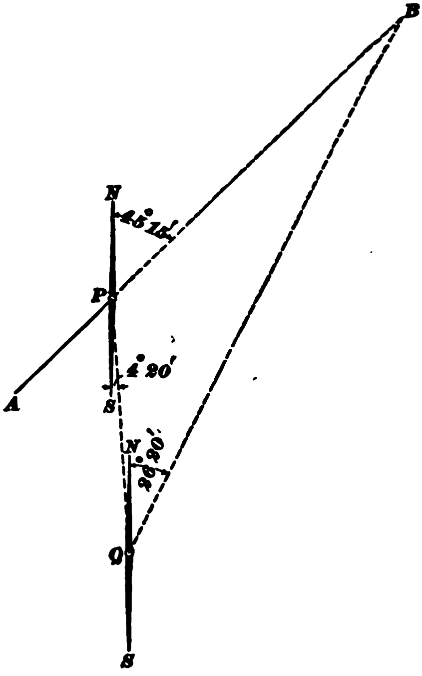


FIG. 19

From *P*, run a line *PQ* in any convenient direction; note its bearing, and make its length approximately equal to *PB*. If, however, *PB* is inconveniently long, PQ may be made equal to about one-half or even one-quarter of it. Theoretically, PQ may have any length; but, in practice, it is preferable not to have it too short as compared with PB. Next, set the compass at Q and take the bearing of QB. From the bearings of AB, PQ, and QB, the angles P and Q can be computed (Art. 25); also, $B = 180° - (P + Q)$. In the triangle PQB, the side PQ and the angles are known, and the side PB can be found by the relation $PB = \frac{PQ}{\sin B} \sin Q$.

EXAMPLE.—Suppose that in Fig. 19 the length of PQ is 100 feet; the bearing of AB is N 45° 15′ E; that of PQ, S 4° 20′ E; that of QB, N 26° 20′ E; what is the distance from P to B?

SOLUTION.—The angles of the triangle PQB can be determined from the bearings as follows: In Fig. 19, NS represents the meridian. It is evident that, in the triangle QPB, angle P is equal to $180° - (45° 15' + 4° 20') = 130° 25'$; angle Q is equal to $4° 20' + 26° 20' = 30° 40'$; and angle B is equal to $180° - (130° 25' + 30° 40') = 18° 55'$. Then, $PB = \frac{100}{\sin 18° 55'} \sin 30° 40' = 157.3$ ft. Ans.

EXAMPLES FOR PRACTICE

1. If in Fig. 18 the bearing of AB is S 63° 15′ E; that of PQ, S 43° 50′ W; that of QB, N 84° 20′ E; and the distance PQ is 150 feet, what is the distance from P to B? Ans. 181.7 ft.

2. If in Fig. 18 the bearing of AB is N 45° 15′ W; that of PQ, N 50° 30′ E; that of QB, N 87° 25′ W; and the distance PQ is 200 feet, what is the distance from P to B? Ans. 199.7 ft.

USE OF THE COMPASS IN RAILROAD SURVEYING

33. Preliminary Railroad Surveys.—The compass is of great value in running preliminary railroad lines where local attraction is absent or very slight. In case a small obstruction, such as a tree or a mass of rock, is encountered, the compass can be quickly moved to the opposite side and the line continued without delay. Should the line as thus produced be a foot or two off the true one, no serious disadvantage will result, since it will be parallel to the true line, and the error will be immaterial for the purpose of preliminary information. In the early days of railroad building, some lines were surveyed and located by means of the compass alone.

34. Station Numbers.—In railroad surveying, the line of survey is divided into **stations**, which are usually 100 feet apart. At each station a stake is driven and marked with a number corresponding to the number of hundreds of feet that the station is distant from the starting point, which is numbered 0. Any point marked between regular stations is sometimes called a **substation**, or **plus**, and the stake is marked by the number of the immediately preceding station plus the number of feet between that station and the substation in question. Thus, a point between Stations 5 and 6, distant 47 feet from Station 5, is marked 5 + 47, and referred to as Substation 5 + 47. More usually, however, the term *station* is applied to both regular stations and substations: the Substation 5 + 47, for instance, is referred to as Station 5 + 47.

35. Organization of Party.—A well-organized compass party consists of a chief of party, a compassman, two chainmen, a flagman, two or more axmen, if the country is thickly wooded, and a stakeman. If possible, stakes of light, well-seasoned wood should be provided. For preliminary lines, where stakes do no permanent service, pine is best. A convenient size is about 2 feet in length by 2 inches in width and 1 inch in thickness. A strong, active stakeman will carry a large number of these stakes, besides the ax with which to drive them. In a timbered country, however, it is often more expeditious to have the axmen make the stakes as they are needed, always keeping a few stakes on hand ahead of the requirements. The stakes are usually either made from saplings about 2 inches in diameter or split from larger pieces, sharpened at the lower end for driving and blazed at the upper end for marking. Both chainmen should always be provided with marking crayons. For this purpose, ordinary red chalk is very satisfactory. It is bought in a crude state, but the pieces are easily cut to suitable form. This material makes a deep red mark, which will stand exposure for years. Specially prepared crayon that is excellent for marking stakes can also be obtained.

36. The Actual Work.—The party, having been organized substantially as described in the preceding article, is prepared to begin work. The general order of procedure is usually about as follows: The compassman sets up the compass at the starting point, or Station 0. The chief of party goes ahead with the flagman and directs the latter to set a flag at the end of the first line, or at some intermediate point. The compassman sights on the point thus marked and the chainmen commence measuring the distance, as in chain surveying. The head chainman marks the stakes and should always keep a number of stakes marked in advance, so as to avoid delay. Of these he need carry only a few, leaving the others to be carried by the stakeman. As soon as the direction of the line is indicated by the flag, the axmen should begin to clear whatever obstacles lie in the way of rapid chaining. At each station, and the moment the rear chainman has put the head chainman in line, the former should carefully note the number of the station at which he stands and call the number to the head chainman, who must answer by repeating the number next in notation. Thus, if the rear chainman stands at Station 25, he must call "Station 25," and the head chainman must reply, "Station 26."

It is good practice for the flagman to carry, besides his flag, a number of light stakes at least 8 feet in length and some strips of red flannel to attach to the stakes so that they may serve as pickets, or temporary flags. If the view for the compass is open, as soon as the compass is in line and the flagman has a signal to that effect, he should replace the flag by one of the stakes with a piece of flannel attached, and join the chief of party, who, unless the line is to be continued farther in the same direction, has gone ahead to select another point for the flag. As soon as the compassman has recorded the bearing of the line, he should take the compass and walk rapidly to the point marked by the flag, or picket, and if in full view of the chainman, should remove the flag, or picket, and set up the compass and be prepared to take the next bearing. As soon as the chainmen reach the compass and have measured the "plus" from the last full station, the rear chainman calls out the full station and plus, which the head chainman marks on the stake and the compassman records as the length of the course run. If the same line is to be continued without change of direction, the compass is set at the same bearing as that of the course just run, and the chainmen, measuring forwards, are lined in by the compassman (see Art. 27).

37. Example of the Use of the Compass in Railroad Work.—Suppose *C A D*, Fig. 20, to be the line of a railroad in operation and that it has been decided to run a compass

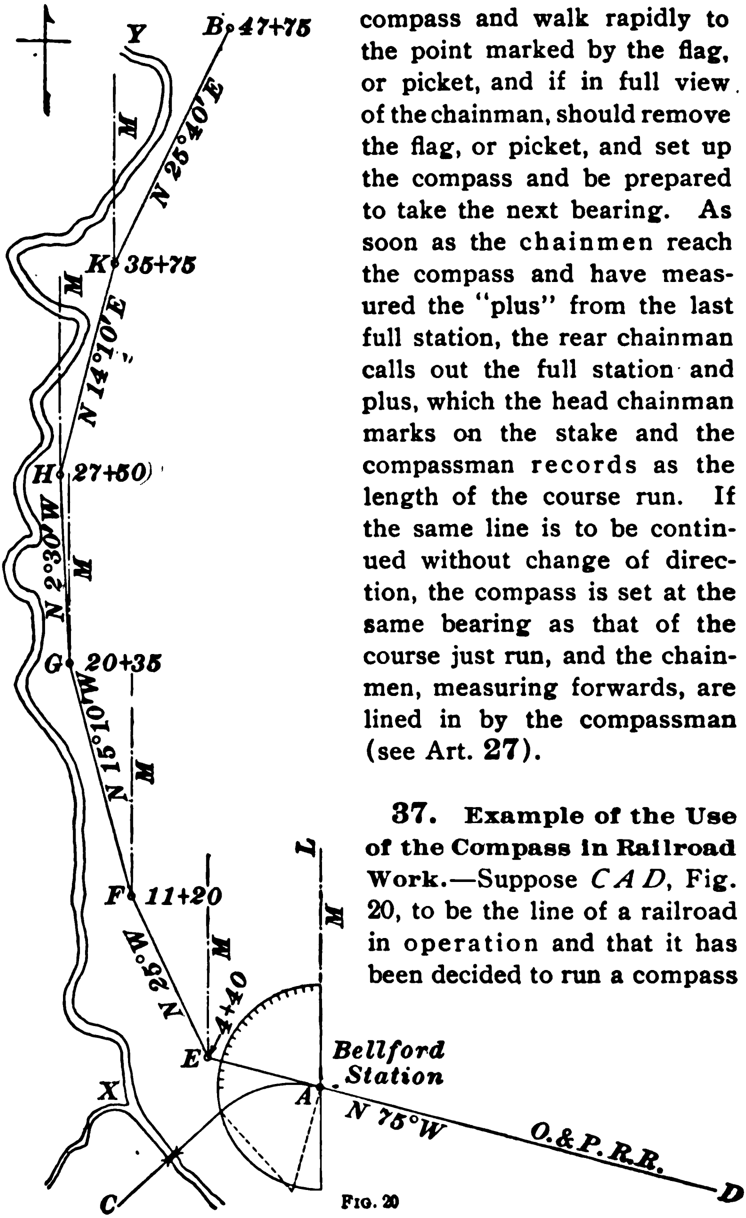


FIG. 20

line from the point A, at the end of the curve, along the valley of the stream XY to the point B. The bearing of the straight line AD cannot be determined by setting up the compass at A, on account of the attraction of the rails. The direction of this line, however, for the purpose of prolonging it, can be obtained by setting up the compass at A and sighting to the flag held at D. The point A, which is the starting point of the line to be run, is marked 0. Producing DA a distance of 440 feet, the point E is reached, which has been previously indicated by the chief of party as a proper place for changing the direction of the line. The compass being set up at E, the bearing of the line AE, which is the prolongation of the line DA, is found by backsighting to A, or, what is preferable, to the point D, if that point can be seen from E. It should be remembered that the bearing of AE, or DE, is the reverse of that of EA, or ED. The number of Station E, namely, 4 + 40, and the bearing of AE are then recorded by the compassman. By this time the chief of party has located the point F and the flag is in place for sighting. The axmen, if there is work for them to do, are put in line by the head chainman, clearing only so much as will interfere with rapid chaining. The bearing of the line EF being recorded, the compass is moved to F, replacing the picket left by the flagman, leveled up and directed toward the point G. When the chainmen reach F, the number *11 + 20* of this station is recorded by the compassman, the instrument is directed to G, and the work continued as before.

FORMS FOR COMPASS FIELD NOTES

38. Form No. 1, given on next page, is a common and convenient form for keeping the record, or **notes**, of a compass survey. The left-hand half of the diagram represents the left-hand page of the notebook; the right-hand half, the right-hand page. The notes are supposed to apply to the field $ABCDE$, Fig. 21. The corner, or station, A is the starting point of the survey, the courses being run from A to B, from B to C, etc. As in chain surveying, the notes read from bottom to top. Opposite the letter denoting a corner is given the bearing of the course running from that corner to the following one, in the order in which the survey was

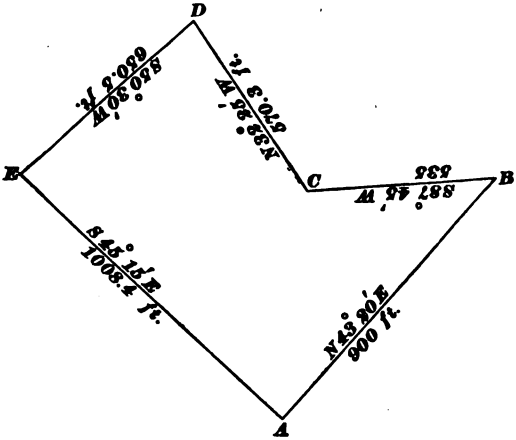
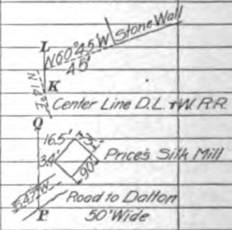


FIG. 21

made. For instance, the bearing N 43° 20′ E horizontally opposite *A* denotes the bearing of the course *A B*. The number opposite a corner in the column of distances is the distance of this corner from the preceding one.

COMPASS NOTES—FORM NO. 1

Station	Bearing	Distance Feet				Remarks	
A		1008.4					
E	S45°15'E	650.5					
D	S50°30'W	570.3					
C	N32°25'W	535.0					
		446.0					
		176.6					
		100.0					
B	S87°45'W	900.0					
A	N43°20'E					Southwest corner of Dr. Peabody's house	



The right-hand page is used for remarks and sketches. When no objects are to be located along the line, as in the case from *A* to *B*, no sketch is necessary. Between *B* and *C*, a sketch is drawn showing the location of a road and mill with respect to the line *B C*. The line being run is usually represented by the center line on the right-hand page, unless objects are to be located at great distances on one side of the former line, in which case it is represented by a vertical line drawn near the right or the left edge of the page, as may be necessary. This is illustrated by the lines *P Q* and *K L*, which represent parts of *B C* and *D E*, respectively. A number written in the column of distances between two letters denoting corners, indicates the distance at which the point horizontally opposite it in the sketch is from the immediately preceding station or corner. Thus, the number 100, horizontally opposite *P*, indicates that the distance from *B* to *P* is 100 feet.

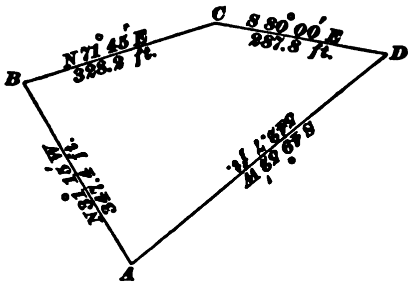


Fig. 22

39. Form No. 2 shows a method of keeping compass notes that is very clear and explicit, and is especially suited to surveys in which it is not necessary to locate any points or objects outside the main line. The notes here given are for the four-sided field represented in Fig. 22.

COMPASS NOTES—FORM NO. 2

Course	Bearing	Distance Feet				Remarks
D A	S 49° 52′ W	542.7				
C D	S 80° 00′ E	287.8				
B C	N 71° 45′ E	328.2				
A B	N 31° 15′ W	347.4				*A* is northeast corner of reservoir wall.

The bearing and distance horizontally opposite any of the courses in the first column are, respectively, the bearing and length of that course. Thus, the bearing and length of *CD* are, respectively, S 80° 00′ E and 287.8 feet.

The lines are supposed to be run in the directions indicated by the letters: *CD* is run from *C* to *D*, not from *D* to *C*.

40. In simple cases similar to the one just illustrated, the best way to keep the notes is generally to make a sketch of the field and write along each line its length and bearing. See Fig. 22.

41. Form No. 3 is a modification of No. 2, and is mainly used in railroad surveying. The notes here shown are those for the preliminary survey described in Art. **37.** Each principal station is denoted by a whole number, as explained in Art. **34,** and each substation by an integer followed by a decimal, the integer denoting the number of

COMPASS NOTES—FORM NO. 3

Station From	Station To	Bearing	Distance	Remarks
35.75	47.75	N 25° 40′ E	12.00	Sta. 47.75 at end of line.
27.50	35.75	N 14° 10′ E	8.25	
20.35	27.50	N 2° 30′ W	7.15	Heavy timber.
11.20	20.35	N 15° 10′ W	9.15	
4.40	11.20	N 25° 00′ W	6.80	Open prairie.
0	4.40	N 75° 00′ W	4.40	Sta. 0 is at P. C. of 14° curve to left at Bellford Sta. on O. & P. R. R.
			47.75	

the station immediately preceding, and the decimal, the number of feet from that station to the substation in question. Thus, Station 27.50 is the same as Station 27 + 50. Distances are given in hundreds of feet. Thus, the distance from 27.50 to 35.75, which is equal to (35.75 − 27.50) stations, is (8.25 × 100) feet, or 825 feet.

COMPASS SURVEYING

(PART 2)

LATITUDE AND LONGITUDE RANGES

DEFINITIONS AND GENERAL FORMULAS

1. Reference Lines.—For the purposes of platting and calculation, it is often convenient to locate all the points of a survey by their coordinates with reference to two coordinate axes perpendicular to each other. In Fig. 1, $T T'$ and $G G'$ are two coordinate axes intersecting at O. The point P is located with reference to these axes by its perpendicular distances $H P$ and $H' P$, or $H P$ and $O H$, from $G G'$ and $T T'$, respectively.

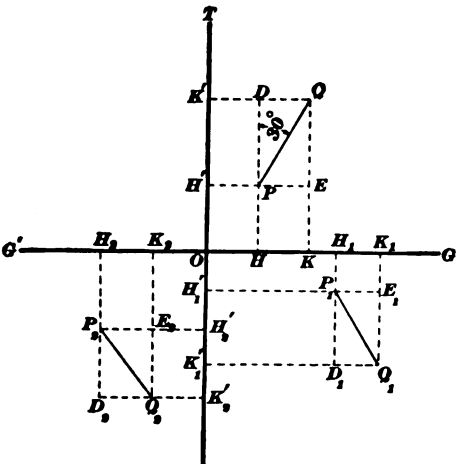


FIG. 1

2. In surveying, $T T'$ usually represents a meridian, either true or magnetic, according as the bearings of the courses are referred to the true or to the magnetic meridian. Such a meridian is called a **reference**

meridian, or principal meridian, while the perpendicular GG', which is an east-and-west line, is called a **reference parallel of latitude, or principal parallel of latitude.** The reference meridian and parallel are generally, though not necessarily, taken through the most westerly corner or station of the tract surveyed.

3. Latitude and Longitude.—The **latitude** of a point is the distance of the point from the reference parallel of latitude. Latitudes are measured along north-and-south lines. In Fig. 1, the latitudes of the points P, P_1, P_2 are, respectively, HP ($= OH'$), H_1P_1 ($= OH_1'$), and H_2P_2 ($= OH_2'$). The latitude of a point is a **north latitude** or a **south latitude** according as the point is north or south of the reference parallel. For the purposes of combining latitudes by algebraic addition, north latitudes are considered positive, and south latitudes, negative. Thus, if HP is 360 feet, and H_2P_2 is 525 feet, the latitude of P is said to be +360 feet, or simply 360 feet, while that of P_2 is said to be −525 feet.

4. The **longitude** of a point is the distance of the point from the reference meridian. Longitudes are measured along east-and-west lines. The longitude of a point is an **east longitude** or a **west longitude** according as the point is east or west of the reference meridian. East longitudes are considered positive; west longitudes, negative. In Fig. 1, the longitude of P is $H'P$ ($= OH$), and that of P_2 is $-H_2'P_2$ ($= -OH_2$).

5. Beginning and End of a Line.—The point from which a line is run and measured is called the **beginning** of the line. The other extremity is called the **end** of the line. If, for instance, the line PQ, Fig. 1, is measured from P to Q, and its bearing is taken by sighting from P toward Q, then P is the beginning and Q the end of the line. If the line is run from Q to P, the beginning is Q, and the end, P. In the former case, the bearing of the line is N 30° E; in the latter, S 30° W.

6. Latitude and Longitude Ranges.—The algebraic difference obtained by subtracting the latitude of the beginning of a line from the latitude of the end of the line will here be called the **latitude range** of the line. Likewise, the algebraic difference between the longitude of the end and the longitude of the beginning of the line will be called the **longitude range** of the line. It should be kept in mind that in obtaining these differences, the coordinate (latitude or longitude) of the end of the line is the minuend, and that of the beginning of the line, the subtrahend; also, that the subtraction is algebraic, latitudes and longitudes having the signs explained in Arts. 3 and 4.

Referring again to Fig. 1, the latitude range of PQ is $KQ - HP = HD - HP = PD$. The longitude range is $K'Q - H'P = K'Q - K'D = DQ$. If the line had been run from Q to P, its latitude range would have been $HP - KQ = KE - KQ = -EQ = -PD$; and its longitude range, $H'P - K'Q = H'P - H'E = -EP = -DQ$.

As will be observed, any line, as PQ, is the hypotenuse of a right triangle whose legs are the latitude range and the longitude range of the line (PE and EQ, in the case of PQ). The latitude range indicates how far the end of the line is north or south of the beginning; and the longitude range, how far the end of the line is east or west of the beginning, or of the meridian passing through the beginning. The latitude range is positive, and is called a **north latitude-range**, or a **northing**, whenever the line bears north; it is negative, and called a **south latitude-range**, or a **southing**, whenever the line bears south. The longitude range is positive, and is called an **east longitude-range**, or an **easting**, when the line bears east; it is negative, and called a **west longitude-range**, or a **westing**, when the line bears west. Thus, the latitude and the longitude range of PQ are, respectively, $+PD$ and $+DQ$; those of QP are $-QE$ and $-EP$. Likewise, the latitude range of P_1Q_1 is $-P_1D_1$, because the end of the line is south of the beginning. The longitude range is $+D_1Q_1$, because the end of the line is east of the beginning. These values may be verified by observing that the latitudes of P_1 and Q_1 are, respectively, $-H_1 P_1$ and $-H_1 D_1$, whose algebraic difference is $-H_1 D_1 - (-H_1 P_1) = -H_1 D_1 + H_1 P_1 = -P_1 D_1$; and that the longitudes of P_1 and Q_1 are, respectively, $+H_1' P_1$ and $+K_1' Q_1$, whose difference is equal to $D_1 Q_1$.

NOTE.—In older books, and in some modern books, the term *latitude* is applied to what has here been called *latitude range;* while what is here called *longitude range* is in them called *departure*. The expressions *latitude difference* and *longitude difference* are sometimes used instead of *latitude range* and *longitude range*, respectively.

7. General Formulas.—Let AB, Fig. 2, be a course whose length is l, and whose bearing is G. In the right triangle AMB, in which AM is the direction of the meridian through A, the latitude range AM and longitude range MB are denoted by t and g, respectively. From trigonometry, we have,

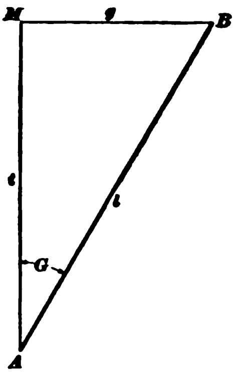


FIG. 2

$$t = l \cos G \qquad (1)$$
$$g = l \sin G \qquad (2)$$

These formulas serve to compute the ranges when the length and bearing of the course have been measured. Special care should be taken to give t and g their proper signs, t being positive when G is north (that is, either northeast or northwest) and g being positive when G is east (that is, either northeast or southeast). When G is south (that is, either southeast or southwest), t is negative; and when G is west (that is, either northwest or southwest), g is negative.

If t and g are given, G is found by the formula

$$\tan G = \frac{g}{t} \qquad (3)$$

and l by either of the formulas following:

$$l = \frac{g}{\sin G} \qquad (4)$$

$$l = \sqrt{t^2 + g^2} \qquad (5)$$

In applying formulas **3** and **5**, the signs of t and g should be disregarded; that is, both t and g should be treated as positive.

EXAMPLE 1.—The length of a course is 896.7 feet, and its bearing N 39° 15′ W; what are the ranges of the course?

SOLUTION.—Here l = 896.7 ft., and G = 39° 15′. Since the bearing is northwest, its latitude range is positive, and its longitude range, negative. We have, then, applying formulas 1 and 2,

$$t = 896.7 \cos 39° 15' = 694.4 \text{ ft. Ans.}$$
$$g = -896.7 \sin 39° 15' = -567.4 \text{ ft. Ans.}$$

In calculations of this kind, logarithmic functions should be employed in preference to natural functions. The work is conveniently arranged by writing first the logarithm of the length of the course, then writing the logarithmic sine of the bearing over it, and the logarithmic cosine of the bearing under it, and adding upwards in one case and downwards in the other, as follows:

$$\begin{aligned} \log g &= 2.75385;\ g = 567.4 \text{ ft. Ans.} \\ \log \sin 39° 15' &= \bar{1}.80120 \\ \log 896.7 &= 2.95265 \\ \log \cos 39° 15' &= \bar{1}.88896 \\ \log t &= 2.84161;\ t = 694.4 \text{ ft. Ans.} \end{aligned}$$

EXAMPLE 2.—The latitude range and the longitude range of a course are, respectively, −13.71 and −9.38 chains; find the bearing and length of the course.

SOLUTION.—Since both ranges are negative, the course bears southwest. Neglecting signs, we have, by formulas 3 and 4,

$$\tan G = \frac{9.38}{13.71}, \text{ whence } G = 34° 23'. \text{ Ans.}$$

$$l = \frac{9.38}{\sin 34° 23'} = 16.61 \text{ ch. Ans.}$$

The logarithmic work is conveniently arranged as follows:

$$\begin{aligned} \log l &= 1.22036;\ l = 16.61. \text{ Ans.} \\ \log \sin 34° 23' &= \bar{1}.75184 \\ \log 9.38 &= 0.97220 \\ \log 13.71 &= 1.13704 \\ \log \tan G &= \bar{1}.83516;\ G = 34° 23'. \text{ Ans.} \end{aligned}$$

The logarithm of 9.38 is written first, and under it that of 13.71. The difference between these two gives the logarithmic tangent of G, from which G is determined. At the same time that G is taken out of the table, its logarithmic sine is taken and written above that of 9.38; it is then subtracted from the latter logarithm to obtain the logarithm of l. For the purpose of finding l, it is better to take G to the nearest minute, as this does not involve any additional work; but, as bearings are taken to the nearest quarter of a degree, the bearing of the line would be stated as S 34° 30′ W.

EXAMPLES FOR PRACTICE

NOTE.—Bearings are given to the nearest quarter of a degree.

1. The length of a course is 19.83 chains, and its bearing N 18° 15′ E; find the ranges of the course. Ans. $\begin{cases} t = 18.83 \text{ ch.} \\ g = 6.21 \text{ ch.} \end{cases}$

2. A line 649 feet long bears S 5° 45′ E; find its ranges. Ans. $\begin{cases} t = -645.7 \text{ ft.} \\ g = 65.0 \text{ ft.} \end{cases}$

3. Find the ranges of a course 3.33 chains long and bearing N 73° 30′ W. Ans. $\begin{cases} t = .95 \text{ ch.} \\ g = -3.19 \text{ ch.} \end{cases}$

4. The length of a course is 197 feet and its bearing is S 53° 45′ W; find its ranges. Ans. $\begin{cases} t = -116.5 \text{ ft.} \\ g = -158.9 \text{ ft.} \end{cases}$

5. The latitude range and longitude range of a course are, respectively, −3.17 and −4.25 chains; find the length l and the bearing G of the course. Ans. $\begin{cases} l = 5.30 \text{ ch.} \\ G = \text{S } 53° \ 15' \text{ W} \end{cases}$

BALANCING THE COMPASS SURVEY OF A CLOSED FIELD

ERROR OF CLOSURE

8. Definitions.—Let $ABCDE$, Fig. 3, be a closed field, OT a reference meridian, and OG a reference parallel of latitude. It is obvious that, if all the courses and bearings were determined with absolute exactness, and a plat of the field made, the end (say A) of the last line would coincide with the beginning of the first. Under such conditions, the survey is said to **close**. As, however, no measurements are free from error, and as the compass is read only to the nearest quarter degree, a survey never closes. When the notes are platted the end of the last line does not coincide with the beginning of the first. This condition is shown in Fig. 4, where AB is the first line as platted from the notes, and EA_1 the last.

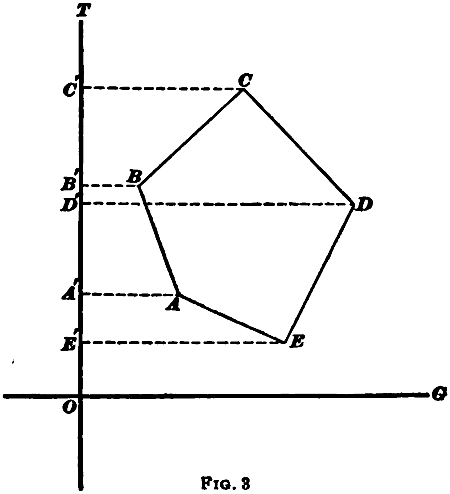


FIG. 3

The distance A_1A from the end of the last line to the beginning of the first is called the **total error of closure**. The ratio of the total error of closure to the sum of the lengths of all the courses expresses the error per unit of length, and will here be called the **relative error of closure**, or the **rate of error**. For example, if the sum of the lengths of the sides of a survey is 3,575 feet, and the total error of closure A_1A, Fig. 4, is 6.5 feet, the relative error is

$$\frac{6.5}{3{,}575} = \frac{1}{550}$$

or 1 in 550.

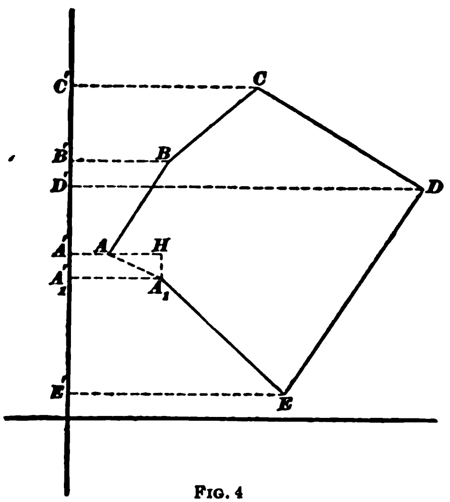


FIG. 4

In ordinary compass surveying, a relative error of 1 in 500, or .002, should not usually be exceeded.

9. Conditions of Closure.—Referring again to Fig. 3, it will be observed that the sum $E'C'$ of the northings, or north latitude-ranges, $E'A'$, $A'B'$, and $B' C'$ is numerically equal to the sum $C' E'$ of the southings, or south latitude-ranges, $C' D'$ and $D' E'$. It may be shown in like manner that the sum of the eastings is numerically equal to the sum of the westings. This is otherwise evident, since, in moving around the field from one point back to the same point, one must move as far in one direction as in the opposite.

In order that a survey may close, it is, therefore, necessary and sufficient that the sum of the northings should be equal to the sum of the southings, and that the sum of the eastings should be equal to the sum of the westings; or, what is the same thing, that the arithmetical difference between the sum of the northings and the sum of the southings should be zero, and the arithmetical difference between the sum of the eastings and the sum of the westings should be zero.

If the south latitude-ranges are considered negative, the arithmetical difference between the sum of the northings and the sum of the southings is simply the algebraic sum of all the latitude ranges. Likewise, if the west longitude-ranges are considered negative, the arithmetical difference between the sum of the eastings and the sum of the westings is the same as the algebraic sum of all the longitude ranges. We may, therefore, say that, in order that a survey may close, it is necessary and sufficient that the algebraic sum of the longitude ranges and that of the latitude ranges should both be zero.

10. Let S_l be the algebraic sum of all the latitude ranges and S_g the algebraic sum of all the longitude ranges. The conditions of closure are expressed algebraically by the following equations.

$$S_l = 0, S_g = 0$$

11. Formula for Relative Error of Closure.—As stated in Art. 8, the conditions $S_l = 0$, $S_g = 0$ never obtain in practice. In Fig. 4, for example, the sum of the north latitude-ranges is $E' A_1' + A' C' = E' C' - A_1' A'$, and that of the south latitude-ranges, $-(C' D' + D' E') = -E' C'$. In this case,

$$S_l = E' C' - A_1' A' + (-E' C') = -A_1' A'.$$

It will be observed that S_t is numerically equal to the latitude range A_1H of the closing line A_1A, but has the opposite sign. Likewise, S_g is numerically equal to the longitude range HA of A_1A, but has the opposite sign; that is, $S_g = -HA$. It should be borne in mind that HA, being a westing, is negative, and that, therefore, $-HA$, or S_g, is positive. This is evident from the figure, which shows that the sum of the eastings is greater than that of the westings.

It is thus seen that the closing line is a course whose latitude range is equal to the algebraic sum of all the latitude ranges with its sign changed, and whose longitude range is equal to the algebraic sum of all the longitude ranges with its sign changed.

The right triangle A_1HA gives,

$$A_1A = \sqrt{A_1H^2 + HA^2} = \sqrt{S_t^2 + S_g^2}$$

The signs of S_t and S_g may be disregarded in this equation, since the square of a negative quantity is always positive. If the sum of the lengths of all the courses is denoted by S_l, and the relative error of closure by e, we have,

$$e = \frac{A_1A}{S_l} = \frac{\sqrt{S_t^2 + S_g^2}}{S_l} = \sqrt{\frac{S_t^2 + S_g^2}{S_l^2}}$$

or, finally

$$e = \sqrt{\left(\frac{S_t}{S_l}\right)^2 + \left(\frac{S_g}{S_l}\right)^2}$$

The application of this formula, and the advantage of having it in this form, will be illustrated presently.

FIRST METHOD OF BALANCING A COMPASS SURVEY

12. Definition.—For the purposes of platting the notes and calculating the contents of a closed field, it is necessary to adjust the notes so that the plat shall close; that is, so that the algebraic sum of the latitude ranges and the algebraic sum of the longitude ranges shall both be zero. The process of effecting this adjustment consists in applying certain corrections to the ranges, and is called **balancing the survey**, although it might more properly be called *balancing the notes*.

13. Correcting the Ranges.—When all the lines have been measured under similar circumstances, and there is no reason to think that certain lines are more likely to be in error than others, the total error in the sum of the ranges of either kind (latitude or longitude) is distributed among the ranges of the various courses by means of the following principle:

The algebraic sum of all the ranges of either kind is to the sum of all the courses as the correction to be applied to the corresponding range of any course is to the length of that course.

Let S_t, S_g, S_l be, as before, the sum of the latitude ranges, that of the longitude ranges, and that of the courses, respectively; l the length of a course, t its latitude range, g its longitude range, and c_t and c_g the corrections to be applied to t and g, respectively. The principle stated above may be expressed algebraically as follows:

$$\frac{S_t}{S_l} = \frac{c_t}{l}; \quad \frac{S_g}{S_l} = \frac{c_g}{l}$$

From these equations are derived the following convenient working formulas:

$$c_t = l \times \frac{S_t}{S_l} \qquad (1)$$

$$c_g = l \times \frac{S_g}{S_l} \qquad (2)$$

In applying these formulas, the value of $\frac{S_t}{S_l}$ and that of $\frac{S_g}{S_l}$ are first calculated; then the length of each course is multiplied by the value of $\frac{S_t}{S_l}$ in order to obtain the corrections for the latitude ranges, and by the value of $\frac{S_g}{S_l}$, in order to obtain the corrections for the longitude ranges. The ratios $\frac{S_g}{S_l}$ and $\frac{S_t}{S_l}$ serve also to compute, by formula of Art. **11**, the relative error of closure, and to determine the corrections to be applied to the lengths of the courses, as will be explained presently.

In all cases, the corrections c_t and c_g are to be subtracted algebraically from t and g, respectively, it being understood that S_t and S_g have their proper signs, and that all the operations are performed algebraically.*

14. Correcting the Lengths of the Courses.—The corrections applied to the ranges will generally alter the lengths of the courses. Since any course is the hypotenuse of a right triangle whose legs are the ranges of the course, its corrected length may be obtained by taking the square root of the sum of the squares of the corrected ranges.

If t' and g' denote, respectively, the corrected latitude range and longitude range of a course, we have, for the corrected length l' (see formula **5**, Art. **7**),

$$l' = \sqrt{t'^2 + g'^2}$$

The corrected length of a course may, however, be more readily computed by means of the following formula:†

---

*The reason for this will be readily seen by considering the correction c_t to be applied to the latitude ranges. If the arithmetical sum of the northings is greater than that of the southings, the northings must be decreased and the southings increased, arithmetically; but, as the southings are negative latitude ranges, increasing them arithmetically is decreasing them algebraically. Thus, if a southing is changed from -12 to -12.06, it is increased arithmetically by .06, and decreased algebraically by .06, since $-12.06 = -12 - .06$. In the case under consideration, S_t is positive, and so is $l \times \frac{S_t}{S_l} = c_t$, since l and S_l are always positive. Therefore, subtracting c_t from a northing will make the northing smaller arithmetically, and subtracting it from a southing will make the southing smaller algebraically, but greater arithmetically.

If the arithmetical sum of the southings is greater than that of the northings, the southings must be decreased arithmetically and the northings increased. In this case, S_t is negative, and so is $l \times \frac{S_t}{S_l} = c_t$. Now, subtracting the negative quantity c_t from a northing, which is positive, makes the northing greater; and subtracting it from a southing, which is negative, makes the southing algebraically greater, but arithmetically smaller. Thus, if the value of c_t is -1.5, and the latitude range to which the correction is to be applied is -123.5, the result is $-123.5 - (-1.5) = -123.5 + 1.5 = -122$, which is algebraically greater, but arithmetically less, than -123.5.

†This formula was first given and derived by Antonio Llano, in *Engineering News* for November 23, 1899.

$$c_l = t \times \frac{S_t}{S_l} + g \times \frac{S_g}{S_l}$$

in which c_l = correction to be subtracted algebraically from measured length of course, as recorded in field notes;

t = latitude range of course;

g = longitude range of course;

S_t, S_g, S_l, the same as in previous articles.

The student should not forget that t, S_t, g, and S_g must be taken with their proper signs.

EXAMPLE.—The table given below contains the bearings and lengths of the courses of a compass survey. The lengths, as measured, and the ranges, as calculated from the measured lengths and bearings, are printed in large type horizontally opposite the letters denoting the corresponding corners. Above these numbers are written in parentheses, and in smaller type, the corrected values of the lengths and ranges. The problem is to determine these corrected values and the relative error of closure.

NOTE.—In practice, the corrected values are written in red ink above the original values, the latter being crossed out.

Courses	Bearings	Lengths Chains	Latitude Ranges		Longitude Ranges	
			N, +	S, −	E, +	W, −
		(10.62)	(6.57)		(8.34)	
A B	N 52° 00′ E	10.63	6.54		8.38	
		(4.08)		(3.55)	(2.01)	
B C	S 29° 45′ E	4.10		3.56	2.03	
		(7.69)		(6.51)		(4.08)
C D	S 31° 45′ W	7.69		6.54		4.05
		(7.16)	(3.49)			(6.27)
D A	N 61° 00′ W	7.13	3.46			6.24
		29.55	10.00	10.10	10.41	10.29
		(= S_l)	−10.10		−10.29	
			−.10		+.12	
			(= S_t)		(= S_g)	

SOLUTION.—(*a*) *To determine the corrected latitude ranges.* Here the sum of the courses, or S_l, is 29.55. The sum of the northings is 10.00, and that of the southings is −10.10. For the algebraic sum of the latitude ranges we have, therefore,

$$S_t = 10.00 + (-10.10) = -.10$$

To apply formula **1**, Art. **13**, we have

$$\frac{S_t}{S_l} = -\frac{.10}{29.55} = -\frac{10}{2{,}955}$$

As the corrections are very small, and the work need not be absolutely accurate, we may write 3,000 instead of 2,955. This gives,

$$\frac{S_t}{S_l} = -\frac{10}{3{,}000} = -.003$$

Applying formula **1**, Art. **13**,

$$c_t \text{ for } AB = 10.63 \times -.003 = -.03$$

It will be observed that $\frac{1}{1000}$ of 10.63 is .01063, and therefore 10.63 × .003 = .01063 × 3. As the calculation is not to be carried beyond links, or hundredths, it is sufficient to multiply .0106, or say .011, by 3. A little attention to these details of calculation will shorten the work considerably. Likewise,

$$c_t \text{ for } BC = 4.10 \times -.003 = -.01$$
$$c_t \text{ for } CD = 7.69 \times -.003 = -.02 \quad \text{(See below)}$$
$$c_t \text{ for } DA = 7.13 \times -.003 = -.02$$

The sum of these corrections should be equal to S_t, or −.10, but is only −.08. We must, therefore, apply a correction of .01 to two of the ranges. As the lengths of the third and fourth courses are nearly equal, we shall add arithmetically 1 link to the correction for CD and that for DA, writing

$$c_t \text{ for } CD = -.03$$

and

$$c_t \text{ for } DA = -.03$$

As stated above, this kind of work is comparatively rough, and therefore does not require very exact calculations.

Subtracting algebraically the corrections just found from the corresponding latitude ranges, the corrected ranges are found to be,

$$\text{for } AB, \quad 6.54 - (-.03) = 6.54 + .03 = 6.57$$
$$\text{for } BC, \quad -3.56 - (-.01) = -3.56 + .01 = -3.55$$
$$\text{for } CD, \quad -6.54 - (-.03) = -6.54 + .03 = -6.51$$
$$\text{for } DA, \quad 3.46 - (-.03) = 3.46 + .03 = 3.49$$

These are the corrected values written in parentheses above the original values. In practice, the correction can always be applied mentally.

(*b*) *To determine the corrected longitude ranges.* Here the sum of the eastings is 10.41, and that of the westings, −10.29. Therefore,

$$S_g = 10.41 - 10.29 = .12$$

and

$$\frac{S_g}{S_l} = \frac{.12}{29.55} = \frac{12}{3{,}000} \text{ (nearly)} = .004$$

Applying formula **2**, Art. **13**,

$$c_g \text{ for } AB = 10.63 \times .004 = .04$$
$$c_g \text{ for } BC = 4.10 \times .004 = .02$$
$$c_g \text{ for } CD = 7.69 \times .004 = .03$$
$$c_g \text{ for } DA = 7.13 \times .004 = \underline{.03}$$
$$.12$$

The corrected longitude ranges are,

$$\text{for } AB, \quad 8.38 - .04 = 8.34$$
$$\text{for } BC, \quad 2.03 - .02 = 2.01$$
$$\text{for } CD, \quad -4.05 - .03 = -4.08$$
$$\text{for } DA, \quad -6.24 - .03 = -6.27$$

(*c*) *To determine the corrected lengths of the courses.* To apply formula of Art. **14**, we have,

$$\frac{S_l}{S_l} = -.003, \quad \frac{S_g}{S_l} = .004$$

Then, remembering that all the quantities in the formula are algebraic, and that southings and westings are negative,

$$c_l \text{ for } AB = 6.54 \times -.003 + 8.38 \times .004 = .01$$
$$c_l \text{ for } BC = -3.56 \times -.003 + 2.03 \times .004 = .02$$
$$c_l \text{ for } CD = -6.54 \times -.003 - 4.05 \times .004 = .00, \text{ practically}$$
$$c_l \text{ for } DA = 3.46 \times -.003 - 6.24 \times .004 = -.03$$

Subtracting algebraically these corrections from the lengths of the courses, we find,

$$\text{corrected length of } AB = 10.63 - .01 = 10.62$$
$$\text{corrected length of } BC = 4.10 - .02 = 4.08$$
$$\text{corrected length of } CD = 7.69 - .00 = 7.69$$
$$\text{corrected length of } DA = 7.13 - (-.03) = 7.16$$

(*d*) *To determine the relative error of closure.* The formula of Art. **11** gives

$$e = \sqrt{.003^2 + .004^2} = \sqrt{.000025} = .005$$

This error is 5 in 1,000, or 1 in 200, and is greater than would be allowed in any but exceedingly rough work.

EXAMPLE FOR PRACTICE

The table given below contains the lengths and bearings of the sides of a five-sided field, and also the values of the ranges as computed from the original notes, the corrected values of the ranges, and those of the lengths. Verify: (*a*) the original values of the ranges; (*b*) the corrected values of the ranges; (*c*) the corrected values of the lengths. (*d*) Calculate the relative error of closure. Ans. (*d*) e = .002

Course	Bearing	Length Chains	l	g	Corrected l	Corrected g	Corrected Length
AB	N 41° 30′ E	10.47	+ 7.84	+ 6.94	+ 7.85	+ 6.96	10.49
BC	N 75° 15′ E	11.86	+ 3.02	+11.47	+ 3.03	+11.49	11.89
CD	S 20° 45′ E	11.64	−10.88	+ 4.12	−10.87	+ 4.14	11.64
DE	S 57° 45′ W	15.78	− 8.42	−13.35	− 8.40	−13.32	15.74
EA	N 48° 00′ W	12.52	+ 8.38	− 9.30	+ 8.39	− 9.27	12.51

SECOND METHOD OF BALANCING A COMPASS SURVEY

15. Weighting the Courses.—The rule given in Art. **13** should not be followed when there is reason to believe that some courses are more likely to be in error than others, as when some of the lines have been measured over rough and broken country, while the others have been measured over smooth and open ground. In such a case the greater part of the error will probably occur in the lines measured under unfavorable conditions, and consequently a greater correction should be applied to those lines. The best method of balancing a survey of this kind is to state, in numbers, the probability of error in each course, as compared with the probability of error in any one of the courses, whose probability of error is taken as unity. These numbers are called **weights** of the courses. Usually, the line that has been run under the most favorable conditions is assumed to have a weight of 1. Then, if another line of the survey has been run under such conditions that it is probable that three times as much error has been made as in the same distance on the line whose weight is 1, the second line will have a weight of 3; and, similarly for the other courses. This assigning of weights to the courses is termed **weighting the courses.**

It should be done during the field work or immediately after it is finished, while an accurate judgment can be formed concerning the difficulties in the measurement of each line. The weights should be entered in the notes.

16. Balancing a Survey by Weighted Courses. When the courses of a survey have been weighted, the survey is balanced by formulas similar to those given in Arts. **13** and **14**; but, whenever the length of a course occurs in those formulas, it should be replaced by the product of that length by the weight of the course.

Let l and w be, respectively, the length and weight of a course; S_{lw} the sum obtained by multiplying the length of each course by its weight and adding the results. The other letters having the same meanings as in previous articles, we have:

$$c_t = l\,w \times \frac{S_t}{S_{lw}} \qquad (1)$$

$$c_g = l\,w \times \frac{S_g}{S_{lw}} \qquad (2)$$

The correction c_l to be applied to the length of the course is given by the formula

$$c_l = \left(t \times \frac{S_t}{S_{lw}} + g \times \frac{S_g}{S_{lw}}\right) w \qquad (3)$$

These corrections are to be subtracted algebraically from the corresponding quantities.

CALCULATING AREA OF A COMPASS SURVEY

PRELIMINARY OPERATIONS

17. To Determine the Most Westerly Corner of a Field.—It is sometimes convenient to take the reference meridian through the most westerly corner of the field surveyed. This corner is very easily determined from the corrected longitude ranges as follows:

Let $ABCDEFGA$, Fig. 5, be the field. Imagine a reference meridian SN passed through any of the corners, as B. The longitude of the next corner C is $C_1 C$, which is equal to the longitude range of BC. The longitude of D is $D_1 D$, which is equal to $D_1 C'$, or $C_1 C$, plus $C' D$; that is, the longitude of D is equal to the longitude of C plus the longitude range

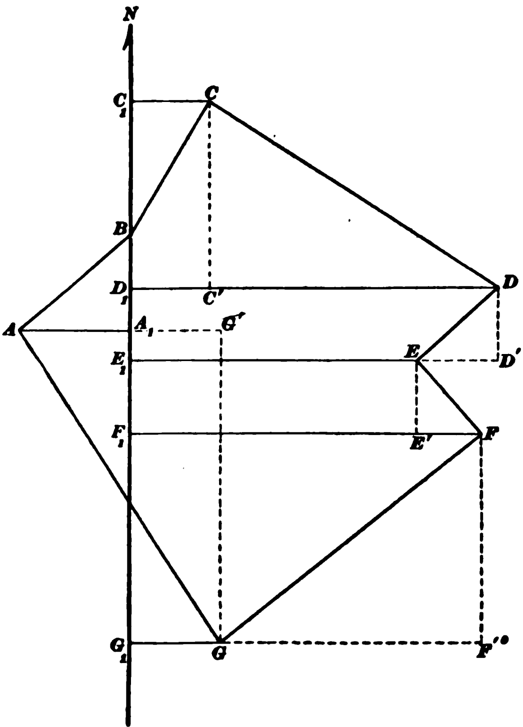


FIG. 5

of CD. The longitude of E is $E_1 E$, which is equal to $E_1 D'$, or $D_1 D$, minus $D' E$, or to $D_1 D + (-D' E)$; that is, the longitude of E is equal to the longitude of D plus the longitude range of DE.

In general, the longitude of the end of any course is equal to the algebraic sum of the longitude of the beginning of the course and the longitude range of the course. The longitude of A, for instance, is $-A_1A$; that of G is G_1G, and we have $-A_1A = -(G'A - A_1G') = -(G'A - G_1G) = G_1G - G'A = G_1G + (-G'A)$.

To determine the longitudes of all the corners, the longitude range of the first course BC from the auxiliary reference meridian is written down as the longitude of the second corner C; to this is added, algebraically, the longitude range of the next course CD, to obtain the longitude of the third corner D. To the result is added the longitude range of the next course DE to obtain the longitude of the fourth corner E; and so on.

Having determined the longitudes of all the corners, the corner whose west longitude is arithmetically greatest is the most westerly. Should it happen that the longitudes of all the corners are positive, this would indicate that the corner through which the auxiliary reference meridian was passed is the most westerly. In the actual application of this rule, it is not necessary to draw the auxiliary reference meridian SN, Fig. 5, nor even to make a diagram at all. Any course may be called the "first course," and the longitudes of the corners may be determined as above, by a simple process of successive additions. As a check on the work, the longitude of the last corner A, Fig. 5, should be arithmetically equal to the longitude range of the last course, but have the opposite sign. Thus, in Fig. 5, the longitude of A with reference to SN is $-A_1A$, while the longitude range of AB is $+A_1A$.

Example.—The longitude ranges of a six-sided field being as given below, it is required to determine the most westerly corner of the field.

Course	Longitude Range
AB	+37.19
BC	−43.16
CD	−18.94
DE	+29.18
EF	−24.73
FA	+20.46

SOLUTION.—Starting with AB as the first course, the operations are arranged as follows (Long. stands for *longitude*, and R. for *range*):

$$
\begin{array}{rrl}
 & +37.19 & = \text{Long. of } B \\
\text{Long. R. of } BC = & -43.16 & \\
 & -\ \ 5.97 & = \text{Long. of } C \\
\text{Long. R. of } CD = & -18.94 & \\
 & -24.91 & = \text{Long. of } D \\
\text{Long. R. of } DE = & +29.18 & \\
 & +\ \ 4.27 & = \text{Long. of } E \\
\text{Long. R. of } EF = & -24.73 & \\
 & -20.46 & = \text{Long. of } F \\
 & & = -(\text{Long. R. of } FA)
\end{array}
$$

As the longitude of D is the (arithmetically) greatest westing, D is the most westerly corner. Ans.

In the majority of cases, the most westerly corner can be at once ascertained from an inspection of the longitude ranges. Thus, in the present case, it is seen that since the easting of AB is about 37, and the westing of BC about 43, the corner C is about 6 chains (= 43 − 37) west of A. As D is about 19 chains west of C, it is about 25 chains west of A. From D, we go about 29 + 20 (ranges of DE and FA) or 49 chains toward the east, and only go back about 25 chains (range of EF) toward the west, so that we still remain east of D. This corner is, therefore, the most westerly.

EXAMPLE FOR PRACTICE

Determine the most westerly corner of the eight-sided tract whose longitude ranges are as given below:

Course	Long. R.	Course	Long. R.
AB	−11.34	EF	+16.64
BC	+12.03	FG	−19.50
CD	− 9.76	GH	− 9.35
DE	+ 8.31	HA	+12.97

Ans. The corner H.

18. Longitude and Double Longitude of a Course. The **longitude** *of a line* is the longitude of its middle point. In Fig. 6, where SN is the reference meridian, and K, L, P, Q are the middle points of AB, BC, DE, EA, respectively, the longitudes of these courses are K_1K, L_1L, P_1P, Q_1Q.

The **double longitude** of a line is, as the name implies, twice the longitude of the line.

19. Again referring to Fig. 6, it will be noticed that, since BB_1C_1C is a trapezoid, and L_1L bisects BC and is parallel to C_1C and B_1B, we have

$$L_1L = \frac{B_1B + C_1C}{2}.$$

and, therefore, $2\,L_1L = B_1B + C_1C$

That is, the double longitude of BC is equal to the sum of the longitudes of its extremities. The same principle evidently applies to all other courses.

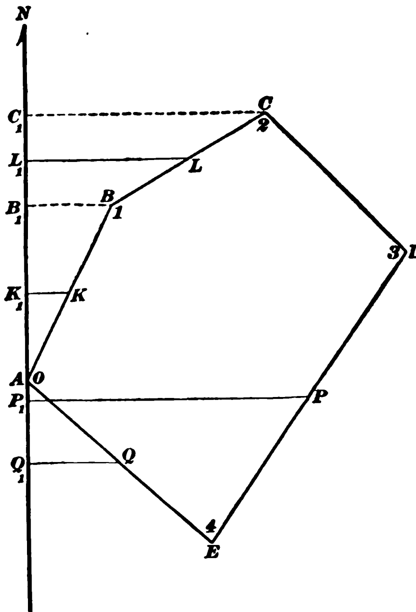


FIG. 6

Let the longitude ranges of the courses AB, BC, CD, etc. be denoted, respectively, by g_1, g_2, g_3, etc., their double longitudes by M_1, M_2, M_3, etc., and the longitudes of the corners B, C, D, etc. by $m_1\,m_2\,m_3$, etc. The numbers on the corners will serve to make this notation clear and easily remembered.

Taking the course DE, the longitudes of whose ends are m_3 and m_4, and whose longitude range is g_4, we have (see Art. **17**)

$$m_4 = m_3 + g_4$$

Likewise, $\quad m_3 = m_2 + g_3$

Adding these two equations,

$$m_3 + m_4 = m_2 + m_3 + g_3 + g_4$$

But $m_3 + m_4$ is the double longitude M_4 of DE, and m_2

$+ m_3$ is the double longitude M_3 of CD. Substituting these values in the preceding equation, the result is

$$M_4 = M_3 + g_3 + g_4$$

As DE is any course, this formula may be stated in words thus:

The double longitude of any course is equal to the double longitude of the preceding course, plus the longitude range of the preceding course, plus the longitude range of the course considered.

This principle affords a very simple method of computing the double longitudes of the courses. The double longitude of the first course AB is obviously equal to the longitude range $B_1 B$, or g_1, of that course. The other double longitudes can be calculated by arranging the work in the general manner indicated below (D. Long. stands for *double longitude*):

$$\begin{aligned}
g_1 &= \text{D. Long. of } AB \\
+g_1 & \\
+g_2 & \\
\hline
M_2 &= \text{D. Long. of } BC \\
+g_2 & \\
+g_3 & \\
\hline
M_3 &= \text{D. Long. of } CD \\
+g_3 & \\
+g_4 & \\
\hline
M_4 &= \text{D. Long. of } DE \\
+g_4 & \\
+g_5 & \\
\hline
M_5 &= \text{D. Long. of } EA \; (= -g_5)
\end{aligned}$$

The additions are, as usual, performed algebraically. As a check, the double longitude of the last course must be equal to the longitude range of that course.*

*The reason for this is that, to obtain M_5, all the longitude ranges are added twice, except the last; that is,

$$M_5 = 2g_1 + 2g_2 + 2g_3 + 2g_4 + g_5$$

By adding and subtracting g_5, this may be written

$$M_5 = 2(g_1 + g_2 + g_3 + g_4 + g_5) - g_5$$

As the quantity in parentheses is the algebraic sum of the adjusted longitude ranges, it is equal to zero (Art. 12), and therefore M_5 is equal to $-g_5$.

EXAMPLE.—Find the double longitudes of the courses of a five-sided field from the following notes, the reference meridian passing through the most westerly corner A:

Courses	Longitude Ranges Feet	
	E, +	W, −
AB	9.73	
BC	8.41	
CD		7.18
DE		.79
EA		10.17

SOLUTION.—Denoting the longitude ranges of AB, BC, etc. by g_1, g_2, etc., and arranging the work as indicated in this article, we have:

$$
\begin{array}{lrl}
g_1 = & 9.73 & = \text{D. Long. of } AB \\
g_1 = & 9.73 & \\
g_2 = & 8.41 & \\
\hline
 & 27.87 & = \text{D. Long. of } BC \\
g_2 = & 8.41 & \\
g_3 = & -\ 7.18 & \\
\hline
 & 29.10 & = \text{D. Long. of } CD \\
g_3 = & -\ 7.18 & \\
g_4 = & -\ .79 & \\
\hline
 & 21.13 & = \text{D. Long. of } DE \\
g_4 = & -\ .79 & \\
g_5 = & -\ 10.17 & \\
\hline
 & 10.17 & = \text{D. Long. of } EA \\
 & & = -(-10.17) = -g_5. \quad \text{Ans.}
\end{array}
$$

In adding several numbers having different signs, those having one sign are added first, then those having the opposite sign, and then the two results are combined by algebraic addition.

EXAMPLE FOR PRACTICE

Verify the values of the double longitudes given in the following table, the reference meridian passing through *A*:

Courses	Longitude Ranges Feet	Double Longitudes Feet
AB	+ 724.5	+ 724.5
BC	−1,002.0	+ 447.0
CD	−1,276.0	−1,831.0
DE	− 372.5	−3,479.5
EF	+4,431.0	+ 579.0
FA	−2,505.0	+2,505.0

COMPUTING THE AREA

20. General Rule.—Let *SN*, Fig. 7, be a reference meridian passing through the most westerly corner of the field *ABCDE*. The lines BB_1, CC_1, KK_1, etc. are perpendicular to *SN*. The courses are supposed to have been run in the order of the letters; that is, from *A* to *B*, then to *C*, and so on. Their latitude ranges are therefore equal, respectively, to $-AB_1$, B_1C_1, C_1D_1, $-D_1E_1$, and $-E_1A$. The middle point of *DE* being *K*, the length of KK_1 is the longitude of *DE*.

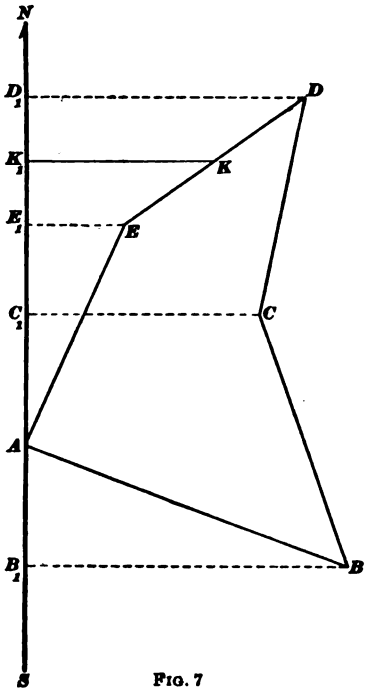


FIG. 7

The area of *ABCDEA* is obviously equal to the sum of the areas of the trapezoids BCC_1B_1 and CDD_1C_1 minus the sum of the triangles ABB_1 and EAE_1, and the trapezoid DEE_1D_1. The area of the latter trapezoid is equal to $K_1K \times D_1E_1$; or, disregarding signs, equal to the longitude of DE multiplied by the latitude range of the same course. It is easy to see that the same rule applies to every other trapezoid, such as BCC_1B_1, and to the triangles ABB_1 and EAE_1. The area of ABB_1, for instance, is equal to the altitude AB_1 (= latitude range) multiplied by $\frac{1}{2}B_1B$ (= longitude of AB).

If the area of the field is denoted by S, and the double longitudes of AB, BC, etc. by M_1, M_2, etc., the longitudes of these courses will be $\frac{M_1}{2}, \frac{M_2}{2}$, etc., and we shall have,

$$S = BCC_1B_1 + CDD_1C_1 - ABB_1 - EAE_1 - DEE_1D_1$$
$$= B_1C_1 \times \frac{M_2}{2} + C_1D_1 \times \frac{M_3}{2} - AB_1 \times \frac{M_1}{2} - E_1A \times \frac{M_5}{2} - D_1E_1 \times \frac{M_4}{2}$$
$$= \frac{1}{2}(B_1C_1 \times M_2 + C_1D_1 \times M_3 - AB_1 \times M_1 - E_1A \times M_5 - D_1E \times M_4)$$

As B_1C_1, C_1D_1, $-AB_1$, $-E_1A$, etc. are the latitude ranges of the courses, the quantity in parentheses is the algebraic sum of the products obtained by multiplying the latitude range of each course by the double longitude of that course; hence the following general method for determining the area of a field:

Rule.--*Multiply the latitude range of each course by the double longitude of the course, giving to the product its proper sign, according to the signs of the factors. Add these products algebraically, and divide the sum by 2.*

The products just referred to are called **double areas**, as each represents twice the area of a trapezoid or a triangle. Thus, the product $AB_1 \times M_1$ represents twice the area of the triangle ABB_1.

The sign of the final result may be negative; but, as it simply indicates the relative positions of the positive and negative areas, it may be disregarded, and the arithmetical difference taken between the sum of the positive and that of the negative products, subtracting the smaller from the larger.

21. Arrangement of the Work.—The work is conveniently arranged as in the table given below, which refers to a six-sided field, the reference meridian passing through the most westerly corner *A*.

Courses	Longitude Ranges Chains	Double Longitudes	Latitude Ranges Chains	Double Areas	
				+	−
AB	+27.4	+ 27.4	+27.2	745.28	
BC	+63.2	+118.0	−23.8		2,808.40
CD	+13.1	+194.3	−37.5		7,286.25
DE	−33.1	+174.3	−33.3		5,804.19
EF	−50.1	+ 91.1	+24.1	2,195.51	
FA	−20.5	+ 20.5	+43.3	887.65	

3,828.44 15,898.84
3,828.44
2) 12,070.40
S = 6,035.20 sq. ch.
= 603.52 A.

The double longitudes are calculated as explained in Art. **19.** Each double longitude is multiplied by the latitude range on the right of it and the product placed in the double-area column under the proper sign.

22. Position of the Reference Meridian.—In calculating areas, it is not necessary to take the meridian through the most westerly corner. The double longitudes may be computed from any corner, provided that they are given their proper signs, and that the signs of the double areas are given their algebraic significance according to the signs of the double longitudes and of the latitudes. The advantage of passing the meridian through the most westerly corner is that the double longitudes are all positive.

EXAMPLE FOR PRACTICE

Compute the area of the field to which the following notes refer:

Ans. 216.2 A.

Courses	Latitude Ranges Feet	Longitude Ranges Feet
AB	−2,343.0	+ 235
BC	−3,416.0	+2,342
CD	+ 876.5	−1,964
DE	− 674.5	−2,878
EA	+5,557.0	+2,265

PLATTING BY LATITUDES AND LONGITUDES

23. Latitudes of Corners.—If a reference parallel of latitude is passed through any corner of a field, the latitudes of all the corners, with respect to that parallel, are determined by the same general rule used for the determination of the longitudes of the corners (Art. **19**); that is, if, starting at the corner through which the parallel passes, the courses are taken in the order in which they were run, the latitude of the end of the first course is equal to the latitude range of that course; the latitude of the end of the second course is equal to the latitude of the end of the first course plus the latitude range of the second course; and, in general, the latitude of the end of any course is equal to the latitude of the beginning of the course plus the latitude range of the course. Usually, the reference meridian and the reference parallel of latitude are passed through the same corner.

EXAMPLE.—To determine the latitudes and longitudes of the corners of the field to which the following notes refer, the reference meridian and parallel passing through the corner *A*:

Courses	Latitude Ranges	Longitude Ranges
AB	+ 48.27	− 41.73
BC	+ 17.66	− 37.18
CD	− 30.25	− 14.92
DE	−106.67	+121.17
EF	+ 96.49	+ 75.85
FA	− 25.50	−103.19

SOLUTION.—The operations and results are indicated by the following arrangement (Lat. stands for *latitude* and R. for *range*):

LATITUDES

Lat. R. of *AB* =	+ 48.27	= Lat. of *B*
Lat. R. of *BC* =	+ 17.66	
	+ 65.93	= Lat. of *C*
Lat. R. of *CD* =	− 30.25	
	+ 35.68	= Lat. of *D*
Lat. R. of *DE* =	− 106.67	
	− 70.99	= Lat. of *E*
Lat. R. of *EF* =	+ 96.49	
	+ 25.50	= Lat. of *F*
Lat. R. of *FA* =	− 25.50	
	00.00	= Lat. of *A*

LONGITUDES

Long. R. of *AB* =	− 41.73	= Long. of *B*
Long. R. of *BC* =	− 37.18	
	− 78.91	= Long. of *C*
Long. R. of *CD* =	− 14.92	
	− 93.83	= Long. of *D*
Long. R. of *DE* =	+ 121.17	
	+ 27.34	= Long. of *E*
Long. R. of *EF* =	+ 75.85	
	+ 103.19	= Long. of *F*
Long. R. of *FA* =	− 103.19	
	000.00	= Long. of *A*

It will be observed that both the latitude and longitude of *A*, as calculated from the preceding latitudes and longitudes, should be zero. This is a check on the calculations.

24. Application to Platting.—Having computed the latitudes and longitudes of the different courses, a plat of the field is very conveniently made as follows:

Draw a light pencil line *SN*, Fig. 8, to represent the reference meridian, in such position that the plat will be as nearly in the center of the sheet as can be estimated from an inspection of the notes, or from a rough sketch previously made. On this meridian, mark the corner *A* from which latitudes and longitudes are reckoned; that is, through which the reference parallel of latitude is supposed to pass. From

A, lay off on SN distances AB_1, AC_1, etc. equal to the latitudes of B, C, etc., upwards, if the latitudes are positive; downwards, if they are negative. Through the points B_1, C_1, etc., draw light pencil lines B_1B', C_1C', etc. perpendicular to SN, and on them lay off distances B_1B, C_1C, etc. equal to the longitudes of the corners: to the right for positive longitudes, and to the left for negative longitudes. The

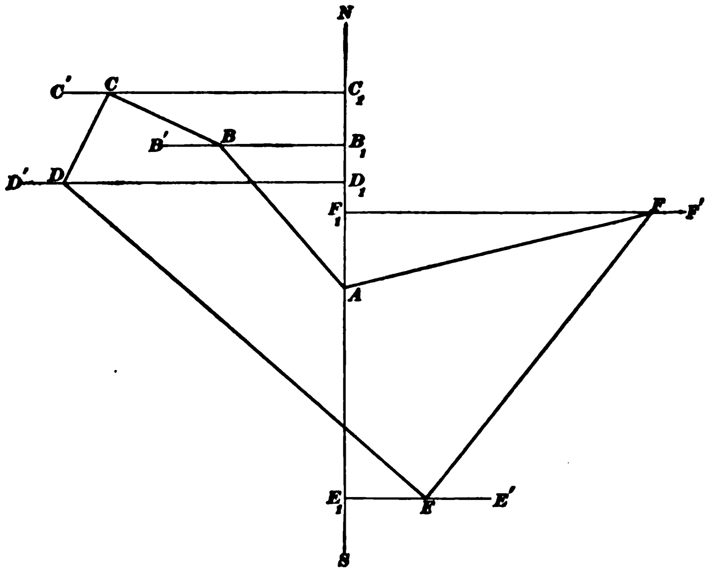


FIG. 8

polygon $ABCDEFA$ formed by joining the points A, B, C, etc. is the required plat of the field.

The reference meridian and parallel are often passed through the most westerly corner of the field. When, however, the field extends a great distance east and west, and the scale is large, it is preferable to pass the reference lines through a corner about half way between the most westerly and the most easterly, as illustrated in Fig. 8.

VARIATION OF THE NEEDLE

25. Declination of the Needle.—It has been stated that the magnetic meridian is the direction of the magnetic needle, that the true meridian is a true north-and-south line, and that these two meridians do not coincide except in a few localities. The angle that the magnetic meridian forms with the true meridian, or, what is the same thing, the angle that the magnetic needle makes with a true north-and-south line, is called the **declination** of the needle. In stating the declination of the needle from the true meridian, the north end of the needle is referred to, and the declination is said to be east or west according as the north end of the needle is deflected to the right (east) or to the left (west) of the true meridian.

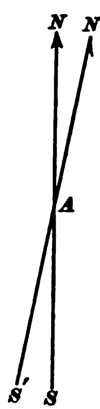


FIG. 9

In Fig. 9, let NS be the true meridian for any given place, and $N'S'$ the magnetic meridian. The angle NAN' is the declination of the needle for that place; it is shown as east declination. Usually, the direction of the magnetic north is indicated by half an arrowhead on one end of a line representing the magnetic meridian, while the true north is indicated by a full arrowhead, as shown in Fig. 9.

The declination of the needle has different values in different localities, and also varies from year to year in a given locality. The differences in the values of the declination for different localities are not regular nor constant, though in a general way they follow a more or less regular system.

The declination of the needle in any locality is ascertained by taking the magnetic bearing of a line whose true bearing is known. The difference between the two bearings is the declination.

In nearly all large cities, the direction of the true meridian is established by astronomical methods, and a line having that direction, to be used for reference, is marked by permanent monuments. The magnetic bearing of such line, taken at any time, gives the declination at that time.

The true meridian can be determined by astronomical observations with the compass; but the methods are too cumbersome and crude. The instrument that the surveyor should use for that purpose is the transit, in a manner to be explained in another Section of this Course.

26. On account of its many imperfections, the compass is not now used for any important final surveys. In case a surveyor has to employ it for a final survey, as that of a farm to be sold, and he does not know the declination at the place, the best thing for him to do is to set permanent marks on one of the lines of the survey, and describe those marks in his notes. Later the true bearing of the line can be determined, and this will give the declination at the time the survey was made; or, if the lines are to be rerun with a compass, the difference between the magnetic bearing of the line, as originally recorded, and the new bearing will give the change in the declination during the interval, which is a correction to be applied to all the bearings. The important thing that the surveyor should keep in mind is that the positions and directions of the lines of a permanent survey must be so determined that there shall be no difficulty in rerunning them at any time; and that, on account of the changes in the direction of the magnetic needle, that end cannot be attained by simply stating the magnetic bearings of the lines at a certain time.

27. Isogonic and Agonic Lines.—A line connecting all points where the declination of the needle is the same is called an **isogonic line.** An **agonic line** is a line connecting all points where the declination is zero; that is, where the needle points in a true north-and-south direction. Both isogonic and agonic lines are extremely irregular, and keep changing their positions almost constantly. Charts showing these lines are published for different years by the United States Coast and Geodetic Survey. Outside of the agonic line, the north end of the needle inclines toward that line; in other words, the declination at any place is east or west according as the place is west or east of the agonic line.

28. Secular Variation.—The declination of the magnetic needle is not only different in different localities on the earth's surface, but in a given locality it undergoes gradual change. This change in the declination of the needle during any given period of time is now commonly called the **variation** of the needle, although the term *variation* was formerly, and is still sometimes, applied to the declination. There is a general variation of the needle that seems to be of a periodic character; that is, it is progressive or continues in the same direction through a long period—over a century—at the end of which, movement in the opposite direction begins. This variation in the declination of the needle is called **secular variation,** and the change in the direction of the variation is called the **secular change.**

29. The Yearly Change.—In the United States, at the present time (1906), the agonic line is moving slowly westwards; in localities east of this line the west declination is increasing, and in localities west of it the east declination is diminishing. The change in the declination is not at all uniform, however, but differs in different localities and at different times in a given locality. Its values in the United States vary between 1 and 5 or 6 minutes per year.

The amount of the yearly variation, as observed at different points in the United States, is written on the isogonic chart published by the Coast and Geodetic Survey. It is expressed in minutes and tenths of a minute per year. These values for the yearly variation of the needle are approximately correct for a number of years following the date for which the chart is constructed, and may be used for obtaining roughly approximate values of the declination in future years.

30. Daily Variation of the Needle.—The needle has also a quite regular daily change, commonly called the **diurnal variation.** It swings daily through an arc that, in the northern part of the United States, varies from about 5 minutes in winter to about 11 minutes in summer. The north end of the needle reaches its extreme eastern position about 8 A. M. It then moves westwards and reaches its extreme western position about 1:30 P. M., when it begins to move eastwards again. It has its mean position, or position giving the true declination, about 10:30 A. M. and between 7 and 8 P. M.

The amount of the change and the times of the extreme positions differ somewhat in different localities and are also subject to a more or less systematic change during the year. The deviations of the needle from its mean daily position, for different hours during the day, are given in the following table; they are expressed in minutes.

Two sets of figures are given, designated as for northern localities and southern localities, respectively. The former apply to latitudes between about 37° and 49° north, and the latter to latitudes between about 25° and 37° north.

31. To Reduce an Observed Bearing to the Daily Mean.—The daily variation of the needle is so small that in the ordinary work of the surveyor it may be neglected entirely. When determining the declination of the needle, however, by directing the line of sight of the compass along a true meridian that has been established, if the observation is not made about 10:30 A. M., when the needle has its mean position for the day, it is well to reduce the declination as observed to its mean value for the day. A bearing observed at any hour may be reduced to its mean value for the day by means of the table given on the opposite page, which contains the corrections to be applied to any observed bearing, in order to reduce it to the mean bearing. The way in which these corrections are applied is the same, in principle, as that employed for reducing magnetic to true bearings, or vice versa, which will presently be explained.

32. Magnetic disturbances affecting the direction of the magnetic needle may occur at any time and cannot be predicted. When of considerable amount and large extent,

TABLE

FOR FINDING MEAN DECLINATION FOR THE DAY

Season and Position in Latitude	Needle Points East of the Mean Magnetic Meridian						Needle Points West of the Mean Magnetic Meridian							
	A. M.	A. M.	A. M.	A. M.	A. M.	A. M.	A. M.	M.	P. M.	P. M.	P. M.	P. M.	P. M.	P. M.
	h. 6	h. 7	h. 8	h. 9	h. 10	h. 11	h. 11	h. 12	h. 1	h. 2	h. 3	h. 4	h. 5	h. 6
December, January, February:	'	'	'	'	'	'	'	'	'	'	'	'	'	
Northern locality	1	1	2	2	2	0	0	2	3	3	2	1	1	0
Southern locality	0	0	1	2	2	1		1	2	2	2	1	0	0
March, April, May:														
Northern locality	3	4	4	4	1	2		4	5	5	4	3	1	1
Southern locality	2	3	3	3	1		1	2	3	3	2	2	1	1
June, July, August:														
Northern locality	4	6	6	5	2	2		4	6	6	5	3	1	1
Southern locality	2	4	4	3	1		2	3	3	3	2	2	1	0
September, October, November:														
Northern locality	2	3	3	3	1	2		3	4	3	2	1	1	0
Southern locality	1	2	3	2	1		1	2	2	2	1	1	0	0

they are known as **magnetic storms.** Their presence is generally indicated by rapid and apparently aimless fluctuations of the needle. They cause deflections of the needle that often amount to $\frac{1}{4}$ degree, and occasionally to $\frac{1}{2}$ and even 1 degree. The compass should not be used during a magnetic storm, as its indications are then uncertain and unreliable.

33. To Change the Bearing of a Line From Magnetic to True or Vice Versa.—When the declination of the needle is known, the true bearing of a line can be very easily determined from its magnetic bearing and vice versa.

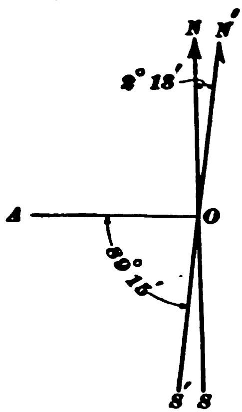


FIG. 10

In a great many cases one can picture in one's mind the relative positions of the true and the magnetic meridian, and of the line whose bearing is to be changed either from magnetic to true or from true to magnetic. Thus, if the declination of the needle is 2° east, and the magnetic bearing of a line is N 43° 00′ E, it is easy to see that the true bearing of the line is N 45° 00′ E, since the line is 43° east of the magnetic north, and the latter is 2° east of the true north. Usually, however, it is better to make a sketch, from which the change from one bearing to another can be readily made without any danger of confusion or error.

Suppose, for example, that it is required to find the true bearing of a line whose magnetic bearing is S 89° 15′ W, the declination being 2° 18′ east. In Fig. 10, the true and the magnetic meridian are shown by the lines *SN* and *S′N′*, respectively, the magnetic north being 2° 18′ east of the true north. The given line is represented by *OA*, making an angle of 89° 15′ with *OS′*. It is evident, from the figure, that the angle that the line makes with the true meridian, or the angle *AOS*, is equal to the sum of the angles *AOS′* and *S′OS*, or 89° 15′ + 2° 18′ = 91° 33′. Since this angle is greater than 90° and bearings are not reckoned above 90°, the true bearing is northwest and is equal to the angle NOA. The angle NOA is equal to 180° minus the angle AOS, or $180° - 91° 33' = 88° 27'$. The true bearing of the line OA is, therefore, N 88° 27′ W.

34. Declination Vernier.—In some compasses, the line passing through the zeros of the needle circle has not a fixed position coinciding permanently with the line of the sights, but can be turned about the center of the instrument through any required small angle to either side of the line of sight, the amount of the deviation being measured by a graduated arc attached to the needle circle that turns along a graduated arc or limb fixed to the main plate of the compass. By means of this device, called variously a **declination vernier, declination arc,** and **variation arc,** the line of zeros can be set at an angle with the line of the sights equal to the declination of the needle at any given place and time, so that the needle will read zero when the compass is sighted on a true north-and-south line. This makes it unnecessary to allow for the declination of the needle in reading the compass, saving both the trouble of doing it and the liability to error.

EXAMPLES FOR PRACTICE

1. The declination being 3° 20′ west, what is the true bearing of a line whose magnetic bearing is: (*a*) N 28° 55′ W? (*b*) S 37° 10′ W?

Ans. { (*a*) N 32° 15′ W; (*b*) S 33° 50′ W

2. When the declination is 5° 30′ east, what is the true bearing of a line whose magnetic bearing is: (*a*) S 12° 20′ E? (*b*) N 27° 55′ E?

Ans. { (*a*) S 6° 50′ E; (*b*) N 33° 25′ E

3. The magnetic bearing of a line is N 2° 30′ E; the declination is 4° 25′ west; what is the true bearing of the line? Ans. N 1° 55′ W

4. The declination being 5° 20′ east, what is the true bearing of a line whose magnetic bearing is N 86° 45′ E? Ans. S 87° 55′ E

5. In a northern locality, the bearing of a line as observed during the month of July, at 3 P. M., is N 48° 21′ E; reduce the bearing to its daily mean value. Ans. N 48° 16′ E

6. In latitude 34° north, the bearing of a line as observed at 8 A. M. on a day in October is S 12° 20′ E; what is the daily mean value of the bearing? Ans. S 12° 17′ E

7. As observed at 9 A. M. on a day in March, the bearing of a line in latitude 45° north is found to be S 14° 58′ W; what is the daily mean value of the bearing? Ans. S 15° 2′ W

8. As observed at 2 P. M. on a day in August, the bearing of a line in latitude 32° north is found to be N 49° 33′ W; reduce the bearing to its daily mean value. Ans. N 49° 36′ W

TRANSIT SURVEYING

(PART 1)

THE TRANSIT

GENERAL DESCRIPTION

1. The Engineers' Transit.—The instrument that is now used most extensively in surveying is called the **engineers'**, or **surveyors', transit.** It combines to an unusual degree the features of accuracy and convenience. It generally has a magnetic needle and a graduated needle circle, and can, therefore, be used as an ordinary compass. In the transit, however, the line of sight, instead of being given by a pair of sights, is defined by the axis of a telescope. The transit is primarily intended for measuring angles in a horizontal plane independently of the needle, but some transits have also a vertical circle or arc for measuring angles in a vertical plane. Usually, transits are so graduated that they measure angles, or "read," to the nearest minute; but there are some that read to the nearest 30, 20, or even 10 seconds; while in geodetic work, improved transits are used by which an angle can be measured to the nearest second.

2. The **telescope** *T*, Fig. 1, revolves in a vertical plane on the axis *a*, which is called the **axis of revolution**, or **transverse axis**, of the telescope, and is supported by the **standards** *D*. These are attached to the **upper plate**, or **vernier plate**, *U*, to which is also attached one of the small **plate levels** *l*. Another plate level *l'* is attached to the vernier plate, as shown in Fig. 2, or to one pair of standards, as shown in Fig. 1. The **object glass,** or **objective lens,** of the telescope is at *o*; it is protected from the direct rays of the sun by the **sunshade** *d*, and is focused by the

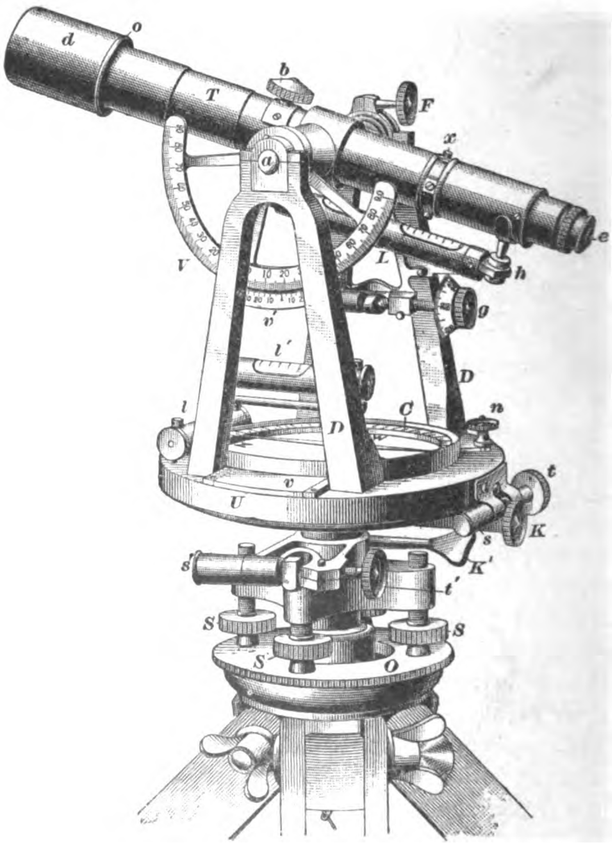


Fig. 1

milled thumb wheel *b*, which is here shown on top of the telescope, though it is commonly placed on the right-hand side.

3. A spirit level *L* is attached to the telescope longitudinally. This consists of a glass tube, curved slightly upwards, and nearly filled with alcohol or a mixture of alcohol and ether, the rest of the space inside being occupied by a small air bubble, called the **level bubble.** The tube is protected by a brass case having an opening in the top, through which the bubble can be seen. In order that the position of the bubble can be distinguished accurately, graduations are marked either on the top of the glass tube or on a small brass scale placed just above the opening. When a level bubble is in the center of its tube, as shown by the graduations, it is said to be **centered.** Each end of the brass case is attached to the telescope tube by means of a small stud, as shown at *h*, and can be raised or lowered by means of two capstan-head nuts that screw on the stud. This level is often called the **attached level.**

4. The graduated arc *V*, called the **vertical limb,** or **vertical arc,** and the **vernier** *v'*, are for measuring vertical angles. The telescope can be clamped and held at any inclination to its axis of revolution by means of the **clamp screw** *F*; it can then be turned slowly on that axis by means of the **tangent screw** *g*, to which is attached a circular scale called a **gradienter.** In some transits, the vertical arc is attached rigidly to the transverse axis of the telescope, as shown in Fig. 1, while in others it connects by a joint that allows it to turn freely on that axis. This joint, however, can be clamped by means of a clamp screw similar to that shown at *F*; and then it turns with the telescope. The vernier *v'*, by which the vertical arc is read, is attached to the standards, and in some transits is made adjustable, so that its zero mark can readily be made to coincide with the zero mark of the vertical limb when the telescope is perfectly horizontal. In some transits the vertical arc is attached to one pair of standards, and its vernier to the transverse axis of the telescope.

5. A **plain transit,** Fig. 2, is one that has no vertical limb nor attached level. The transit here illustrated is of a different make, and consequently of somewhat different form from that shown in Fig. 1. Corresponding parts are designated by the same letters in both figures, that they may be readily compared. In Fig. 2 the objective *o* is covered by the cap *c* instead of by the sunshade *d* shown in Fig. 1. This protects the objective when the instrument is not in use.

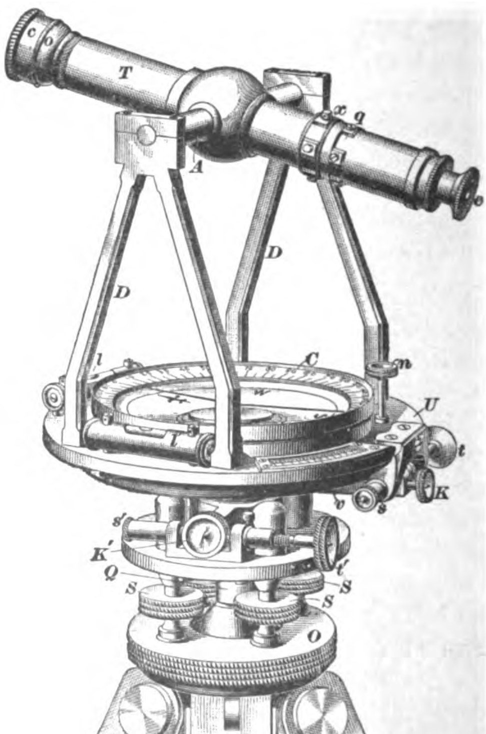


FIG. 2

6. The Plates.—All transits have two concentric plates rotating independently on the same axis, which is called the **axis of the instrument.** When the axis of the instrument is vertical, these plates rotate in a horizontal plane. The lower of the two plates carries a graduated circle, called the **horizontal limb,** or **horizontal circle.** To the upper or **vernier plate**, which rotates within and above the other, are attached two verniers, which travel along the graduated circle of the lower plate. By means of these plates, carrying, respectively, the graduated circle and the verniers, horizontal angles are measured. The graduated circle is entirely covered and concealed by the upper plate, except at two small openings where the verniers are attached to the upper plate. Through each opening, which is protected by glass, can be seen the vernier and a short arc of the graduated circle. The opening through which one vernier is read is shown at v, Fig. 2; the other is diametrically opposite. The verniers are usually distinguished by marking one A and the other B. The vernier plate can be clamped firmly to the lower plate by means of the clamp screw K, called the **upper clamp**, or **vernier clamp**; and by means of the screw t, called the **upper tangent screw**, it can be revolved slowly on the lower plate, moving the vernier along the divided circle, so that the instrument can easily be set at any given angle. In order to prevent any lost motion, the tangent screw operates against the opposing spring s.

At C is shown the **compass circle**, or **needle circle**, which is attached to the upper plate. The needle-lifting screw is shown at n. These are the same as in a compass.

7. The Centers.—The upper plate U, Fig. 3, is attached to an accurately turned and slightly conical axis or spindle Q that extends down nearly to the tripod head. In transits of the most modern construction, this axis revolves within a socket that is controlled by the **leveling screws** $S S$ and about the upper portion of which revolves a socket that extends down from the lower plate, forming what is called a **compound center**. The centers, which control the entire instrument above the leveling screws, can be clamped against rotation by means of the clamp screw K', and the instrument can then be revolved slowly by means of the tangent screw t'. This clamp screw is called variously the **lower clamp**, **clamp to the centers**, or **clamp to the lower plate**, and the tangent screw is designated by corresponding terms. The centers are connected with the plate O, sometimes called the **lower leveling plate**, by means of a hemispherical or ball-and-socket **joint**, shown at M, Fig. 3. The centers and the entire instrument above them are supported in position by the four leveling screws, which serve also to level the instrument. The plate O screws on the tripod head. This plate and the leveling screws, considered together, are sometimes spoken of as the **leveling head.**

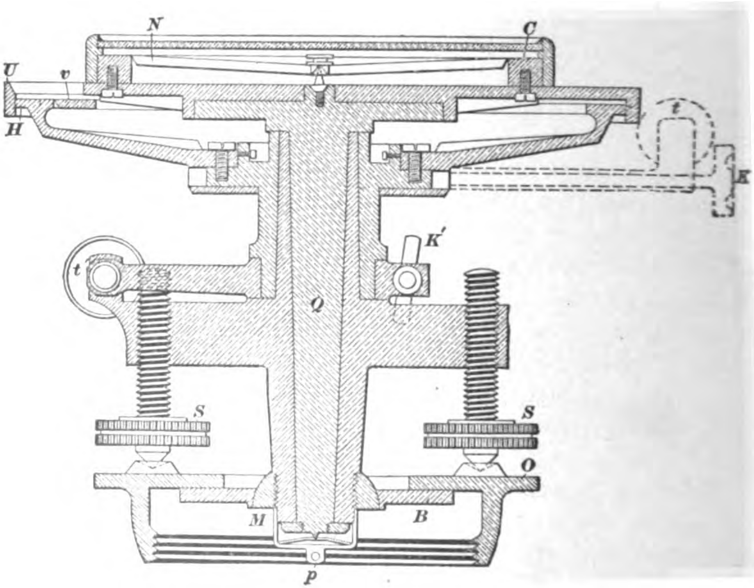


FIG. 3

8. Shifting Center.—The exact position of the instrument over a point is indicated by a **plumb-bob** suspended from a small chain or loop attached to the lower end of the centers at the central point p, Fig. 3. It will be noticed that the centers do not connect directly with the lower leveling plate O, but, by means of the hemispherical joint M, connect with a plate B, which bears against the under side of the leveling plate. By loosening the leveling screws, the plate B and the leveling screws that support the entire instrument can be moved or shifted a small distance in any direction on the lower leveling plate, thus permitting the instrument to be easily and exactly centered over a point, where it is held firmly by merely tightening the leveling screws. This arrangement is called a **shifting center** or **shifting tripod head.**

PRINCIPAL PARTS OF THE TRANSIT

THE TELESCOPE

9. Lenses.—In general, a **lens** is a transparent body with curved surfaces. In passing through a lens, rays of light are bent, or **refracted,** their direction being thus changed. The lenses commonly used for optical instruments have either two spherical surfaces, as shown in Figs. 4 and 5, or one spherical and one plane surface, as shown in Fig. 6.

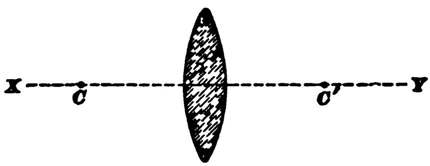


FIG. 4

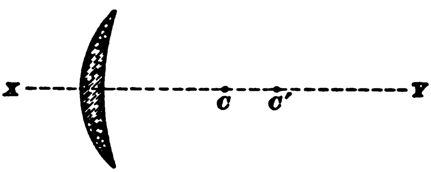


FIG. 5

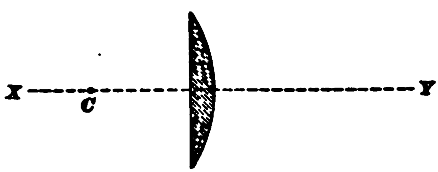


FIG. 6

The **principal axis,** or simply the **axis,** of a lens having two spherical surfaces is the line XY, Figs. 4 and 5, passing through the centers C, C' of those surfaces. In the case of a lens having one plane and one spherical surface, the axis is the line XY, Fig. 6, passing through the center C of the spherical surface of the lens and perpendicular to the plane surface. In surveying and other optical iustruments, several lenses are often joined together, forming a **compound lens.** The axes of these lenses are always in the same line, which is called the axis of the compound lens.

The **optical center** of a lens is a point within, and in the principal axis of the lens, its position being such that every ray of light passing through it is refracted equally and in opposite directions at the opposite surfaces of the lens, so that the ray leaves the lens in a direction parallel to that in which it enters.

10. General Description of the Telescope.—One of the most important parts of a transit is the telescope, by means of which the line of sight from the point occupied by the instrument to any observed point is defined. It consists of two combinations of lenses placed in a tube. One of these combinations, called the **object glass**, or **objective**, receives the rays of light coming from any object toward which the telescope is directed, deflects them, and makes them form an inverted image of the object. This image appears to be in a plane surface meeting the axis of the objective in a point called the **focus of the objective.** The rays pass from this image to the observer's eye through another combination of lenses, called the **eyepiece.** The main object of this combination is to magnify the image of the object observed.

There are two general classes of eyepieces; namely, the **positive eyepiece**, which magnifies the image directly in its inverted position, thus showing objects upside down; and the **compound eyepiece**, which reinverts the image and shows objects in their natural positions. Telescopes having eyepieces of the former class are called **inverting telescopes**, and those having eyepieces of the latter class are called **erecting telescopes.** Most all ordinary American transits have erecting telescopes.

11. The Cross-Hairs.—In order that the line of sight may be brought to bear precisely on any point of an object within the field of view of the telescope, two fine material lines, or threads, called **cross-hairs**, **cross-wires**, or **cross-lines**, are placed with their intersection at the common focus of the object glass and the eyepiece. These cross-hairs, as seen through the eyepiece, seem to be in the same position as the image of the distant object, to which they appear to be attached.

The cross-hairs are fastened to a substantial brass ring, or diaphragm, placed within the telescope and held in position by capstan-headed screws that pass through the telescope tube and screw into the diaphragm, as shown in Fig. 7. They are commonly placed at right angles to each other, one being vertical and the other horizontal. The ring, together with the cross-hairs that are attached to it, is sometimes called the **reticle or reticule.** By means of the four capstan-headed screws (of which three are shown at x, Fig. 1) that control it, the reticle can be moved and the cross-hairs adjusted, as will be explained elsewhere. The cross-hairs are either of platinum wire, drawn very fine, or of spider webs. Some engineers prefer the platinum cross-wires, because they are less affected by changes of temperature; others prefer spider webs, because they are more opaque, and also because they do not gather moisture nor dust.

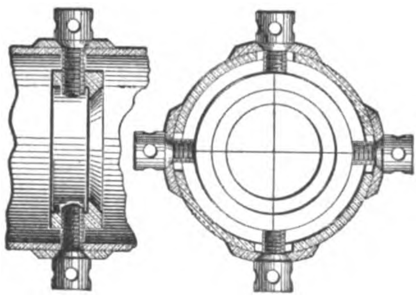


FIG. 7

12. Line of Sight or of Collimation.—Considered in a general sense and without reference to the instrument, a **line of sight** is a line joining any observed point with the eye of the observer. With reference to the telescope of a transit, the **line of sight,** or **line of collimation,** is a line passing through the optical center of the object glass and the intersection of the cross-wires. When this line is directed to any external point, the image of the point appears to coincide with the intersection of the cross-hairs, and the instrument itself, or the telescope, is said to be directed to the point observed. In this position, the line of collimation defines, or coincides with, the line between the point of observation and the point observed.

THE HORIZONTAL LIMB

13. The horizontal limb, or graduated circle, is divided into 360 equal parts, or degrees, and each degree is further divided into two or three equal parts, as the case may be. If the degree is divided into two equal parts, each part will measure 30 minutes of arc, and if into three equal parts, each part will measure 20 minutes. Each subdivision of a degree is marked by a line somewhat shorter than the regular degree graduations, while each fifth degree is indicated by a longer line of division, and each tenth degree is marked by a still longer line and is also numbered.

The graduated circle is numbered in various ways, three systems of numbering being employed. These may be described as

1. The **azimuth system**, in which the figures extend from 0 continuously around the entire circle to 360.

2. The **transit system**, in which the figures extend from 0 in opposite directions through the adjacent semicircles to 180 at the point diametrically opposite the zero point.

3. The **compass system**, in which the figures extend each way from two 0 points diametrically opposite each other through the adjacent quadrants to the 90° points.

There are usually two rows of figures extending around the graduated circle of a transit, each row being numbered according to one of the above described systems. In some transits, both rows are of the first, or azimuth, system, extending in opposite directions around the circle. The different systems are also combined in various ways.

THE VERNIER

14. General Principle of the Vernier.—A **vernier** is an auxiliary scale used for the purpose of measuring, on another scale, fractional parts of the smallest subdivisions of the latter scale. Thus, if a scale is divided into feet and tenths, the vernier may be used to measure on that scale tenths of tenths, or hundredths, of a foot. The manner in which this is done is shown by the illustrations that follow.

In Fig. 8, *A B* is a scale graduated to inches, and *M N* is a vernier that can be slid along *A B* in either direction. In this case, the graduated part of the vernier is 7 inches long, as shown by the scale, but is divided into eight equal parts,

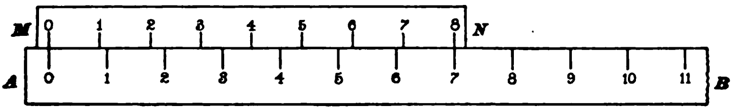


FIG. 8

numbered from *0* to *8*; that is, the vernier is divided into a number of equal parts (eight) that is one more than the number of divisions (seven) it covers on the scale. Each division of the vernier being one-eighth of the total length of the vernier, is one-eighth of 7 inches, or $\frac{7}{8}$ inch; and, therefore, the difference between one division on the scale and one on the vernier is $\frac{1}{8}$ inch.

Suppose, now, that the vernier is slid along the scale until it occupies the position shown in Fig. 9, with its zero between the 2-inch and the 3-inch mark of the scale. For convenience, the division lines on the vernier will here be marked and denoted by the letters a_0, a_1, a_2, etc. The zero of the vernier being at a_0, its distance *2–*a_0 from the 2-inch

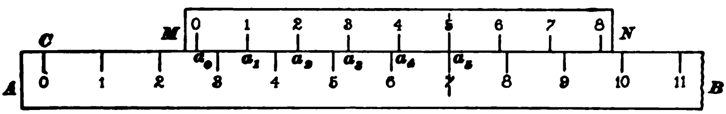


FIG. 9

mark of the scale is ascertained as follows: It will be noticed that the division line a_5, or *5*, of the vernier, coincides with a division line of the scale. It should be very particularly noticed that it is not necessary to observe the number of the division line of the scale that coincides with a division line of the vernier. As one division of the scale is greater by $\frac{1}{8}$ inch than one division of the vernier, the distance *6–*a_4 on the scale is $\frac{1}{8}$ inch; as two divisions of the scale are greater by $\frac{2}{8}$ inch than two divisions of the vernier, the distance $5–a_5$ is $\frac{2}{8}$ inch. Likewise, $4–a_4 = \frac{3}{8}$ inch; $3–a_3 = \frac{4}{8}$ inch; and $2–a_2 = \frac{5}{8}$ inch. The distance $2–a_0$ is, therefore, equal to a number of eighths (five) equal to the number of the division line of the vernier that coincides with a division line of the scale.

To measure the length of a line with the scale *AB* and the vernier *MN*, the zero mark of the scale is placed at the beginning of the line, and the vernier is slid along the scale until its zero mark coincides with the end of the line. The number of the division mark of the scale immediately preceding the end of the line, or the zero mark of the vernier, will give the number of whole inches in the line, and the additional number of eighths is given by the number of the division mark of the vernier that coincides with a division mark of the scale. Thus, if in Fig. 9 the line to be measured is $C–a_0$, the zero mark of the scale is placed at *C*; and the vernier is slid along the scale until its zero mark coincides with the end a_0 of the line. The length of the line is evidently 2 inches plus $2–a_0$, or $2\frac{5}{8}$ inches. This is also the distance from the zero of the scale to the zero of the vernier.

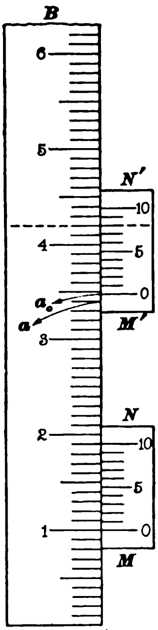


FIG. 10

15. In Fig. 10 is shown a vertical scale *AB* with a vernier *MN*; this may be taken to represent a leveling rod. The divisions marked *1*, *2*, etc. on the scale are tenths of a foot, and each of these is divided into ten equal parts, or hundredths of a foot. The graduated part of the vernier *MN* covers nine of the smallest subdivisions of the scale, and is, therefore, .09 foot long; it is divided into ten equal spaces, and each space is one-tenth of .09, or .009 foot. The difference between one division on the scale and one on the vernier is .01 – .009, or .001 foot.

To measure a distance Ca_0, the zero of the scale is placed at *C*, and the vernier is slid up the scale to the position

$M'N'$, in which its zero coincides with the end of the distance to be measured. The scale shows that the distance Ca_0 is made up of the two parts Ca and aa_0, the former of which is $0.3 + .04 = .34$ foot. The distance aa_0 is determined by the vernier thus: As will be noticed, the eighth division mark of the vernier coincides with a division mark (it is not necessary to note which one) of the scale. Since one division of the scale is greater by .001 than one division of the vernier, the seventh mark of the vernier is .001 foot from the scale division mark immediately below it; the sixth mark of the vernier is .002 foot from the scale division mark immediately below it, etc. In this manner it is found that the zero mark of the vernier is .008 foot from the scale division mark a immediately below it. The distance aa_0 is, therefore, .008, and $Ca_0 = .34 + .008 = .348$ foot. Here the number of tenths and hundredths in the required distance is read off the scale, and the number of thousandths is equal to the number of the vernier division mark (eight) that coincides with a scale division mark.

16. Verniers are made in various forms and are graduated in various manners; but the general principle of those that are commonly used in engineering instruments is the same, and may be stated as follows: The length of the vernier is made equal to a certain number of the smallest divisions or subdivisions of the scale, and is then divided into equal parts, the number of these parts being greater by one than the number of subdivisions that the vernier covers on the scale. Thus, in Fig. 8, the vernier covers seven of the smallest divisions (inches) of the scale, and is divided into seven plus one, or eight, equal parts; in Fig. 10, the vernier covers nine of the smallest divisions (hundredths) of the scale, and is divided into nine plus one, or ten, equal parts. The difference between one division of the scale and one of the vernier is called the **least reading** of the vernier, and its value is a fractional part of the smallest division on the scale equal to one divided by the number of divisions of the vernier. In Fig. 8, the smallest division on the scale is 1 inch; the vernier is divided into eight equal parts, and its least reading is one-eighth of 1 inch, or ⅛ inch. In Fig. 10, the number of divisions of the vernier is ten, and the least reading is one-tenth of one of the smallest subdivisions on the scale; that is, one-tenth of .01, or .001 foot.

The distance from the zero of the vernier to the division mark of the scale immediately preceding it is indicated by the number of the vernier division mark that coincides with a division mark of the scale. This number, multiplied by the least reading, gives the distance in question. In Fig. 9, the distance $2\text{–}a_0$ was found to be ⅝ inch, or 5 (coinciding vernier division mark) × ⅛ (least reading). In Fig. 10, the distance $a\,a_0$ was found to be .008, or 8 (coinciding vernier division mark) × .001 (least reading). The number thus obtained by multiplying the number of the coinciding division mark by the least reading is called **the reading of the vernier**, while the number of divisions and subdivisions of the scale preceding the zero of the vernier is called the **reading of the scale.** In Fig. 9, the reading of the scale is 2 inches; in Fig. 10, .34 foot. The sum of the reading of the scale and that of the vernier expresses the distance of the zero of the vernier from the zero of the scale, and is called the **reading of the instrument.** Sometimes the expression *reading of the vernier* is, for convenience, used instead of *reading of the instrument*, and likewise *to read the vernier* is used in the sense of *to read the instrument*, it being understood that the reading of the scale is to be taken first, and then increased by the reading of the vernier.

17. Formulas Relating to the Vernier.

Let s = length of one of smallest subdivisions of scale;
n = number of equal parts into which vernier is divided;
r = least reading of vernier;
m = number of vernier division mark coinciding with division mark of scale;
R = reading of vernier.

Then, from what was said in the last article,

$$r = \frac{s}{n} \qquad (1)$$

$$R = rm = \frac{s}{n} \times m \qquad (2)$$

EXAMPLE 1.—A scale is divided into inches and half inches; the vernier is divided into eight equal parts, which cover seven of the half-inch divisions of the scale. (*a*) What is the least reading of the vernier? (*b*) What is the reading of the vernier when its third mark coincides with a division mark of the scale?

SOLUTION.—(*a*) Here $s = \frac{1}{2}$ in., and $n = 8$. Therefore, by formula **1**,

$$r = \frac{\frac{1}{2}}{8} = \frac{1}{16} \text{ in. Ans.}$$

(*b*) Here $r = \frac{1}{16}$ in. and $m = 3$. Therefore, by formula **2**,

$$R = \frac{1}{16} \times 3 = \frac{3}{16} \text{ in. Ans.}$$

EXAMPLE 2.—The scale of a barometer is divided into inches and fiftieths. (*a*) What must be the length of a vernier, and how must the vernier be divided, that its least reading may be .002 inch? (*b*) What is the reading of the vernier when its seventh mark coincides with a division mark on the scale?

SOLUTION.—(*a*) Formula **1**, solved for n, gives, $n = \frac{s}{r}$. In this case, $s = \frac{1}{50} = .02$, and $r = .002$; hence,

$$n = \frac{.02}{.002} = 10$$

Therefore, the vernier must be divided into ten equal parts, covering nine of the small divisions of the scale, which makes it $\frac{9}{50}$ inch long. Ans.

(*b*) To apply formula **2**, we have, $r = .002$, $m = 7$, and therefore

$$R = .002 \times 7 = .014 \text{ in. Ans.}$$

18. The Circular Vernier.—For the purpose of measuring angles, a circle graduated to degrees and fractions of a degree (usually halves, thirds, or sixths) is employed as a scale. Smaller fractions of a degree than those into which the circle is divided are measured by means of a vernier, substantially as has been explained. An arrangement of this kind is shown in Fig. 11, in which *AB* is the scale, or graduated circle, and *MN* is the vernier. The circle is divided into degrees and half degrees, so that here $s = \frac{1}{2}°$

= 30′. As the vernier is divided into six equal parts, its length, expressed in degrees, is equal to five (= 6 − 1) of the small divisions of the circle, or $2\frac{1}{2}°$. The least reading of the vernier, expressed in minutes, is (formula **1**, Art. **17**) $\frac{30}{6}$, or 5′. In the case shown in the figure, the second division mark of the vernier coincides with a division mark of the circle. The reading of the vernier, or the arc $a\,a_v$, is, therefore, 2 × 5 = 10′. The reading of the circle to a is 51° 30′. If the center of the circle is at the vertex of an angle, one of whose sides passes through the zero of the graduated circle and the other through a_v, the angle is equal to the reading of the circle plus the reading of the vernier, or 51° 30′ + 10′ = 51° 40′.

Usually, the division lines on the vernier are marked by numbers indicating the corresponding readings of the vernier when those lines coincide with lines on the graduated

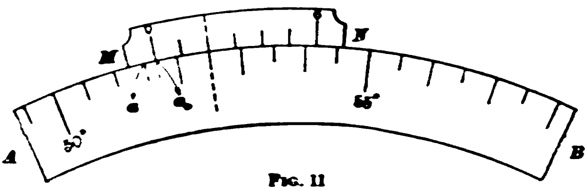


FIG. 11

circle. Thus, in the case illustrated in Fig. 11, the last division line would be marked and called 30 (= 6 × 5); the one immediately preceding, 25 (= 5 × 5); and so on. This makes it unnecessary to perform the multiplication indicated in formula **2** of Art. **17**.

19. The Transit Verniers.—Generally, the horizontal circle of a transit is divided into degrees and halves, and the vernier reads to minutes. As in this case, $s = \frac{1}{2}° = 30'$, and $r = 1'$, formula **1**, Art. **17**, gives $n = \frac{30}{1} = 30$, which shows that the vernier must be divided into thirty equal parts covering twenty-nine of the half-degree divisions of the graduated circle. In Fig. 12, *A B* represents part of the horizontal circle, and *M N* one vernier, there being another vernier *M N′* on the left of *M*. The vernier *M N* is used when angles are turned to the left; that is, when the zero of the vernier slides in the direction *A B*, and the degrees are indicated by the upper figures (*60*, *70*, *80*, etc.) on the graduated circle. The vernier *M N'* is used when angles are turned to the right, and the degrees are indicated by the lower figures (*90*, *100*, *110*, etc.) on the graduated circle. Nearly all transits have two combinations of verniers similar to *N N'*, the zeros of which are 180° apart. Each of these combinations, although it really consists of two verniers, is referred to as one vernier, one of them being called vernier *A* and the other vernier *B*. For very accurate work, both verniers are read, and if they do not agree, the mean of the two readings is taken as the true reading.

Again referring to Fig. 12, suppose that, the center of the graduated circle being over the vertex of an angle to be

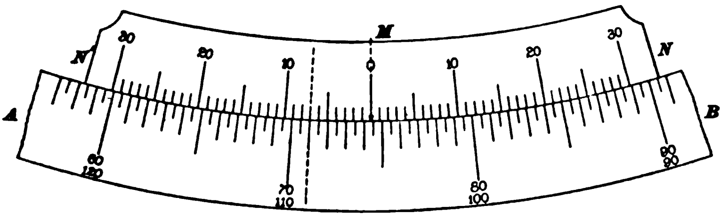


FIG. 12

measured, and its zero on one of the sides, the vernier has been slid to the right along the graduated circle until the other side of the angle passes through the zero mark of the vernier, and that the vernier has then the position shown in the figure. Since the vernier has moved to the right, the side *M N* is to be read. The twenty-third mark of the vernier coincides with a division mark of the scale, and, as the least reading of the vernier is 1′, its reading, in this case, is 23′. The reading of the scale, up to the division mark immediately preceding the zero of the vernier, is 74°. The reading of the instrument, or the measure of the angle, is, therefore, 74° + 23′ = 74° 23′.

20. The vertical circle or arc *V*, Fig. 1, is often graduated to degrees and halves, and the vernier *v'*, Fig. 1, which is double, like the vernier of the horizontal circle, reads either to single minutes or to 5 minutes. If the vernier is attached to the standards, it is stationary, and instead of it sliding along the vertical arc, the vertical arc slides on it. Care should be taken always to read that side of the vernier whose numbers increase in the same direction as those by which degrees are measured on the graduated circle.

EXAMPLES FOR PRACTICE

1. Each division of the scale of a barometer is .05 inch. (*a*) If the vernier is divided into twenty-five equal parts, what is its least reading? (*b*) If the reading of the scale is 29.10 inches, and the thirteenth division mark of the vernier coincides with a division mark of the scale, what is the reading of the barometer?

Ans. { (*a*) .002 in.; (*b*) 29.126 in. }

2. The graduated circle of a transit is divided into degrees and thirds, and the vernier, into forty equal parts; what is the least reading of the vernier? Ans. 30″

3. The graduated circle of a transit is divided into 10-minute spaces; how many of these spaces must the vernier cover, that its least reading may be 10″? Ans. 59

4. What is the reading of the instrument in Fig. 12, the angles being measured from *B* toward *A*; that is, assuming that the vernier has slid along the circle in the direction *NN*′? Ans. 105° 37′

MOTIONS OF THE TRANSIT

21. The Plate Levels.—Before an angle can be measured with the transit, the plates carrying the graduated circle and verniers must be made horizontal. This is effected by means of the four leveling screws *S*, Figs. 1 and 2, the horizontal position of the plates being indicated by the two plate levels *l*, *l*′, which are placed at right angles to each other. Each of these levels is substantially the same as the large level attached to the telescope, Art. **3**, but smaller. When they are adjusted properly and each air bubble is exactly in the center of its tube, the plate to which they are attached is perfectly level, and will revolve in a horizontal plane.

22. The Leveling Screws.—The leveling screws *S*, Fig. 3, have milled heads by which they are turned, and are arranged 90° apart about the axis of the instrument; they are thus in pairs, the two screws in each pair being on opposite sides of the center, and the vertical plane through one pair being at right angles to that through the other pair. Their upper ends screw into arms, or into a solid circular plate projecting from the centers, and their lower ends rest on the lower leveling plate. By turning the two screws in either pair, one in and the other out, the level plate can be tipped in the necessary direction to make it horizontal. A more complete description of this operation will be given later.

23. The Axes of Rotation.—The design and construction of the transit is such as to provide two axes of rotation. The more important of these is the axis of the instrument. (Art. 6), which must be truly vertical when the level plate is horizontal; this axis is also called the **vertical axis of the telescope.** The other is the transverse or horizontal axis of the telescope (Art. 2). This axis, when adjusted properly, is at right angles to the vertical axis, and is therefore horizontal when the instrument is level. The point where these two axes intersect is the **center of the instrument;** that is, the point from which all measurements are taken, and is the point that is always understood when the center (not centers) of the instrument is referred to. In a properly constructed transit, the line of collimation should pass exactly through this point.

24. Rotation in Two Planes.—The line of sight has two distinct turning motions, namely:

1. *Horizontal Rotation About the Vertical Axis of the Instrument.*—As has been explained, the centers of the transit are so made that this motion can be effected in two ways—either by revolving the upper plate only, or by revolving both plates together. Since measurements of azimuth are made by means of the rotation of the instrument about its vertical axis, this motion is commonly spoken of as **rotation in azimuth,** and the convenient phrases *to revolve in*

azimuth, *to reverse in azimuth*, etc. are often used with reference to it. The operation of revolving the telescope on its vertical axis through one-half of one revolution, or 180°, so that it will point in the opposite direction, is sometimes called **reversing in azimuth.**

2. *Vertical Rotation About the Transverse Axis of the Telescope.*—The operation of revolving the telescope in a vertical plane, that is, turning it on its transverse axis, so that it will point in the opposite direction, is called **plunging** the telescope. In general, **to reverse** the telescope is to turn it through 180° about either axis so that it will point in the opposite direction. The term is sometimes used in the sense of revolving the telescope 180° in azimuth, but oftener in that of plunging the telescope. In order to avoid ambiguity, it is preferable to employ the terms *reverse in azimuth* and *plunge*, or else the self-explaining expressions *reverse the telescope on its vertical axis*, *reverse the telescope on its horizontal axis.*

TRANSIT FIELD WORK

PLACING THE INSTRUMENT IN POSITION

25. Setting Up the Instrument.—Since much of the work of an engineering party is suspended while the instrument is being set up, it is highly important that facility in performing this operation be acquired. In setting up a transit over a point from which measurements are to be taken, three preliminary conditions should be satisfied as nearly as possible, viz.:

1. The tripod feet should be planted firmly.
2. The plate on which the leveling screws rest should be approximately level.
3. The plumb-bob should be directly over the mark (as a tack) by which the point is defined or indicated.

The first condition is very essential in order that the instrument may maintain its position unchanged. It is desirable that the second condition be obtained approximately, so as to avoid turning the leveling screws to their extreme positions. The third condition must be obtained perfectly, and this is rendered comparatively easy by the shifting center with which most modern transits are provided. (See Art. 8.) The string by which the plumb-bob is suspended from the centers should be so adjusted that the point of the bob almost touches the mark over which it is desired to set it.

If the instrument has no shifting center, the plumb-bob must be brought directly over the point by moving the legs of the tripod. The direction in which they must be moved will be clearly indicated by the position of the plumb-bob. In soft soil, the transitman, after laying down the instrument, goes around it, pressing the legs into the ground, so as to insure steadiness. The plumb-bob can usually be brought over the point by exerting a little extra pressure on one or two of the legs.

26. Leveling the Instrument.—The next operation is to make the level plate horizontal. This is called **leveling the instrument,** and is accomplished as follows: Loosen the lower clamp and turn the instrument on its vertical axis so that one of the plate levels is parallel to the line passing through a pair of opposite leveling screws. As the plate levels are at right angles to each other, when one level is parallel to the line passing through one pair of leveling screws, the other level is also parallel to the line passing through the other pair of leveling screws, so that one is leveled with one pair of screws and the other is leveled with the other pair. By grasping the milled heads of the two leveling screws in either pair, one between the thumb and forefinger of each hand, turn one screw in and the other out until the bubble of the corresponding level is centered approximately, then manipulate the other pair of leveling screws in the same manner and repeat with each pair of leveling screws alternately until each bubble is centered exactly. The leveling screws being all right-handed, if the thumbs of both hands move toward the center of the instrument in turning either pair of screws, the right-hand side of the instrument will be elevated and the left-hand side depressed, so that the bubble will move to the right, while if the thumbs move outwards, or from the center, the bubble will move to the left. Turning both screws of a pair to the right will loosen them on the lower leveling plate, so that, if both pairs are loosened, the instrument can be shifted on the tripod head, as described in Art. **8**, while turning both screws of a pair toward the left will again tighten them and hold the instrument firmly in place. Both pairs of screws should be brought to firm bearings, but should not be turned so tightly as to cause any strain.

27. Focusing the Telescope.—The operation of bringing the object observed and the cross-hairs distinctly into the field of view is called **focusing** the telescope. The object glass is attached to a slide that fits into the object end of the main telescope tube and is caused to slide in or out by means of a rack and pinion operated by a milled thumb wheel *b*, Fig. 1. By this means, the object glass is moved in or out, according as the object viewed is farther from or nearer to the instrument. The telescope being directed to any object to be observed, the object glass is focused by turning the screw *b* one way or the other until the object appears distinctly in the field of view. As stated above, the nearer the object is, the farther out should the object glass be slid.

The eyepiece is focused on the cross-hairs in a similar manner by sliding it in or out along the eye end of the main telescope tube. In some instruments, the eyepiece is controlled by a milled thumb wheel similar to that which controls the object-glass slide; in others, it is controlled by a milled ring encircling the eyepiece, which, by a rotary motion, operates a screw that carries the eyepiece in or out, while in other instruments the slide is merely pushed in or pulled out with the hand. The eyepiece is adjusted so as to bring the cross-hairs properly into focus for the eye of the observer; it is then allowed to remain in such position and its focus seldom needs to be changed for the same observer. To bring the cross-hairs into proper focus, point the telescope toward the sky and move the eyepiece in or out until the cross-hairs appear perfectly sharp and distinct. When in proper focus, the cross-hairs will appear to be attached to, or be a part of, whatever object is observed, and will show no movement, but will retain exactly the same position on the object regardless of how much the position of the eye may be changed.

THE WORK OF THE TRANSIT

28. Transit Points.—A point over which the transit is set is called an **instrument point**, or **transit point**. The stakes that are set to mark the instrument points of a transit survey differ materially from the regular station stakes described in *Compass Surveying*, Part 1. The flag, held at the measured distance, is lined in by the instrument, and a stake is driven at the point thus determined until its top is about 1½ feet above the ground. When a transit point is set, the position for the stake is found in the same manner. But in this case the stake has usually a level top and is driven nearly flush with the ground. Such a stake is called a **hub** or **plug**. After a plug has been driven at the required point, a pointed pole or rod (better still, a pencil or a nail, with the point downwards), is held on it and lined in by the transitman. The exact point thus determined is marked by a flat-headed tack driven flush with the top of the hub. After the tack is driven, the flag (or pencil, or nail) is held on it and the instrument again directed to it as a check.

This operation of marking the exact point on the hub is called **centering the hub**; and when the hub is so marked, it is said to be **centered**. That the hub may be easily found, a stake similar to a station stake is driven at a distance of about a foot from the hub, at right angles to the line, and marked with the number of the station on the side facing the hub. Such a stake is called a **guard stake**.

29. How to Prolong a Straight Line.—Let AB, Fig. 13, be a straight line whose position on the ground is fixed by stakes set at A and B, and let it be required to prolong the line to C. This can be done in two ways; namely, by **foresight only**, or by **backsight and foresight**, the latter method being commonly called **backsight**.

1. *By Foresight*.—The transit is set over the point at A, and the line of sight directed to a flag held at B; if the point C is to be set at a given distance from B, the chainmen

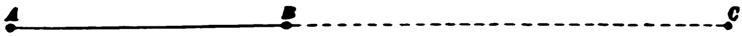


FIG. 13

measure the required distance, the head chainman being kept in line by the transitman. When the required distance has been measured, the point C, which evidently lies in the prolongation of AB, is marked by a stake or otherwise.

2. *By Backsight*.—Set the transit over the point at B and sight on a flag held at A. Plunge the telescope, which will then be directed along the prolongation of AB. Any required distance BC may then be measured from B in the direction indicated by the line of sight.

30. Review of the Functions of the Clamps and Tangent Screws.—The manner in which the clamps and tangent screws work has already been explained. The subject, however, is here restated in a different form, as a thorough understanding of it is of the greatest importance for an intelligent manipulation of the transit.

The lower clamp and the lower tangent screw control the motion of the lower plate about its vertical axis. When the lower clamp is set, the lower plate, and therefore the instrument as a whole, cannot be revolved except very slowly by means of the lower tangent screw.

The upper clamp and tangent screw control the sliding motion of the upper plate over the lower. When the upper clamp is set, the upper plate cannot be revolved over the lower except very slowly by means of the upper tangent screw. When the instrument is to be revolved as a whole, the upper clamp should be set, and the lower loosened. This motion does not alter the reading of the instrument.

When the upper plate is to be revolved over the lower, the lower clamp should be set, and the upper loosened.

This motion, which makes the vernier slide along the graduated circle, naturally changes the reading of the instrument.

31. Directing the Line of Sight, or the Telescope, to a Given Mark.—Suppose that, the instrument being set up and leveled at some point, with both clamps set, it is desired to direct the line of sight toward a certain mark, as a pole held at one of the stations of a survey.

In order to do this, one of the clamps must be loosened. If the reading of the vernier is to remain unchanged, the lower clamp should be loosened; otherwise, the upper clamp. We shall assume that the upper clamp is loosened. The plate is then revolved horizontally, or in azimuth, one hand being placed on each side of it. The transitman walks

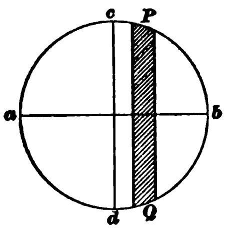


FIG. 14

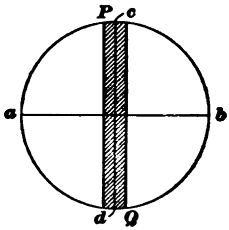


FIG. 15

around the instrument, looking over the telescope, until the latter points to the flagpole to be observed, as nearly as he can estimate by the eye. He then looks through the telescope, which, if necessary, he turns to one side or the other, and up or down, until the pole appears in the field of view. Still turning the plate with his hands, he brings the image of the flagpole nearly into coincidence with the vertical cross-hair. This is shown in Fig. 14, where *a b* and *c d* are the cross-hairs, and *P Q* the image of the flagpole. He now sets the upper clamp, and turns the upper tangent screw until the intersection of the cross-hairs exactly bisects the pole, as shown in Fig. 15. This completes the operation. The process is the same when the lower clamp is loosened before revolving the telescope, except that in that case the lower instead of the upper tangent screw is used.

Whenever such expressions as "Loosen the lower clamp and direct the telescope (or the line of sight) to Station A" occur, it should be understood that an operation similar to the one just described is to be performed.

32. Measurement of Horizontal Angles.—The horizontal circle of the transit, like that of the compass, measures only horizontal angles; that is, angles between the horizontal projections of the lines of sight. This subject was fully explained in *Compass Surveying*, Part 1.

Let *A B* and *A C*, Fig. 16, be two lines on the ground the angle between which it is desired to measure with the transit. Set up the instrument precisely over the vertex *A*, level it carefully, loosen the upper clamp, and turn the upper plate until the zero of the vernier to be read (say vernier *A*) nearly coincides with the zero of the graduated circle. Clamp the plates, and by turning the upper tangent screw bring the zero of the vernier exactly in line with that of the limb. This operation is called **setting the vernier at zero.** Loosen the lower clamp (if it is not already loosened), and direct the telescope to a flag held at *B* (see Art. **31**). Next, loosen the upper clamp, and direct the telescope to a flag held at *C*. The arc of the graduated circle traversed by the zero point of the vernier will measure the angle *B A C*, whose value can be determined by reading the instrument; that is, by adding the reading of the vernier to that of the limb (see Arts. **16** and **19**). The arc is here described as turned clockwise, or in a direction corresponding to the movement of the hands of a watch when face upwards, and in this case has a value of 143° 30′, which will be the reading of the instrument.

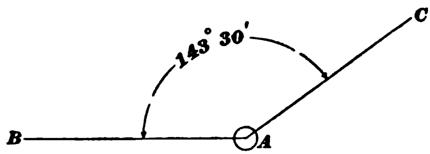


FIG. 16

It is not always necessary nor convenient to set the vernier at zero before measuring an angle. The upper clamp being set, whatever the position of the vernier may be, the telescope is directed to *B*, as explained above, and the reading of the instrument taken. The upper clamp is then loosened, the telescope directed to *C*, and the instrument read again. The difference between the two readings is the value of the angle.

TRAVERSING

DIFFERENT METHODS OF TRAVERSING

33. In surveying, a **traverse** is a series of consecutive courses whose lengths and directions are determined by measurement. For determining the directions of the courses of a traverse, three methods are commonly employed, namely:

1. *By Bearings.*—In this method, which was treated in *Compass Surveying*, Part 1, the directions of the courses of the traverse are determined by their magnetic bearings.

2. *By Azimuths.*—In this method, the directions of the courses of the traverse are determined by their azimuths.

3. *By Deflection Angles, or by Deflection.*—In this method, the relative directions of the courses of the traverse are determined by measuring the angle by which the direction of each course is deflected from the prolongation of the immediately preceding course.

TRAVERSING BY AZIMUTH

34. On Azimuth.—As explained in *Compass Surveying*, Part 1, the **azimuth** of a line is the angle that the line makes with the meridian. It was also explained that azimuth is measured from 0° to 360°, either from the south in the direction west—north—east, or from the north in the direction east—south—west. Sometimes, a line that is neither a true nor a magnetic meridian is used as a line of reference from which azimuths are measured in the same manner as if the line were a meridian. Such a line of reference is called an **assumed meridian**, or, for shortness, a **meridian.**

When the directions of courses are given by their azimuths, a transit is used with its horizontal circle graduated from

0° to 360°, as explained in Art. **13.** It often happens that, by the addition of certain angles, an azimuth greater than 360° is obtained. An azimuth greater than 360° is equal to the same azimuth diminished by 360°. Thus, referring to Fig. 17, where $S N$ is the meridian, it is immaterial whether we say that the azimuth of $A B$ is measured by the arc $M O P Q M O$ or by the arc $M O$; that is, whether we call it 400° (= 360° + 40°), or 40°. In other words, whether the telescope is deflected from the meridian by 400° or by 40°, it will, in its final position, point in the direction of $A B$. As the limb of the transit reads only to 360°, whenever an azi-

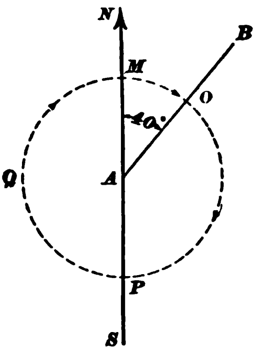


FIG. 17

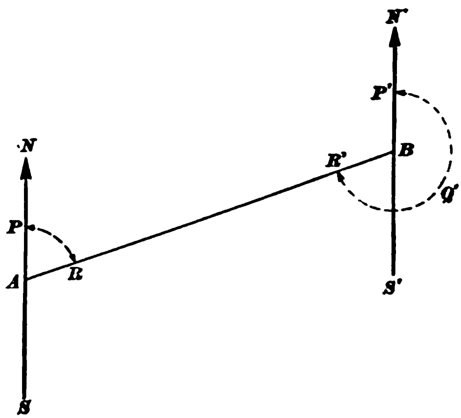


FIG. 18

muth greater than 360° occurs, 360° should be subtracted from it, in order that it may be turned off with the transit.

35. Forward Azimuth and Back Azimuth.—When the azimuth of a line is referred to, the vertex of the angle measuring it is assumed to be at the beginning of the line. If a backsight is taken along the line, from the end to the beginning, the azimuth of this backsight is called the **back azimuth** of the line, while the azimuth proper is called the **forward azimuth.** In Fig. 18, where A is the beginning and B the end of a line, and $S N$ and $S' N'$ are meridians, the forward azimuth of the line is measured by the arc $P R$, and the back azimuth by the arc $P' Q' R'$.

It will be noticed that the back azimuth $P'Q'R'$ is equal to $S'BR' + 180°$, or $NAR + 180°$; that is, equal to the forward azimuth plus 180°. In all cases, the back azimuth of a line is equal to the forward azimuth of the line plus 180°. Should the sum be greater than 360°, it must be replaced by the difference between it and 360°, as explained in the last article. Thus, in Fig. 19, the forward azimuth of AB is 290°; its back azimuth is 290° + 180°, or 470°, or, subtracting 360° from this sum, 470° − 360°, or 110°.

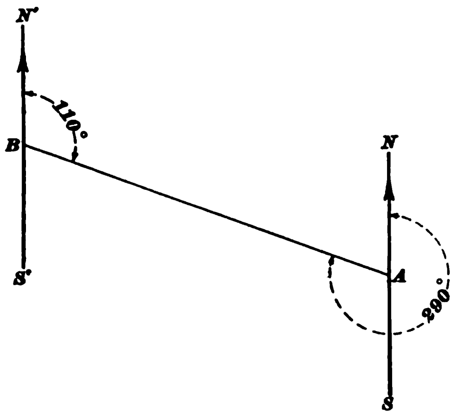


FIG. 19

36. Orienting the Transit.—To **orient** a transit is to set its vernier (usually vernier A) to read the azimuth of a given line, and then direct the telescope along that line. It is evident that, if the upper plate is then unclamped and revolved until the vernier reads zero, the telescope will be directed toward the north or toward the south, according to which point the azimuths are reckoned from. For this reason, the orienting of the transit is usually said to consist in so setting the instrument that the vernier of its horizontal limb reads zero when the telescope is directed toward the north if azimuths are reckoned from the north, and toward the south if azimuths are reckoned from the south.

37. The Process of Azimuth Traversing.—Suppose that A, Fig. 20, is a given point on a line AF previously surveyed, and that it is desired to connect this point with the point E by a traverse following the contour of the surface in such a manner as to give about the minimum rise and fall. The true bearing of AF, as previously determined, is N 42° 36′ W; therefore, its azimuth, counted from the north, is

360° − 42° 36′ = 317° 24′. The points *B*, *C*, and *D* are chosen in advance of the survey, in such positions as will fulfil the required conditions as nearly as can be judged, each point being selected while the instrument is being moved forwards, set up, and oriented at the preceding point. The instrumental operations in running this traverse are as follows:

The transit is first set up at *A* and oriented by setting the vernier (say vernier *A*), at 317° 24′ (azimuth of *A F*) and directing the telescope, with the upper plate clamped, along *A F*, the point *F* being marked by a flag. The vernier clamp is now loosened, the telescope is turned in azimuth and

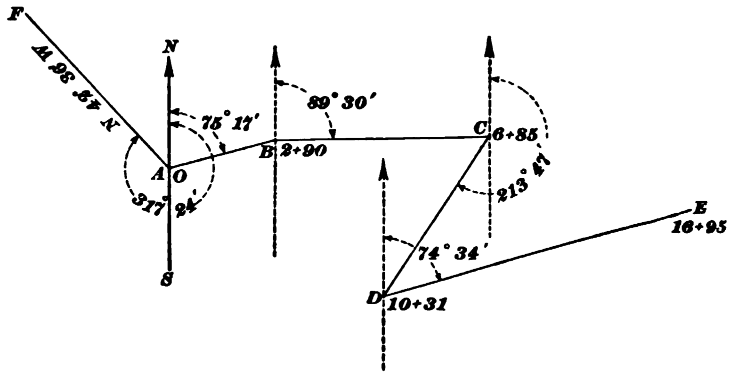


Fig. 20

directed to a flag held at *B*, and the vernier is read. The reading, which in this case is 75° 17′, is recorded as the azimuth of *A B*. As *A* is the initial point of the survey, complete information as to how the instrument is oriented should be described by means of a sketch or a written statement. As a check, the magnetic bearing of *A F* and that of the last line should be taken and recorded. Suppose the magnetic bearing of *A F* to be N 40° 10′ W; as the true bearing is N 42° 36′ W, the declination is 2° 26′ west. This value should be noted.

The instrument is now moved forwards, set up at *B*, and oriented by making the reading of the vernier equal to the azimuth of *B A*, which is equal to that of *A B* plus 180°; that is, 75° 17′ + 180° = 255° 17′ (see Art. 35). The upper clamp being set, the telescope is directed to *A*; the lower clamp is set, the upper clamp loosened, the telescope directed to *C*, and the vernier read. The reading is found to be 89° 30′, which is recorded as the azimuth of *B C*. The instrument is then moved to *C*, and the azimuth of *CD* is determined as explained for *B C*. This azimuth is found to be 213° 47′. The instrument is moved to *D* and oriented by backsighting on *C*. The forward azimuth of *CD* being 213° 47′, the back azimuth is 213° 47′ + 180° = 393° 47′, or 393° 47′ − 360° = 33° 47′ (see Art. 34). Setting the vernier at this reading, and directing the telescope to *C*, the transit is oriented at *D*. The upper clamp is now loosened, the telescope directed to *E*, and the vernier read again, the reading being the azimuth of *DE*.

The magnetic bearing of *D E* is now taken; suppose it to be N 77° 15′ E. As the declination is 2° 26′ west, the approximate true bearing of *D E*, as obtained from the compass, is N 74° 49′ E. Since the line has an azimuth of 74° 34′, its true bearing is evidently N 74° 34′ E, which agrees with that given by the compass within the limit of accuracy of the latter instrument, with which angles are read to the nearest quarter of a degree. In a traverse consisting of many lines, it is advisable to take the magnetic bearing of every third or fourth line, and compare it with the true bearing obtained from the azimuth of the line.

38. Field Notes of an Azimuth Traverse.—The notes on the next page are those of the azimuth traverse shown in Fig. 20. They are similar to those used in compass surveying, and do not require any explanation. The distances, which are obtained by merely subtracting the number of each instrument station from the number of the succeeding instrument station, are recorded in the fourth column. This is usually done in the office.

39. Latitude and Longitude Ranges Computed From Azimuths.—The ranges of a line whose azimuth is known are calculated by formulas similar to those used in

Compass Surveying, Part 2, the azimuth of the line being used instead of the bearing. When azimuths are reckoned from

Station	Azimuth	True Bearing	Distances	Remarks
16 + 95	74° 34′	N 74° 49′ E		End of line.
10 + 31	213° 47′			
6 + 85	89° 30′			
2 + 90	75° 17′			
F, 64	317° 24′	N 42° 36′ W		
0				Station 0 is at Sta. 58 + 60 of surveyed line of O. & B. R. R. Oriented by forward azimuth on Inst. Point *F*, at Sta. 64 of same. True bearing N 42° 36′ W. Declination 2° 26′ west.

the north, these formulas have the advantage that they give both the numerical value and the algebraic sign of each range. In Fig. 21, *S N* is the meridian, *A B* is a line whose

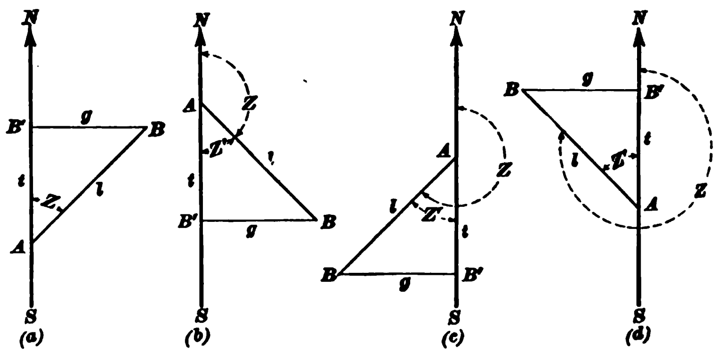


Fig. 21

azimuth is *Z*, length *l*, and ranges *t* and *g*. The general formulas for finding *t* and *g* are, whatever may be the value of *Z*,

$$t = l \cos Z \qquad (1)$$
$$g = l \sin Z \qquad (2)$$

In Fig. 21 (*a*), Z is acute, and the formulas at once follow from the right triangle $A B B'$.

In Fig. 21 (*b*), we have, since south latitude ranges are negative, and $\cos Z' = -\cos (180 - Z') = -\cos Z$, and $\sin Z' = \sin (180° - Z') = \sin Z$,

$$t = -B' A = -l \cos Z' = -l \times (-\cos Z) = l \cos Z$$
$$g = l \sin Z' = l \sin Z$$

In Fig. 21 (*c*), both t and g are negative, and, since $\cos Z = \cos (180° + Z') = -\cos Z'$, and $\sin Z = -\sin Z'$, we have,

$$t = -B' A = -l \cos Z' = l \times (-\cos Z') = l \cos Z$$
$$g = -B B' = -l \sin Z' = l \times (-\sin Z') = l \sin Z$$

Let the student verify that the formulas apply likewise to Fig. 21 (*d*).

40. Platting an Azimuth Traverse.—The most accurate method of platting a traverse is by latitude and longitude ranges. This method was fully explained in *Compass Surveying*, Part 2. The ranges are calculated by the formulas given in the preceding article.

The plat may be made with some degree of expedition by means of an ordinary circular or semicircular protractor, a decimally divided scale, and some means for drawing parallel lines, such as a parallel ruler moving on rollers, a **T** square and triangle, or a pair of triangles. The position of the initial point A, Fig. 20, and the direction of the meridian are chosen so that the whole traversed line may be platted on the sheet. Whenever convenient, the meridian should be made vertical on the plat.

It may or it may not be desirable to show the line by which the instrument is first oriented, as the line $A F$; when shown, it is platted in substantially the same manner as any other line of the survey; in the present case, this line has a bearing N 42° 36′ W. The protractor is placed on the left-hand (west) side of the meridian line, with its straightedge coinciding with this line and its center at the initial point A, and a point is marked on the paper at the graduation mark for 42° 36′ on the graduated semicircular edge of the protractor. The protractor is then removed and a line drawn through the initial point *A* and the point marked. It will be noticed that the direction of the line *AF* is laid off by its bearing. When the direction of a line is laid off by its azimuth, however, the angle is laid off from the north point in the direction of the movement of the hands of a watch; that is, in the direction east—south—west—north, up to 360°. If the protractor used is a complete circle, it is placed on the paper with its center at the initial point of the line and its zero point on the meridian, north of the initial point, and the azimuth angle is laid off on the graduated circular edge in the direction just described. A similar method is employed with a semicircular protractor if the azimuth is not greater than 180°.

When a semicircular protractor is used, however, and the azimuth exceeds 180°, it cannot be laid off directly since the protractor does not measure angles greater than 180°. In such a case, the azimuth is subtracted from 360° and the remainder is laid off toward the west of north; that is, from the north in the direction north—west—south, by placing the protractor on the left-hand (west) side of the meridian with its center at the initial point of the line and its zero on the meridian north of the vertex of the angle.

Protractors are usually graduated to degrees and half degrees. Minutes must be estimated by the eye.

TRAVERSING BY DEFLECTIONS

41. The Process.—The relative directions of the courses of a traverse may be determined by measuring, with a transit, the angle by which each line is deflected from the prolongation of the one immediately preceding. Such a traverse is sometimes called an **angle line**. The method by azimuths is far preferable, and the student is advised to use it exclusively. In the field notes, an angle is sometimes recorded *R* or *L*, according as it is turned to the right or to the left of the line from which it is measured.

The process of running a traverse by deflections is as follows: Let the line to be run be *A B C D E*, Fig. 22, which is the same as that shown in Fig. 20. The transit is set up at the starting point *A*, and, with the vernier set at zero, the telescope is directed to a flag held at *F*, the position of the line *A F* being supposed known from a previous survey. Or, *A F* may be a line defined by the point *A* and some permanent object, such as a church spire or a tree. The important thing is to have a fixed reference line from which to start the survey. The vernier plate is then unclamped, the telescope turned in azimuth and directed toward a flag

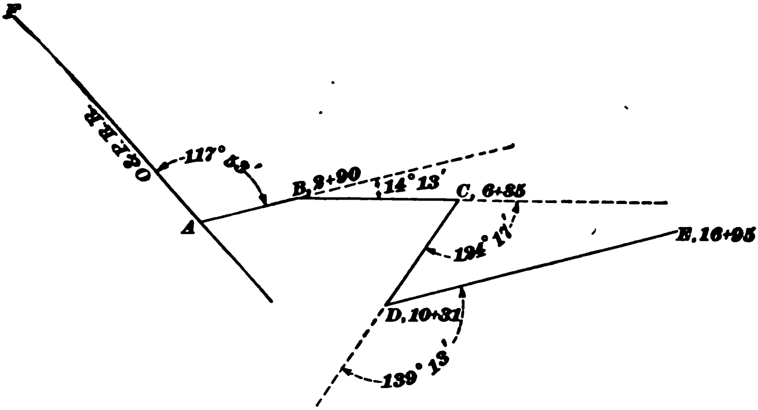


Fig. 22

held at the next point *B*, and the vernier read. The reading, which in this case is 117° 53′, is recorded as the deflection from *A F*. As this deflection is to the right of *A F*, it is marked *R*.

The instrument is now moved forwards and set up at *B*, and the flag that was at *F* is moved to *A*. With the vernier again set at zero, the telescope is directed to the flag at *A*, and then plunged. The upper clamp is loosened, the telescope is directed to *C*, and the vernier read. The reading, which is 14° 13′, is the angle made by *B C* with the prolongation of *A B*. As it is turned to the right of that prolongation, it is marked *R*. The instrument is then moved to *C*, and the deflection to *D* measured in a similar manner. The same operation is repeated at each of the other angle points of the line. The distance between *A* and *B* is measured when the transit is at *A*, the transitman keeping the head chainman in line; the distance between *B* and *C*, when the transit is at *B*, etc.

42. Checking by the Needle—Deduced Bearings. Notwithstanding the greatest care, errors in the reading and recording of angles will occur. The best check to such errors is obtained by reading the magnetic needle. Although this is not an exact check, owing to the lack of

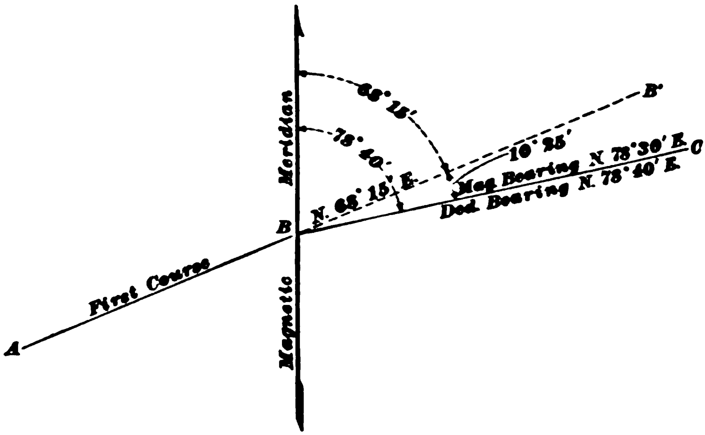
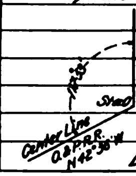


FIG. 23

precision in reading the needle and to the influence of local attraction, yet it is very convenient and is used very generally. The bearing of each line is taken and recorded immediately after the deflection angle has been read off the vernier.

The method is illustrated in Fig. 23, in which the lines *A B* and *B C* represent the first two lines of a deflection traverse. The bearing of *A B*, as determined from *B* (not from *A* on account of the local attraction at *A*) is supposed to be S 68° 15′ W. The angle deflected from *B* to *C* is 10° 25′ to the right. The bearing of the line *A B* is N 68° 15′ E, or 68° 15′ to the right of north. Since both the bearing of the first course and the angle turned are to the right, the sum of the bearing and angle should be the bearing of the second course. The bearing of a course, as thus computed from the bearing of the adjacent course and the angle between the courses, is called its **deduced or calculated bearing.** The deduced bearing of the course *B C* is, therefore, N (68° 15′ + 10° 25′) E, or N 78° 40′ E. The magnetic bearing of the course *B C*, as read from the needle, is N 78° 30′ E, which agrees with the deduced bearing within the limits of accuracy of the compass.

43. Field Notes.—A good form for the field notes of a deflection traverse like that described in Art. **41** is given below. The first four columns are supposed to cover some, or all, of the left-hand page of the notebook. The last

Station	Deflection	True Bearing	Ded. Bearing	Remarks
16 + 95				End of Line
10 + 31	L 139° 13′	N 74° 49′ E	N 74° 34′ E	
6 + 85	R 124° 17′	S 33° 49′ W	S 33° 47′ W	
2 + 90	R 14° 13′	N 89° 19′ E	N 89° 30′ E	
0			N 75° 17′ E	Sta. 0 H.B. of Switch
				at Benton Sta.
				Declination 2° 26′ W

column represents the entire right-hand page, which is used for remarks and sketches. The sketch shown locates Station *0* and describes the direction of the first course by means of the angle that it makes with the center line of the railroad at this station.

Another way of recording the work is simply to make a sketch on which angles and distances are marked, with whatever explanations are necessary.

TRANSIT SURVEY OF A CLOSED FIELD

THE FIELD WORK

44. Measuring the Angles.—The corners of the field to be surveyed having been marked, the relative directions of the boundary lines are determined as for a traverse, setting the instrument at the different corners in succession, and using either the method of azimuths or that of deflection; the former is by far the better of the two. If the direction of the true meridian is known, azimuths are referred to it. If not, a reference line, defined by permanent marks or monuments, may be assumed as a meridian, to which all azimuths are referred.

Another method consists in setting the instrument at the different corners in succession and measuring directly the angle between the two lines meeting at each corner.

45. Field With Irregular Boundary.—When one or more of the boundary lines are not straight, as in fields bounded by rivers or lakes, a traverse line is run as close to the boundary line as possible, and offsets are taken from the traverse line to those points at which the direction of the boundary line changes.

46. Checking the Angles.—After the work is finished, the accuracy of the measurement of the angles is checked by one of the following methods, according to the method used in measuring the angles:

(*a*) *When the Angles Are Measured Directly.*—As explained in *Geometry*, Part 1, the sum S of the interior angles of a polygon of n sides is given by the formula

$$S = 180° \times (n - 2)$$

It should be borne in mind, in applying this formula, that reentrant angles, as that at A, Fig. 24, are greater than 180°.

The angle A should be called 260°, not 100°. The sum of the measured angles should satisfy the formula within about 2 minutes per angle.

(*b*) *When the Deflection Method is Used.*—Each deflection angle, being the angle made by a side with the prolongation of the preceding, is an exterior angle of the polygon. The sum of all these angles should be equal to 360°, within the limits mentioned above.

(*c*) *When Azimuths Are Used.*—The azimuth of the first line is measured, or laid off, when the transit is set at the first corner, and then the azimuth of each line is determined from that of the preceding, as already explained. In this manner, the azimuth of the first line may be determined from that of the last, by again setting the transit at the starting corner. This second determination of the azimuth of the first line should be approximately equal to the one with which the survey was started. Besides, the sum of the angles of the polygon should satisfy the formula already given. The manner of determining the angle between two lines whose azimuths are known is similar to that used when the bearings are known, and does not require special explanation.

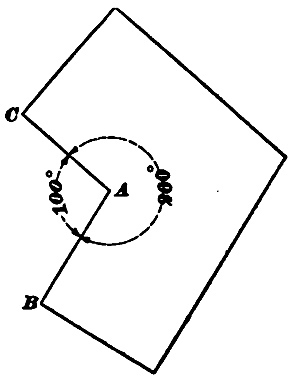


FIG. 24

BALANCING THE SURVEY

I. WHEN THERE IS A SMALL ANGULAR ERROR

47. Latitude and Longitude Ranges.—The latitude and longitude ranges are computed by the formulas in Art. **39.** If the azimuths have not been measured directly and the azimuth of one of the sides is known, the azimuths of the other sides can be readily determined from the measured angles. If no azimuth has been measured, any of the sides is assumed as a meridian, and azimuths and ranges are referred to it.

48. General Remark on Balancing.—As in the case of compass surveying, it will generally be found that a transit survey does not close. Here, however, the method of balancing explained in *Compass Surveying*, Part 2, cannot be applied, as that method assumes the possibility of larger errors in the angles than are likely to occur in transit work.

49. Case Where the Angular Error Is Small.—When, in testing the angular measurement of a survey that does not close, it is found that the angular error, though sufficiently great to be considered, is small as compared with the error of closure, the following rule may be applied for correcting the ranges:

Rule.—*As the arithmetical sum of all the ranges of one kind is to the corresponding range of any course, so is the total error in the sum of the ranges of that kind to the correction to be applied to the corresponding range of that course.*

In order to express this by a formula,

Let R = arithmetical sum of ranges of one kind;

r = corresponding range of any course;

E = total error in latitude or longitude ranges;

c = correction in latitude or longitude range, to be applied to r.

Then, $R : r = E : c$; whence,

$$c = \frac{E\,r}{R}$$

If it is desired to weight the courses when using this rule, the ranges of each separate course are multiplied by the weight assigned to that course, and the ratio between each weighted range and the arithmetical sum of all the weighted ranges of the same kind is used in distributing the error. In other words, the weighted range is substituted for r and the arithmetical sum of the weighted ranges for R in the above formula.

The student should notice that only arithmetical values are used in applying the formula. The correction for any range is added arithmetically to the original range, if that will diminish the total error; otherwise, it should be subtracted. Thus, if the sum of the northings is greater than that of the southings, the northings should be diminished, and the southings increased.

II. WHEN THERE IS NO ANGULAR ERROR

50. General Consideration.—The foregoing rule for balancing a survey is founded on the assumption that the error of closure is due partly to erroneous angles and partly to errors in chaining. When, however, the angles are measured accurately, the error of closure may be assumed to be mainly due to erroneous chaining. In this case, the survey must be balanced by correcting the lengths of the sides, which must be so adjusted as to give a closed figure approaching the true conditions as nearly as possible. In adjusting the lengths of the lines due consideration should be given to the following principles:

Principle I.—*Measurements made either up or down a slope are likely to be too long as compared with measurements made under similar conditions on level ground.*

Principle II.—*Error in chaining is more likely to occur in lines measured over rough ground or under unfavorable conditions than in lines measured over smooth ground and under favorable conditions.*

These principles may serve as a guide in balancing a transit survey, an operation that must be done by trial, as no exact method has yet been devised.

51. Trial Method.—Let $ABCDEF$, Fig. 25, represent the plat of a survey that contains no angular error and does not close, and in which the line AF represents the error of closure.

From a mere inspection of the plat it is usually easy to determine which lines must be lengthened and which lines shortened, in order to close the survey. Should any difficulty be experienced in determining this, prolong the closing line, in both directions if necessary, and draw perpendiculars to it from the extreme angles of the plat. The angles from which the perpendiculars are drawn will lie between lines that are to be lengthened and those that are to be shortened. Thus, in the figure, the closing line FA is prolonged in both directions and the perpendiculars BM and EN drawn to it from the extreme angles of the plat. AM and NF are evidently the projections of the lines that must be lengthened, and

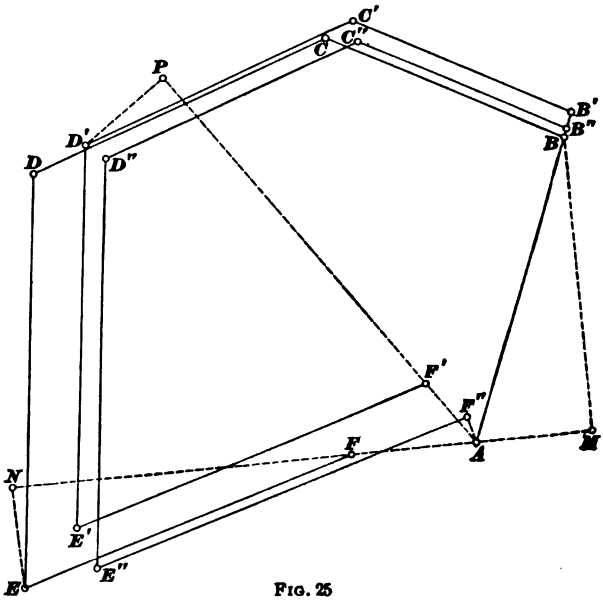


FIG. 25

MN the projection of the lines that must be shortened, in order to close the survey. Consequently, the lines AB and EF must be lengthened and the lines BC, CD, and DE, shortened; the angles B and E lie between the lines that must be lengthened and those that must be shortened.

Measure the error of closure AF to the scale of the drawing, and divide the amount of this error by the sum of the lengths of all the sides. Multiply each line that is to be lengthened by one plus the quotient, and each line that is to be shortened by one minus the quotient. Then, beginning with a line adjacent to the error of closure and proceeding in order around the plat, lay off each line of a length equal to the product thus obtained and in a direction parallel to its former position. When this has been done accurately, it will be found that the error of closure is much diminished, and in some cases wholly corrected, but in most cases a new error of closure, as $F'A$, will be formed.

The new error of closure is corrected by what is substantially a repetition of the process just described. The line AF' representing the new error of closure is divided by the sum of the lengths of the new sides, and each side is multiplied by one plus the quotient or one minus the quotient, according as it is to be lengthened or shortened. From an inspection of the figure, it is found that the lines AB', $B'C'$, and $C'D'$ are to be shortened, and the lines $D'E'$ and $E'F'$ are to be lengthened. It will be noticed that the line AB that was lengthened in the first trial is shortened in the second trial, and that the line DE that was shortened in the first trial is lengthened in the second. The second trial gives the smaller error of closure $F''A$, and by repeating the operation, the error of closure can be made smaller and smaller. As this method is partly graphic, and therefore only approximate, it will be necessary to make the last slight adjustments arbitrarily, in order to cause the survey to balance accurately.

52. Graphic Method.—When a survey having no angular error does not close, it can be balanced by arbitrarily distributing the error among all the lines of the survey by means of the following graphic construction:

Let $ABCDEA'$, Fig. 26, represent the traverse of the boundary of a tract of land, in which the error of closure AA' is to be corrected by adjusting the lengths of the sides. Draw the line AA', representing the error of closure, divide it into a number of equal parts that is one less than the figure has sides, and, beginning at A', number consecutively the points of division, including the extremities of the line, as shown. From each angle of the tract, draw a line parallel to *A A′* and in the same direction from the angle that *A′* is from *A*; make each line of the same length as *A A′*, divide it into the same number of equal parts, and number the points of division in the same manner. Now, beginning at *A′* draw the lines *1–1*, *2–2*, *3–3*, *4–4*, and *5–1*, joining corresponding points in succession around the figure, prolonging

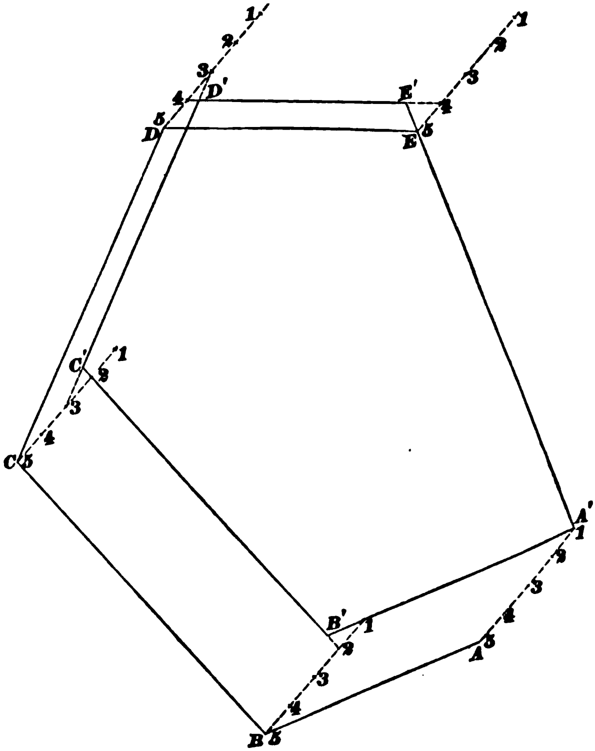


Fig. 26

each two adjacent lines to an intersection. The lines so drawn will be parallel, respectively, to the original sides of the plat, the last line coinciding with the last side *E A′*, and will form a closed figure. This method merely distributes the error arbitrarily among all the lines of the survey, in such manner as to affect to the greatest degree those lines that are most nearly parallel to the error of closure.

ADJUSTMENTS OF THE TRANSIT

53. Definitions.—The high degree of accuracy attained in measurements made with a transit requires delicacy and exactness in the construction of certain essential parts, and extreme precision in their arrangement. Such parts are liable to become disarranged or disturbed from their true positions by shocks and other causes; this does not necessarily injure the instrument itself, but it impairs its accuracy. In order that such parts, when disturbed, can be rearranged, they are made movable, or **adjustable,** by means of accurately fitted screws. An adjustable part is said to be **in adjustment** or **out of adjustment,** according as it is or is not in its proper position. The instrument itself is said to be in adjustment or out of adjustment, according as its parts are or are not in adjustment. **To adjust,** or **make the adjustments,** of an instrument, is so to arrange its parts that they shall have their proper relative positions.

54. Conditions That Must be Fulfilled by a Transit in Adjustment.—When a transit is in perfect adjustment, it must, after being leveled, fulfil the following conditions:

1. The centers must revolve on a truly vertical axis, so that the plate levels will remain centered during a complete revolution.

2. The line of collimation must be perpendicular to the transverse axis of the telescope, so that it will be in the same straight line when the telescope is plunged.

3. The axis of revolution (the transverse axis) of the telescope must be horizontal, and, therefore, perpendicular to the vertical axis of the instrument.

When a transit has a level and a vertical arc or circle attached to the telescope, it should fulfil the following additional conditions:

4. The line of collimation must be parallel to a line

tangent to the tube of the telescope level at its middle point, so that the line of collimation will be horizontal when the bubble of the telescope level stands at the middle of its tube.

5. The vernier of the vertical arc or circle must read zero when the line of collimation is horizontal.

The adjustments establishing these conditions should be made in the order in which the conditions are stated. The best time for adjusting an instrument is on a cloudy day, or in the early morning before the air has become heated and the sun dazzling. An open and nearly level space affording an unobstructed sight for at least 400 feet from the transit in opposite directions should be chosen for making the adjustments. In setting up the instrument, the feet of the tripod should be planted firmly in solid ground that is not subject to jars from heavy machinery or other causes, so that its position will not be disturbed.

55. First Adjustment.—To make the axes of the plate levels perpendicular to the vertical axis of the instrument, so that when the bubbles are centered by the leveling screws the axis of the centers will be truly vertical and the plates will revolve in a horizontal plane.

This adjustment is substantially the same as for the compass, and is performed as follows:

With the upper clamp set and the lower clamp loose, turn the instrument so that the plate levels will be, respectively, parallel to the lines determined by the two pairs of leveling screws, and bring each bubble to the middle of its tube by means of the corresponding pair of leveling screws. Next, turn the instrument half way around; that is, revolve it in azimuth through 180°, so that each level will be in the reverse position with respect to the same pair of leveling screws. If the levels are in adjustment, the bubbles will remain in the centers of the tubes. If the bubbles do not remain so, but run to either end, bring them half way back to the middle of the tubes by means of the capstan-headed screws attached to the ends of the tubes, and the rest of the way back by the leveling screws. Then revolve the instrument again through 180° and observe the positions of the bubbles. Sometimes this adjustment is made by one trial, but it is usually necessary to repeat the operation.

56. Second Adjustment.—To make the line of sight perpendicular to the transverse axis of the telescope.

The manner of performing this adjustment is illustrated in Fig. 27. Set and level the instrument at a point *A*, and direct the telescope to some well-defined point *B* a few hundred feet distant. Both clamps being set, plunge the telescope and set another point, as a marking pin or a tack, in the top of a stake, a few hundred feet away, on the opposite side of the instrument from *B*. If the line of sight is truly perpendicular to the transverse axis of the telescope, this point will be in the prolongation of *B A*. In order to ascertain if this is the case, loosen either clamp, turn the

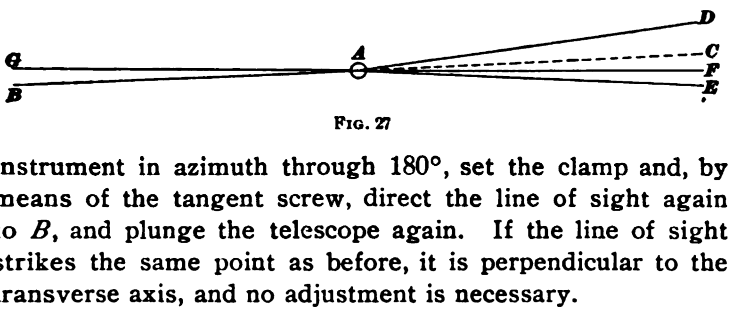


FIG. 27

instrument in azimuth through 180°, set the clamp and, by means of the tangent screw, direct the line of sight again to *B*, and plunge the telescope again. If the line of sight strikes the same point as before, it is perpendicular to the transverse axis, and no adjustment is necessary.

But, suppose that the point set after the first plunging is at *D*, and that the point set after the second plunging is at *E*, to one side of *D*. This will show that the line of sight must be adjusted. In order to make this adjustment, measure the distance *D E* (the points *D* and *E* are set at the same distance from *A*, as nearly as can be estimated by the eye), and set a mark at *F*, making the distance *E F* equal to one-fourth *D E*. Move the cross-hairs by means of the capstan-headed screws until the vertical hair exactly covers the mark at *F*, being careful to move them in the opposite direction to that in which it would appear they should move. In order to move the cross-hairs, loosen the screw on the side of the telescope tube away from which they are to be moved, and then tighten the screw on the opposite side. Bring the screws to a firm bearing, but do not turn them so tight as to cause any strain. The cross-hairs having been thus moved and the telescope plunged back, the line of sight will not fall on the point *B*, but on a point *G*, at a distance from *B* equal to *E F*. By means of either tangent screw, bring the line of sight again on the point *B*, then plunge the telescope. If the adjustment is perfect, the line of sight will strike the point *C*, which is in the prolongation of the line *B A* and midway between *D* and *E*. The adjustment should be tested by reversing the instrument again in azimuth, then plunging the telescope and sighting forwards as before. It may be necessary to repeat the operation several times in order to obtain an exact adjustment.

57. Third Adjustment.—To make the transverse axis of the telescope perpendicular to the vertical axis of the instrument, so that when the instrument is leveled the transverse axis of the telescope will be horizontal.

Suspend a fine, smooth plumb-line from a rigid support at as high an elevation as convenient and at a distance from the instrument not exceeding the length of the line. The weight should be suspended in a pail of water, care being exercised that it does not touch the bottom of the pail and that the line is not exposed to wind. With both plate bubbles in the middle of their tubes, direct the line of sight to the upper end of the plumb-line; then, turning the telescope slowly downwards, notice whether the intersection of the cross-hairs exactly follows the line throughout its length. If it does follow it, the line of collimation revolves in a vertical plane. The plumb-line will usually vibrate slightly, but its mean position can be estimated by the eye. If the intersection of the cross-hairs does not coincide with the plumb-line throughout its length, but diverges to one side as it approaches the bottom of the line, the error must be corrected by raising or lowering one end of the transverse axis of the telescope, which is adjustable by means of screws placed in one of the standards. If the intersection of the cross-hairs diverges on the side of the plumb-line toward the adjustable end of the transverse axis, this end is to be lowered; if on the opposite side, it is to be raised.

This adjustment can also be tested and made in the following manner:

Level the instrument, and direct the telescope to some well-defined point on a church spire or other high object, as the point *A*, Fig. 28. Having set both the upper and the lower clamp, depress the object end of the telescope and set a point in the line of sight on the ground at the base of the object; loosen the upper clamp, reverse the instrument in azimuth, plunge the telescope, sight again on the high point, again turn the telescope downwards, and notice if the line of sight strikes the same point as before. If it does, the transverse axis of the telescope is horizontal. If the point first set is the point *B*, and the second line of sight passes through *D*, instead of *B*, the transverse axis is not horizontal, and must be adjusted. The adjustment is made by raising or lowering one end of the transverse axis (in this case the right-hand end would have to be lowered), and again repeating the test, until the points *B* and *D* coincide (that is, until the line of sight, when the telescope is depressed, strikes the same point, as *C*, both before and after reversal).

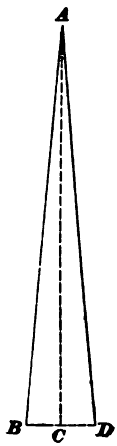


FIG. 28

58. Fourth Adjustment.—To make the bubble of the telescope level stand in the middle of its tube when the line of sight is horizontal. This adjustment makes the transit adapted to leveling work. It is the same as that of a regular level, and will be described in another section of this Course.

59. Fifth Adjustment.—To make the vernier of the vertical arc or circle read zero when the line of sight is horizontal. To perform this adjustment, level the instrument, turn the telescope on its transverse axis until the bubble in the attached level is nearly in the middle of its tube; clamp the telescope, and center the bubble of the attached level exactly by means of the gradienter screw *g*, Fig. 1. If the vernier of the vertical limb does not read zero, set it so that it will read zero by means of the capstan-headed screws that control it.

This adjustment is not strictly necessary, provided that the reading of the vernier when the telescope is horizontal is observed and noted. This reading is called the **index error** of the vertical circle or vernier and should be allowed for in reading vertical angles.

60. An Important Test.—For convenience in directing the telescope to a signal, it is desirable that the vertical cross-hair should be truly vertical, and the other truly horizontal. The two cross-hairs are attached to the adjustable diaphragm exactly at right angles to each other, so that when one is vertical the other is horizontal. In order to test the vertical cross-wire, sight on any sharply defined point, focusing the telescope perfectly and bringing the point exactly in range with either end of the vertical cross-wire. Then turn the telescope on its transverse axis slowly and notice if the point sighted remains on the cross-wire throughout the motion. Should any deviation be discernible, loosen the capstan-headed screws that control the cross-hairs, and by the pressure of the hand, or by tapping lightly against the heads of the screws outside the telescope tube, rotate the cross-hairs very carefully in the direction opposite that in which they should apparently be rotated, until the point sighted remains on the cross-hair throughout the motion of the telescope. Then tighten the screws sufficiently to bring them to a firm bearing without straining them, and repeat the test.

This test should be applied before performing the third adjustment for the line of collimation. If the plate levels are in perfect adjustment, it can also be made by sighting at a plumb-line suspended at a suitable height and distance, with the plate levels centered perfectly, and observing whether the vertical cross-hair coincides exactly with the plumb-line.

TRANSIT SURVEYING

(PART 2)

DETERMINATION OF TRUE MERIDIAN

PRELIMINARY DEFINITIONS AND EXPLA-NATIONS

TERRESTRIAL CIRCLES–LATITUDE AND LONGITUDE

1. The Earth's Axis and Poles.—The earth revolves, or turns, once in 24 hours about an imaginary north-and-south line called its **axis.** The points where this line meets the earth's surface are called the earth's **poles,** one being the **north pole** and the other the **south pole.**

2. Equator and Meridians.—The **equatorial plane,** or **plane of the equator,** is a plane passing through the center of the earth, and perpendicular to the earth's axis. It cuts the surface of the earth in a circle called the **terrestrial equator.**

Any plane containing the earth's axis is called a **meridian plane,** and the circle in which it intersects the earth's surface is called a **meridian line,** or simply a **meridian.**

In Fig. 1, SN is the axis of the earth; S is the south pole; N, the north pole; $EQRT$, the earth's equator; and the circles $NRSE$, NGS, and NFS are meridians. All meridian planes are perpendicular to the plane of the equator. It is evident that through any place on the earth's surface a meridian may be passed; such a meridian is called the meridian of that place.

It will be noticed that meridians are circles, not straight lines, and that they all meet at the poles. But, for the purposes of ordinary surveying, where only comparatively small areas are dealt with, meridians are treated as parallel straight north-and-south lines; or, rather, all points within the area surveyed are assumed to have the same meridian. The error arising from this mode of treatment is too small to be considered in such work.

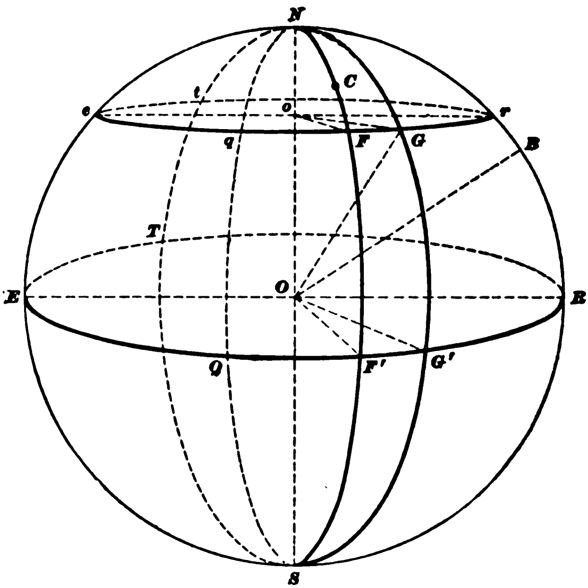


FIG. 1

3. Latitude.—The **latitude** of a point on the earth's surface is the angle that the radius of the earth passing through that point makes with the plane of the equator. In Fig. 1, the latitudes of B and G are, respectively, the angles BOR and GOG'. These angles are measured by the arcs RB and $G'G$ of the meridians passing through the two points. The latitude of a point may, therefore, be also defined as the angular distance of the point from the equator, that distance being measured on the meridian through the point, and being the number of degrees in the arc of the meridian included between the equator and the point. When it is stated that the latitudes of the points *B*, *G*, *F*, are, respectively, *R B*, *G' G*, *F' F*, it should be understood that these arcs are to be expressed in degrees, latitude being an angular quantity, not a linear quantity.

The latitude of a point is said to be north or south according as the point is north or south of the equator.

4. Parallels of Latitude.—Any plane parallel to the equator, that is, perpendicular to the earth's axis, cuts the earth's surface in a circle called a **parallel of latitude,** or simply a **parallel.** The circle *e q r t*, Fig. 1, whose plane is parallel to that of *E Q R T*, is a parallel of latitude. All points, as *r*, *G*, *F*, on a parallel have the same latitude, as the arcs *R r*, *G' G*, *F' F* are evidently equal.

5. Longitude.—The **longitude** of a point on the earth's surface is the angle between the meridian plane through that point and another meridian plane assumed as a plane of reference. This angle is also referred to as the angle between the meridian passing through the given point and the meridian determined by the plane of reference. The meridian of Greenwich, England, is generally taken as a reference meridian. Suppose that *N G S*, Fig. 1, is the meridian of Greenwich; the longitude of the point *C*, referred to that meridian, is the angle between the planes of the meridians *N G S* and *N C S*. This angle is the same as *G' O F'*, or *G O F*, and is measured by the arc *G' F'* of the equator, or by the arc, as *G F*, of any parallel included between the meridian of Greenwich and the meridian through the point considered. The longitude of a point may also be defined as the angular distance of the point west or east of the reference meridian.

Longitude is usually counted from the reference meridian toward the west, from 0° to 360°.

NOTE.—It will be noticed that some terms, as *latitude* and *longitude*, are here used in a sense somewhat different from that given to them in previous Sections of this Course. The circumstances under which these terms are employed, however, always indicate plainly in what sense they should be taken.

THE CELESTIAL SPHERE

6. Definition.—The sun and stars seem to be and move on the inner surface of an immense sphere. This sphere is called the **celestial sphere, or the heavens.**

7. The Celestial Poles.—The points where the axis of the earth, produced, meets the celestial sphere are called the **celestial poles.** The one nearer the north pole of the earth is called the **north pole;** the other, the **south pole.**

8. The Celestial Equator and Meridians.—The **celestial equator** is the circle in which the plane of the terrestrial equator intersects the celestial sphere.

A **celestial meridian** is a circle in which a meridian plane intersects the celestial sphere.

9. The Zenith.—If a vertical line at any point on the earth's surface is produced upwards, the point where it pierces the heavens is called the **zenith** of that point.

10. The Celestial Horizon.—The **celestial horizon** of any point is the circle in which a horizontal plane through that point intersects the celestial sphere.

11. Altitude.—The **altitude** of a point, with regard to a point on the horizon, is the angle that a line from the latter to the former point makes with the plane of the horizon.

12. Latitude and Longitude Measured on Celestial Circles.—In Fig. 2 are represented the earth *O*—*n r s e* and the celestial sphere *O*—*N R S E*. The dimensions of the earth, as compared with those of the sphere, are shown very much exaggerated. In reality, the earth is so small, compared with the celestial sphere, that for all astronomical purposes it is treated as a point. The earth's axis *s n*, when produced, meets the sphere in the celestial poles *N* and *S*. The meridian planes *n a s*, *n g s*, *n r s*, when extended, intersect the sphere in the celestial meridians *N A S*, *N G S*, and *N R S*, respectively. The celestial equator *E Q R* is the intersection of the celestial sphere and the plane of the terrestrial equator *e q r*.

Assuming *n g s*, or *N G S*, to be the meridian of Greenwich (or any other reference meridian), and *n a s*, or *N A S*, the meridian of a point *a* on the earth's surface, the longitude of that point is measured by the arc $a' g'$, on the terrestrial equator, or by the arc $A' G'$, on the celestial equator. The latitude of *a* is the angle $a\,O\,a'$, or $A\,O\,A'$, measured by either $a'\,a$ or $A'\,A$. Since $O\,a$ is a radius of the earth, or a vertical

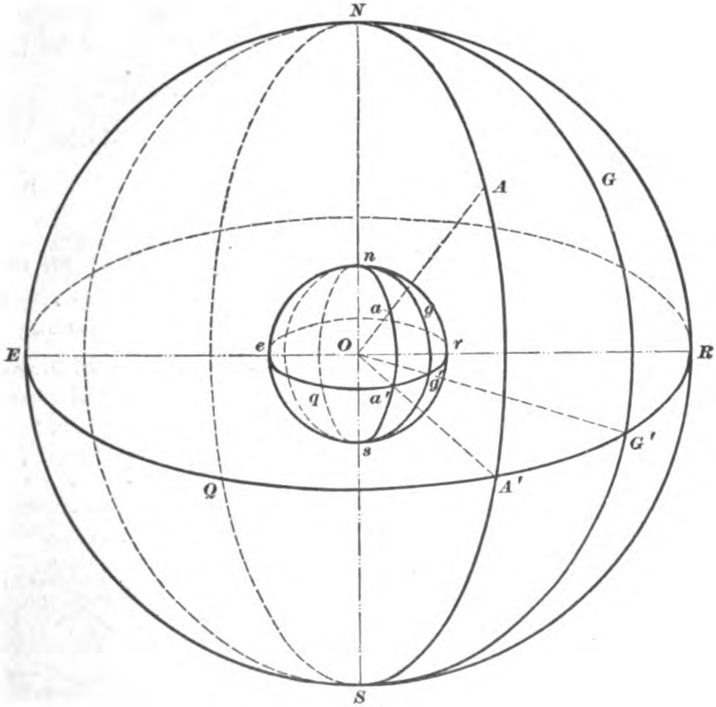


FIG. 2

line, the point *A* in which that line produced meets the heavens is the zenith of the point *a*. Therefore, the latitude of any point on the earth's surface is measured by the arc $A'\,A$ of the celestial meridian of that point included between the equator and the zenith of that point. Hence, the latitude of a point on the earth's surface may be defined as the angular distance of the zenith of that point from the celestial equator.

TIME

13. Culmination.—All celestial bodies have an apparent motion from east to west around the axis of the celestial sphere. In this motion, they cross every meridian twice in 24 hours. The passage of a celestial body across the meridian of a place is called the **culmination**, or **transit**, of that body, with respect to that meridian; and the culmination of a celestial body is called **upper culmination** or **lower culmination**, **upper transit** or **lower transit**, according as the body crosses the meridian above or below the pole.

14. Sidereal Time.—The interval of time that elapses between two successive upper or lower transits of a star over the same meridian is called a **sidereal day**. This day is divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. It begins, for any place, when a special point of the equator, called the **vernal equinox**, crosses the meridian above the pole. This instant is called **sidereal noon**. Sidereal hours, minutes, and seconds are reckoned from 0 to 24 hours, starting from sidereal noon. Time expressed in sidereal days and fractions (hours, minutes, seconds) is called **sidereal time**.

15. True Solar, or Apparent, Time.—The interval between two successive upper transits of the sun is called a **true solar day**, or an **apparent day**. Like the sidereal day, the apparent day is divided into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. Time expressed in these units is called **apparent time**.

On account of the fact that the sun does not, like the stars, move in a plane perpendicular to the axis of the heavens, and that its motion is not uniform, a solar day is not equal to a sidereal day; nor are all solar days of equal duration. Apparent time is therefore not convenient to use for the ordinary affairs of life.

16. Mean Solar Time.—The earth makes one complete revolution around the sun in 366.2422 sidereal days, or 365.2422 true solar days. This motion constantly changes the relative positions of the earth, sun, and stars, and has the effect of making the sun seem to move among the stars, making a complete circuit of the heavens in 366.2422 sidereal days. In this apparent motion, the sun crosses the equator every spring at the point defined in Art. **14** as the vernal equinox. The **mean sun** is an imaginary body supposed to start from the vernal equinox at the same time as the true sun, and to move uniformly on the equator, returning to the vernal equinox with the true sun. The time between two successive upper transits of the mean sun is called a **mean solar day**, and time expressed in mean solar days is called **mean solar time**, or simply **mean time**. This is the time shown by ordinary clocks and watches.

A mean solar day is, as its name implies, the mean of the duration of all the true solar days in a year (a year being the time in which either the true or the mean sun makes a complete circuit of the heavens). As there are 365.2422 true solar days and 366.2422 sidereal days in a year, we have,

$$1 \text{ mean solar day} = \frac{366.2422}{365.2422} = 1.0027379 \text{ sidereal days}$$

$$= 24^h\ 3^m\ 56.55^s, \text{ sidereal time}$$

Likewise,

$$1 \text{ sidereal day} = \frac{365.2422}{366.2422} = .99726957 \text{ mean solar day}$$

$$= 23^h\ 56^m\ 4.09^s, \text{ mean solar time}$$

These values are given here as a matter of general information. In practice, sidereal time is converted into mean solar, and mean solar into sidereal, by means of tables.

17. Civil Time.—In the ordinary way of reckoning time, the day begins at 12 o'clock at night, and hours are counted up to 12 at noon, and, beginning again at noon, up to 12 at midnight. The day is thus divided into two intervals of 12 hours each, the first interval being known as A. M. time, the second as P. M. time.* When time is reckoned in this manner, it is called **civil time.**

*These abbreviations mean, respectively, *ante meridiem* (before meridian transit, before noon) and *post meridiem* (after meridian transit, after noon).

18. Astronomical Time.—For astronomical purposes, the day is considered to begin at noon, and hours are counted from 0 to 24. When time is reckoned in this manner, it is called **astronomical time.**

The astronomical day begins 12 hours later than the civil day, and this fact should be borne in mind when converting astronomical time into civil, or vice versa. For instance, the date October 17, $7^h\ 14^m\ 3^s$ (7 hours, 14 minutes, 3 seconds), astronomical time, means $7^h\ 14^m\ 3^s$ after noon of the civil date October 17, and is in civil time, $7^h\ 14^m\ 3^s$ P. M. The astronomical date February 20, $18^h\ 6^m\ 12^s$ means $18^h\ 6^m\ 12^s$ after noon of the civil date February 20, or $6^h\ 6^m\ 12^s$ after midnight of February 20; that is, February 21, $6^h\ 6^m\ 12^s$ A. M.

19. Relation Between Longitude and Time.—The earth revolves about its axis from west to east once in 24 hours. This causes the sun to appear to move from east to west, crossing all meridians in succession. The mean sun is supposed to have a similar apparent motion, and to describe a complete circle around the earth in 24 mean solar hours. Now, there being 360° in a complete circle, the sun moves over $\frac{360°}{24}$, or 15°, every hour. If the difference between the longitudes of two places A and B is 15°, and B is west of A, the sun will reach the meridian of A one hour earlier than the meridian of B; so that, when it is 0^h (noon) at B, it is 1^h at A. In general, if the difference between the longitudes of A and B, expressed in degrees, is L, the time at A will always be $\frac{L}{15}$ greater than the time at B. Thus, if $L = 25°\ 15'\ 30''$ ($= 25.2583°$) and the time at B is $13^h\ 12^m$, the time at A will be

$$13^h\ 12^m + \frac{25.2583}{15} = 13^h\ 12^m + 1.684^h = 14^h\ 53^m\ 2^s$$

In order that operations of this kind may be readily performed, longitude is often expressed in hours, minutes, and seconds, instead of degrees. For instance, a longitude of

15° is expressed as 1^h, one of 30°, as 2^h, etc. When longitude is so expressed, the difference between the hours of any two places is simply the difference between their longitudes. If, for instance, the longitude of *A* is $14^h\ 17^m\ 12^s$, and that of *B* is $19^h\ 37^m\ 45^s$, the difference, $5^h\ 20^m\ 33^s$, must be added to the time at *B* to find the corresponding time at *A*.

Let the longitudes of *A* and *B*, expressed in time (hours, minutes, etc.), be g_a and g_b; and let T_a be the time at *A* when the time at *B* is T_b. Then,

$$T_a = T_b + (g_b - g_a)$$

EXAMPLE 1.—The longitude of Washington, west of Greenwich, is $5^h\ 8^m\ 1^s$; that of San Francisco, $8^h\ 9^m\ 47^s$. (*a*) What is the time at Washington when it is $9^h\ 3^m$ at San Francisco? (*b*) What is the time at San Francisco, when it is $19^h\ 54^m\ 30^s$ at Washington?

SOLUTION.—(*a*) Here *A*, the eastern locality, is Washington, and *B* is San Francisco, and the data are, $g_b = 8^h\ 9^m\ 47^s$, $g_a = 5^h\ 8^m\ 1^s$, $T_b = 9^h\ 3^m$; applying the formula,

$$T_a = 9^h\ 3^m + (8^h\ 9^m\ 47^s - 5^h\ 8^m\ 1^s) = 12^h\ 4^m\ 46^s. \quad \text{Ans.}$$

(*b*) Since here the time at *A* is given, and what is required is the time at *B*, the formula must be solved for T_b:

$$T_b = T_a - (g_b - g_a)$$

or, substituting the given value,

$$T_b = 19^h\ 54^m\ 30^s - (8^h\ 9^m\ 47^s - 5^h\ 8^m\ 1^s) = 16^h\ 52^m\ 44^s. \quad \text{Ans.}$$

EXAMPLE 2.—The longitudes of Havana (Cuba) and Salt Lake City (Utah) being, respectively, $5^h\ 29^m\ 26^s$ and $7^h\ 28^m\ 24^s$, what is the time at Havana when it is $11^h\ 42^m\ 45^s$ P. M. at Salt Lake City?

SOLUTION.—Here $g_a = 5^h\ 29^m\ 26^s$, $g_b = 7^h\ 28^m\ 24^s$, and $T_b = 11^h\ 42^m\ 45^s$. Therefore, by the formula,

$$T_a = 11^h\ 42^m\ 45^s + (7^h\ 28^m\ 24^s - 5^h\ 29^m\ 26^s) = 13^h\ 41^m\ 43^s \text{ P. M.}$$

As this is civil time, in which the day ends at 12 P. M., the true value of T_a is the excess of $13^h\ 41^m\ 43^s$ over 12^h; that is, $13^h\ 41^m\ 43^s - 12^h = 1^h\ 41^m\ 43^s$; but this time evidently belongs to the following day; so that, if the given time at Salt Lake City is for Monday, the time at Havana will be $1^h\ 41^m\ 43^s$, Tuesday. Ans.

20. Rules for Converting Degrees of Longitude Into Time, or Vice Versa.—When it is necessary to pass from degrees to time, or from time to degrees, it is not expedient to express minutes and seconds as decimals.

Since 1 hour is equivalent to 15°, it follows that 1 minute of time is equivalent to $\frac{15°}{60} = 15'$; and 1 second of time, to $\frac{15'}{60} = 15''$. From these relations the following rules are derived:

Rule I.—*To pass from time to degrees: (1) Multiply the seconds of time by 15, and divide the product by 60; the integral part of the quotient will be minutes of arc, and the remainder, seconds of arc. (2) Multiply the minutes of time by 15, add to the product the minutes of arc obtained in (1), and divide the result by 60; the integral part of the quotient will express degrees, and the remainder, minutes of arc. (3) Multiply the hours by 15, and to the product add the degrees found in (2); the result will be degrees.*

Rule II.—*To pass from degrees to time: (1) Divide the number of degrees by 15; the integral part of the quotient will be hours. (2) Multiply the remainder by 60, add the product to the given minutes of arc, and divide the result by 15; the integral part of the quotient will be minutes of time. (3) Multiply the remainder by 60, add the product to the given number of seconds of arc, and divide the sum by 15; the result (carried, if necessary, to one or more decimal places) will be seconds of time.*

EXAMPLE 1.—To express, in degrees, a difference of longitude of $3^h\ 47^m\ 34^s$.

SOLUTION.—The operation is disposed as shown below. Multiplying 34, the number of seconds of time, by 15, gives 510 sec. of arc. Dividing 510 by 60, the quotient 8 expresses minutes of arc, and the remainder, 30, expresses seconds. Multiplying 47^m by 15 gives 705′, to which must be added the 8 minutes obtained from the previous operation.

3^h	47^m	34^s
15	15	15
45	705	60) 510 (8
11	8	480
56°	60) 713 (11	30″
	660	
	53′	

The result is 713′, which divided by 60 gives 11°, and 53′ over. Multiplying 3^h by 15, and adding to the product the 11° previously obtained, the result is 56°. The answer is, therefore, 56° 53′ 30″.

EXAMPLE 2.—To express, in hours, a difference of longitude of 21° 28′ 17″.

SOLUTION.—The work is disposed as shown below. This arrangement, studied in connection with rule II, makes further explanation

```
15 ) 21 ( 1h
     15
     --
      6
    ×60
    ---
    360
add  28
    ---
15 ) 388 ( 25m
     375
     ---
      13
     ×60
     ---
     780
 add  17
     ---
15 ) 797 ( 53.13s
```

unnecessary. The result, to the nearest hundredth of a second, is 1^h 25^m 53.13^s. Ans.

EXAMPLES FOR PRACTICE

(1) What is the civil date corresponding to the astronomical date, January 5, 7^h 35^m? Ans. January 5, 7^h 35^m P. M.

(2) What is the civil date corresponding to the astronomical date, March 24, 15^h 7^m 30^s? Ans. March 25, 3^h 7^m 30^s A. M.

(3) What is the astronomical date corresponding to the civil date, July 4, 6^h 30^m A. M.? Ans. July 3, 18^h 30^m

(4) The longitudes of New York and Sacramento are, respectively, 74° 00′ 03″ and 121° 27′ 44″ west of Greenwich. (*a*) Express these longitudes in time. (*b*) What is the astronomical time at New York, when the time at Sacramento is 13^h 6^m 20^s? (*c*) What is the civil time at Sacramento when it is 11^h 28^m 15^s A. M. at New York?

Ans. { (*a*) 4^h 56^m; 8^h 5^m 51^s
(*b*) 16^h 16^m 11^s
(*c*) 8^h 18^m 24^s A. M.

(5) Express in degrees, minutes, and seconds of arc a difference of longitude of 2^h 39^m 37^s. Ans. 39° 54′ 15″

(6) Express, in time, a difference of longitude of 49° 59′ 43″. Ans. 3^h 19^m 58.87^s

21. Local Time.—The time dealt with in the foregoing articles refers to the meridian of the place to which the time refers, and is counted from the instant the mean sun crosses the meridian of that place, either above the pole (astronomical time) or below the pole (civil time). This time is called **local time,** or the time of the place considered. Thus, when it is said that Chicago's local astronomical time is $3^h\ 45^m$, it is meant that it is $3^h\ 45^m$ since the mean sun crossed the meridian of Chicago. All places on the same meridian have the same local time.

22. Standard Time.—If watches and clocks showed local time at every place, it would be a complicated and cumbersome operation to compare those times for the regulation of the ordinary affairs of life. For this reason, watches and clocks are set, within certain longitudes, to keep

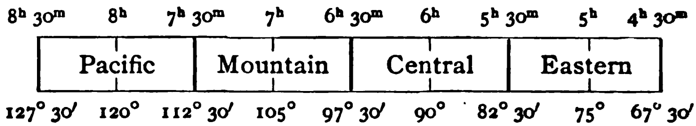


Fig. 3

time referred to a certain meridian between those longitudes; that is, the time shown by a clock or watch between those longitudes is the local time of all the places on the meridian of reference. Time thus reckoned is called **standard time.**

The United States is divided into four zones, or sections, of standard time, whose reference meridians are, respectively, 75°, 90°, 105°, and 120° west of Greenwich; or, in hours, 5^h, 6^h, 7^h, and 8^h west of Greenwich. Each of these meridians passes through the center of a zone of standard time, and, therefore, controls the watch time of all places within $7\frac{1}{2}°$ on either side of it. This is shown in Fig. 3. Time referred to the 75° meridian is called **eastern time;** to the 90° meridian, **central time;** to the 105° meridian, **mountain time;** and to the 120° meridian, **Pacific time.**

23. To Change Standard Into Local Time.—This is an operation with which the student should familiarize himself, as it is of constant occurrence in the applications of astronomy to surveying. When the standard time at any point within one of the zones is given, the local time is determined by the formula in Art. **19.** But in order to apply this formula it is necessary to know the longitude of the place. When the longitude is given in degrees, minutes, and seconds of arc, it must first be reduced to time.

EXAMPLE.—The civil time, by a watch, at a place whose longitude is 81° 37′, is $9^h\ 37^m\ 45^s$ A. M.; what is the local time at that instant?

SOLUTION.—Since the longitude is 81° 37′, the place lies within the zone of 75° meridian, or eastern, time. To apply the formula in Art. **19**, we have,

$$T_a = 9^h\ 37^m\ 45^s$$

The difference $g_b - g_a$ is obtained by reducing to time the difference 81° 37′ − 75° = 6° 37′ between the longitude of the given place and that of the reference meridian. Therefore,

$$T_b = T_a - (g_b - g_a) = 9^h\ 37^m\ 45^s - 26^m\ 28^s = 9^h\ 11^m\ 17^s. \text{ Ans.}$$

EXAMPLES FOR PRACTICE

1. The longitude of Cincinnati is 84° 29′ 31″; what is the local time when the standard (central) time is $4^h\ 24^m\ 17^s$ P. M.?
Ans. $4^h\ 46^m\ 19^s$ P. M.

2. The longitude of a place being 113° 49′, what is the local time at the place when the time shown by a watch is $11^h\ 58^m\ 30^s$ P. M.?
Ans. $23^m\ 14^s$ after midnight

3. The time by a watch at Los Angeles being $1^h\ 44^m$ A. M., and the longitude of the place being 118° 10′ 44″, what is the astronomical local time at that instant?
Ans. $13^h\ 51^m\ 17^s$

DETERMINATION OF THE MERIDIAN BY OBSERVATIONS OF POLARIS

POLARIS

24. Position of Polaris.—There is a star called *Polaris*, the *north star*, or the *pole star*, the proximity of which to the north pole makes it very convenient for the determination of the true meridian. Its position can be very easily ascertained by means of a constellation, or group of stars, commonly known as the *Dipper*, and called by astronomers *Ursa Major*, or the *Great Bear*. This constellation consists of seven stars, a line joining which forms the outline of a dipper, as shown in Fig. 4. Although its position with reference to both the meridian and the horizon varies with the date and time of night, as well as with the observer's position, it can, on account of its peculiar form, be easily identified on any clear night. The two stars in the bowl that are farther from the handle are known to astronomers as α (*alpha*) and β (*beta*), and commonly called the **pointers**, from the fact that

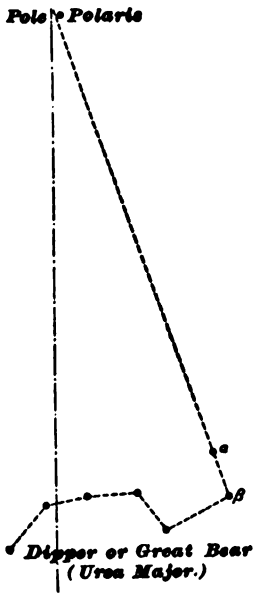


FIG. 4

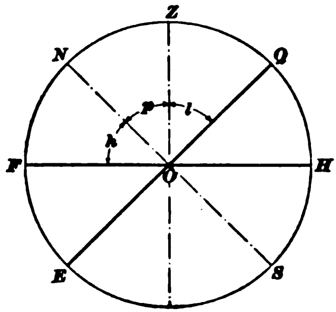


FIG. 5

they point almost directly to the north pole. By following toward the north the line determined by the pointers, the north star, which lies very nearly on that line, and at the same time very nearly due north, can be readily located. (See Fig. 4.)

25. Approximate Determination of Latitude From Polaris.—In nearly all methods of determining the true meridian, the latitude of the place of observation must be known, at least approximately. In the majority of cases, the latitude can be taken from a map or book of reference. In case this cannot be done, a sufficiently close value may be obtained by measuring with a transit the altitude of Polaris, which is very nearly (within about 1°) equal to the latitude of the place.

This method of determining latitude is founded on the following very simple and very useful principle:

The latitude of any place on the earth's surface is equal to the altitude of the pole with respect to the horizon of that place.

Let O, Fig. 5, be any point on the earth's surface; $ESQN$, the meridian; FH, the horizon; EQ, the equator; Z, the zenith; N, the north pole. The latitude of the place is l (Art. 3), and the altitude of the pole is h ($= FON$). Now, since SN is perpendicular to EQ, and OZ to FH, we have

$$p + h = 90°;\ p + l = 90°;\ \text{whence } h = l$$

DETERMINATION OF THE MERIDIAN BY OBSERVING POLARIS AT CULMINATION

26. Times of Upper Culmination of Polaris.—In Table I are given the local astronomical times of upper culmination of Polaris for different dates, between the beginning of the year 1905 and the end of the year 1911. This table contains the times of upper culmination for the dates given. The extreme right-hand column of the table contains the difference between the times of culmination for any two succeeding days. Each difference applies to any day between the date horizontally opposite that difference in the left-hand column, and the following date. Thus, the difference 3.95^m, which is horizontally opposite January 1, indicates that, between January 1 and January 15, the time of culmination decreases by 3.95 minutes per day. For instance, the time of culmination on January 8 is obtained by subtracting from the time of culmination for January 1 the product $3.95^m \times 7$, or 27.65, the number of days elapsed from January 1 to January 8 being 7.

In order, however, to facilitate calculation, Table II gives the corrections to be applied for all intermediate dates. Suppose, for instance, that it is desired to find the time of

TABLE I

LOCAL MEAN ASTRONOMICAL TIME OF UPPER CULMINATION OF POLARIS

	1905		1906		1907		1908		1909		1910		1911		Difference for 1 Day Minutes
	h	m	h	m	h	m	h	m	h	m	h	m	h	m	
January 1 . . .	6	41.4	6	42.8	6	44.3	6	45.7	6	43.2	6	44.7	6	46.1	3.95
January 15 . . .	5	46.1	5	47.5	5	49.0	5	50.4	5	47.9	5	49.4	5	50.8	3.95
February 1 . .	4	39.0	4	40.4	4	41.9	4	43.3	4	40.8	4	42.3	4	43.7	3.95
February 15 . .	3	43.8	3	45.2	3	46.7	3	48.1	3	45.6	3	47.1	3	48.5	3.95
March 1	2	48.5	2	49.9	2	51.4	2	48.9	2	50.3	2	51.8	2	53.2	3.94
March 15 . . .	1	53.4	1	54.8	1	56.3	1	53.8	1	55.2	1	56.7	1	58.1	3.94
April 1	0	46.5	0	47.9	0	49.4	0	46.8	0	48.3	0	49.8	0	51.2	3.94
April 15	23	47.5	23	48.9	23	50.4	23	47.8	23	49.3	23	50.8	23	52.2	3.93
May 1	22	44.7	22	46.1	22	47.6	22	45.1	22	46.5	22	48.0	22	49.4	3.93
May 15	21	49.7	21	51.1	21	52.6	21	50.1	21	51.5	21	53.0	21	54.4	3.92
June 1	20	43.1	20	44.5	20	46.0	20	43.5	20	44.9	20	46.4	20	47.8	3.92
June 15	19	48.3	19	49.7	19	51.2	19	48.7	19	50.1	19	51.6	19	53.0	3.91
July 1	18	45.7	18	47.1	18	48.6	18	46.1	18	47.5	18	49.0	18	50.4	3.91
July 15	17	50.9	17	52.3	17	53.8	17	51.3	17	52.7	17	54.2	17	55.6	3.92
August 1 . . .	16	44.3	16	45.7	16	47.2	16	44.7	16	46.1	16	47.6	16	49.0	3.92
August 15 . . .	15	49.5	15	50.9	15	52.4	15	49.9	15	51.3	15	52.8	15	54.2	3.92
September 1 . .	14	42.9	14	44.3	14	45.8	14	43.3	14	44.7	14	46.2	14	47.6	3.92
September 15 .	13	48.0	13	49.4	13	50.9	13	48.4	13	49.8	13	51.3	13	52.7	3.93
October 1 . . .	12	45.2	12	46.6	12	48.1	12	45.6	12	47.0	12	48.5	12	49.9	3.93
October 15 . . .	11	50.2	11	51.6	11	53.1	11	50.6	11	52.0	11	53.5	11	54.9	3.93
November 1 . .	10	43.3	10	44.7	10	46.2	10	43.7	10	45.1	10	46.6	10	48.0	3.93
November 15 . .	9	48.3	9	49.7	9	51.2	9	48.7	9	50.1	9	51.6	9	53.0	3.94
December 1 . .	8	45.2	8	46.6	8	48.1	8	45.6	8	47.0	8	48.5	8	49.9	3.94
December 15 .	7	50.0	7	51.4	7	[illegible]	7	50.1	7	51.8	7	[illegible]	7	54.7	3.95

TABLE II

CORRECTION FOR INTERMEDIATE DATES

Day of the Month	Difference for 1 Day in Minutes				
	3.91	3.92	3.93	3.94	3.95
2 or 16	3.9	3.9	3.9	3.9	3.9
3 or 17	7.8	7.8	7.9	7.9	7.9
4 or 18	11.7	11.8	11.8	11.8	11.8
5 or 19	15.6	15.7	15.7	15.8	15.8
6 or 20	19.5	19.6	19.6	19.7	19.7
7 or 21	23.5	23.5	23.6	23.6	23.7
8 or 22	27.4	27.4	27.5	27.6	27.6
9 or 23	31.3	31.4	31.4	31.5	31.6
10 or 24	35.2	35.3	35.4	35.5	35.5
11 or 25	39.1	39.2	39.3	39.4	39.5
12 or 26	43.0	43.1	43.2	43.3	43.4
13 or 27	46.9	47.0	47.2	47.3	47.4
14 or 28	50.8	51.0	51.1	51.2	51.3
29	54.7	54.9	55.0	55.2	55.3
30	58.6	58.8	58.9	59.1	59.2
31	62.6	62.7	62.9	63.0	63.2

upper culmination on October 9, 1905. The difference for 1 day horizontally opposite October 1 in Table I is 3.93. Looking for this difference in the vertical columns of Table II, 31.4^m is found in that column horizontally opposite the given date 9 (9 or 23). Subtracting 31.4^m from $12^h\ 45.2^m$, which is the time of culmination for October 1, 1905, the required time is found to be $12^h\ 13.8^m$.

27. Between April 12 and April 15, the correction given in Table II cannot, in some cases, be subtracted, the minuend being less than the subtrahend. In this case, $23^h\ 56.1^m$, which is the mean solar time between two consecutive culminations (Art. **16**), should be added to the minuend before performing the subtraction.

28. It should be borne in mind that the times given in Table I are mean local times counted in the astronomical way; that is, from 0^h to 24^h. In applying the table, care should be taken to reduce the civil standard time shown by the observer's watch or clock to the corresponding local astronomical time.

29. Time of Lower Culmination.—Since Polaris makes a complete revolution around the pole in $23^h\ 56.1^m$, the interval of time between an upper and the next lower culmination is $\frac{23^h\ 56.1^m}{2}$, or $11^h\ 58^m$, nearly. If the time of upper culmination for any given date, as found from the table, is less than $11^h\ 58^m$, the latter quantity is added to it, in order to obtain the time of lower culmination. If the time of upper culmination is greater than $11^h\ 58^m$, the latter quantity should be subtracted from it, in order to obtain the time of lower culmination for the same date. This will be readily understood by reference to Fig. 6, in which the circle *A B* represents the path of Polaris, and *L U*, the meridian. The star describes either of the semicircles *L A U* or *U B L* in about $11^h\ 58^m$. Let this interval of time be denoted by *T*. If, when Polaris is at *U* (upper culmination), the time is less than *T*, this indicates that, at the beginning of the day, the star was at some point *S* to the right of *L*. Before the end of that day, the star will describe the path *U B L S*, crossing the meridian at *L* (lower culmination) *T* hours after its passage through *U*. If, when Polaris is at *U*, the time is greater than *T*, this shows that, at the beginning of the day, the star was at some point *S'* to the left of *L*; in moving from *S'* to *U*, the star evidently reached *L*, its lower culmination, *T* hours before it reached *U*; so that, in this case, the time of lower culmination is obtained by subtracting *T* from the time of upper culmination.

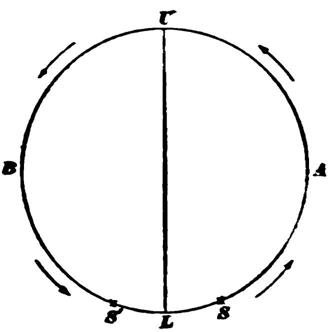


FIG. 6

30. Observing Polaris and Marking the Meridian. Select a date on which Polaris is at either lower or upper culmination during the night (preferably during the early part of the evening). Determine, by means of Table I, the exact time of culmination, being careful to reduce the tabular values to standard civil time. It is safer, in order to avoid confusion, for the observer to set his watch to show local time (Art. 23). About 15 minutes before the time of culmination, set the transit in such a position that an unobstructed view toward the north may be obtained for a distance of between 300 and 500 feet. Drive a stake, and mark by a tack the exact point occupied by the instrument. About 5 minutes before the time of culmination, direct the telescope to the star, holding a lamp in front and a little toward one side of the objective glass to illuminate the cross-wires. Set both clamps, and with either tangent screw set the vertical cross-wire exactly on the star. The star will appear to be moving toward the left or toward the right according as it is approaching upper or lower culmination. Follow it in its motion by turning the tangent screw until the exact time of culmination (which, preferably, should be called out by an assistant). This completes the observation of the star. Now depress the telescope, direct it to a point on the ground about 400 or 500 feet from the instrument, and have an assistant drive a tack on top of a stake in line with the line of sight; this completes the operation. The line between the two stakes is a true north-and-south line, or true meridian.

EXAMPLE 1.—To find the time of upper culmination of Polaris on September 12, 1907.

SOLUTION.—From Table I,

Upper culmination, September 1, 1907	$14^h\ 45.8^m$
Difference for 1 day, 3.92.	

From Table II,

Correction under 3.92, opposite 12 ("12 or 26") .	43.1
Required time	$14^h\ 2.7^m$
	Ans.

EXAMPLE 2.—To find the time of lower culmination of Polaris on April 13, 1908.

SOLUTION.—From Table I,

Time of upper culmination, April 1, 1908	0ʰ 46.8ᵐ
Add 1 sidereal day (Art. **27**)	23 56.1
	24ʰ 42.9ᵐ
Difference for 1 day, 3.94.	
Correction from Table II, under 3.94 and opposite 13	47.3
Time of upper culmination	23ʰ 55.6ᵐ
Subtract (see Art. **29**)	11 58
Time of lower culmination	11ʰ 57.6ᵐ

EXAMPLE 3.—An observer is located in longitude 110° 12′, and his watch keeps standard mountain time; what is the time, by his watch, when Polaris will be at lower culmination on December 21, 1905?

SOLUTION.—From Table I,

Time of upper culmination, December 15, 1905	7ʰ 50ᵐ
Difference for 1 day, 3.95.	
Correction from Table II	23.7
Time of upper culmination, December 21, 1905	7ʰ 26.3ᵐ
Add (Art. **29**)	11 58
Time of lower culmination	19ʰ 24.3ᵐ

As this is astronomical time, it shows the interval that has elapsed since noon of December 21, and corresponds, therefore, to the civil date December 22, 7ʰ 24.3ᵐ A. M. We must, then, take the time of the previous culmination corresponding to the astronomical date, December 20.

Upper culmination, December 15, 1905	7ʰ 50ᵐ
Difference for 1 day, 3.95.	
Correction from Table II	19.7
Upper culmination, December 20	7ʰ 30.3ᵐ
Adding	11 58
Lower culmination, December 20	19ʰ 28.3ᵐ

The local civil time of culmination is, then, 7ʰ 28.3ᵐ A. M. As the observer is in longitude 110° 12′, he is 5° 12′ west of the standard 7-hour meridian. This difference in longitude converted into time is 20.8ᵐ. Then, by the formula in Art. **19**, noticing that here T_b = 7ʰ 28.3ᵐ,

$$T_a = 7^h\ 28.3^m + 20.8^m = 7^h\ 49.1^m$$

which is the required standard time of lower culmination. As stated in Art. **30**, it may be convenient to set the watch on local time beforehand. Ans.

EXAMPLES FOR PRACTICE

1. Find the time of upper culmination of Polaris: (*a*) on February 24, 1908; (*b*) on April 13, 1907.
Ans. { (*a*) 3^h 12.6^m; (*b*) 0^h 2.1^m }

2. Find the time of lower culmination of Polaris: (*a*) on September 5, 1910; (*b*) on April 14, 1906.
Ans. { (*a*) 2^h 32.5^m; (*b*) 11^h 54.8^m }

3. An observer is located in longitude 118° 4′, his watch keeping standard Pacific time; find the time, by his watch, at which Polaris will be at lower culmination: (*a*) on July 4, 1909; (*b*) on November 8, 1907.
Ans. { (*a*) 6^h 30.1^m P. M.; (*b*) 10^h 12.9^m A. M. }

DETERMINATION OF THE MERIDIAN BY OBSERVING POLARIS AT ELONGATION

31. Elongation of a Star.—When in its motion about the axis of the heavens a star reaches its extreme westerly position, it is said to be at **western elongation.** Likewise, the star is at **eastern elongation** when in its extreme eastern position.

32. Azimuths of Polaris.—The azimuth of an object, with respect to a point of observation, is the azimuth of the line between that point and the object. Table III gives the azimuths of Polaris at elongation for different years and latitudes. If a transit is directed to the star when at elongation, and the corresponding azimuth taken from the table, all that is necessary to determine the meridian is to turn the telescope through an angle equal to the azimuth, to the left (west) if the star is at eastern elongation; to the right (east) if the star is at western elongation.

33. Time of Elongation of Polaris.—Polaris is at eastern elongation about 5^h 55^m before it reaches its upper culmination; and at western elongation, 5^h 55^m after upper culmination. The times of elongation can, therefore, be readily determined from those of culmination taken from Table I. It should be noted, however, that the time of elongation does not need to be known exactly; it is only necessary to know it approximately, so that the observer may know about when to make his observations.

TABLE III

AZIMUTHS OF POLARIS AT ELONGATION

Latitude Degrees	Year											
	1905		1906		1907		1908		1909		1910	
	Degrees	Minutes	Degrees	Minutes	Degrees	Minutes	Degrees	Minutes	Degrees	Minutes	Degrees	Minutes
25	1	19.4	1	19.1	1	18.7	1	18.4	1	18.1	1	17.7
26		20.1		19.8		19.4		19.1		18.7		18.4
27		20.8		20.5		20.1		19.8		19.4		19.1
28		21.6		21.3		20.9		20.5		20.1		19.8
29		22.4		22.1		21.7		21.3		20.9		20.5
30	1	23.1	1	22.8	1	22.4	1	22.1	1	21.7	1	21.3
31		24.0		23.6		23.2		22.9		22.5		22.2
32		24.9		24.5		24.1		23.8		23.4		23.1
33		25.9		25.5		25.1		24.7		24.3		24.0
34		26.9		26.5		26.1		25.7		25.3		25.0
35	1	27.9	1	27.5	1	27.1	1	26.8	1	26.4	1	26.0
36		29.0		28.6		28.2		27.9		27.5		27.1
37		30.1		29.7		29.3		29.0		28.6		28.2
38		31.4		31.0		30.6		30.2		29.8		29.4
39		32.7		32.3		31.8		31.4		31.0		30.6
40	1	34.0	1	33.6	1	33.2	1	32.8	1	32.4	1	32.0
41		35.4		35.0		34.6		34.2		33.8		33.4
42		36.9		36.5		36.0		35.6		35.2		34.8
43		38.5		38.1		37.6		37.2		36.8		36.3
44		40.1		39.7		39.2		38.8		38.4		37.9
45	1	41.8	1	41.4	1	40.9	1	40.5	1	40.1	1	39.6
46		43.7		43.2		42.7		42.3		41.9		41.4
47		45.6		45.1		44.6		44.2		43.7		43.3
48		47.7		47.2		46.7		46.3		45.8		45.3
49		49.8		49.3		48.8		48.4		47.9		47.4
50	1	52.0	1	51.5	1	51.0	1	50.6	1	50.1	1	49.6
51		54.4		54.0		53.5		53.0		52.5		52.0
52		57.0		56.4		55.9		55.4		54.9		54.4
53		59.6		59.1		58.6		58.1		57.6		57.1
54	2	2.5	2	2.0	2	1.5	2	0.9	2	0.4		59.9
55	2	5.6	2	5.0	2	4.4	2	3.9	2	3.4	2	2.8
56		8.8		8.2		7.7		7.1		6.6		6.0
57		12.2		11.7		11.1		10.5		10.0		9.4
58		15.9		15.3		14.7		14.2		13.6		13.0
59		19.8		19.2		18.6		18.0		17.4		16.8
60	2	24.0	2	23.4	2	22.8	2	22.1	2	21.5	2	20.9
61		28.5		27.9		27.2		26.6		25.9		25.3
62		33.4		32.7		32.1		31.4		30.8		30.1
63		38.6		38.0		37.3		36.6		35.9		35.2
64		44.3		43.6		42.9		42.2		41.5		40.8
65	2	50.4	2	49.7	2	49.0	2	48.3	2	47.5	2	46.8
66		57.1		56.3		55.6		54.8		54.1		53.3
67	3	4.4	3	3.6	3	2.8	3	2.0	3	1.2	3	0.4
68		12.3		11.5		10.7		9.8		9.0		8.2
69		21.0		20.1		19.3		18.4		17.6		16.7
70	3	30.6	3	29.7	3	28.8	3	27.9	3	27.0	3	26.1
71		41.3		40.3		39.4		38.4		37.5		36.5
72		53.2		52.1		51.1		50.1		49.1		48.1

34. Making the Observation and Marking the Meridian.—Determine the approximate time of elongation as just explained. About 20 minutes before that time, set the transit over a point properly marked, and level it carefully. Set the vernier at zero. Direct the telescope to the star, and, with both clamps set, follow the star by means of the lower tangent screw. If the star is approaching eastern elongation, it will be moving to the right; if western, to the left. About the time of elongation, it will be noticed that the star ceases to move horizontally, and that its image appears to follow the vertical cross-wire of the instrument. The star has then reached its elongation, and the observation is completed. Take the azimuth from Table III. Depress the telescope, and turn it through an angle equal to the azimuth, to the west or to the east, according as the star was at eastern or western elongation. The line of sight will then be directed along the true meridian, and by marking another point 400 or 500 feet from that occupied by the instrument, the direction of the true meridian will be established.

This is the most accurate method of determining the true meridian, and, where possible, should be used in preference to others.

EXAMPLE.—To find the times of elongation of Polaris on May 10, 1906.

SOLUTION.—From Table I,

Upper culmination, May 1, 1906	22^h 46.1^m
Difference for 1 day, 3.93.	
Correction from Table II	35.4
Upper culmination, May 10, 1906	22^h 10.7^m
For western elongation, add	5 55.0
	28^h 5.7^m
As this brings the time on the following day, it is necessary to subtract a sidereal day	23 56
Western elongation (to nearest minute), May 10, 1906	4^h 10^m Ans.
Also,	
Upper culmination, May 10, 1906 (to nearest minute)	22^h 11^m
Subtract	5^h 55
Eastern elongation	16^h 16^m Ans.

EXAMPLES FOR PRACTICE

1. Find the time of eastern elongation of Polaris on August 17, 1907.
Ans. 9^h 49.6^m

2. Find the time of western elongation of Polaris on May 17, 1908.
Ans. 3^h 41.3^m

PROBLEMS ON INACCESSIBLE LINES

TRIANGULATION

35. Definition.—Triangulation is an application of the principles of trigonometry to the determination of distances and angles. It consists essentially in determining by trigonometry the values of all the parts of a triangle from three parts whose values are measured directly. The values of all parts of the triangle being thus determined, any side can be taken as the side of a new triangle, and by measuring the angles at the extremities of that side, the values of the unknown parts of this triangle can be computed. This process can be repeated so as to cover any given area with a network of triangles, all the sides of which, except the first, are determined by calculation.

The purpose for which triangulation is most often employed in ordinary surveying, however, is the determination of the lengths of lines that are inaccessible for direct linear measurement. A few illustrations of a general character will be given here, covering the triangulation problems that occur most often in ordinary field work.

36. Problem I.—*The two ends of a line not wholly accessible being both visible, and one of them accessible, to determine the length of the line.*

A common case in which this problem has to be solved occurs when the line of a survey crosses a stream too wide and deep for direct measurement, and it is required to determine the length of the line intercepted by the stream.

Let A, Fig. 7, be the accessible end of the line, and B the other end, which must be distinctly defined, either by some prominent natural object, such as a tree, or by a flagpole. Set the transit over the point A, set the vernier at zero, and direct the telescope to B. Unclamp the upper plate, and turn off an angle A to the right or to the left, as may be more

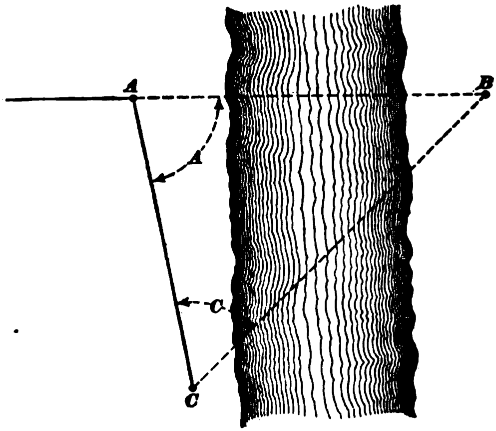


FIG. 7

convenient. A right angle is preferable, and should be used whenever practicable. Measure along the line of sight a distance of about 300 or 400 feet to a point C. Set the transit at C, and measure the angle C. In the triangle ABC, the angle B is found from the relation $B = 180° - (A + C)$, and then the distance AB is calculated by the following formula, which is given in trigonometry:

$$AB = \frac{AC \sin C}{\sin B}$$

If A is a right angle, ABC is a right triangle, and then

$$AB = AC \tan C$$

In this case, it is very convenient to make AC a multiple of 100—say 400—and use the natural tangent of C. When A is not a right angle, logarithmic functions are preferable.

EXAMPLES FOR PRACTICE

1. Suppose that in the triangle *A B C*, Fig. 7, the side *A C* is 425 feet, and the angles *A* and *C* are 79° 00′ and 56° 20′, respectively; what is the distance *A B*? Ans. 503.2 ft.

2. If in the triangle *A B C*, Fig. 7, the side *A C* is 500 feet, the angle *A*, 90°, and the angle *C*, 49° 54′, what is the distance *A B*? Ans. 593.8 ft.

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37. Problem II.—*To determine the length of a line both ends of which are inaccessible.*

Let *A B*, Fig. 8, be the line, the ends *A* and *B* of which are inaccessible. Select two points *P*, *Q* from which both ends of the line can be seen, and at a distance from each other of about 300 or 400 feet (or more, if necessary). After measuring the line *P Q*, set the transit at one of its extremities, as *P*, and measure the angles *K* and *L*. Then take the transit to *Q*, and measure the angles *M* and *N*.

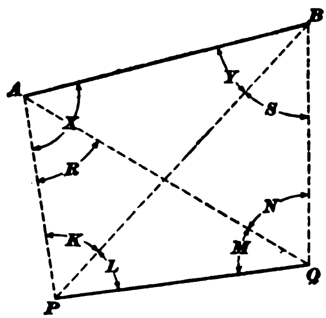

FIG. 8

In the triangle *A P Q*, the angle *R* is computed from the relation $R = 180° - (K + L + M)$, and then

$$AP = \frac{PQ \sin M}{\sin R}$$

Likewise, in the triangle, *B P Q*, $S = 180° - (L + M + N)$, and

$$BP = \frac{PQ \sin (M + N)}{\sin S}$$

In the triangle *A P B*, the sides *A P* and *P B* and their included angle *K* are now known. To find *A B*, we have, from trigonometry,

$$\tan \tfrac{1}{2}(X - Y) = \frac{PB - PA}{PB + PA} \cot \tfrac{1}{2} K$$

$$AB = \frac{(PB - PA) \cos \tfrac{1}{2} K}{\sin \tfrac{1}{2}(X - Y)}$$

EXAMPLE.—If, in Fig. 8, the distance PQ is 400 feet, and the angles, as measured, are $K = 37° 10'$, $L = 36° 30'$, $M = 52° 15'$, $N = 32° 55'$, what is the distance AB?

SOLUTION.—In the triangle APQ, $R = 180° - (37° 10' + 36° 30' + 52° 15') = 54° 05'$, and

$$AP = \frac{400 \sin 52° 15'}{\sin 54° 05'} = 390.53 \text{ ft.}$$

In the triangle BPQ, $S = 180° - (36° 30' + 52° 15' + 32° 55') = 58° 20'$, $M + N = 52° 15' + 32° 55' = 85° 10'$, and

$$BP = \frac{400 \sin 85° 10'}{\sin 58° 20'} = 468.30 \text{ ft.}$$

We have, also, $K = 37° 10'$, $\frac{1}{2} K = 18° 35'$, and

$$\tan \tfrac{1}{2}(X - Y) = \frac{(468.30 - 390.53)}{468.30 + 390.53} \cot 18° 35'$$

whence, $\frac{1}{2}(X - Y) = 15° 04'$, and therefore

$$AB = \frac{(468.30 - 390.53) \cos 18° 35'}{\sin 15° 04'} = 283.58 \text{ ft. Ans.}$$

EXAMPLE FOR PRACTICE

If, in Fig. 8, the distance PQ is 500 feet, and the angles K, L, M, and N are, respectively, 42° 30′, 42° 55′, 45° 30′, and 20° 50′, find the distance AB. Ans. 348.3 ft.

---

38. Problem III.—*To determine the angle between two lines whose point of intersection is inaccessible; also, the distances from a point in each line to the point of intersection.*

This problem is of frequent occurrence in railroad work, the two given lines being the center lines of two tracks that are to be connected by a curve.

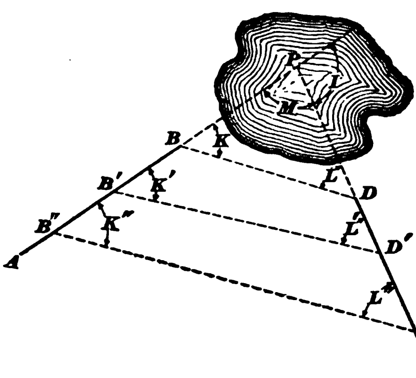

FIG. 9

Let AB and CD, Fig. 9, be the given lines, P their inaccessible point of intersection, and BP and DP the required distances. Measure the distance BD

and the angles K and L. Then, $M = 180° - (K + L)$, and

$$BP = \frac{BD \sin L}{\sin M}, \; DP = \frac{BD \sin K}{\sin M}$$

In railroad work, the external angle I is usually required instead of M. Since I is an exterior angle of the triangle BDP, we have $I = K + L$.

If B is not visible from D, two points are selected, as B' and D', one on each line, that are visible from each other. The distance $B'D'$ and the angles K' and L' are measured, and M, I, $B'P$, and $D'P$ determined as before. The distances $B'B$ and $D'D$ are next measured, and then

$$BP = B'P - B'B, \text{ and } DP = D'P - D'D$$

If the lines are entirely invisible from each other, as when there is an intervening hill, any convenient point B'' is taken on one of them, and a line $B''D''$ is run making any convenient angle K'' with it. The intersection D'' of this line with CD is marked, the distance $B''D''$ and the angle L'' are measured, and the solution completed as before. The distance $B''D''$ is measured when the line is being run. As D'' is not visible from B'', it will be necessary to set the instrument at one or more intermediate points, from each of which the line is prolonged in the manner explained in *Transit Surveying*, Part 1. If it is not convenient to run a straight line between AB and CD, a broken or traverse line may be used, as will be explained elsewhere.

EXAMPLES FOR PRACTICE

1. Referring to Fig. 9, suppose that the angle $K = 25° 10'$, the angle $L = 24° 30'$, and that the distance $BD = 410$ feet. Determine the angle I and the distances BP and DP.

Ans. $\begin{cases} I = 49° 40' \\ BP = 223.05 \text{ ft.} \\ DP = 228.72 \text{ ft.} \end{cases}$

2. Suppose that, in Fig. 9, the angles K'' and L'' are $61° 45'$ and $42° 33'$, respectively, and that the distance $B''D''$ is 312.4 feet. If the distances $B''B$ and $D''D$ are, respectively, 52.2 feet and 36.5 feet, determine the distances BP and DP.

Ans. $\begin{cases} BP = 165.8 \text{ ft.} \\ DP = 247.5 \text{ ft.} \end{cases}$

39. Special Problem.—The following special problem is given as a further illustration of the methods that are used in practice. The student will readily understand that a great variety of such problems are likely to occur, which cannot be

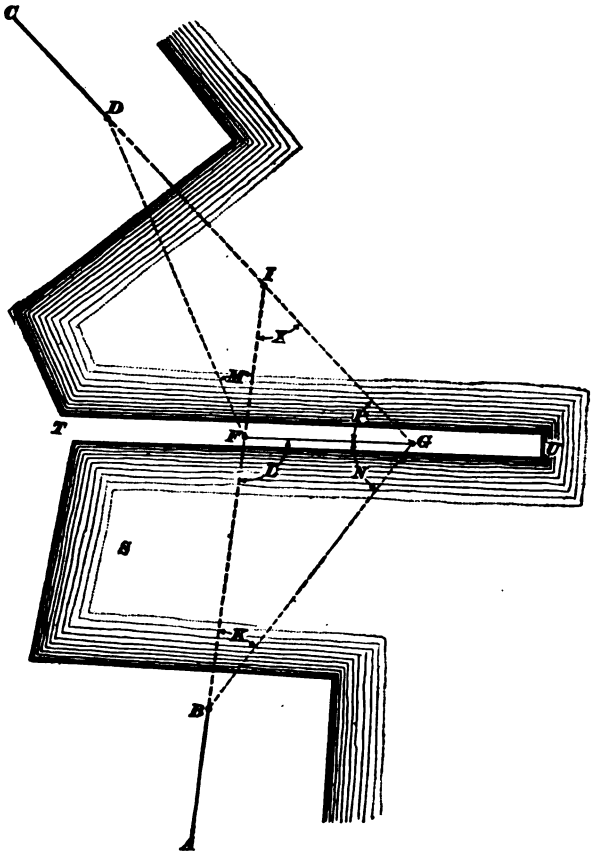

Fig. 10

brought under any classification or general rules. In work of this kind, the surveyor must exercise his ingenuity.

In Fig. 10, *A B* and *C D* represent two lines on shore that intersect at the point *I*, which, being in the water, is inaccessible. The line *A B* when prolonged crosses a dock *S*, too wide for direct measurement, then crosses the wharf or pier TU, and passes over the water to the point of intersection I. It is required to find the distances BF, BI, and DI, and the angle X between the two lines.

With the transit at D, the point G is located on the wharf in the prolongation of the line CD. With the transit at B, the point F, in the prolongation of AB, is located on the wharf, and at the same time the angle K is measured. The transit is then set at F, directed to A, and the angle L between AF and FG is measured. The telescope is again directed to A, plunged, and the angle M between AI and FD is measured. The distance FG is measured, and also the angle P at G.

Since the angle L is an external angle of the triangle FIG, we have,

$$X + P = L, \text{ whence } X = L - P$$

The angles N and D are found from the relations

$$N = 180° - (K + L),\ D = X - M$$

The triangle BFG gives

$$BF = \frac{FG \sin N}{\sin K}$$

The triangle FIG gives

$$FI = \frac{FG \sin P}{\sin X}$$

The sum of BF and FI gives BI. The triangle FDI gives

$$DI = \frac{FI \sin M}{\sin D}$$

EXAMPLE FOR PRACTICE

Suppose that, in Fig. 10, $K = 38° 40'$, $L = 109° 10'$, $M = 21° 50'$, $P = 57° 50'$ and the distance $FG = 180.3$ feet. Determine the distances BF, BI, and DI and the angle X between AB and CD.

$$\text{Ans.} \begin{cases} BF = 153.6 \text{ ft.} \\ BI = 349.1 \text{ ft.} \\ DI = 147.6 \text{ ft.} \\ X = 51° 20' \end{cases}$$

PASSING OBSTACLES

40. A building or other obstacle is often found directly in the line of survey. In such cases the problem is to carry the line around the obstacle so as to continue it in its true position beyond the obstacle, and at the same time determine the length of the portion of the line that cannot be, or is not, measured directly. This can be accomplished in different ways, and in many cases in a very simple manner and without resorting to triangulation. Some of the methods employed are described in *Chain Surveying* and in *Compass Surveying*, Part 1. Here only those methods will be described that are used in transit surveying.

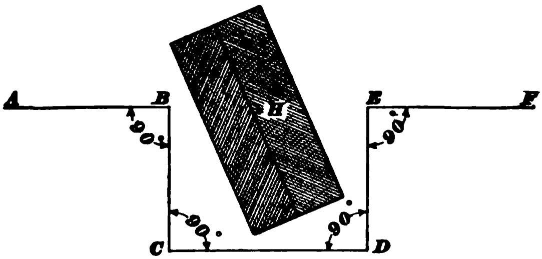

FIG. 11

41. First Method: by Right-Angle Offsets.—Suppose that a building *H*, Fig. 11, stands directly in the line *A B* of a survey and extends a considerable distance on each side of the line, which must be prolonged beyond the building, as at *E F*. The instrument being set up at *B*, the telescope is directed to *A*, and the angle *A B C*, equal to 90°, is turned off on the horizontal circle, in this case to the left, giving the direction of the line *B C*. A distance *B C* is then measured along this line, of sufficient length for the line *C D*, at right angles to *B C*, to clear the end of the building. The transit is then set at *C*, the telescope directed to *B*, and the angle *B C D*, equal to 90°, is turned in the opposite direction to that in which the angle *A B C* was turned. On the line *C D*, a distance *C D* is measured sufficiently long to clear the obstacle. The instrument is then set up at *D*, the telescope is directed to *C*, and the right angle *CDE* turned toward the prolongation of *AB*. On the line *DE*, a distance *DE* is measured equal to *BC*. The instrument is now set up at *E*, the telescope is directed to *D*, and the angle *DEF*, equal to 90°, is turned off. The telescope will then be directed along the continuation *EF* of *AB*. The distance *BE* is equal to the distance *CD*.

When the offset distances are small, the right angles can usually be estimated by the eye closely enough for ordinary purposes, but when a high degree of accuracy is required or the offset is large, the right angles should be given by the transit, as just explained.

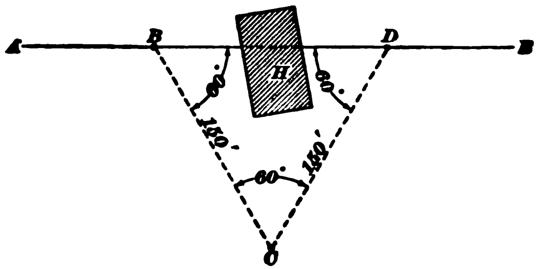

FIG. 12

42. Second Method: by an Equilateral Triangle. Let *AB*, Fig. 12, be the line of survey, and *H* the obstacle to be passed. The instrument being set up at *B*, the telescope is directed to *A*, then plunged, and the forward angle *DBC*, equal to 60°, is turned off. A point *C* is set in the line *BC* at a suitable distance from *B*, say 150 feet. The transit is then set up at *C*, the telescope is directed to *B*, an angle *BCD*, equal to 60°, is turned off, and a distance equal to *BC* is measured to a point *D*. The point *D* will be in the prolongation of *AB*.

The transit is then set up at *D*, directed to *C*, and an angle *CDE*, equal to 120°, is turned to the left. The line of sight will then point along the prolongation *DE* of *AB*. The distance *BD* is equal to *BC*, since the triangle *BCD* is equilateral.

43. Modifications of Second Method.—It may sometimes be convenient to make the angle DBC, Fig. 12, equal to 45° and the angle BCD equal to 90°, in which case the angle CDB is made equal to the angle DBC, or 45°, and the distance CD is made equal to the distance BC. The distance BD will then be equal to $BC \div \sin 45° = 1.4142\, BC$.

In some cases, a triangle whose angles are, respectively, 30°, 60°, and 90° can be employed to advantage. In other cases it may be convenient to use other angles. Any angles whatever may be turned off from B and C. The angle CDE to be turned off at D is always equal to $B + C$. The distance BC is taken so that the line CD run from C will clear the obstacle. The distance CD to be measured from C is computed from the measured angles and the measured distance BC. The distance BD is computed in a similar manner.

44. Third Method: by an Isosceles Triangle.—Suppose that $A'A$, Fig. 13, is the line of survey and that the obstacle is a large tree T, standing directly in the line. The transit being set at A, any small angle $B'AB$ may be turned off in either direction, just large enough to pass the obstacle. A point B is set in the deflected line in any convenient position near or just beyond the obstacle, and the distance AB is measured. The transit is then set up at B, a

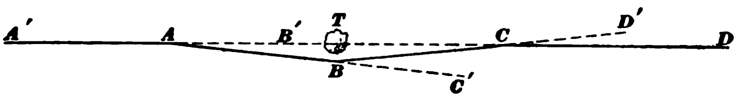

FIG. 13

backsight is taken on A, and a forward angle ABC is turned off equal to 180° minus twice the angle $B'AB$. The distance BC is then measured along this line equal to the distance AB, and the point C is marked. This point will lie in the prolongation of the line $A'A$. The transit is then set at C, directed to B, and a forward angle BCD is turned off equal to 180° minus the first angle $B'AB$. Then the telescope will be directed along the prolongation of the line $A'A$.

That the student may not forget the different stages of the process, he should bear in mind that the object of the operation is to determine an isosceles triangle ABC. Since the acute angles at A and C are equal, the exterior angle CBC' is equal to twice either of them, as A, and therefore ABC is equal to $180° - 2A$, and BCD to $180° - A$. Another way of proceeding is to direct the telescope from B to A, plunge it, and turn off the angle $C'BC$ equal to $2A$. Likewise, after directing the telescope from C to B, it may be plunged, and the angle $D'CD$ turned off equal to A. However, it is always advisable not to plunge the telescope.

The distance AC can be readily determined by solving the triangle ABC, which gives

$$AC = 2\overline{AB}\cos A$$

For nearly all practical work, the cosine of an angle less than 1° may be taken equal to 1, and therefore, when A is less than 1°, AC may be taken equal to $2AB$.

SUPPLYING OMISSIONS

GENERAL METHODS

45. Introductory Statement.—It is sometimes impossible to measure the length and bearing of every side of a closed field, and sometimes, from accident, omissions occur in the notes. In such cases, the parts that are missing must be calculated from the other parts. In a closed survey, any two omissions can be supplied by calculation. The surveyor should make every measurement practicable, however, so as to avoid the necessity of supplying omissions in this manner; for, when omissions are supplied by computation, it must be assumed that the remaining field notes are exactly correct; consequently, there are no means of balancing the work, and all errors are thrown into the part or parts supplied.

In what follows, the directions of the courses will be assumed to be given by azimuths, not by bearings. When, however, they are given by bearings, the same general methods apply.

46. Cases That May Occur.—There are six cases in which omissions can be supplied by calculation, namely:

1. When the azimuth of one side is wanting.
2. When the length of one side is wanting.
3. When the azimuth and length of one side are wanting.
4. When the azimuths of two sides are wanting.
5. When the lengths of two sides are wanting.
6. When the azimuth of one side and the length of another side are wanting.

47. Cases 1, 2, and 3. When the azimuth, the length, or the azimuth and length of any side are wanting, find the ranges of the remaining sides. The algebraic sum of the ranges of either kind, with its sign changed, is equal to the corresponding range of the side in question. Knowing the two ranges of the course, its length l and azimuth Z can be found by the formulas,

$$\tan Z = g \div t$$

$$l = \sqrt{g^2 + t^2}, \text{ or } l = g \div \sin Z$$

EXAMPLE.—The azimuths and lengths of the first three courses of a survey are 32° 15′, 22 chains; 143° 30′, 10 chains; and 164° 15′, 5 chains, respectively. To determine the length and azimuth of the fourth course, which closes the survey.

SOLUTION.—Let g_1, g_2, g_3, g_4 be the longitude ranges, and t_1, t_2, t_3, t_4, the latitude ranges of the courses; also, let l_4 and Z_4 be, respectively, the required length and azimuth of the fourth course.

The ranges of the three given sides are as follows:

$$\begin{aligned} g_1 &= 22 \sin 32° 15' = 22 \times .53361 = 11.74 \\ g_2 &= 10 \sin 143° 30' = 10 \times .59482 = 5.95 \\ g_3 &= 5 \sin 164° 15' = 5 \times .27144 = 1.36 \\ &\qquad\qquad 19.05 \text{ ch.} \end{aligned}$$

$$\begin{aligned} t_1 &= 22 \cos 32° 15' = 22 \times .84573 = 18.61 \\ t_2 &= 10 \cos 143° 30' = 10 \times -.80386 = -8.04 \\ t_3 &= 5 \cos 164° 15' = 5 \times -.96246 = -4.81 \\ &\qquad\qquad +5.76 \end{aligned}$$

Therefore, $g_4 = -19.05$ ch., $t_4 = -5.76$ ch., and $\tan Z_4 = \frac{-19.05}{-5.76}$.

The value of Z_4 may be either the acute angle 73° 11′, or 180° + 73° 11′, since these angles have the same tangent. Notice, however, that, since l_4 is always positive, and $l_4 = g_4 \div \sin Z_4$, $\sin Z_4$ must have the same sign as g_4. In this case, g_4 is negative; therefore, $\sin Z_4$ must be negative, and the value 180° + 73° 11′ = 253° 11′ must be taken for Z_4. Then,

$$l_4 = \frac{-19.05}{\sin 253° 11'} = 19.90 \text{ ch. Ans.}$$

EXAMPLES FOR PRACTICE

1. The azimuths and lengths of the first three courses of a survey are 48° 16′, 25.16 chains; 158° 30′, 14.00 chains; and 208° 13′, 15.62 chains, respectively. Determine: (*a*) the length of the fourth course, which closes the survey; (*b*) the bearing of that course.

Ans. { (*a*) 19.33 ch. (*b*) N 58° 43′ W

2. The lengths of the four sides of a closed field are 286.5 feet, 300 feet, 250 feet, and 573 feet, respectively. The azimuths of the first three sides are 274° 16′, 248° 32′, and 158° 46′; find the azimuth of the fourth side. Ans. 55° 53′

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48. Cases 4 and 5.—(*a*) When the two deficient sides are adjacent, find the ranges of the other sides. Let *D E* and *E F*, Fig. 14, be the two deficient sides. Then, having found the ranges of the other sides, the azimuth and length of the line *D F* joining their extremities can be found as described for one side. Then, in the triangle *D E F*, in Case 4, three sides are known, and in Case 5 one side and two angles are known, from either of which conditions all remaining parts of the triangle can be calculated and the omissions supplied.

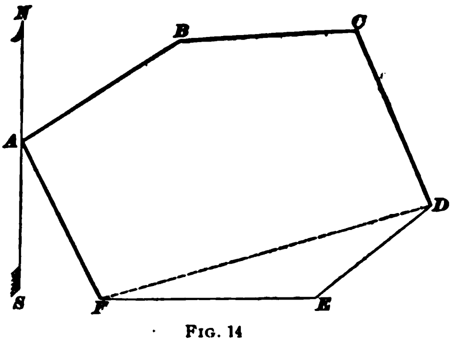

FIG. 14

(*b*) When the two deficient sides are not adjacent, one of these sides must be shifted to a position adjacent to the other. Let *A F* and *D E*, Fig. 15, be the two sides whose azimuths or lengths are wanting. Imagine the side *D E* to be shifted, parallel to itself, to the position *G F*, and the side *E F* to the position *D G*, thus bringing the two deficient sides *A F* and *F G* adjacent and forming the new figure *A B C D G F*. This does not change the values of the lengths or azimuths of the sides, and reduces the solution of the problem to the case in which the deficient sides are adjacent. Draw the line *A G*, joining the two extremities of the known sides. Then, in the figure *A B C D G*, the lengths and azimuths of all the sides except *A G* are known. The length and azimuth of *A G* can be found as described for finding the length and azimuth of one side.

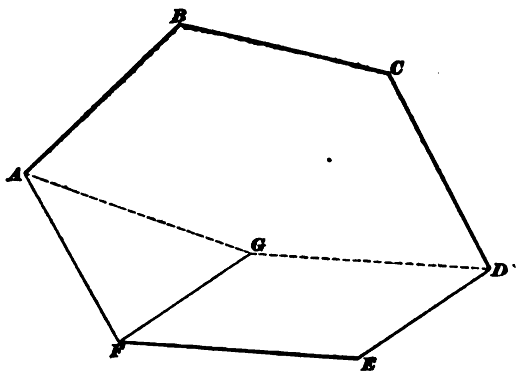

FIG. 15

If the lengths of the two sides *A F* and *D E* are known and their azimuths are wanting, in the triangle *A F G* the lengths of the three sides are known, and the angles can be calculated and the required azimuths determined. If the azimuths of the sides *A F* and *D E* are known and the lengths are wanting, in the triangle *A F G* all the angles and the length of one side are known, and the lengths of the other two sides can be calculated.

In solving all problems of this character, a plat of the traverse should be drawn to scale, showing all the conditions clearly and the triangle that is to be solved. This will prevent mistakes and facilitate the work.

49. Case 6.—When the azimuth of one side and the length of another are wanting, the problem is solved by shifting one deficient side to a position adjacent to the other deficient side, and calculating the line joining the extremities of the known sides as described for Cases 4 and 5. In this case, the two deficient sides together with the calculated closing side form a triangle in which two sides and one angle are known.

EXAMPLE.—Suppose that, in Fig. 15, the azimuths and lengths of the first three and the fifth courses are: *A B*, 43° 50′, 5.76 chains; *B C*, 77° 30′, 5.65 chains; *C D*, 157° 15′, 6.23 chains; and *E F*, 275° 35′, 7.24 chains; and that the length of *D E* is 6.80 chains and the azimuth of *F A* is 330° 36′. To determine the azimuth of *D E* and the length of *F A*.

SOLUTION.—Assume the side *D E* to be shifted parallel to itself to the position *G F*, and the side *E F* shifted to the position *D G*. In the new figure *A B C D G F*, it is required to find the azimuth of *G F* and the length of *F A*. The line *G A* is drawn and its length and azimuth are calculated as described in Art. **48.** Its length is found to be 4.72 ch., and its azimuth, 266° 00′. In the triangle *A G F*, the azimuth of *G A* is 266° 00′, and the azimuth of *F A* is 330° 36′. Therefore, the angle *G A F* is equal to 330° 36′ − 266° 00′ = 64° 36′. In the triangle *G A F*

$$\sin AFG = \frac{AG \sin 64° 36'}{FG} = \frac{4.72 \times .90334}{6.80}$$

whence, *A F G* = 38° 50′.

$$AGF = 180° - (64° 36' + 38° 50') = 76° 34'$$

Then $$FA = \frac{FG \sin 76° 34'}{\sin 64° 36'} = 7.32 \text{ ch. Ans.}$$

The azimuth of *G F* is equal to the azimuth of *G A* minus the angle *A G F*, or 266° 00′ − 76° 34′ = 189° 26′. Ans.

EXAMPLES FOR PRACTICE

1. Suppose that, in Fig. 14, the bearings and lengths of the first three and the sixth courses are N 40° 36′ E, 314.0 feet; N 89° 35′ E, 406.0 feet; S 32° 14′ E, 212.0 feet; and N 26° 15′ W, 196.2 feet; and that the bearings of *D E* and *E F* are S 57° 46′ W, and N 79° 47′ W. Determine the lengths of *D E* and *E F*.

Ans. $\begin{cases} DE = 514.3 \text{ ft.} \\ EF = 204.8 \text{ ft.} \end{cases}$

2. The azimuths and lengths of the first two courses of a survey are 282° 36′, 32.00 chains; and 38° 49′, 14.00 chains, respectively. The length of the third course and the azimuth of the fourth course are,

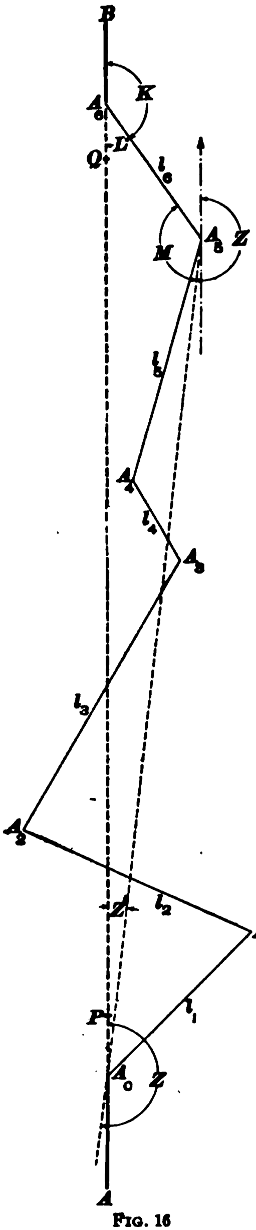

FIG. 16

respectively, 32.52 chains, and 198° 15′. Determine: (*a*) the azimuth of the third course; (*b*) the length of the fourth course.

Ans. { (*a*) 74° 08′
(*b*) 28.20 ch.

APPLICATIONS

50. Passing Obstacles by a Traverse.—Suppose that obstacles make it impossible to measure directly a long portion PQ of a line AB, Fig. 16, and that the conditions make it impossible or inconvenient to apply any of the methods already described for passing obstacles. Starting from a point A_0, on the line, and near the point P where the first obstacle is met, run a traverse $A_0 A_1 A_2 A_3$, etc., selecting the points A_1, A_2, etc. so that the number of lines will be as few as possible, and that their lengths l_1, l_2, l_3, etc. can be conveniently measured. The last point A_5 should be so selected that a line from it shall cross the line of survey at a point A_6 (to be determined) beyond the last obstacle Q. For convenience, the line AB is taken as a meridian; that is, it is treated as if it were a north-and-south line, B being its northern end. The azimuths of l_1, l_2, l_3, etc., referred to this assumed meridian, are measured, and also the lengths of all the lines to A_5. The azimuth of l_6 is also measured, but its length, which determines the point A_6 on the line of survey, must be computed. Here the azimuths of all the lines are known (the azimuth of $A_6 A_0$

being 0°), to determine the lengths of the two sides A_5A_6 and A_6A_0. These lengths are determined as explained in Art. **48**. Since the direction of A_5A_6 is known, the point A_6 is located by measuring from A_5 the calculated length l_6. The transit is then set at A_6, and oriented on A_5, by setting the vernier to read the back azimuth of A_5A_6 and directing the telescope to A_5. The upper plate is then loosened, and the vernier is set at zero. The line of sight will then have the direction A_6B, or AB, since the azimuth of this line is 0°.

EXAMPLE.—To determine A_5A_0 and l_6 from the following measurements, the azimuths being denoted by Z:

l_1 = 325 feet	Z_1 = 45°
l_2 = 400 feet	Z_2 = 294°
l_3 = 500 feet	Z_3 = 30°
l_4 = 150 feet	Z_4 = 330°
l_5 = 400 feet	Z_5 = 16°
	Z_6 = 325°

SOLUTION.—The ranges of l_1, l_2, etc. are as follows:

$$g_1 = 325 \sin 45° = +229.8$$
$$g_2 = 400 \sin 294° = -400 \sin (360° - 294°) = -365.4$$
$$g_3 = 500 \sin 30° = +250.0$$
$$g_4 = 150 \sin 330° = -150 \sin (360° - 330°) = -75.0$$
$$g_5 = 400 \sin 16° = +110.3$$
$$+149.7$$

$$t_1 = 325 \cos 45° = +229.8$$
$$t_2 = 400 \cos 294° = 400 \cos (360° - 294°) = +162.7$$
$$t_3 = 500 \cos 30° = +433.0$$
$$t_4 = 150 \cos 330° = 150 \cos (360° - 330°) = +129.9$$
$$t_5 = 400 \cos 16° = +384.5$$
$$+1339.9$$

The ranges of the closing line A_5A_0 have opposite signs to those of the algebraic sums of the ranges of all the other courses. Therefore, the longitude range g of A_5A_0 is −149.7, and the latitude range t is −1,339.9.

For the azimuth Z of A_5A_0, we have

$$\tan Z = \frac{-149.7}{-1{,}339.9}$$

The value of Z may be either 6° 23′ or 180° + 6° 23′, since these angles have the same tangent. But, since sin Z must have the same sign as the longitude range (see Art. **47**), it must be negative, and since sin $(180° + 6° 23')$ is negative, the value of Z is $180° + 6° 23' = 186° 23'$

The angle Z' is equal to the back azimuth of $A_3 A_4$, or $(180° + 186° 23') - 360° = 6° 23'$. For the length l of $A_3 A_4$, we have

$$l = \frac{g}{\sin Z}$$

It is not necessary to actually calculate l, as will be seen from the transformations given below.

The angle K is equal to the back azimuth of $A_3 A_4$, that is $(325° + 180°) - 360° = 145°$. Also, $L = 180° - K = 180° - 145° = 35°$

$$M = Z_4 - Z = 325° - 186° 23' = 138° 37'$$

The triangle $A_2 A_3 A_4$ now gives

$$l_4 = l\frac{\sin Z'}{\sin L};\ A_4 A_4 = l\frac{\sin M}{\sin L}$$

Substituting the value of l given above, we have

$$l_4 = \frac{g}{\sin Z}\,\frac{\sin Z'}{\sin L};\ A_4 A_4 = \frac{g}{\sin Z}\,\frac{\sin M}{\sin L}$$

But $$\frac{g}{\sin Z} = \frac{-149.7}{\sin(180° + Z')} = \frac{-149.7}{-\sin Z'} = \frac{149.7}{\sin Z'}$$

Then, $$l_4 = \frac{149.7}{\sin Z'}\,\frac{\sin Z'}{\sin L} = \frac{149.7}{\sin L} = 261.0$$

$$A_4 A_4 = \frac{149.7}{\sin Z'}\,\frac{\sin M}{\sin L} = \frac{149.7 \sin 138° 37'}{\sin 6° 23' \sin 35°} = 1{,}551.9. \quad \text{Ans.}$$

51. Inaccessible Intersections.—The intersection P, Fig. 17, of two lines being inaccessible, and the conditions being such that the methods explained in Arts. **38** and **39** cannot be applied, the distances BP and DP and the angle I are determined by running a traverse line $B A_1 A_2 A_3 D$ between two points on the given lines. For convenience, one of the given lines (AB in this case) is taken as a meridian. After measuring the azimuths and lengths of l_1, l_2, l_3, l_4, the transit is set at D, on CD, oriented on DA_3, and the azimuth of DC is measured. The angle I is equal to this azimuth, since AB, or AE, is the meridian.

In the polygon $B A_1 A_2 A_3 D I$, all azimuths and lengths are known, except the lengths of the two sides DP and PB. These sides can be supplied by the methods already explained. It should be carefully borne in mind that the azimuth of DP is equal to the back azimuth of DC, or $180° + I$, and that the azimuth of PB is zero.

EXAMPLE.—To determine DP and PB from the following measurements:

$l_1 = 600$ feet	$Z_1 = 120°$
$l_2 = 300$ feet	$Z_2 = 6°$
$l_3 = 450$ feet	$Z_3 = 75°$
$l_4 = 350$ feet	$Z_4 = 30°$
I = azimuth of DC	$= 107° 39'$

SOLUTION.—The ranges of the traverse are:

$g_1 = 600 \sin 120°$	$= + 519.6$	$t_1 = 600 \cos 120°$	$= -300.0$
$g_2 = 300 \sin 6°$	$= + 31.4$	$t_2 = 300 \cos 6°$	$= +298.4$
$g_3 = 450 \sin 75°$	$= + 434.7$	$t_3 = 450 \cos 75°$	$= +116.5$
$g_4 = 350 \sin 30°$	$= + 175.0$	$t_4 = 350 \cos 30°$	$= +303.1$
	$+1160.7$		$+418.0$

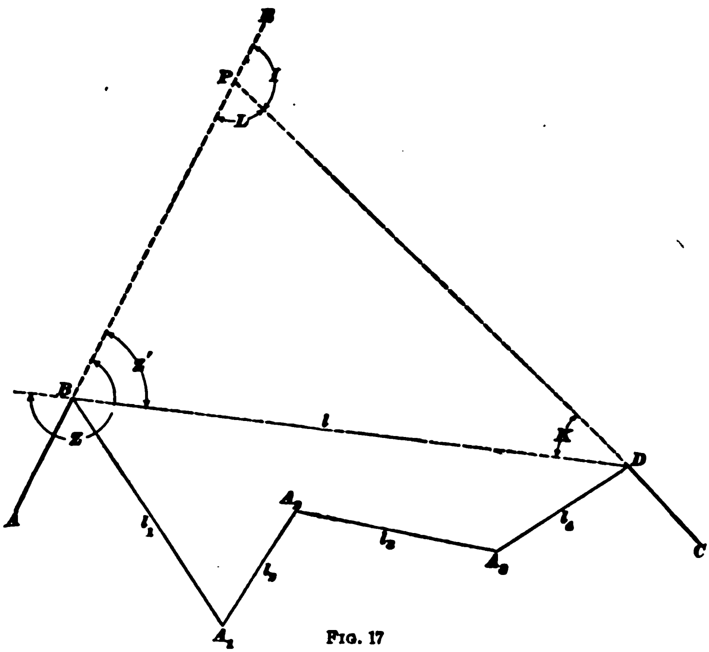

FIG. 17

For the azimuth Z of the closing line DB we have

$$\tan Z = \frac{-1{,}160.7}{-418.0},\ Z = 250° 12'$$

Also,

$$Z' = Z - 180° = 70° 12'$$

$$l = \frac{-1{,}160.7}{\sin Z} = \frac{-1{,}160.7}{\sin(180° + Z')} = \frac{1{,}160.7}{\sin Z'}$$

$$L = 180° - I = 180° - 107° 39' = 72° 21'$$

$$K = I - Z' = 107° 39' - 70° 12' = 37° 27'$$

The triangle BDP gives

$$DP = l\frac{\sin Z'}{\sin L} = \frac{1{,}160.7}{\sin Z'}\,\frac{\sin Z'}{\sin L} = \frac{1{,}160.7}{\sin L} = \frac{1{,}160.7}{\sin 72°\,21'} = 1{,}218.0 \text{ ft.}$$

$$PB = l\frac{\sin K}{\sin L} = \frac{1{,}160.7}{\sin Z'}\,\frac{\sin K}{\sin L} = \frac{1{,}160.7}{\sin 70°\,12'}\,\frac{\sin 37°\,27'}{\sin 72°\,21'} = 787.2 \text{ ft.}$$

EXAMPLES FOR PRACTICE

1. The azimuths of four courses of a deflected traverse that has been run to pass an obstacle in the line of survey are 273° 12′, 49° 56′, 312° 15′, and 42° 45′, respectively. The lengths of the first three courses are 250 feet, 150 feet, and 200 feet, respectively. Determine: (*a*) the length of the fourth course; (*b*) the distance between the two points.

Ans. { (*a*) 416.6 ft. (*b*) 551.1 ft.

2. In order to determine the distances of two points B and D, Fig. 17, on the lines AB and CD, from their point of intersection P, a traverse of five lines was run from B to D. The lengths of these lines and their azimuths, taking AB as the meridian, were as follows:

l_1 = 200 feet Z_1 = 102° 45′
l_2 = 300 feet Z_2 = 38° 36′
l_3 = 150 feet Z_3 = 139° 14′
l_4 = 100 feet Z_4 = 42° 51′
l_5 = 250 feet Z_5 = 32° 23′

Azimuth of DC = 113° 37′

Determine the distances BI and DI and the angle of intersection I.

Ans. { BI = 659.4 ft. DI = 744.5 ft. I = 113° 37′

PROBLEMS ON AREAS

GENERAL REMARKS ON THE DETERMINATION OF AREAS

AREA BOUNDED BY STRAIGHT LINES

52. Division Into Triangles.—When the lengths and bearings (or azimuths) of all the courses of a closed field are known, the best method for calculating the area is by double longitudes, as explained in *Compass Surveying*, Part 2. If the angles at the different corners have been measured, instead of the bearings or azimuths, any of the sides may be assumed as a meridian; the azimuths or bearings of all the sides with reference to that meridian can be easily determined, the ranges calculated, and the method of double longitudes applied.

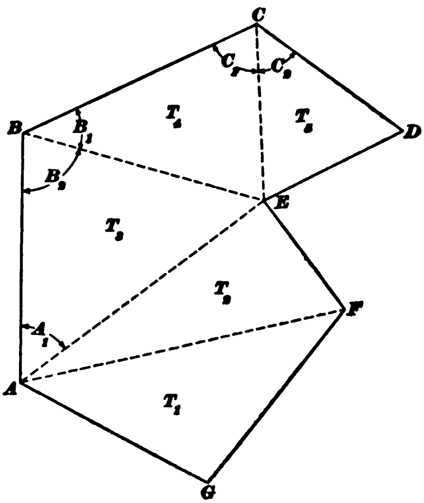

Fig. 18

Another way of computing the area is by dividing the field into triangles. This method is especially adapted to a field surveyed by the chain only, as explained in *Chain Surveying*. As there stated, the division into triangles should, if possible, be so made that the angles will be neither too acute nor too obtuse. In this respect, the surveyor must use his judgment and exercise some ingenuity. Fig. 18 shows a tract divided into triangles by the diagonals *E C*, *E B*, etc. If the chain alone is used, these diagonals must be measured in the field.

If a transit or compass is used, it is not necessary to measure the diagonals *CE*, *B E*, etc., but, the angles C_1, B_1, B_2, etc. having been measured, the areas T_4, T_5, etc. are computed by the formula given in *Plane Trigonometry*, Part 2, for the area of a triangle of which one side and the angles are known.

When two of the sides of a triangle T_6 are parts of the boundary and the angle between them has been measured, the area is equal to one-half the product of the two sides and the sine of the included angle.

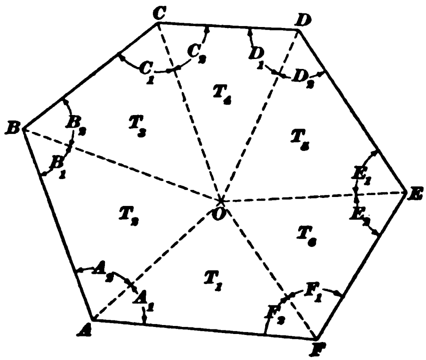

FIG. 19

If there is a point inside the field visible from all the corners, the method illustrated in Fig. 19 may be used. Here *O* is a point inside the field visible from all the corners. The lengths of the sides *A B*, *B C*, etc. having been measured, as well as the angles A_1, A_2, B_1, B_2, etc., the areas of the triangles T_1, T_2, T_3, etc. are computed by the formula of trigonometry just referred to.

For the conditions shown in Fig. 19, a method that is in some cases very expeditious, when it is not necessary to measure the boundary lines of the tract, is to set up the transit at *O* and measure the angles *A O B*, *B O C*, *C O D*, etc. and also measure the radial lines *O A*, *O B*, *O C*, etc. The area of each triangle is equal to one-half the product of the two sides and the sine of the included angle.

53. When not very accurate results are required, the area of a field, or of any other figure bounded by straight lines, may be obtained by making an accurate drawing to scale, dividing it into triangles, and measuring on the drawing itself whatever dimensions are necessary for determining the area of each triangle.

In order to obtain as accurate results as possible, the plat should be as large and the triangles into which the polygon is divided should be as nearly equilateral as the conditions will permit.

Let the irregular polygon *ABCDEF*, Fig. 20, be the outline of a tract of land the area of which is required. The diagonals *BF*, *CF*, and *CE* are drawn, dividing the figure into the four triangles *ABF*, *BCF*, *CEF*, and *CDE*. From the vertexes *A*, *B*, *D*, and *E*, the perpendiculars *AG*,

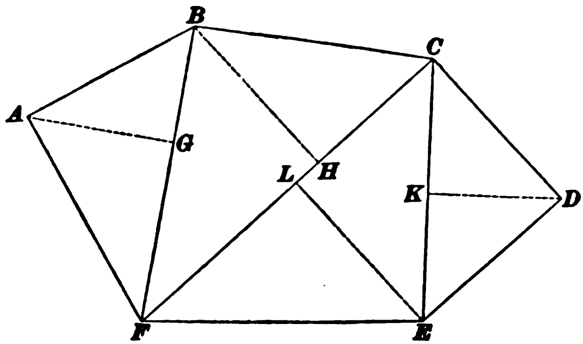

FIG. 20

BH, *DK*, and *EL* are let fall on the opposite bases of the triangles. The lengths of the several bases and altitudes are measured to the scale used in constructing the plat, and the area of each triangle is calculated by multiplying the altitude by one-half the base.

EXAMPLE.—Assuming Fig. 20 to represent the plat of a tract of land, drawn to a scale of 4 chains to the inch, it is required to calculate the area of the tract.

SOLUTION.—In the triangle *ABF*, the base scales 6.25 ch. and the altitude 3.10 ch.; hence, its area is equal to $\frac{1}{2} \times 6.25 \times 3.10 = 9.6875$ sq. ch. The base *FC* of the triangle *BCF* scales 8.20 ch. and the altitude *BH* scales 3.80 ch.; the altitude *LE* of the triangle *CEF*, which has the same base, scales 3.90 ch.; hence, the area of the quadrilateral *BCEF* formed by the two triangles *BCF* and *CEF* is equal to $\frac{1}{2} \times 8.20 \times (3.80 + 3.90) = 31.57$ sq. ch. The base CE of the triangle CDE scales 5.50 ch. and its altitude scales 2.75 ch.; hence, the area of this triangle is equal to $\frac{1}{2} \times 5.50 \times 2.75 = 7.5625$ sq. ch. The area of the entire tract is therefore equal to $9.6875 + 31.57 + 7.5625 = 48.82$ sq. ch. $= 4.882$ A. Ans.

54. Division of the Plat Into Trapezoids.—A plat of the tract having been drawn to scale, the greater portion of the plat can be divided into trapezoids and the remainder into triangles, and the area of each trapezoid and triangle calculated separately. The sum of these different areas will then be the area of the tract. Thus, in Fig. 21, the diagonal AD is drawn as a base, and the lines BB', CC', EE', and FF' are drawn perpendicular to AD, dividing the figure into trapezoids and triangles. The area of the trapezoid $BCC'B'$ is equal to one-half the sum of its bases BB' and CC' multiplied by its altitude $B'C'$. The areas of the other trapezoids are calculated in like manner, the dimensions being scaled from the plat. In the case here illustrated, the area of the polygon is equal to the sum of the trapezoids $BCC'B'$ and $EFF'E'$, and the triangles ABB', CDC', DEE', and AFF'.

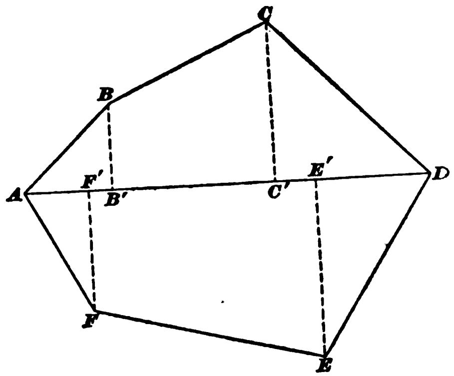

FIG. 21

55. Auxiliary Trapezoids.—If the plat of the tract, when drawn to scale, is an irregular polygon, its area can often be calculated easily by means of the difference between the sums of the areas of two series of trapezoids constructed for the purpose. A straight line is drawn as a line of reference in any position near the polygon, and a perpendicular is drawn to this line from each angle of the polygon. Since these perpendiculars will be parallel to each other, a series of trapezoids will be formed, one side of each trapezoid being formed by a side of the polygon and one side by the line of reference. The area of each trapezoid is then calculated, and the sum of the areas of the trapezoids lying between the line of reference and the nearer sides of the polygon is subtracted from the sum of the areas of those lying between the same line and the farther sides of the polygon; the remainder is the area of the polygon.

Thus, in order to determine the area of the polygon *ABCDEFG*, Fig. 22, the line of reference *HP* is drawn, and from the angles of the polygon the lines *AH*, *BK*, *GL*, etc. are drawn perpendicular to *HP*. The areas of all the trapezoids thus formed are found, and from the sum of the areas of the three trapezoids *ABKH*, *BCNK*, and *CDPN* the sum of the areas of the trapezoids *AGLH*, *GFML*, *FEOM*, and *EDPO* is subtracted. The remainder is the area of the polygon *ABCDEFG*. The line of reference may be drawn in any position near the polygon, as in the figure, or through an angle of the polygon, or may coincide with one of its sides.

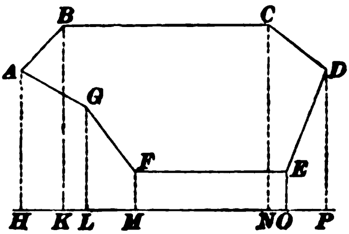

FIG. 22

EXAMPLE.—Assume that the polygon *ABCDEFG*, Fig. 22, represents the plat of a tract of land, drawn to scale, from the angles of which perpendiculars have been drawn to the line of reference *HP*, and that by measuring the lines to scale they are found to have the following lengths, in chains: $AH = 7.20$, $BK = 9.60$, $CN = 9.60$, $DP = 7.40$, $EO = 2.00$, $FM = 2.00$, $GL = 5.40$, $HK = 2.20$, $KN = 10.60$, $NP = 3.00$, $PO = 2.10$, $OM = 7.90$, $ML = 2.50$, and $LH = 3.40$; what is the area of the tract?

SOLUTION.—The areas of the trapezoids lying between the line of reference and the farther sides of the polygon are: $ABKH = \frac{7.20 + 9.60}{2} \times 2.20 = 18.48$; $BCNK = 9.60 \times 10.60 = 101.76$; $CDPN = \frac{9.60 + 7.40}{2} \times 3 = 25.5$; and the sum of these three trapezoids is $18.48 + 101.76 + 25.50 = 145.74$ sq. ch. The areas of the trapezoids lying between the line of reference and the nearer sides of the polygon are: $DPOE = \frac{7.40 + 2.00}{2} \times 2.10 = 9.87$; *EOMF*

$= 2 \times 7.90 = 15.80$; $FMLG = \frac{2.00 + 5.40}{2} \times 2.50 = 9.25$; $GLHA = \frac{5.40 + 7.20}{2} \times 3.40 = 21.42$; and the sum of these four trapezoids is $9.87 + 15.80 + 9.25 + 21.42 = 56.34$ sq. ch. Hence, the area of the polygon is equal to $145.74 - 56.34 = 89.40$ sq. ch. $= 8.94$ A. Ans.

EXAMPLES FOR PRACTICE

1. Assume that Fig. 20 represents the plat of a tract of land drawn to scale, and that by measuring the lines to scale they are found to have the following lengths, in chains: $BF = 7.80$, $AG = 3.85$, $FC = 10.20$, $BH = 4.80$, $LE = 4.90$, $CE = 6.90$, and $KD = 3.45$; what is the area of the tract? Ans. 7.639 A.

2. Assume that Fig. 22 represents the plat of a tract of land drawn to scale, and that by measuring the lines of the plat to scale they are found to have the following lengths, in chains: $AH = 29.00$, $BK = 38.50$, $CN = 38.50$, $DP = 29.50$, $EO = 8.00$, $FM = 8.00$, $GL = 21.50$, $HK = 8.80$, $KN = 42.50$, $NP = 12.00$, $PO = 8.50$, $OM = 31.50$, $ML = 10.00$, and $LH = 13.50$; what is the area of the tract? Ans. 144.15 A.

AREAS BOUNDED BY IRREGULAR LINES

56. The methods of computing the area of a figure bounded by one or more curved lines were fully treated in *Plane Trigonometry*, Part 2. As explained elsewhere, the method that is most commonly employed in surveying is that of selected offsets, which consists in running one or more straight lines close to the boundary, measuring perpendicular offsets from them to the points on the boundary in which the direction of the latter changes, and considering the portion of the boundary between any two consecutive offsets to be a straight line. This is a very convenient and accurate method, and can be applied to any area, whether partly or wholly bounded by irregular lines.

DIVISION AND PARTITION OF LAND

57. The surveyor is frequently required to solve problems relative to the subdivision and partition of land, such as cutting off from a field a piece having a given area, or dividing a tract into parts whose areas shall have a given ratio, by a line having a given direction. The process of dividing a tract in such manners is very simple when the tract is of rectangular form, but is much more difficult in other cases. The solution of problems of this kind depends on a judicious application of the principles of geometry and trigonometry. This subject cannot be treated in a general manner, it being obviously impossible to formulate or give rules for all the cases that may occur in practice. A few common cases are here given, which should be studied carefully, as they may suggest methods for the solution of other similar problems.

58. Problem I.—*To divide a trapezoid into two parts whose areas shall be proportional to two given numbers, by a line parallel to the bases.*

Let it be required to divide the trapezoid $ABCD$, Fig. 23, into two parts that shall be to each other as two given numbers m, n, the dividing line EF (to be determined) being parallel to the bases. Let the length of this line be denoted by x, the area of the trapezoid by S, and the areas of the two parts by S_1 and S_2, as shown. The rest of the notation is plainly shown by the figure. The auxiliary line DB' is parallel to CB.

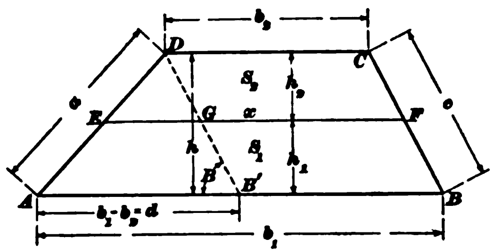

FIG. 23

(*a*) *To determine* S_1 *and* S_2.—By the conditions of the problem, we must have,

$$S_1 : S_2 = m : n; \text{ whence, } S_2 = \frac{n S_1}{m}$$

Also, $$S_1 + S_2 = S$$

or, writing the value of S_2 found above,

$$S_1 + \frac{n S_1}{m} = S$$

whence, clearing of fractions and solving for S_1,

$$S_1 = \frac{m}{m+n} S \qquad (1)$$

Similarly, $$S_2 = \frac{n}{m+n} S \qquad (2)$$

The value of S is to be found by one of the formulas given in *Plane Trigonometry*, Part 2, according to the data.

(*b*) *To determine the length x of the dividing line EF.*—As shown in *Plane Trigonometry*, Part 2,

$$S_1 = \frac{b_1^2 - x^2}{2(\cot A + \cot B)}, \quad S = \frac{b_1^2 - b_2^2}{2(\cot A + \cot B)}$$

Substituting these values in formula 1 and canceling $2(\cot A + \cot B)$, there results

$$b_1^2 - x^2 = \frac{m}{m+n}(b_1^2 - b_2^2)$$

whence, clearing of fractions,

$$m\,b_1^2 + n\,b_1^2 - (m+n)\,x^2 = m\,b_1^2 - m\,b_2^2$$

Transposing and solving for x,

$$x = \sqrt{\frac{m\,b_2^2 + n\,b_1^2}{m+n}} \qquad (3)$$

(*c*) *To determine the distances DE and AE.*—In the similar triangles DAB' and DEG, we have,

$$\frac{DE}{EG} = \frac{DA}{AB'}$$

But, $EG = EF - GF = EF - DC = x - b_2$, $DA = a$, and $AB' = b_1 - b_2$. By the substitution of these values, the preceding proportion becomes

$$\frac{DE}{x - b_2} = \frac{a}{b_1 - b_2}$$

whence, $$DE = \frac{a(x - b_2)}{b_1 - b_2} \qquad (4)$$

Also,

$$AE = AD - DE = a - \frac{a(x - b_2)}{b_1 - b_2} = a\left(1 - \frac{x - b_2}{b_1 - b_2}\right)$$

or, performing the subtraction,

$$AE = \frac{a(b_1 - x)}{b_1 - b_2} \qquad (5)$$

In applying these formulas, the value of x is first found by formula **3.** The calculated distance DE or AE is measured on AD from D or A, as the case may be, and the line EF is then run from E parallel to the bases, and equal in length to the calculated value of x.

(*d*) *To determine the altitudes* h_1 *and* h_2 *of the trapezoids* S_1 *and* S_2.—Any two corresponding sides of the similar triangles DAB' and DEG are to each other as the altitudes of the triangles; that is, as h is to h_2. Therefore,

$$\frac{AB'}{EG} = \frac{h}{h_2}, \text{ or } \frac{b_1 - b_2}{x - b_2} = \frac{h}{h_2}$$

whence, solving for h_2,

$$h_2 = \frac{h(x - b_2)}{b_1 - b_2} \qquad (6)$$

Subtracting h_2 from h, and reducing, we find,

$$h_1 = \frac{h(b_1 - x)}{b_1 - b_2} \qquad (7)$$

EXAMPLE.—Suppose that $ABCD$, Fig. 23, represents a tract of land in which $DC = 50$ chains, $AB = 100$ chains, $AD = 47.50$ chains, and $h = 35$ chains, and that the tract is to be so divided by the line EF that the parts m and n will be as 3 and 2, respectively; required EF, DE, and h_1.

SOLUTION.—By substituting the given values in formula **3**,

$$EF = \sqrt{\frac{3 \times 50^2 + 2 \times 100^2}{5}} = \sqrt{5{,}500} = 74.16 \text{ ch. Ans.}$$

Formula **4**,

$$DE = \frac{47.50 \times (74.16 - 50)}{100 - 50} = 22.95 \text{ ch. Ans.}$$

Formula **7**,

$$h_1 = \frac{35(100 - 74.16)}{100 - 50} = 18.09 \text{ ch. Ans.}$$

EXAMPLES FOR PRACTICE

1. The bases of a trapezoidal field are 46.75 and 25.33 chains. The altitude of the trapezoid is 7.45 chains, and one of the angles (say A, Fig. 23) adjacent to the longer base is 73° 47′. It is desired to divide the field into two trapezoidal tracts by a line parallel to the bases, and so that the tract S_1 adjacent to the longer base shall be two-thirds of

the tract S_2 adjacent to the shorter base; find S_1, S_2, x, AE, DE, h_1, and h_2.

$$\text{Ans.}\begin{cases} S_1 = 10.74 \text{ A.} \\ S_2 = 16.11 \text{ A.} \\ x = 39.60 \text{ ch.} \\ AE = 2.59 \text{ ch.} \\ DE = 5.17 \text{ ch.} \\ h_1 = 2.49 \text{ ch.} \\ h_2 = 4.96 \text{ ch.} \end{cases}$$

2. The area of a trapezoidal field is 102.74 acres; its bases are 39.46 and 53.67 chains; and one of the angles, say D, adjacent to the shorter base is 127° 58′. It is desired to cut off a trapezoidal tract $DEFC$, adjacent to the shorter base, and such that its area shall be 60 acres; find the length x of the dividing line, and also the distance DE and the altitude h_2.

$$\text{Ans.}\begin{cases} x = 48.27 \text{ ch.} \\ DE = 17.36 \text{ ch.} \\ h_2 = 13.68 \text{ ch.} \end{cases}$$

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59. Problem II.—*To cut off a given area by a line starting from a given point on the boundary of a polygonal field.*

Let $ABCDEF$, Fig. 24, be the plat of a field bounded by straight lines, from which it is required to cut off S acres by a line run through a given point G in the boundary. Draw a line GD from G to one of the opposite angles of the plat in such position as to cut off an area nearly equal to the required area. Calculate the length and bearing of GD by the method of supplying omissions already explained. Calculate the area $GBCD$, which will be called S_1. Find the difference between the required area S and the calculated area S_1. If S is greater than S_1, an additional area S' must be found; let GDH be this area.

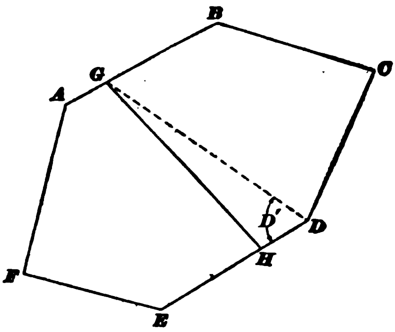


FIG. 24

Then, area $GDH = S - S_1 = S'$. In the triangle GDH, the side GD and the angle D' are known. We have, from trigonometry, $S' = \frac{1}{2} GD \times DH \sin D'$; whence,

$$DH = \frac{2S'}{GD \sin D'}$$

If the required area S is less than S_1, the process is substantially the same, except that the required distance should be calculated and measured from D along the line DC, instead of on the line DE, thus locating the position of GH on the opposite side of GD.

EXAMPLE.—In Fig. 24, assume that the length of the line GD is 8.93 chains, that the angle GDH is 61°, and that the area of $GBCD$ is 3.58 acres; what must be the distance of the point H from the point D in order that the line GH will cut off 5 acres; that is, in order that the area of the figure $GBCDH$ will be 5 acres?

SOLUTION.—The area S' of GDH is equal to $5 - 3.58 = 1.42$ A., or 14.2 sq. ch. Substituting in the formula,

$$DH = \frac{2 \times 14.2}{8.93 \sin 61°} = 3.64 \text{ ch. Ans.}$$

60. Problem III.—*To cut off a given area from a polygonal field by a line running in a given direction.*

Let $ABCDEF$, Fig. 25, be the plat of a field bounded by straight lines, from which it is desired to cut off S acres by a line having a given direction. Let AG be a line parallel to the required line and passing through some definite point in the boundary, as the point A, which line is seen by inspection to be near where the required line must pass in order to cut off the given area. The bearings of AG and GD are known, since AG must have the required direction, and GD is a portion of one side of the plat; and their lengths can be calculated by the methods used for supplying omissions. Calculate the area $ABCG$. Call this area S_1, and find the difference S' between this area and the area S to be cut off. In the present instance, S_1 will be supposed greater than S, so that $S' = S_1 - S$. Let HI, parallel to AG, be the required line of division. This line

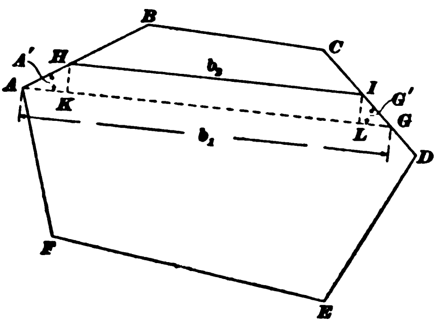


FIG. 25

forms with AG a trapezoid whose area is S'. Let AG be denoted by b_1, and HI by b_2. Then (see *Plane Trigonometry*, Part 2),

$$S' = \frac{b_1^2 - b_2^2}{2(\cot A' + \cot G')}$$

whence,

$$b_2 = \sqrt{b_1^2 - 2\,S'(\cot A' + \cot G')} \qquad (1)$$

For logarithmic functions, $\cot A' + \cot G'$ may be replaced by $\frac{\sin(A' + G')}{\sin A' \sin G'}$. The value of b_2 may then be written

$$b_2 = \sqrt{b_1^2 - 2\,S' \frac{\sin(A' + G')}{\sin A' . \sin G'}} \qquad (2)$$

Also, since $S' = \frac{1}{2}(b_1 + b_2)\,HK$,

$$HK = IL = \frac{2\,S'}{b_1 + b_2} \qquad (3)$$

Finally, $$AH = \frac{HK}{\sin A'} = \frac{2\,S'}{(b_1 + b_2)\sin A'} \qquad (4)$$

$$GI = \frac{IL}{\sin G'} = \frac{2\,S'}{(b_1 + b_2)\sin G'} \qquad (5)$$

Lay off AH from A on the line AB and GI from G on the line CD, thus determining the points H and I, which are points in the boundary at the extremities of the required line.

In any case where the length or direction of a line is calculated, the line should be run on the ground to see if the calculated and measured values agree, so as to check the accuracy of the work.

EXAMPLE.—Suppose that, in Fig. 25, the length of AG is 7.75 chains, the area of $ABCG$ is 9.16 square chains, and the angles HAK and IGL are 28° and 42°, respectively; what must be the lengths of AH and GI in order that the area of $HBCI$ cut off by the line HI will be 5 square chains?

SOLUTION.—The area $AHIG$, or S', is equal to 9.16 − 5 = 4.16 sq. ch. To determine the length of the line HI, the known values are substituted in formula **1**, which gives

$$b_2 = HI = \sqrt{7.75^2 - 2 \times 4.16\,(1.88073 + 1.11061)} = 5.93 \text{ ch.}$$

Substituting this value of HI in formula **3**,

$$IL = HK = \frac{2 \times 4.16}{7.75 + 5.93} = .61 \text{ ch.}$$

Substituting the values of IL and HK in formulas **4** and **5**,

$$AH = \frac{2 \times 4.16}{(7.75 + 5.93) \sin 28°} = 1.30 \text{ ch. } \textbf{Ans.}$$

$$GI = \frac{2 \times 4.16}{(7.75 + 5.93) \sin 42°} = .91 \text{ ch. } \textbf{Ans.}$$

61. Problem IV.—*To cut off a given area from an irregular tract.*

It is sometimes required to cut off a piece having a given area from a tract of irregular outline, by a line having a given direction. The method of procedure in such a case is very similar to that described in the preceding article, the chief difference being that in the present case the tract considered is of irregular form, whereas the tract considered in the preceding article is bounded by straight lines.

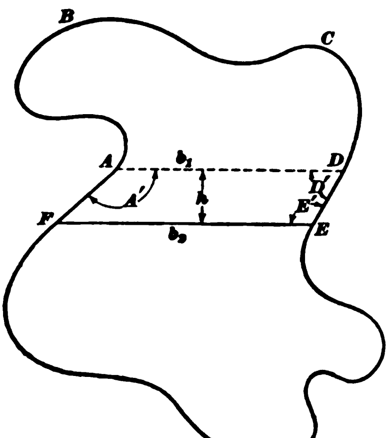


FIG. 26

A trial line should first be run in the required direction and in a position that will cut off an area approximating the required area, and the exact area of the part cut off should be calculated by any of the methods for irregular areas that have been explained. It then becomes an easy problem to determine the position of a line, parallel to this trial line, that will give the required area, since the small piece included between these two parallel lines, representing the excess or deficiency of area, will usually be very closely, if not exactly, of trapezoidal form. The position of the true dividing line, with reference to the trial line, can be calculated, and the line can be located, in substantially the same manner as explained in the preceding article.

Suppose that from the irregular tract shown in Fig. 26 it is required to cut off a piece $BCEF$, having a given area S, by the line FE, having a given direction. The trial line AD is run in a position approximating that of the true dividing line and parallel thereto, and the area $ABCD = S_1$ cut off by the trial line is calculated and compared with the required area. Then, $S' = S - S_1$ will be the area of the small strip $ADEF$ lying between the trial line AD and the true dividing line FE. If the lines AD and FE are reasonably near together, the included strip will be so nearly trapezoidal in form that it may usually be considered to be exactly so.

The angles A' and D' may be determined by measuring them on the plat. With the notation shown in the figure, b_1 is found by formula **1**, Art. **60**; h, by formula **3**, Art. **60**; FA and ED, by formulas **4** and **5**, Art. **60**.

EXAMPLE.—Suppose that, in Fig. 26, the trial line AD is found to measure 12.45 chains, and that the area of $ABCD$ is 14.32 acres. If the angles A' and D' measure 139° and 58°, respectively, what must be the distance h in order that the area of the figure $FBCE$ will be 16 acres?

SOLUTION.—The area S' is equal to 16 − 14.32 = 1.68 A., or 16.8 sq. ch. Substituting known values in formula **1**, Art. **60**,

$$b_1 = \sqrt{12.45^2 - 2 \times 16.8\,(-1.15037 + .62487)} = 13.14 \text{ ch.}$$

Formula **3**, Art. **60**,

$$h = \frac{2 \times 16.8}{12.45 + 13.14} = 1.31 \text{ ch. Ans.}$$

EXAMPLES FOR PRACTICE

1. Suppose that $ABCD$, Fig. 23, represents a tract of land in which DC = 150 chains, AB = 200 chains, and AD = 80.5 chains, and that the tract is to be divided by the line EF so that the area of the part adjacent to AB is to the part adjacent to DC as 5 is to 4; what is the length: (*a*) of EF? (*b*) of AE?

Ans. { (*a*) 174.01 ch. (*b*) 41.84 ch.

2. Suppose that, in Fig. 24, the line GD measures 10.8 chains, the angle GDH measures 70°, and the area $GBCD$ is 6.32 acres; what must be the distance of the point H from the point D in order that the area $GBCDH$ will be 8.45 acres? Ans. 4.20 ch.

A SERIES OF QUESTIONS

RELATING TO THE SUBJECTS
TREATED OF IN THIS VOLUME.

It will be noticed that the questions contained in the following pages are divided into sections corresponding to the sections of the text of the preceding pages, so that each section has a headline that is the same as the headline of the section to which the questions refer. No attempt should be made to answer any of the questions until the corresponding part of the text has been carefully studied.

GEOMETRY

(PART 1)

EXAMINATION QUESTIONS

(1) (*a*) How many degrees are there in one of the interior angles of an equiangular octagon? (*b*) One of the interior angles of an equiangular polygon is 108°; what is the name of the polygon?

Ans. $\begin{cases} (a)\ 135° \\ (b)\ \text{Pentagon} \end{cases}$

(2) How many equal sectors are there in a circle, if each sector measures two-sevenths of a right angle? Ans. 14

(3) One-third of an angle of a certain triangle is 14° 47′ 10″, and one of the other angles is two and one-half times the given angle; what are the angles of the triangle?

Ans. $\begin{cases} 44°\ 21'\ 30'' \\ 110°\ 53'\ 45'' \\ 24°\ 44'\ 45'' \end{cases}$

(4) The exterior angle of an isosceles triangle is 104° 30′ 20″; what is the value of each of the equal opposite-interior angles? Ans. 52° 15′ 10″

(5) If one acute angle of a right triangle is two-fifths of a right angle, what is the value of the other acute angle?

Ans. Three-fifths of a right angle

(6) If two of the adjacent (not opposite) angles of an inscribed quadrilateral are 36° 45′ 36″ and 148° 23′ 50″, respectively, what are the values of the other angles of the quadrilateral?

Ans. $\begin{cases} 143°\ 14'\ 24'' \\ 31°\ 36'\ 10'' \end{cases}$

(7) The sum of two angles of a triangle is 138° 47′ 40″; what is the value of the other angle? Ans. 41° 12′ 20″

(8) What part of a right angle is one of the angles of an equilateral triangle? Ans. Two-thirds of a right angle

(9) If one angle of a triangle is two-thirds of a right angle, and another is one-third of a right angle, what kind of triangle is the triangle? Ans. Right triangle

(10) The exterior angle of a triangle is 120°, and one of the opposite-interior angles is 60°; what kind of triangle is the triangle? Ans. Equilateral triangle

(11) Three points A, B, and C not in a straight line are given; the distance from A to B is 2 inches and the distances from B to C and from C to A are each $1\frac{1}{2}$ inches. Draw a circumference that shall pass through these three points.

(12) (*a*) The angle of intersection of two tangents is 67°; what is the value of each angle formed by the tangents and the chord through the points of contact? (*b*) If a line is drawn from the point of intersection of the tangents to the center of the circle, what is the value of the angle between this line and either of the tangents? Ans. $\begin{cases} (a)\ 33°\ 30' \\ (b)\ 56°\ 30' \end{cases}$

(13) If an inscribed angle contains 48°, how many degrees in the central angle that intercepts the same arc?

Ans. 96°

(14) In a triangle $A\,B\,C$, what is the value of each angle if A is twice, and B three times C? Ans. $\begin{cases} A = 60° \\ B = 90° \\ C = 30° \end{cases}$

(15) In an isosceles triangle $A\,B\,C$, the unequal angle C is equal to twice the sum of the other two angles; what is the value of each angle of the triangle? Ans. $\begin{cases} A = 30° \\ B = 30° \\ C = 120° \end{cases}$

GEOMETRY

(PART 2)

EXAMINATION QUESTIONS

(1) The shortest distance from a point to a line is 9 inches; the distances from this point to the two extremities of the line are 12 inches and 15 inches; what is the length of the line? Ans. 19.94 in.

(2) The chord of an arc of a circle whose radius is 6 inches, is 4 inches long; what is the length of the chord of half the arc? Ans. 2.03 in.

(3) The length of a perpendicular from the center of a circle to a chord is $5\frac{3}{4}$ inches; if the diameter of the circle is 17 inches, what is the length of the chord? Ans. 12.52 in.

(4) If the perimeter of a regular inscribed octagon is 24 inches, and the length of the perpendicular from the center to one of the sides is 3.62 inches, what is the diameter of the circle in which the octagon is inscribed? Ans. 7.84 in.

(5) The area of a circle is 89.42 square inches; what is: (*a*) the diameter of the circle? (*b*) the circumference? (*c*) What is the length of a side of a regular hexagon inscribed in this circle?

Ans. (*a*) 10.67 in.; (*b*) 33.52 in.; (*c*) 5.335 in.

(6) The outside and inside diameters of a cast-iron spherical shell are 16 inches and 12 inches; what is its weight, if a cubic inch of cast iron weighs .261 pound? Ans. 323.61 lb.

(7) The length of an arc of a circle is $5\frac{13}{32}$ inches, by measurement; if the number of degrees in the arc is 27, what is the diameter of the circle? Ans. 22.95 in.

(8) (*a*) What is the area of a circle whose diameter is $17\frac{1}{64}$ inches? (*b*) What is the length of an arc 16° 7′ 21″ in the above circle?

Ans. $\begin{cases} (a)\ 227.4 \text{ sq. in.} \\ (b)\ 2.394 \text{ in.} \end{cases}$

(9) The sides of a triangle ABC are $AB = 13.8$, $BC = 15.6$, and $AC = 19.8$ chains, respectively; the side $A'B'$ of a triangle $A'B'C'$, similar to the triangle ABC, is 17.8 chains; what are the lengths of the sides $B'C'$ and $A'C'$ of the triangle $A'B'C'$?

Ans. $\begin{cases} B'C' = 20.1 \text{ ch.} \\ A'C' = 25.5 \text{ ch.} \end{cases}$

(10) (*a*) What is the convex area of a cone whose base is 7 inches in diameter and whose altitude is 11 inches? (*b*) What is the entire area?

Ans. $\begin{cases} (a)\ 126.92 \text{ sq. in.} \\ (b)\ 165.4 \text{ sq. in.} \end{cases}$

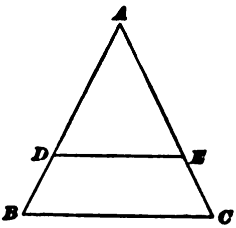


Fig. I

(11) The area of a triangular field ABC, Fig. I, is 20 acres and the length of the side AB is 22 chains; it is desired to divide the field into two parts by a line DE parallel to BC; what should be the length of AD in order that the area of the triangle ADE may be 10 acres? Ans. 15.56 ch.

(12) The altitude of a rectangular solid is 18 inches, its base is a square, one edge of which measures $5\frac{1}{4}$ inches; what is: (*a*) its convex area? (*b*) its entire area? (*c*) its volume?

Ans. $\begin{cases} (a)\ 378 \text{ sq. in.} \\ (b)\ 433.125 \text{ sq. in.} \\ (c)\ 496.125 \text{ cu. in.} \end{cases}$

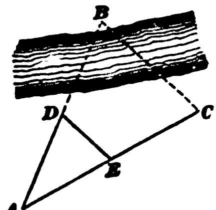


Fig. II

(13) In Fig. II, ED is parallel to CB. If AC measures 10 chains, AE, 4 chains, and AD, 6 chains, what is the distance of the inaccessible point B from the point A?

Ans. 15 ch.

(14) If the horizontal and inclined lines of sight of a transit intercept 3 feet on a rod held at a distance of 100 feet, at what distance will the same lines of sight intercept 6.75 feet on the rod? Ans. 225 ft.

(15) What is the area of a regular hexagon whose sides are 8 inches long? Ans. 166.3 sq. in.

(16) What is the height of a cylinder that has the same volume and diameter as a sphere 12 inches in diameter?
Ans. 8 in.

(17) One diagonal of a trapezium is 11 inches; the lengths of the perpendiculars from the opposite vertexes on this diagonal are $4\frac{1}{4}$ inches and 7 inches; what is the area of the trapezium? Ans. 61.875 sq. in.

(18) How many acres are there in a triangular field whose sides are 15, 17, and 18 chains, respectively?
Ans. 11.83 A.

(19) The chord of an arc of a segment is 48 inches and the height of the segment is 12 inches; what is the area of the segment? Ans. 402 sq. in.

(20) A railway cutting is 600 feet long; the areas, in square yards, of cross-sections taken every 100 feet are: 225, 213, 196, 182, 187, 200, 208. Find the number of cubic yards in the cutting as calculated: (*a*) by the prismoidal formula, considering the alternate sections as end sections; (*b*) by average end areas.
Ans. (*a*) 39,766 cu. yd.; (*b*) 39,817 cu. yd.

(21) Find, by the prismoidal formula, the volume of a frustum of a rectangular pyramid, the dimensions of the lower base being 4.5 feet and 2.5 feet, those of the upper base, 2.25 feet and 1.25 feet, and the altitude, 12 feet.
Ans. 78.75 cu. ft.

(22) What is the length of a railroad circular curve having a radius 2,940 feet and subtending an angle of 52° 30′ at the center? Ans. 2,693.9 ft.

(23) Find the area of a trapezoidal field whose parallel sides are 18.75 and 12.5 chains, the perpendicular distance between them being 15 chains.

Ans. 23.44 A.

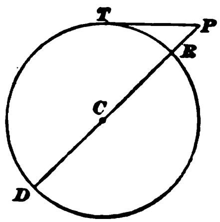


Fig. III

(24) If, in Fig. III, the distance of the point P from the center of the circle is 97 feet, and the radius of the circle is 72 feet, what is the length of the tangent TP? Ans. 65 ft.

(25) If the area of a triangle is 240 square inches, what is the area of a triangle whose dimensions are three times as great? Ans. 15 sq. ft.

PLANE TRIGONOMETRY

(PART 1)

EXAMINATION QUESTIONS

(1) Find the sine: (*a*) of 22° 43′; (*b*) of 44° 56′; (*c*) of 49° 17′; (*d*) of 79° 23′ 30″.

(2) Find the tangent: (*a*) of 11° 37′; (*b*) of 19° 0′ 25″; (*c*) of 64° 6′ 45″; (*d*) of 78° 45′ 50″.

(3) Find the logarithmic cosines of the angles given in example 1.

(4) Find the logarithmic cotangent: (*a*) of 25° 15′ 23″; (*b*) of 5° 41′ 26″; (*c*) of 77° 37′ 27″; (*d*) of 45° 1′ 48″.

Ans. (*a*) .32627; (*b*) 1.00153; (*c*) $\bar{1}.34128$; (*d*) $\bar{1}.99954$

(5) In a right triangle, the two legs are 437 and 792 feet in length; find the hypotenuse and the two acute angles.

Ans. 904.56 ft.; 28° 53′ 19″; 61° 6′ 41″

(6) Find: (*a*) the angle whose tangent is .13476; (*b*) the angle whose cotangent is .32323.

Ans. (*a*) 7° 40′ 30″; (*b*) 72° 5′ 15″

(7) What is the angle: (*a*) whose sine is .92112? (*b*) whose cosine is .55570?

Ans. (*a*) 67° 5′ 25″; (*b*) 56° 14′ 28″

(8) What is the angle: (*a*) whose logarithmic cosine is $\bar{1}.12575$? (*b*) whose logarithmic tangent is .06323?

Ans. $\begin{cases} (a)\ 82^\circ\ 19'\ 24'' \\ (b)\ 49^\circ\ 9'\ 23'' \end{cases}$

(9) Construct an angle, draw a perpendicular from one side of the angle to the other, and express the functions of the angle in terms of the sides of the right triangle thus formed.

(10) The base and the altitude of a triangle are $9\frac{1}{2}$ and 12 inches, respectively. If the angle that one side makes with the base is 79° 22′, find the perimeter of the triangle.

Ans. 35.73 in.

(11) The base of an isosceles triangle is 125 feet, and the opposite angle 130° 51′; find the length of the equal sides.

Ans. 68.73 ft.

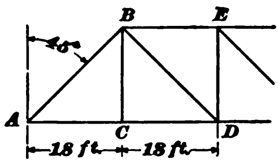


FIG. I

(12) The end post *A B* of a bridge, Fig. I, is inclined at an angle of 45° to the vertical, and the panel length *A C* is 18 feet; the member *A C* being horizontal, determine the length of the end post.

Ans. 25.46 ft.

(13) If the stress in the member *A C*, Fig. I, is known, a triangle *A′ B′ C′*, similar to the triangle *A B C*, can be constructed, in which *A′ C′* will represent the stress in the member *A C*, and *A′ B′* the stress in the member *A B*. The stress in *A C* being 18,200 pounds, determine the stress in *A B* to the nearest 10 pounds.

Ans. 25,740 lb.

(14) A street has a grade of 6 per cent. (that is, the street rises or falls 6 feet in every hundred feet measured horizontally); find the angle of inclination of the street to the horizontal.

Ans. 3° 26′ 2″

(15) A chord of a circle is 100 feet long. If the chord is 3.5 feet distant from the middle point of the subtended arc, find: (*a*) the number of degrees and minutes in the arc; and (*b*) the radius of the circle.

Ans. $\begin{cases} (a)\ 16^\circ\ 0'\ 56'' \\ (b)\ 358.8\text{ ft.} \end{cases}$

(16) The angle of elevation of a railway between two points *A* and *B* is 8° 12′ and between the points *B* and *C* is 7° 26′; if the horizontal distance between *A* and *B* is 1,250 feet and that between *B* and *C* is 375 feet, what is the vertical height of the point *C* above the point *A*?

Ans. 229.0 ft.

(17) In order to prolong a line of survey through an obstaclé, as indicated in Fig. II, an angle *D B C*, equal

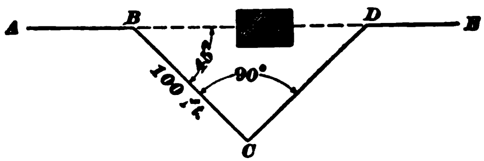


FIG. II

to 45°, is turned and a distance *B C* equal to 100 feet is measured; at *C*, an angle *B C D* equal to 90° is turned: (*a*) What must be the distance measured from *C* to *D* in order that *D* may be in the prolongation of *A B*? (*b*) What is the distance between the points *B* and *D*?

Ans. { (*a*) 100 ft.
(*b*) 141.42 ft.

(18) (*a*) If, in the preceding example, the angle *D B C* had been made equal to 30°, the angle *B C D* and the distance *B C* remaining the same, what would the distance *C D* have been? (*b*) Determine the distance between the points *B* and *D*.

Ans. { (*a*) 57.735 ft.
(*b*) 115.47 ft.

PLANE TRIGONOMETRY

(PART 2)

EXAMINATION QUESTIONS

(1) Find: (*a*) the logarithmic sine of 2° 32′ 54″; (*b*) the logarithmic tangent; (*c*) the logarithmic cotangent.

Ans. (*a*) $\bar{2}.64799$; (*b*) $\bar{2}.64842$; (*c*) 1.35158

(2) Find: (*a*) the logarithmic cosine of 87° 23′ 16″; (*b*) the logarithmic cotangent.

Ans. (*a*) $\bar{2}.65873$; (*b*) $\bar{2}.65919$

(3) What is the angle (*a*) whose logarithmic sine is $\bar{2}.37889$? (*b*) whose logarithmic cotangent is $\bar{2}.38692$?

Ans. (*a*) 1° 22′ 16″; (*b*) 88° 36′ 14″

(4) Find the logarithmic cotangent of 0° 58′ 42″.

Ans. 1.76759

(5) Find the logarithmic tangent of 88° 2′ 45″.

Ans. 1.467

(6) Find the angle whose logarithmic cotangent is 1.92456.

Ans. 0° 40′ 54″

(7) The three sides of a triangle are a = 16, b = 32, and c = 40 feet, respectively; determine the angles.

Ans. A = 22° 19′ 55″; B = 49° 27′ 30″; C = 108° 12′ 36″

(8) The three sides of a triangular field ABC are as follows: $AB = 40.2$ chains, $BC = 38.8$ chains, and $CA = 25.4$ chains; assuming AC as the base of the triangle, determine the altitude. Ans. 37.3 ch.

(9) Two transits at the ends of a base line AB sight on a sounding boat C at the same instant. The angle between the base line and the line of sight AC is 50° 17′, and that between the base line and the line of sight BC is 68° 24′; the length of the base line being 500 feet, determine the distance of the boat from each end of the base line.

Ans. $\begin{cases} AC = 529.9 \text{ ft.} \\ BC = 438.4 \text{ ft.} \end{cases}$

(10) In a triangle ABC, the lengths of the sides AB and AC are 26.75 feet and 39.64 feet, respectively, and the included angle A is 36° 20′ 43″; find the remaining parts.

Ans. $\begin{cases} C = 41° \ 13' \ 27'' \\ B = 102° \ 25' \ 50'' \\ BC = 24.06 \text{ ft.} \end{cases}$

(11) In a triangle ABC, the side $AB = 16$ feet 5 inches, the side $BC = 13$ feet 6½ inches, and the angle $A = 54° \ 54' \ 54''$; find the remaining parts.

Ans. $\begin{cases} B = 42° \ 19' \ 36'', \text{ or } 27° \ 50' \ 36'' \\ C = 82° \ 45' \ 30'', \text{ or } 97° \ 14' \ 30'' \\ AC = 11.14 \text{ ft., or } 7.73 \text{ ft., nearly} \end{cases}$

(12) In a triangle ABC, the angle $A = 29° \ 21'$, the angle $C = 76° \ 44' \ 18''$, and the side $AC = 35.86$ feet; find the other three parts.

Ans. $\begin{cases} BC = 18.29 \text{ ft.} \\ AB = 36.33 \text{ ft.} \\ B = 73° \ 54' \ 42'' \end{cases}$

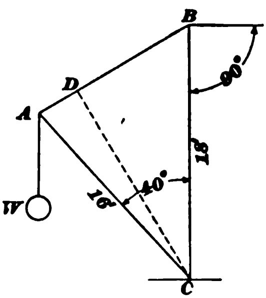


FIG. I

(13) In order to determine, by the method of moments, the stress in the rope AB of the derrick shown in Fig. I, it is necessary to know the distance CD from C to AB; from the data shown in the figure, determine that distance.

Ans. 15.715 ft.

(14) The angles from two stations M and N, Fig. II, to two inaccessible points P and Q, being as shown, and the distance MN being 500 feet, find the distance PQ.

Ans. 215.5 ft.

(15) From the data shown in Fig. III, determine the altitude of the trapezoid $ABCD$.

Ans. 345.5 ft.

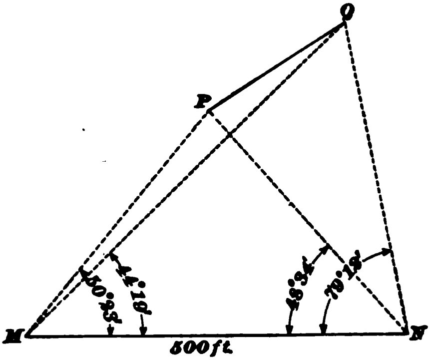


FIG. II

(16) The parallel sides of a trapezoid are 536.17 and 216.18 feet, and the other two sides, 474.3 and 300 feet; find the area S and the altitude h of the trapezoid.

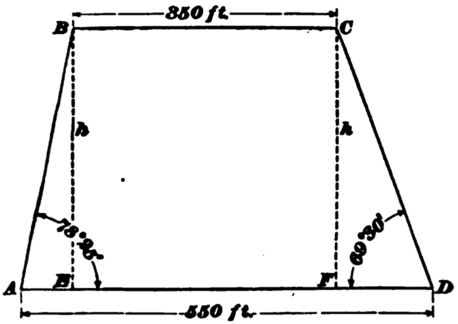


FIG. III

Ans. $\begin{cases} S = 2.55 \text{ A.} \\ h = 295.66 \text{ ft.} \end{cases}$

(17) What must be the radius r and the side l of a regular octagon, that its area may be 35 acres?

Ans. $\begin{cases} r = 11.12 \text{ ch.} \\ l = 8.51 \text{ ch.} \end{cases}$

(18) What is the area of a triangular field in which one side measures 14 chains and the two adjacent angles measure 63° 20′ and 58° 40′, respectively? Ans. 8.82 A.

CHAIN SURVEYING

EXAMINATION QUESTIONS

Note.—In all questions marked with an asterisk (*), the student is required to use logarithms and send his logarithmic work as part of his solutions.

(1) *The horizontal distance between a point at the base and one at the top of a hill was measured by breaking the chain, as explained in Art. **17**, and found to be 364 feet. The distance between the same points, measured along the (uniform) slope, was found to be 403 feet. Find the height of the hill. Ans. 173.0 ft.

(2) The last measurement along a line was 17 links. The rear chainman had then nine pins, there being one in the ground, marking the end of the line. If four tallies were recorded, what was the length of the line? Ans. 48.17 ch.

(3) The recorded length of a line, measured with a 50-foot tape that was .05 foot too short, was 979 feet; what was the correct length of the line? Ans. 978 ft.

(4) Describe how to erect a perpendicular to a line AB, Fig. I, by taking the distance BC equal to 27 links; give the distances BD and CD, and explain how the position of D is determined.

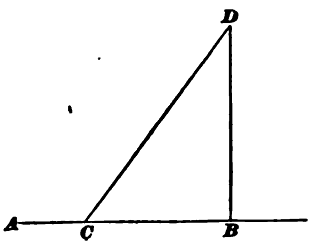


Fig. I

(5) *Two lines, each 60 feet long, were measured from the vertex of an angle along the sides of the angle. The distance between the extremities of these lines was 89.5 feet; determine the angle. Ans. 96° 28′

(6) Give an example showing how to determine the distance from a given point to an inaccessible point; draw a sketch, assume the conditions and the results of the necessary measurements, and compute the required distance.

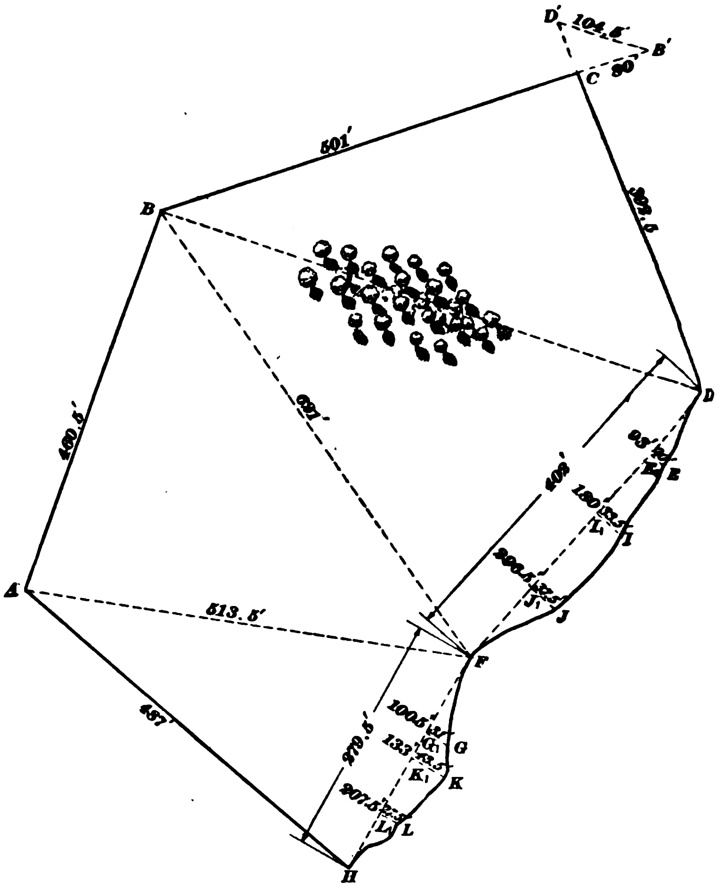


FIG. II

(7) In Fig. II, the full lines represent a closed field, the dimensions being as shown. The length of the diagonal BD was determined by a tie-line $B'D'$. The side BC was produced 80 feet to B'; find: (a) the distance CD' that must

have been measured along the prolongation of DC; (*b*) the length of the diagonal BD.

Note.—Distances are given to the nearest half foot.

$$\text{Ans.}\begin{cases} CD' = 62.7 \text{ ft.} \\ BD = 654.4 \text{ ft.} \end{cases}$$

(8) *Compute the area, in acres, of the field referred to in the preceding question. [There are 43,560 square feet in an acre.] Ans. 9.60 A.

COMPASS SURVEYING

(PART 1)

EXAMINATION QUESTIONS

NOTE.—The student is required to send both the answers and the methods by which he has obtained them. He is also required to work out all trigonometric problems by logarithms, and send in the details of his calculations.

(1) The bearing of a line *A B*, as taken from *A*, was found to be N 89° 00′ W. A foresight from *A* on an external point *O* (make a sketch placing *O* wherever you choose) gave S 53° 45′ E as the bearing of *A O*, while a backsight from *O* on *A* gave N 57° 30′ W as the bearing of *O A*; there was no local attraction at *O*. What was the correct bearing of *A B*? Ans. S 87° 15′ W

(2) The bearings of two lines are, respectively, S 75° 15′ E and N 13° 30′ W; what is the angle between the lines?

Ans. 118° 15′

(3) The bearing of a line *A B* is S 76° 15′ W; what must be the bearing of another line that shall run from *A*, on the right of *A B*, and make with the latter line an angle of 106° 45′?

Ans. N 3° 00′ E

(4) A line *A B* makes with another line *A D* an angle of 65° 30′, *A B* lying on the left of *A D*; if the bearing of *A D* is S 89° 45′ E, what is the bearing of *A B*?

Ans. N 24° 45′ E

(5) Find the length of *A B*, Fig. I, from the following data: *A P* = 275 feet; *P Q* = 150 feet; bearing of *A P*, S 52° 15′ E; bearing of *P Q*, S 17° 30′ W; bearing of *Q B*, S 79° 45′ E. Ans. *A B* = 597.2 ft.

(6) The bearings of the lines *A B* and *A C*, Fig. II, are, respectively, N 86° 30′ E and S 12° 00′ W, and their lengths,

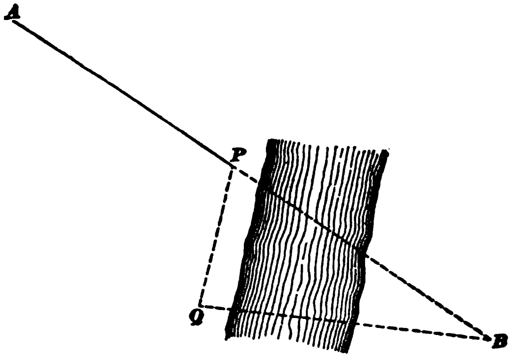


Fig. I

586.5 feet and 728.5 feet. Find: (*a*) the length of the line *B C* joining their extremities; (*b*) its bearing.

Ans. { (*a*) 1,050.3 ft.
(*b*) S 44° 33′ W

(7) In running a railroad line with regular stakes at intervals of 100 feet, how would a point 569 feet from the starting point be designated or referred to?

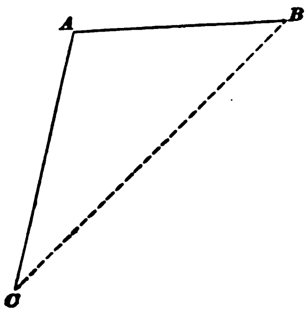


Fig. II

(8) With main stations at intervals of 100 feet, what is the distance from Substation 9 + 89 to Substation 14 + 04? Ans. 415 ft.

(9) How would you determine the bearing of the center line of a railway, the rails being laid? Assume any figures for an example, and explain the operations through which you would go.

(10) The true bearing of a line is N 30° 30′ E, and its magnetic bearing N 32° 15′ E; what is the angle between the true and the magnetic meridian? Ans. 1° 45′

COMPASS SURVEYING

(PART 2)

EXAMINATION QUESTIONS

NOTE.—Angles are given to the nearest quarter of a degree. The student should use logarithms where they can be used to advantage.

(1) The length and bearing of a course are, respectively, 19.43 chains and S 87° 45′ W; find the ranges of the course.

Ans. $\begin{cases} t = -.76 \text{ ch.} \\ g = -19.42 \text{ ch.} \end{cases}$

(2) The ranges of a course are $t = -23.16$ chains and $g = 11.97$ chains; find the length l and the bearing G of the course.

Ans. $\begin{cases} l = 26.07 \text{ ch.} \\ G = \text{S } 27° \ 15' \text{ E} \end{cases}$

(3) The algebraic sum of the latitude ranges of a survey is −.07 chain, that of the longitude ranges is .13 chain, and the sum of the lengths of all the courses is 41.73 chains; find: (*a*) the total error of closure; (*b*) the relative error of closure.

Ans. $\begin{cases} (a) \ .15 \text{ ch.} \\ (b) \ .004 \end{cases}$

(4) In the following table are given the bearings and lengths of the sides of a field; also, the calculated and corrected values of the ranges and the corrected lengths. Verify these values and give the different steps taken for the construction of such a table.

Courses	Bearings	Lengths Feet	Calculated Latitude Ranges	Calculated Longitude Ranges	Corrected Latitude Ranges	Corrected Longitude Ranges	Corrected Lengths
A B	S 67° 15′ W	740.5	− 286.4	− 682.9	−284.9	− 683.6	740.6
B C	S 13° 45′ W	369.5	− 358.9	− 87.8	−358.2	− 88.2	368.9
C D	N 72° 30′ E	1,064.0	+ 320.0	+1,014.8	+322.1	+1,013.5	1,063.6
D A	N 37° 00′ W	400.9	+ 320.2	− 241.3	+321.0	− 241.7	401.8

(5) What is the relative error of closure in question 4?

Ans. .002

(6) Determine the most westerly corner of the field to which the following notes refer:

Courses	Longitude Ranges Chains
AB	9.76
BC	−13.41
CD	−16.83
DE	21.78
EA	− 1.30

Ans. The corner *D*

(7) Compute, by latitude ranges and double longitudes, the area of the field to which the following notes refer. Construct a table similar to that given in Art. **21**, and explain how the different quantities are obtained.

Courses	Latitude Ranges Feet	Longitude Ranges Feet
AB	+570.5	+255.0
BC	+ 99.0	−558.5
CD	−352.5	−449.0
DE	−449.0	+217.5
EA	+132.0	+535.0

Ans. 11.71 A.

(8) As observed at 11 A. M. on a day in September, the bearing of a line in latitude 42° north was found to be S 76° 25′ E; reduce the bearing to its daily mean value.

Ans. S 76° 23′ E

(9) Compute the latitudes and longitudes of the corners of the field referred to in question 7, the reference meridian and parallel passing through the corner *E*.

Ans.

Corner	Latitude Feet	Longitude Feet
E	000.0	000.0
A	+132.0	+535.0
B	+702.5	+790.0
C	+801.5	+231.5
D	+449.0	−217.5

(10) The declination of the needle is 3° 15′ east; explain how the true bearings of the following courses are obtained from their magnetic bearings:

Courses	Magnetic Bearings	True Bearings
AB	N 15° 45′ E	N 19° 00′ E
BC	N 88° 30′ E	S 88° 15′ E
CD	S 20° 30′ W	S 23° 45′ W
DE	N 50° 15′ W	N 47° 00′ W

(11) The declination of the needle being 5° 10′ west, explain how the following true bearings are obtained from the magnetic bearings:

Courses	Magnetic Bearings	True Bearings
AB	N 7° 30′ W	N 12° 40′ W
BC	N 45° 15′ E	N 40° 5′ E
CD	S 15° 45′ E	S 20° 55′ E
DE	S 2° 30′ W	S 2° 40′ E

(12) (*a*) If the magnetic bearing of a line at the time of a survey made 20 years ago was N 40° 00′ W and its present magnetic bearing is N 38° 45′ W, what is the yearly variation of the needle? (*b*) What is the present bearing of a line in

the same survey whose original bearing was N 65° 00′ E? (*c*) Of a line whose bearing was S 20° 30′ E? (*d*) If the declination of the needle at the time of the original survey was 5° 25′ east, what is the present declination?

Ans. { (*a*) 0° 4′ W
(*b*) N 66° 15′ E
(*c*) S 19° 15′ E
(*d*) 4° 10′ E

(13) Make a plat of the field to which the following notes refer; use a scale of 2 chains to the inch, and explain all the operations in detail.

Courses	Latitude Ranges Chains	Longitude Ranges Chains
AB	+3.50	−2.82
BC	− .34	−4.06
CD	+3.49	+3.68
DE	+ .95	+6.21
EA	−7.60	−3.01

TRANSIT SURVEYING

(PART 1)

EXAMINATION QUESTIONS

NOTE.—The student is required to send in not only his answers, but also his methods of obtaining them.

(1) A scale is divided into eighths of an inch. (*a*) Into how many equal parts must the vernier be divided that the least reading may be $\frac{1}{64}$ inch? (*b*) What is the length of the vernier?

Ans. $\begin{cases} (a)\ 8 \\ (b)\ \frac{7}{8}\text{ in.} \end{cases}$

(2) A scale is so graduated that its smallest division is .05 foot. (*a*) What is the least reading of the vernier, if the vernier is divided into ten equal parts that cover nine of the divisions on the scale? (*b*) What is the reading of the vernier when its eighth mark coincides with a division on the scale?

Ans. $\begin{cases} (a)\ .005\text{ ft.} \\ (b)\ .04\text{ ft.} \end{cases}$

(3) The horizontal circle of a transit is divided into degrees, and the vernier is divided into ten equal spaces that cover nine spaces on the circle; what is the least reading of the vernier? Ans. 6′

(4) If in example 3, the ninth mark of the vernier coincides with a division mark on the circle, what is the reading of the vernier? Ans. 54′

(5) Make a sketch of part of a scale graduated to quarter-inches with vernier reading to $\frac{1}{32}$ inch, and explain how the vernier is read.

(6) If the forward north azimuth of a line is 276° 57′, find: (*a*) the true bearing of the line; (*b*) the back azimuth.

Ans. $\begin{cases} (a) \text{ N } 83° \ 03' \text{ W} \\ (b) \ 96° \ 57' \end{cases}$

(7) Explain, by an example not given in the text, what is meant by calculated or deduced bearings.

(8) Take a deflection traverse of three lines and explain how to run it with a transit. Assume whatever values are necessary.

(9) Find the latitude and longitude ranges: (*a*) of a line whose azimuth is 343° 41′ and whose length is 569 feet; (*b*) of a line whose azimuth is 233° 04′ and whose length is 1,026 feet.

Ans. $\begin{cases} (a) \ t = 546.1; \ g = -159.9 \\ (b) \ t = -616.5; \ g = -820.1 \end{cases}$

(10) The azimuth of a line is 354° 29′, and the true bearing of another, intersecting the first, is S 83° 18′ W; what is the angle between the two lines? Ans. 91° 11′

(11) The magnetic bearing of a line is N 55° 15′ E, and an angle of 15° 17′ is turned to the right; what is the calculated bearing of the second line? Ans. N 70° 32′ E

(12) The magnetic bearing of a line is N 13° 15′ W and an angle of 40° 20′ is turned to the left; what is the calculated bearing of the second line? Ans. N 53° 35′ W

(13) Take an azimuth traverse consisting of five lines; explain how it is run, assuming the necessary values; write down the notes, and make a plat.

(14) The longitude range of a course is −1,215.8 feet, and the latitude range is +1,427.8 feet; what is the azimuth of the line? Ans. 319° 35′

TRANSIT SURVEYING

(PART 2)

EXAMINATION QUESTIONS

(1) An observer is located in longitude 95° 15′, his watch keeping standard time; find the time, by his watch, at which Polaris will be in lower culmination on February 12, 1905.

Ans. 4^h 18.5^m A. M.

(2) The longitude of Albany, N. Y., is 73° 44′ 49″, and that of Los Angeles, Cal., is 118° 10′ 44″; what is the time at Albany when it is 9^h 41^m A. M. at Los Angeles?

Ans. 12^h 38^m 44^s P. M.

(3) Find the times of elongation of Polaris on December 31, 1908.

Ans. { Eastern elongation, 0^h 52.2^m
Western elongation, 12^h 42.2^m

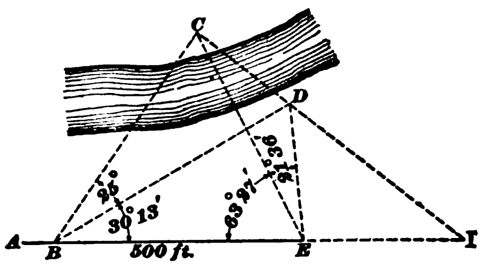


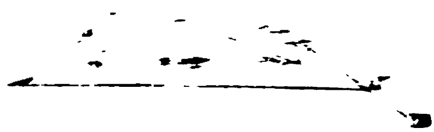
FIG. I

(4) A line is to pass through two inaccessible points *C* and *D*, Fig. I, and meet another line *A B* produced. From two points *B* and *E*, on *A B*, angles to the points *C* and *D*

whose

respectively

intersection and



Ans. $BE = 437.1$ ft
$EC = 509.7$ ft

(7) Supply the missing parts from the following notes:

Side	Length Feet	Azimuth
A B		33° 45′
B C	300	86° 23′
C D		169° 36′
D E	450	243° 54′
E A	268	317° 30′

Ans. { *A B* = 398.8 ft.
C D = 355.9 ft.

(8) Supply the missing parts from the following notes:

Side	Length Feet	Azimuth
A B	298.0	
B C	458.7	29° 48′
C D		48° 32′
D E	632.4	150° 46′
E A	729.4	249° 12′

Ans. { Azimuth of *A B* = 274° 13′
C D = 590.2 ft.

(9) A traverse $A_0 A_1 A_2 A_3 A_4$, Fig. III, was run between two points A_0, A_4 of a line $A B$, the part $A_0 A_4$ of which

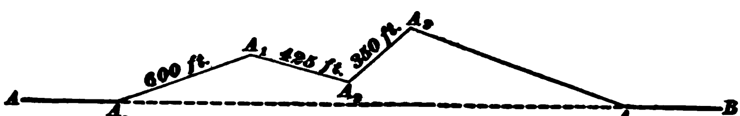


Fig. III

was inaccessible; the distances of the traverse were as shown. The azimuth of the lines $A_0 A_1$, $A_1 A_2$, $A_2 A_3$, $A_3 A_4$, referred to $A B$ as a meridian, B being the north end, were, respectively, 340°, 15°, 318°, and 20°. Compute the distance $A_0 A_4$, and explain how to prolong the line beyond A_4.

Ans. $A_0 A_4$ = 2,140.1 ft.

(10) In order to determine the inaccessible point of intersection of two lines AB and CD, Fig. IV, the traverse $B A_1 A_2 D$ was run. The angles and distances being as

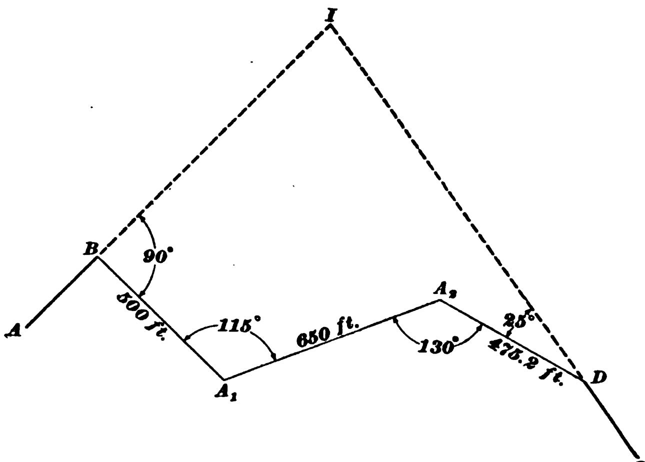


FIG. IV

shown, find: (*a*) the angle I between the two lines; (*b*) the distances DI and IB.

HINT.—Take AI as a meridian, and find the azimuths of the other lines from the measured angles.

Ans. $\begin{cases} (a)\ I = 80° \\ (b)\ DI = 1252.7 \text{ ft.}, IB = 929.8 \text{ ft.} \end{cases}$

(11) In the preceding example, what are the length and azimuth of the line DB (not drawn in the figure), the line AB being taken as a meridian? (Do not forget that the azimuth of DB is not the same as that of BD.)

Ans. $\begin{cases} DB = 1424.6 \text{ ft.} \\ \text{Azimuth} = 240° \end{cases}$

(12) In Fig. V, AB = 58 chains; DC = 32 chains, and AD = 24 chains. It is required to divide the trapezoid $ABCD$ into two parts by a line EF parallel to the

bases and having such position that the area adjacent to the longer base shall be to the area adjacent to the shorter base as 7 is to 4; what will be the length: (*a*) of *EF*? (*b*) of *DE*?

Ans. { (*a*) 43.3 ch.
(*b*) 10.43 ch.

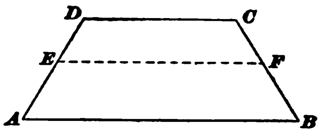


Fig. V

(13) The area of the triangle *ABC*, Fig. VI, is 76.31 acres. The lengths of *AC* and *AB* being as shown, it is required to cut off an area S_1 = 30 acres by a line *EF* parallel to *AC*; find the lengths of *AE* and *EF*.

Ans. { *AE* = 8.95 ch.
EF = 38.52 ch.

(14) Assume that, in Fig. VII, *FC* measures 9.32 chains, the angle *FCG* measures 73°, and the area of *FABC* is 4.56 acres; what must be the distance of the point *G* from the point *C* in order that the line *FG* will cut off 6 acres, that is, in order that the area of the figure *FABCG* will be 6 acres? Ans. 3.23 ch.

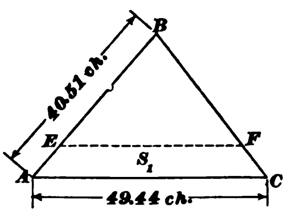


Fig. VI

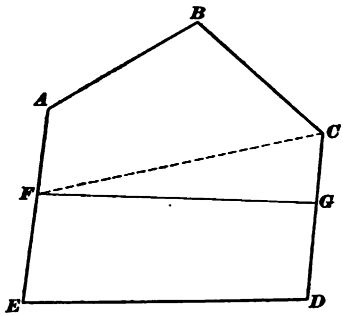


Fig. VII

A KEY

TO ALL THE QUESTIONS AND EXAMPLES
INCLUDED IN THE
EXAMINATION QUESTIONS IN THIS VOLUME.

It will be noticed that the Keys have been given the same section numbers as the Examination Questions to which they refer. All article references refer to the Instruction Paper bearing the same section number as the Key in which they occur, unless the title of some other Instruction Paper is given in connection with the references.

GEOMETRY

(PART 1)

(1) (*a*) The number of sides of an octagon is eight; hence, applying the formula in Art. **115**, the sum of the interior angles of the octagon is $S = 2 \times (8 - 2) = 12$ right angles. Since the octagon is equiangular, one of the interior angles is equal to $\frac{1}{8} \times 12 = 1\frac{1}{2}$ right angles, or $90° \times 1\frac{1}{2} = 135°$. Ans.

(*b*) If one of the angles of an equiangular polygon is 108°, or $\frac{108}{90} = \frac{6}{5}$ right angles, their sum S is equal to $\frac{6}{5} \times n = \frac{6n}{5}$. But from the formula in Art. **115**, we have $S = 2n - 4$. Therefore, $\frac{6n}{5} = 2n - 4$; whence, solving for n, $n = 5$. A polygon of five sides is a pentagon. Ans.

(2) Since every circumference contains 360° (Art. **155**) or four right angles, if one of the equal sectors measures two-sevenths of a right angle, there are $4 \div \frac{2}{7} = 14$ equal sectors in the circle. Ans.

(3) Since one-third of an angle is equal to 14° 47′ 10″, the angle is 14° 47′ 10″ × 3 = 44° 21′ 30″. Ans. One of the other angles is two and one-half times this angle, or 44° 21′ 30″ × 2½ = 110° 53′ 45″. Ans. The sum of the angles of a triangle is equal to 180° (Art. **69**); therefore, the third angle is equal to 180° − (44° 21′ 30″ + 110° 53′ 45″) = 24° 44′ 45″. Ans.

(4) An exterior angle of a triangle is equal to the sum of the opposite-interior angles (Art. **68**). The sum of the opposite-interior angles is, therefore, 104° 30′ 20″, and as the opposite-interior angles are, in this case, equal, the value of each angle is one-half of 104° 30′ 20″, or 52° 15′ 10″.

(5) The sum of the acute angles of a right triangle is equal to one right angle (Art. **70**). Therefore, if one of the acute angles is equal to two-fifths of a right angle, the other acute angle is equal to $\frac{5}{5} - \frac{2}{5} = \frac{3}{5}$ of a right angle. Ans.

(6) The opposite angles of an inscribed quadrilateral are supplementary (Art. **164**). Therefore, one of the required angles is equal to $180° - 36° 45' 36'' = 143° 14' 24''$, and the other angle is equal to $180° - 148° 23' 50'' = 31° 36' 10''$. Ans.

(7) The sum of the angles of a triangle is 180° (Art. **69**); therefore, the required angle is equal to $180° - 138° 47' 40'' =$ **41° 12' 20''**. Ans.

(8) An equilateral triangle is also equiangular; therefore, one of the angles of an equilateral triangle is equal to one-third of two right angles, or two-thirds of one right angle (Art. **70**).

(9) The sum of the given angles of the triangle is $\frac{2}{3} + \frac{1}{3} = 1$ right angle; therefore, the triangle is a right triangle (Art. **70**).

(10) If the exterior angle of a triangle is 120° and one of the opposite-interior angles is 60°, the other opposite-interior angle is equal to $120° - 60° = 60°$, and the third angle $= 180° - (60° + 60°) = 60°$. The triangle is therefore equilateral.

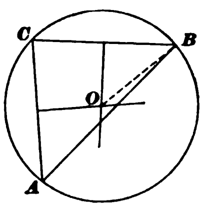


FIG. 1

(11) In Fig. 1, the distance from A to B is 2 inches, from B to C is $1\frac{1}{4}$ inches, and from C to A $1\frac{1}{4}$ inches. At the middle points of AC and BC, erect perpendiculars and prolong them until they intersect at O. With O as a center and OB as a radius, draw a circle, which will pass through the three points A, B, C (Art. **145**).

(12) (*a*) Since the tangents from a point to a circle are equal (Art. **150**), the triangle ABC, Fig. 2, formed by the tangents and a chord through the points of contact is isosceles, and the angle A is equal to the angle B (Art. **76**). The angle BCD, being an exterior angle to the triangle ABC, is equal to the sum of A and B. Therefore, the angles A and B are each equal to one-half of $67° = 33° 30'$.

(*b*) A line joining the point of intersection of the tangents to the center of the circle bisects the angle between the tangents (Art. **151**). The angle ACB is equal to $(180° - 67°) = 113°$. Therefore, the angle $OCB = \frac{113°}{2} = 56° 30'$. Ans.

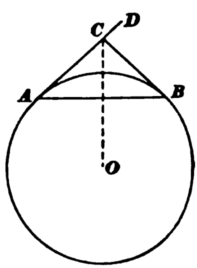


FIG. 2

(13) A central angle is measured by the intercepted arc (Art. **154**), and an inscribed angle is measured by one-half the intercepted arc (Art. **156**). Therefore, the central angle contains $48 \times 2 = 96°$.

(14) Let x = value of angle C;
$2x$ = value of angle A;
and $3x$ = value of angle B.

The sum of the angles of a triangle = 180°; therefore. $6x = 180°$, whence $x = 30°$, $2x = 60°$, and $3x = 90°$. Therefore, $A = 60°$, $B = 90°$; and $C = 30°$. Ans.

(15) Let x = value of angle A or B;
Then $2x$ = their sum;
and $4x$ = unequal angle C.
Their sum $6x = 180°$
whence, $x = 30°$ and $4x = 120°$. Therefore, $A = 30°$, $B = 30°$, and $C = 120°$. Ans.

GEOMETRY

(PART 2)

(1) The shortest distance from a point to a line is a perpendicular to the line. In Fig. 1, $BD = 9$, $AB = 12$, and $BC = 15$ in. $AD = \sqrt{\overline{AB}^2 - \overline{BD}^2} = \sqrt{12^2 - 9^2} = 7.94$ in., $DC = \sqrt{\overline{BC}^2 - \overline{BD}^2} = \sqrt{15^2 - 9^2} = 12$ in. Therefore, $AC = AD + DC = 7.94 + 12 = 19.94$ in. Ans.

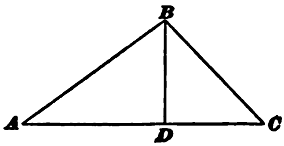


FIG. 1

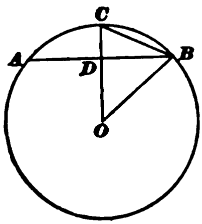


FIG. 2

(2) The radius perpendicular to a chord bisects it. (See *Geometry*, Part 1.) Therefore, in the right triangle DOB, Fig. 2, $DB = \frac{1}{2}$ of $4 = 2$ in., and $OB = 6$ in. $DO = \sqrt{6^2 - 2^2} = 5.66$ in. In the right triangle CDB, $CD = 6 - 5.66 = .34$ in., and $DB = 2$ in. BC, which is the chord of half the arc, is equal to $\sqrt{.34^2 + 2^2} = 2.03$ in. Ans.

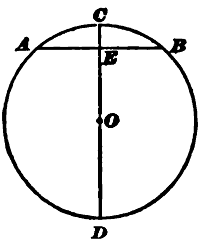


FIG. 3

(3) In Fig. 3, the radius $CO = \frac{17}{2} = 8.5$ in., and $EO = 5.75$, whence $CE = 8.5 - 5.75 = 2.75$ in., and $ED = 17 - 2.75 = 14.25$ in. Then, $CE:AE = AE:ED$ (Art. 28); $AE = \sqrt{CE \times ED} = \sqrt{2.75 \times 14.25} = 6.26$ in., and $AB = 6.26 \times 2 = 12.52$ in. Ans.

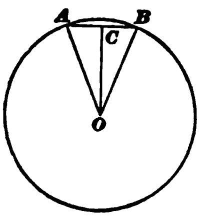


FIG. 4

(4) One side of the octagon is equal to one-eighth of 24, or 3 in. In Fig. 4, AB represents one side of the octagon. Then, $CB = \frac{3}{2} = 1.5$ in., and in the right triangle COB, the radius OB

$= \sqrt{3.62^2 + 1.5^2} = 3.92$ in., and the diameter is equal to $3.92 \times 2 = 7.84$ in. Ans.

(5) (*a*) From formula **2**, Art. **77**,

$$d = \sqrt{\frac{A}{.7854}} = \sqrt{\frac{89.42}{.7854}} = 10.67 \text{ in. Ans.}$$

(*b*) Applying formula of Art. **71**,

$$c = 3.1416 \times 10.67 = 33.52 \text{ in. Ans.}$$

(*c*) The length of the side of a regular hexagon inscribed in a circle is equal to the radius, $\frac{10.67}{2} = 5.335$ in. Ans.

(6) The volume of the shell is equal to the difference in the volume of a sphere 16 in. in diameter and one 12 in. in diameter (see Art. **114**). Volume of shell is $.5236\ (16^3 - 12^3) = 1{,}239.9$ cu. in. Weight of shell is $1{,}239.9 \times .261 = 323.61$ lb. Ans.

(7) From formula of Art. **73**,

$$r = \frac{180\, l}{\pi\, n}, \text{ and } d = \frac{360}{\pi\, n}$$

Substituting known values,

$$d = \frac{360 \times 5\frac{13}{32}}{3.1416 \times 27} = 22.95 \text{ in. Ans.}$$

(8) (*a*) $17\frac{1}{64} = 17.016$ in. Applying formula **2**, Art. **77**,

$$A = .7854 \times (17.016)^2 = 227.4 \text{ sq. in. Ans.}$$

(*b*) $16° 7' 21'' = 16.1225°$. Applying formula of Art. **73**,

$$l = \frac{3.1416 \times 8.508 \times 16.1225}{180} = 2.394 \text{ in. Ans.}$$

(9) Similar triangles have their sides proportional; therefore,

$$A B : A' B' = B C : B' C'$$

$$13.8 : 17.8 = 15.6 : B' C'$$

whence, $$B' C' = \frac{17.8 \times 15.6}{13.8} = 20.1 \text{ ch. Ans.}$$

Also, $$A B : A' B' = A C : A' C'$$

$$13.8 : 17.8 = 19.8 : A' C'$$

whence, $$A' C' = \frac{17.8 \times 19.8}{13.8} = 25.5 \text{ ch. Ans.}$$

(10) (*a*) The slant height of the cone is the hypotenuse of a right triangle whose legs are the radius of the base and the altitude of the cone.

$$s = \sqrt{3.5^2 + 11^2} = 11.543 \text{ in.}$$

The perimeter of the base is equal to $3.1416 \times 7 = 21.9912$ in. Applying formula of Art. **101**,

$$c = \frac{21.9912 \times 11.543}{2} = 126.92 \text{ sq. in. Ans.}$$

(*b*) The area of the base is equal to $7^2 \times .7854 = 38.48$ sq. in. The entire area is equal to $126.92 + 38.48 = 165.4$ sq. in. Ans.

(11) To divide the triangle ABC into two equal parts, we have, $AD = .70711\ AB$ (Art. **55**). Then,

$$AD = .70711 \times 22 = 15.56 \text{ ch.} \quad \text{Ans.}$$

(12) The perimeter of the base is equal to $5\frac{1}{4} \times 4 = 21$ in.

(*a*) Applying formula of Art. **95**,

$$c = 21 \times 18 = 378 \text{ sq. in.} \quad \text{Ans.}$$

(*b*) The areas of the two bases are $5\frac{1}{4} \times 5\frac{1}{4} \times 2 = 55.125$ sq. in. The entire area is equal to $378 + 55.125 = 433.125$ sq. in. Ans.

(*c*) Applying formula of Art. **96**,

$$V = 5\frac{1}{4} \times 5\frac{1}{4} \times 18 = 496.125 \text{ cu. in.} \quad \text{Ans.}$$

(13) The sides of the triangle are proportional to the segments into which they are divided by the line ED (Art. **10**); therefore,

$$AB : AC = AD : AE$$

$$AB : \ 10 = 6 : 4$$

whence,

$$AB = \frac{10 \times 6}{4} = 15 \text{ ch.} \quad \text{Ans.}$$

(14) In Fig. 5,

$$AC : AD = BC : DE$$

$$100 : AD = \ 3 : 6.75$$

whence, $AD = \dfrac{100 \times 6.75}{3} = 225$ ft. Ans.

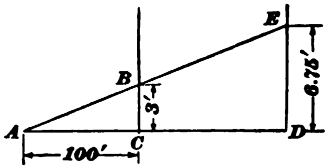


FIG. 5

(15) The area of a regular hexagon whose side is 1 in. is 2.5981 sq. in. (Art. **68**). Applying formula of Art. **68**,

$$A = 2.5981 \times 8^2 = 166.3 \text{ sq. in.} \quad \text{Ans.}$$

(16) The volume of a sphere is, by formula of Art. **114**, $\frac{1}{6}\pi d^3$, and the volume of a cylinder is, by formula of Art. **96**, writing $\frac{1}{4}\pi d^2$ for A $\frac{\pi}{4} d^2 h$. Since the volume and diameter of the sphere are equal to the volume and diameter of the cylinder, we have

$$\frac{1}{6}\pi d^3 = \frac{1}{4}\pi d^2 h$$

whence,

$$h = \frac{2}{3} d = \frac{2}{3} \times 12 = 8 \text{ in.} \quad \text{Ans.}$$

(17) The area of the trapezium is equal to the sum of the areas of two triangles, whose common base is 11 in., and whose altitudes are $4\frac{1}{4}$ in. and 7 in., respectively.

$$\frac{11 \times 4\frac{1}{4}}{2} + \frac{11 \times 7}{2} = 61.875 \text{ sq. in.} \quad \text{Ans.}$$

(18) $s = \dfrac{15 + 17 + 18}{2} = 25$. Applying formula of Art. **47**,

$$A = \sqrt{25(25-15)(25-17)(25-18)} = 118.3 \text{ sq. ch.} = 11.83 \text{ A.}$$

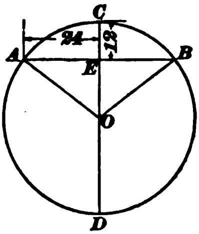


Fig. 6

(19) In Fig. 6, $ED : AE = AE : EC$ (Art. **28**), or $ED : 24 = 24 : 12$; whence, $ED = \frac{24 \times 24}{12} = 48$ in. The diameter $CD = 48 + 12 = 60$ in. and the radius is equal to $\frac{60}{2} = 30$ in. Also, $EO = 30 - 12 = 18$ in. The length of the arc ACB is, according to formula of Art. **74**,

$$l = \frac{4\sqrt{48^2 + 4 \times 12^2} - 48}{3} = 55.6 \text{ in.}$$

According to formula of Art. **79**, the area of the sector is equal to

$$A' = \frac{1}{2} \times 30 \times 55.6 = 834 \text{ sq. in.}$$

The area of the triangle $AOB = \frac{48 \times 18}{2} = 432$ sq. in. The area of the segment is, then (Art. **80**),

$$834 - 432 = 402 \text{ sq. in. Ans.}$$

(20) (*a*) The height of each prismoid, considering the alternate sections as end sections, is $\frac{200}{3}$ yd. The volume of the first prismoid is, by formula of Art. **126**, $(225 + 4 \times 213 + 196)\frac{200}{3 \times 6} = 14{,}144$ cu. yd.; of the second, $(196 + 4 \times 182 + 187)\frac{200}{3 \times 6} = 12{,}344$ cu. yd.; and of the third, $(187 + 4 \times 200 + 208)\frac{200}{3 \times 6} = 13{,}278$ cu. yd. The total volume of the cutting is $14{,}144 + 12{,}344 + 13{,}278 = 39{,}766$ cu. yd. Ans.

(*b*) By average end areas, the height of each section is $\frac{100}{3}$ yd. By formula of Art. **128**:

Volume of first section is $(225 + 213)\frac{100}{2 \times 3} = 7300$ cu. yd.

Volume of second section is $(213 + 196)\frac{100}{2 \times 3} = 6817$ cu. yd.

Volume of third section is $(196 + 182)\frac{100}{2 \times 3} = 6300$ cu. yd.

Volume of fourth section is $(182 + 187)\frac{100}{2 \times 3} = 6150$ cu. yd.

Volume of fifth section is $(187 + 200)\frac{100}{2 \times 3} = 6450$ cu. yd.

Volume of sixth section is $(200 + 208)\frac{100}{2 \times 3} = 6800$ cu. yd.

The total volume is 39817 cu. yd.

Ans.

(21) The dimensions of the middle section are $\frac{4.5 + 2.25}{2} = 3.375$ ft. and $\frac{2.5 + 1.25}{2} = 1.875$ ft. To apply formula of Art. **126**, we have $A = 4.5 \times 2.5 = 11.25$ sq. ft., $A' = 2.25 \times 1.25 = 2.8125$ sq. ft., $4M = 4(3.375 \times 1.875) = 25.3125$ sq. ft. Applying formula of Art. **126**,

$$V = (11.25 + 2.8125 + 25.3125)\frac{12}{6} = 78.75 \text{ cu. ft. Ans.}$$

(22) Applying formula of Art. **73**,

$$l = \frac{3.1416 \times 2{,}940 \times 52.5}{180} = 2{,}693.9 \text{ ft. Ans.}$$

(23) Applying formula of Art. **50**,

$$A = (18.75 + 12.5)\frac{15}{2} = 234.4 \text{ sq. ch.} = 23.44 \text{ A. Ans.}$$

(24) The total length of the secant, Fig. 7, is $97 + 72 = 169$ in. The external segment of the secant is $97 - 72 = 25$ in. From Art. **29**, we have

$$DP : PT = PT : RP$$
$$169 : PT = PT : 25$$
$$PT = \sqrt{169 \times 25} = 65 \text{ ft. Ans.}$$

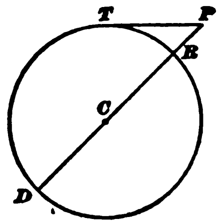


Fig. 7

(25) The ratio between the corresponding sides of the triangles is 1 : 3; therefore (Art. **58**),

$$240 : x = 1^2 : 3^2$$
$$x = 240 \times 9 = 2{,}160 \text{ sq. in., or 15 sq. ft. Ans.}$$

PLANE TRIGONOMETRY

(PART 1)

(1) (*a*) Since the given angle is less than 45°, the sine is taken from the table as explained in Art. **18.** Therefore, sin 22° 43′ = .38617. Ans.

(*b*) Sin 44° 56′ = .70628. Ans.

(*c*) Since the given angle is greater than 45°, the sine is taken from the table as explained in Art. **19**; therefore, sin 49° 17′ = .75794. Ans.

(*d*) See rule, Art. **20.** The difference between sin 79° 23′ and sin 79° 24′ is .00006. $\frac{30}{60} \times .00006 = .00003$. Adding this correction to sin 79° 23′, sin 79° 23′ 30″ = .98291. Ans.

(2) (*a*) According to Art. **18,** tan 11° 37′ = .20557. Ans.

(*b*) See rule, Art. **20.** The difference between tan 19° 0′ and tan 19° 1′ is .00032. $\frac{25}{60} \times .00032 = .00013$. Adding this correction to tan 19° 0′, tan 19° 0′ 25″ = .34446. Ans.

(*c*)
$$\begin{aligned} \tan 64°\ 7' &= 2.06094 \\ \tan 64°\ 6' &= \underline{2.05942} \\ \text{Correction} &= .00152 \times \frac{45}{60} = .00114 \end{aligned}$$

Adding this correction to tan 64° 06′,

tan 64° 6′ 45″ = 2.06056. Ans.

(*d*)
$$\begin{aligned} \tan 78°\ 46' &= 5.03499 \\ \tan 78°\ 45' &= \underline{5.02734} \\ \text{Correction} &= .00765 \times \frac{50}{60} = .00638 \end{aligned}$$

Adding this correction to tan 78° 45′,

tan 78° 45′ 50″ = 5.03372. Ans.

(3) According to the rule given in Art. **26,** the logarithmic cosines of the angles given in example 1, are as follows:

(*a*) log cos 22° 43′ = $\bar{1}$.96493. Ans.

(*b*) log cos 44° 56′ = $\bar{1}$.84999. Ans.

(*c*) log cos 49° 17′ = $\bar{1}$.81446. Ans.

(*d*) Since this angle contains odd seconds, the correction is obtained by applying the rule given in Art. **28**,

$$\log \cos 79° \, 23' = \bar{1}.26538; \; D = 68$$

In the column of proportional parts, the correction found opposite 30 (number of seconds) and under 68 is $34 = \frac{s}{60} \times D$. Since it is the logarithmic cosine that is required, this correction must be subtracted. See rule, Art. **27**.

$$\log \cos 79° \, 23' = \bar{1}.26538$$
$$\frac{s}{60} \times D = \phantom{\bar{1}.265}34$$
$$\log \cos 79° \, 23' \, 30'' = \bar{1}.26504. \quad \text{Ans.}$$

(4) (*a*) See Arts. **27** and **28**.

log cot 25° 15′ = .32640; D = 33
p. p. for 20 = 11
p. p. for 3 = 2
p. p. for 23 = 13 13
log cot 25° 15′ 23″ = .32627. Ans.

(*b*) log cot 5° 41′ = 1.00209; D = 128.
p. p. for 26 = 56
log cot 5° 41′ 26″ = 1.00153. Ans.

(*c*) log cot 77° 37′ = $\bar{1}$.34155; D = 60
p. p. for 27 = 27
log cot 77° 37′ 27″ = $\bar{1}$.34128. Ans.

(*d*) log cot 45° 1′ = $\bar{1}$.99975; D = 26
p. p. for 48 = 21
log cot 45° 1′ 48″ = $\bar{1}$.99954. Ans.

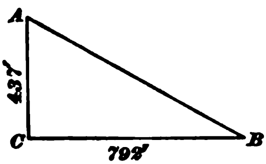


FIG. 1

(5) In Fig. 1, $\tan B = \frac{437}{792} = .55177$. See Art. **35**.

$B = 28° \, 53' \, 19''$. Ans.
$A = 90° - 28° \, 53' \, 19'' = 61° \, 06' \, 41''$. Ans.

By formula **5**, Art. **35**, $AB = \frac{437}{\sin B} = \frac{437}{.48311} = 904.56$ ft. Ans.

(6) (*a*) See rule, Art. **22**. Looking in the table it is found that the given tangent lies between .13461 = tan 7° 40′ and .13491 = tan 7° 41′. The difference is .00030. The difference between the given tangent and the tangent of the smaller angle is .00015. The number of seconds in the angle is therefore

$$\frac{15}{30} \times 60 = 30''. \quad \text{Therefore, } .13476 = \tan 7° \, 40' \, 30''. \quad \text{Ans.}$$

(*b*)

cot 72° 5′ =	.32331	.32331
cot 72° 6′ =	.32299	.32323
	32	8

The number of seconds in the angle is $\frac{8}{32} \times 60 = 15''$. Therefore, .32323 = cot 72° 5′ 15″. Ans.

(7) (*a*) See Art. **22.**

sin 67° 6′ =	.92119	.92112
sin 67° 5′ =	.92107	.92107
	12	5

$\frac{5}{12} \times 60 = 25''$. Therefore, .32323 = sin 67° 5′ 25″. Ans.

(*b*)

cos 56° 14′ =	.55581	.55581
cos 56° 15′ =	.55557	.55570
	24	11

$\frac{11}{24} \times 60 = 28''$. Therefore, .55570 = cos 56° 14′ 28″. Ans.

(8) (*a*) See rule, Art. **30.**

log cos 82° 19′ =	$\bar{1}.12612$	$\bar{1}.12612$
log cos 82° 20′ =	$\bar{1}.12519$	$\bar{1}.12575$
	93	37

$\frac{37}{93} \times 60 = 24''$. Therefore, $\bar{1}.12575$ = log cos 82° 19′ 24″. Ans.

(*b*) See Art. **31.** The given logarithm lies between .06313 = log tan 49° 09′ and .06339 = log tan 49° 10′. D = 26. .06323 − .06313 = 10. The proportional part under 26 and next lower than 10 is 8.7, horizontally opposite 20. 10 − 8.7 = 1.3. The number 1.3 is found opposite 3. Therefore, the number of seconds is 23 and .06323 = log tan 49° 9′ 23″. Ans.

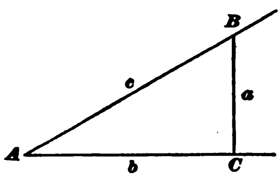


Fig. 2

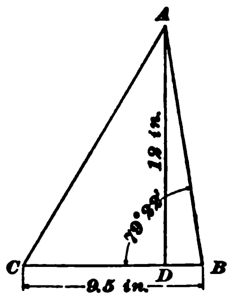


Fig. 3

(9) See Art. **4.**

$$\sin A = \frac{a}{c} \qquad \cot A = \frac{b}{a}$$

$$\tan A = \frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\cos A = \frac{b}{c} \qquad \csc A = \frac{c}{a}$$

(10) In the right triangle $A\,D\,B$, Fig. 3, $A\,B = \frac{12}{\sin 79° \; 22'} = 12.21$ in. $D\,B = 12 \cot 79° \; 22' = 2.25$; $C\,D = 9.5 - 2.25 = 7.25$ in. In the right triangle $A\,D\,C$, $A\,C = \sqrt{12^2 + 7.25^2} = 14.02$ in. The perimeter of the triangle is therefore

$$12.21 + 9.5 + 14.02 = 35.73 \text{ in. Ans.}$$

(11) The perpendicular $A\,D$ bisects the angle $B\,A\,C$ and the base $B\,C$, Fig. 4. In the right triangle $A\,D\,B$, the angle $B\,A\,D = \frac{130° \; 51'}{2} = 65° \; 25' \; 30''$ and $D\,B = \frac{125}{2} = 62.5$ ft. Then $A\,B = \frac{62.5}{\sin 65° \; 25' \; 30''} = 68.73$ ft. Ans.

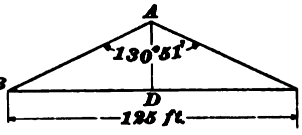


FIG. 4

(12) In the right triangle $A\,B\,C$, Fig. 5 (*a*), the angle $B\,A\,C$ is equal to $90° - 45° = 45°$, and $A\,B = \frac{18}{\cos 45°} = 25.46$ ft. Ans.

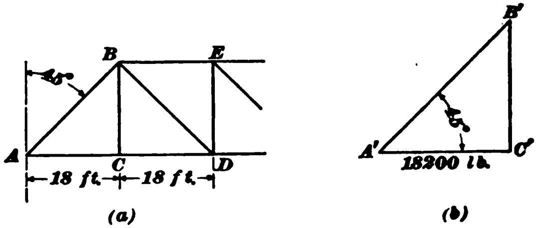


FIG. 5

(13) The triangle $A'\,B'\,C'$, Fig. 5 (*b*), is similar to the triangle $A\,B\,C$. The corresponding angles are therefore equal, and $B'\,A'\,C' = 45°$. If $A'\,C'$ represents the stress of 18,200 lb. in the member $A\,C$, $A'\,B'$ will represent the stress in the member $A\,B$.

$$A'\,B' = \frac{A'\,C'}{\cos 45°} = \frac{18{,}200}{.70711} = 25{,}740 \text{ lb. Ans.}$$

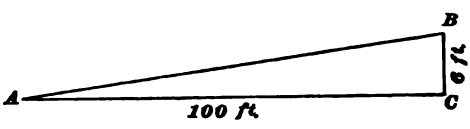


FIG. 6

(14) $\tan A = \frac{6}{100} = .06000$; whence, $A = 3° \; 26' \; 2''$. Ans.

(15) (*a*) In Fig. 7, $AD = \frac{100}{2} = 50$ ft., and $\tan CAD = \frac{3.5}{50} = .07000$; angle $CAD = 4° 0' 14''$. The angle CAB, being an inscribed angle, is measured by one-half the arc BC; therefore, the arc $BC = 4° 0' 14'' \times 2 = 8° 0' 28''$. But the arc BC is one-half the arc ACB; therefore the arc ACB is equal to $8° 0' 28'' \times 2 = 16° 0' 56''$. Ans.

(*b*) In the right triangle AOD, the radius $AO = \frac{AD}{\sin AOD} = \frac{50}{\sin 8° 0' 28''} = 358.8$ ft. Ans.

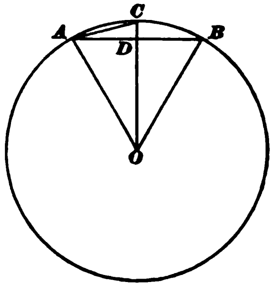


FIG. 7

(16) The vertical height of the point B above the point A, Fig. 8, is equal to $BB' = 1{,}250 \tan 8° 12' = 48.9$ ft. The vertical height of the point C above the point B is equal to $CC' = 375 \tan 7° 26' = 180.1$ ft.

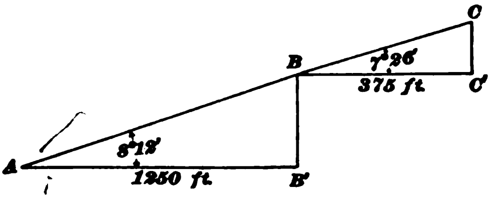


FIG. 8

Therefore, the vertical height of the point C above the point A is $48.9 + 180.1 = 229.0$ ft. Ans.

(17) (*a*) In the right triangle BCD, Fig. II, of the Examination Questions, $CD = 100 \tan 45° = 100$ ft. Ans.

(*b*) $BD = \frac{CD}{\sin 45°} = \frac{100}{.70711} = 141.42$ ft. Ans.

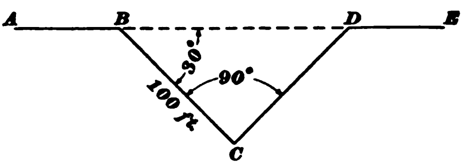


FIG. 9

(18) (*a*) In the right triangle BCD, Fig. 9,

$CD = BC \tan 30° = 100 \times .57735 = 57.735$ ft. Ans.

(*b*) $BD = \frac{CD}{\sin 30°} = \frac{CD}{.5} = 115.47$ ft. Ans.

PLANE TRIGONOMETRY

(PART 2)

(1) (*a*) See rule, Art. **2**.

$$\begin{aligned} \log A'' = \log 9{,}174 &= 3.96256 \\ S &= \underline{\bar{6}.68543} \\ \log \sin A &= \bar{2}.64799 \end{aligned}$$

Therefore, log sin 2° 32′ 54″ = $\bar{2}$.64799. Ans.

(*b*)

$$\begin{aligned} \log A'' = \log 9{,}174 &= 3.96256 \\ T &= \underline{\bar{6}.68586} \\ \log \tan A &= \bar{2}.64842 \end{aligned}$$

Therefore, log tan 2° 32′ 54″ = $\bar{2}$.64842. Ans.

(*c*) See rule, Art. **3**.

$$\begin{aligned} C &= 5.31414 \\ \log A'' = \log 9{,}174 &= \underline{3.96256} \\ \log \cot A &= 1.35158 \end{aligned}$$

Therefore, log cot 2° 32′ 54″ = 1.35158. Ans.

(2) Since the given angle lies between 87° and 90°, the simplest way to determine the cosine and cotangent is to subtract the angle from 90° and find the corresponding cofunction. See Art. **4**.

$$90° - 87° 23' 16'' = 2° 36' 44'' = A_1$$

(*a*) log cos 87° 23′ 16″ = log sin 2° 36′ 44″.

$$\begin{aligned} \log A_1'' = \log 9{,}404 &= 3.97331 \\ S &= \underline{\bar{6}.68542} \\ \log \sin A_1 &= \bar{2}.65873 \end{aligned}$$

Therefore, log cos 87° 23′ 16″ = $\bar{2}$.65873. Ans.

(*b*) log cot 87° 23′ 16″ = log tan 2° 36′ 44″.

$$\begin{aligned} \log A_1'' = \log 9{,}404 &= 3.97331 \\ T &= \underline{\bar{6}.68588} \\ \log \tan 2° 36' 44'' &= \bar{2}.65919 \end{aligned}$$

Therefore, log cot 87° 23′ 16″ = $\bar{2}$.65919. Ans.

(3) (*a*) The angle whose logarithmic sine is $\bar{2}$.37889 is found by means of the rule stated in Art. **5**.

The nearest logarithmic sine is $\bar{2}.37750$, and the correction opposite this in the S column is $\bar{6}.68553$.

$$\begin{aligned} \log \sin A &= \bar{2}.37889 \\ S &= \underline{\bar{6}.68553} \\ \log A'' &= 3.69336 \\ A &= 4{,}936'' = 1^\circ\ 22'\ 16''. \quad \text{Ans.} \end{aligned}$$

(*b*) The nearest logarithmic cotangent found in the table is $\bar{2}.38809$, which lies in the column marked log cot at the bottom; therefore, the given logarithmic cotangent is to be treated as though it were a tangent, and the angle corresponding to this tangent is to be subtracted from 90°. See Art. **5**. The correction in the T column opposite $\bar{2}.38809$ is $\bar{6}.68566$.

$$\begin{aligned} \log \sin A_1 &= \bar{2}.38692 \\ T &= \underline{\bar{6}.68566} \\ \log A_1'' &= 3.70126 \\ A_1 &= 5{,}026'' = 1^\circ\ 23'\ 46'' \end{aligned}$$

Therefore, $A = 90^\circ - 1^\circ\ 23'\ 46'' = 88^\circ\ 36'\ 14''$. Ans.

(4) See Art. **5**. The correction under C and horizontally opposite 59′ is 5.31438.

$$\begin{aligned} & 5.31438 \\ \log A'' = \log 3{,}522 = & \underline{3.54679} \\ \log \cot A = & 1.76759 \end{aligned}$$

Therefore, log cot 0° 58′ 42″ = 1.76759. Ans.

(5) See Art. **4**. $90^\circ - 88^\circ\ 2'\ 45'' = 1^\circ\ 57'\ 15'' = A_1$. The correction under C and horizontally opposite 57 is 5.31426; $A_1'' = 7{,}020 + 15 = 7{,}035$.

$$\begin{aligned} C &= 5.31426 \\ \log A_1'' = \log 7{,}035 &= \underline{3.84726} \\ \log \cot A_1 &= 1.46700 \end{aligned}$$

Since log cot A_1 = log tan A, log tan A = 1.46700. Ans.

(6) The logarithmic cotangent nearest 1.92456 is 1.92347; the C correction horizontally opposite 1.92347 is 5.31440.

$$\begin{aligned} C &= 5.31440 \\ \log \cot A &= \underline{1.92456} \\ \log A'' &= 3.38984 \\ A'' &= 2{,}454'' = 0^\circ\ 40'\ 54'' \end{aligned}$$

Therefore, 1.92456 = log cot 0° 40′ 54″. Ans.

(7) Applying formula 1, Art. **23**,

$$\cos A = \frac{32^2 + 40^2 - 16^2}{2 \times 32 \times 40} = .92500$$

$$A = 22^\circ\ 19'\ 55''. \quad \text{Ans.}$$

$$\cos B = \frac{16^2 + 40^2 - 32^2}{2 \times 16 \times 40} = .65000$$

$$B = 49^\circ\ 27'\ 30''. \quad \text{Ans.}$$

$$\cos C = \frac{16^2 + 32^2 - 40^2}{2 \times 16 \times 32} = -\ .31250$$

The angle whose cosine is .3125 is 71° 47′ 24″. The cosine of C being negative indicates that the angle C is obtuse. Therefore,

$$C = 180° - 71° 47' 24'' = 108° 12' 36''. \quad \text{Ans.}$$

(8) The angle A, Fig. 1, is determined by applying formula **2**, **Art. 23**,

$$s = \tfrac{1}{2}(40.2 + 38.8 + 25.4) = 52.2$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(52.2 - 25.4)(52.2 - 40.2)}{52.2\,(52.2 - 38.8)}}$$

$$\begin{aligned} \log (52.2 - 25.4) &= \log 26.8 = 1.42813 \\ \log (52.2 - 40.2) &= \log 12 \quad = 1.07918 \\ &\qquad\qquad\quad 2.50731 \\ \log (52.2 - 38.8) &= \log 13.4 = 1.12710 \\ \log 52.2 &= 1.71767 \\ &\qquad\qquad\quad 2.84477 \\ \log \tan \tfrac{1}{2} A &= \bar{1}.66254 \div 2 = \bar{1}.83127 \end{aligned}$$

whence $\tfrac{1}{2} A = 34° 8' 22''$, and $A = 34° 8' 22'' \times 2 = 68° 16' 44''$.
The altitude $BD = AB \sin A = 40.2 \sin 68° 16' 44''$.

$$\begin{aligned} \log 40.2 &= 1.60423 \\ \log \sin A &= \bar{1}.96802 \\ \log BD &= 1.57225 \\ BD &= 37.3 \text{ ch.} \quad \text{Ans.} \end{aligned}$$

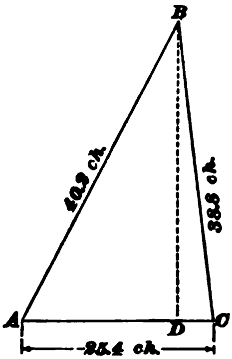


FIG. 1

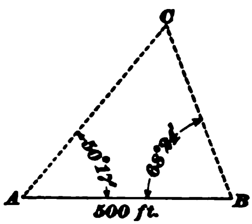


FIG. 2

(9) The angle C, Fig. 2, is equal to $180° - (50° 17' + 68° 24') = 61° 19'$.

$$M = \frac{500}{\sin 61° 19'}$$

$$b = \frac{500}{\sin 61° 19'} \sin 68° 24'; \text{ and } c = \frac{500}{\sin 61° 19'} \sin 50° 17'$$

$$\begin{aligned} \log 500 &= 2.69897 \\ \log \sin 61° 19' &= \bar{1}.94314 \\ \log M &= 2.75583 \end{aligned}$$

$$\log b = 2.72421; \quad b = 529.9 \text{ ft. Ans.}$$
$$\log \sin 68° 24' = \bar{1}.96838$$
$$\log M = 2.75583$$
$$\log \sin 50° 17' = \bar{1}.88605$$
$$\log a = 2.64188; \quad a = 438.4 \text{ ft. Ans.}$$

(10) Applying formula **1**, Art. **21**,

$$\tan \tfrac{1}{2}(A - B) = \frac{39.64 - 26.75}{39.64 + 26.75} \cot \frac{36° 20' 43''}{2}$$
$$\log (39.64 - 26.75) = \log 12.89 = 1.11025$$
$$\log 39.64 + 26.75 = \log 66.39 = 1.82210$$
$$\bar{1}.28815$$
$$\log \cot 18° 10' 22'' = .48379$$
$$\log \tan \tfrac{1}{2}(A - B) = \bar{1}.77194$$
$$\tfrac{1}{2}(A - B) = 30° 36' 12''$$
$$A = 90° - 18° 10' 22'' + 30° 36' 12'' = 102° 25' 50''$$
$$B = 180° - (36° 20' 43'' + 102° 25' 50'') = 41° 13' 27''$$

Applying formula **2**, Art. **21**,

$$B\,C = \frac{(39.64 - 26.75) \cos 18° 10' 22''}{\sin 30° 36' 12''}$$
$$\log (39.64 - 26.75) = \log 12.89 = 1.11025$$
$$\log \cos 18° 10' 22'' = \bar{1}.97778$$
$$1.08803$$
$$\log \sin 30° 36' 12'' = \bar{1}.70679$$
$$\log B\,C = 1.38124$$
$$B\,C = 24.06 \text{ ft. Ans.}$$

(11) 16 ft. 5 in. = 16.417 ft.; 13 ft. $6\tfrac{1}{2}$ in. = 13.542 ft. The modulus of the triangle is $M = \dfrac{13.542}{\sin 54° 54' 54''}$; $A\,B = M \sin C$; whence $\sin C = \dfrac{A\,B}{M} = \dfrac{16.417}{13.542} \sin 54° 54' 54''$.

$$\log 16.417 = 1.21529$$
$$\log 13.542 = 1.13168$$
$$0.08361$$
$$\log \sin 54° 54' 54'' = \bar{1}.91291$$
$$\log \sin C = \bar{1}.99652$$

$\bar{1}.99652$ is the logarithmic sine of 82° 45′ 30″, or of 180° − 82° 45′ 30″ = 97° 14′ 30″. Therefore,

$$C = 82° 45' 30'', \text{ or } 97° 14' 30''$$
$$\text{and } B = 180° - (82° 45' 30'' + 54° 54' 54'') = 42° 19' 36''$$
$$\text{or } B = 180° - (97° 14' 30'' + 54° 54' 54'') = 27° 50' 36''$$

Taking $B = 42° 19' 36''$, $AC = M \sin B = \frac{13.542}{\sin 54° 54' 54''} \sin 42° 19' 36''$ $= 11.14$ ft. Ans.

$$\begin{aligned} \log 13.542 &= 1.13168 \\ \log \sin 54° 54' 54'' &= \bar{1}.91291 \\ \log M &= 1.21878 \\ \log \sin 42° 19' 36'' &= \bar{1}.82824 \\ & \quad 1.04702 = \log 11.143 = 11.14 \text{ ft.} \end{aligned}$$

Taking $B = 27° 50' 36''$, $AC = \frac{13.542}{\sin 54° 54' 54''} \sin 27° 50' 36'' = 7.73$ ft., nearly. Ans.

$$\begin{aligned} \log M &= 1.21878 \\ \log \sin 27° 50' 36'' &= \bar{1}.66936 \\ & \quad .88814 = \log 7.7293 = 7.73 \text{ ft., nearly.} \end{aligned}$$

(12) The angle $B = 180° - (76° 44' 18'' + 29° 21') = 73° 54' 42''$.

$$M = \frac{35.86}{\sin 73° 54' 42''}; \quad AB = \frac{35.86}{\sin 73° 54' 42''} \sin 76° 44' 18''$$

$$BC = \frac{35.86}{\sin 73° 54' 42''} \sin 29° 21'$$

$$\begin{aligned} \log 35.86 &= 1.55461 \\ \log \sin 73° 54' 42'' &= \bar{1}.98265 \\ \log M &= 1.57196 \end{aligned}$$

$\log AB = 1.56022$; $AB = 36.33$ ft. Ans.

$$\begin{aligned} \log \sin 76° 44' 18'' &= \bar{1}.98826 \\ \log M &= 1.57196 \\ \log \sin 29° 21' &= \bar{1}.69032 \end{aligned}$$

$\log BC = 1.26228$; $BC = 18.29$ ft. Ans.

(13) To find the angle ABC, apply formula **1**, Art. **21**:

$$\tan \tfrac{1}{2}(A - B) = \frac{18 - 16}{18 + 16} \cot \frac{40°}{2} = \frac{\cot 20°}{17}$$

Whence $\frac{1}{2}(A - B) = 9° 10' 49''$, and $B = 90° - 20° - 9° 10' 49''$ $= 60° 49' 11''$.

In the right triangle DBC, $CD = BC \sin ABC = 18 \sin 60° 49' 11'' = 15.715$ ft. Ans.

(14) In the triangle PMN, the angle $MPN = 180° - (50° 23' + 48° 34') = 81° 3'$.

$$PM = \frac{500}{\sin 81° 3'} \sin 48° 34' = 379.48$$

$$\begin{aligned} \log 500 &= 2.69897 \\ \log \sin 48° 34' &= \bar{1}.87490 \\ & \quad 2.57387 \\ \log \sin 81° 3' &= \bar{1}.99468 \\ & \quad 2.57919 = \log 379.48. \end{aligned}$$

In the triangle MQN the angle $MQN = 180° - (44° 19' + 79° 12') = 56° 29'$.

$$MQ = \frac{500}{\sin 56° 29'} \sin 79° 12' = 575.69$$

$$\begin{aligned} \log 500 &= 2.69897 \\ \log \sin 79° 12' &= \bar{1}.99224 \\ &\overline{2.69121} \\ \log \sin 56° 29' &= \bar{1}.92102 \\ \log MQ &= \overline{2.77019}; \; MQ = 589.1 \end{aligned}$$

In the triangle PMQ, the angle $PMQ = 50° 23' - 44° 19' = 6° 4'$. Let the angles MPQ and MQP be represented by A and B, respectively. By formula **1**, Art. **21**,

$$\tan \tfrac{1}{2}(A - B) = \frac{(589.10 - 379.48)}{589.10 + 379.48} \cot \frac{6° 4'}{2}$$

$$\begin{aligned} \log (589.10 - 379.48) = \log 209.62 &= 2.32143 \\ \log (589.10 + 379.48) = \log 968.58 &= 2.98613 \\ &\overline{\bar{1}.33530} \\ \log \cot 3° 2' &= 1.27580 \\ \log \tan \tfrac{1}{2}(A - B) &= \overline{0.61110} \\ \tfrac{1}{2}(A - B) &= 76° 14' 30'' \end{aligned}$$

By formula **2**, Art. **21**,

$$PQ = \frac{(589.10 - 379.48) \cos 3° 2'}{\sin 76° 14' 30''}$$

$$\begin{aligned} \log (589.10 - 379.48) = \log 209.62 &= 2.32143 \\ \log \cos 3° 2' &= \bar{1}.99939 \\ &\overline{2.32082} \\ \log \sin 76° 14' 30'' &= \bar{1}.98736 \\ \log PQ &= \overline{2.33346} \end{aligned}$$

$PQ = 215.5$ ft. Ans.

(15) In Fig. III, $AE = h \cot 78° 25'$; $FD = h \cot 69° 30'$. But $AE + FD = 550 - 350 = 200$ ft. Therefore, $h \cot 78° 25' + h \cot 69° 30' = 200$ ft., whence $h = \dfrac{200}{\cot 78° 25' + \cot 69° 30'} = \dfrac{200}{.20497 + .37388}$ $= 345.5$ ft. Ans.

(16) The area of the trapezoid is found by applying the formula of Art. **34**.

$$d = 536.17 - 216.18 = 319.99; \; b_1 + b_2 = 752.35$$

$$s = \frac{1}{2}(474.30 + 300.00 + 319.99) = 547.15$$

$$s - a = 72.85, \; s - c = 247.15, \; s - d = 227.16$$

Substituting in formula of Art. **34**,

$$S = \frac{752.35}{319.99}\sqrt{547.15 \times 72.85 \times 247.15 \times 227.16} = 111{,}220 \text{ sq. ft.}$$

$$\frac{111{,}220}{43{,}560} = 2.55 \text{ A. Ans.}$$

The value of h is found from formula of Art. **32** to be

$$h = \frac{2S}{b_1 + b_2}$$

Substituting in this equation,

$$h = \frac{2 \times 111{,}220}{752.35} = 295.66 \text{ ft. Ans.}$$

(17) From formula **1**, Art. **35**, we have,

$$r = \sqrt{\frac{2S}{n \sin \frac{360^\circ}{n}}} \quad (1)$$

An octagon has eight sides; $\sin \frac{360^\circ}{8} = .70711$.

$$S = 35 \text{ A.} = 350 \text{ sq. ch.}$$

Substituting in equation (1),

$$r = \sqrt{\frac{2 \times 350}{8 \times .70711}} = 11.12 \text{ ch. Ans.}$$

Substituting in formula **2**, Art. **35**,

$$l = 2 \times 11.12 \sin \frac{180^\circ}{8} = 8.51 \text{ ch. Ans.}$$

(18) To find the area of the triangle, apply formula **2**, Art **29**:

$$S = \frac{14^2}{2(\cot 63^\circ 20' + \cot 58^\circ 40')} = 88.2 \text{ sq. ch.} = 8.82 \text{ A. Ans.}$$

CHAIN SURVEYING

(1) In this case, we have given the hypotenuse and one side of a right triangle to find the other side. The height of the hill is equal to $\sqrt{403^2 - 364^2} = 173.0$ ft. Ans. See Art. **13**.

(2) See Art. **16**. The four tallies represent $4 \times 10 = 40$ ch.; eight of the pins held by the rear chainman represent one chain each, and the other one, 17 li., or .17 ch. Therefore,

$$\text{length of line} = 40 + 8 + .17 = 48.17 \text{ ch. Ans.}$$

(3) The correct length of the line is found by formula of Art. **21**. Here, $L = 979$ ft., and $e = \frac{.05}{50} = .001$ ft. per ft. The tape was too short, therefore the length was recorded too long, and the error is to be subtracted.

$$L_o = 979 - .001 \times 979 = 978 \text{ ft. Ans.}$$

(4) See Art. **26**. If BC, Fig. I, is taken equal to $27 = (3 \times 9)$ li., BD must be equal to $4 \times 9 = 36$, and CD, to $5 \times 9 = 45$ li. Since $BD + CD = 36 + 45 = 81$ li., the position of D may be determined as follows: Fix one end of the chain at B and the eighty-first link at C; then, taking hold of the thirty-sixth link, pull the chain until both parts of it are taut. The point D is at the end of the thirty-sixth link. Ans.

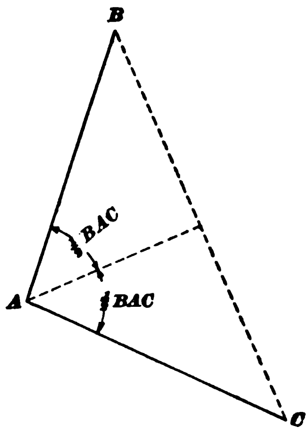


Fig. 1

(5) See Art. **29**. In Fig. 1, AB and AC are each equal to 60 ft., and BC is 89.5 ft. The angle BAC can be determined from the relation

$$\sin \frac{1}{2} BAC = \frac{\frac{1}{2} BC}{AC} = \frac{\frac{1}{2} \times 89.5}{60}$$

Hence, using logarithms,

$$\frac{1}{2} BAC = 48° \ 14', \text{ and } BAC = 96° \ 28'. \text{ Ans.}$$

(6) See Art. **32**.

(7) See Art. **38**. (*a*) The distance CD' can be found from the relation $CD' : CD = CB' : BC$, which gives

$$CD' = \frac{CD \times CB'}{BC}$$

Therefore, $CD' = \frac{392.5 \times 80}{501} = 62.7$ ft. Ans.

(*b*) The length of the diagonal BD can be found from the relation $BD : B'D' = BC : CB'$, which gives

$$BD = \frac{B'D' \times BC}{CB'} = \frac{104.5 \times 501}{80} = 654.4 \text{ ft. Ans.}$$

(8) The area of the field shown in Fig. II is determined by adding the areas of the parts into which the figure is divided. The areas of the triangles AHF, ABF, BFD, and BCD are found by applying the formula of Art. **46**. In the triangle AHF, $s = \frac{487 + 513.5 + 279.5}{2} = 640$; $640 - 487 = 153$; $640 - 513.5 = 126.5$; $640 - 279.5 = 360.5$.

Area of triangle $AHF = \sqrt{640 \times 153 \times 126.5 \times 360.5} = 66{,}825$ sq. ft.

In the triangle ABF, $s = \frac{513.5 + 621 + 460.5}{2} = 797.5$; $797.5 - 513.5 = 284.0$; $797.5 - 621 = 176.5$; $797.5 - 460.5 = 337.0$.

Area of triangle ABF

$$= \sqrt{797.5 \times 284 \times 176.5 \times 337} = 116{,}070 \text{ sq. ft.}$$

In the triangle BDF, $s = \frac{621 + 654.4 + 402}{2} = 838.7$; $838.7 - 621 = 217.7$; $838.7 - 654.4 = 184.3$; $838.7 - 402 = 436.7$.

Area of triangle BDF

$$= \sqrt{838.7 \times 217.7 \times 184.3 \times 436.7} = 121{,}230 \text{ sq. ft.}$$

In the triangle BCD, $s = \frac{501 + 392.5 + 654.4}{2} = 773.95$; $773.95 - 501 = 272.95$; $773.95 - 392.5 = 381.45$; $773.95 - 654.4 = 119.55$.

Area of triangle BCD

$$= \sqrt{773.95 \times 272.95 \times 381.45 \times 119.55} = 98{,}150 \text{ sq. ft.}$$

The area of the part FKH is equal to the sum of the areas of the triangles FG_1G and HL_1L and those of the trapezoids G_1GKK_1 and K_1KLL_1.

From the measurements given on the figure, we have $FG_1 = 100.5$; $G_1K_1 = 133 - 100.5 = 32.5$; $K_1L_1 = 207.5 - 133 = 74.5$; and $L_1H = 279.5 - 207.5 = 72.0$.

Area of triangle $FG_1G = \frac{100.5 \times 31}{2} = 1{,}557.8$ sq. ft.

Area of trapezoid $G_1GKK_1 = \frac{(43.5 + 31)\ 32.5}{2} = 1{,}210.6$ sq. ft.

Area of trapezoid K_1KLL_1

$$= \frac{(43.5 + 27.5) \times 74.5}{2} = 2{,}644.8 \text{ sq. ft.}$$

Area of triangle $H L_1 L = \frac{72 \times 27.5}{2} = 990$ sq. ft.

The area of the part DIF is equal to the sum of the areas of the triangles $D E_1 E$ and $F J_1 J$ and those of the trapezoids $E_1 E I I_1$ and $I_1 I J J_1$. From the measurements given we have, $D E_1 = 93$; $E_1 I_1 = 180 - 93 = 87$; $I_1 J_1 = 296.5 - 180 = 116.5$; and $J_1 F = 402 - 296.5 = 105.5$.

Area of triangle $D E_1 E = \frac{93 \times 20}{2} = 930$ sq. ft.

Area of trapezoid $E_1 E I I_1 = \frac{(20 + 33.5)\,87}{2} = 2{,}327.3$ sq. ft.

Area of trapezoid $I_1 I J J_1 = \frac{(33.5 + 37.5)\,116.5}{2} = 4{,}135.8$ sq. ft.

Area of triangle $J_1 J F = \frac{105.5 \times 37.5}{2} = 1{,}978.1$ sq. ft.

The areas just found and their sum are given below:

$$
\begin{array}{r}
66825.0 \\
116070.0 \\
121230.0 \\
98150.0 \\
1557.8 \\
1210.6 \\
2644.8 \\
990.0 \\
930.0 \\
2327.3 \\
4135.8 \\
1978.1 \\
\hline
43560\,)\,418049.4 \text{ sq. ft.} \\
\hline
9.60 \text{ A. Ans.}
\end{array}
$$

COMPASS SURVEYING

(PART 1)

(1) The conditions are represented in Fig. 1. $N' S'$ represents the position of the needle when the compass is at A, and $N S$ represents the correct position of the needle. It is evident, from the figure, that the effect of the local attraction at A is to deflect the needle to the west of north and decrease the northwest bearings by the amount of the

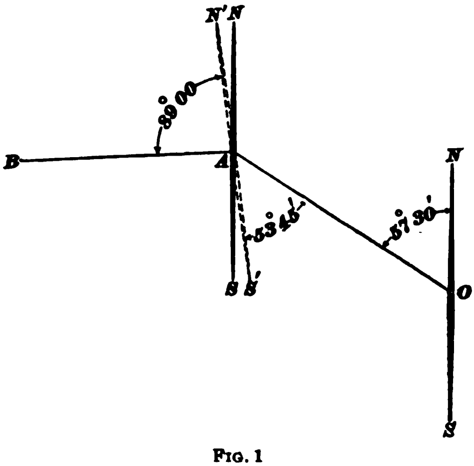


FIG. 1

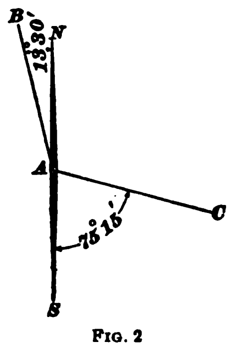


FIG. 2

angle $S A S'$. This angle is equal to the difference between 57° 30' and 53° 45', or 3° 45'. The correct magnetic bearing of the line $A B$ is equal to the sum of 89° 00' and 3° 45', or 92° 45'. Since this is greater than 90°, the bearing is 180° − 92° 45' = S 87° 15' W.

(2) The angle between the lines $A B$ and $A C$, Fig. 2, is equal to the sum of $B A N$ and $N A C$. The angle $N A C$ is equal to 180° − 75° 15' = 104° 45'. Therefore, $B A C$ = 13° 30' + 104° 45' = 118° 15'. Ans.

(3) In Fig. 3, the bearing of the line $A B$ is S 76° 15′ W, and the line $A C$ runs from A on the right of $A B$ and makes an angle of

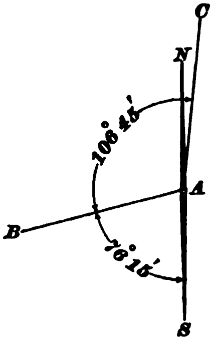


Fig. 3

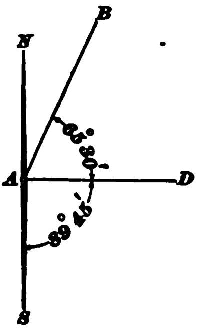


Fig. 4

106° 45′ with $A B$. The bearing of $A C$ is evidently equal to the sum of $B A S$ and $B A C$ minus 180°. Now [106° 45′ + 76° 15′] − 180° = 3° 00′; therefore, the bearing is N 3° 00′ E. Ans.

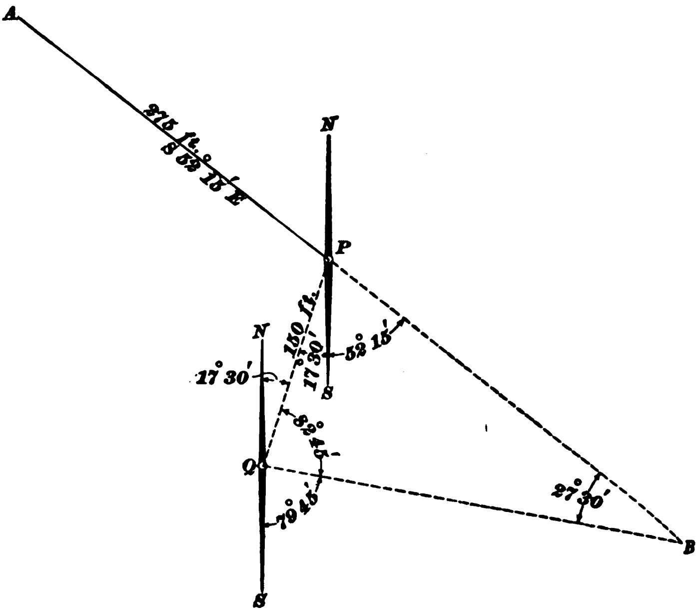


Fig. 5

(4) The bearing of $A D$, Fig. 4, is S 89° 45′ E, and the line $A B$ to the left of $A D$ makes an angle of 65° 30′ with $A D$. The angle

$N A B$ is evidently equal to $180° - [65° 30' + 89° 45'] = 24° 45'$, and the bearing of the line $A B$ is N 24° 45′ E. Ans.

(5) In the triangle $P Q B$, Fig. 5, the angle P is equal to $17° 30' + 52° 15' = 69° 45'$; the angle Q is evidently equal to $180° - [17° 30' + 79° 45'] = 82° 45'$; the angle $B = 180° - (P + Q) = 180° - (82° 45' + 69° 45') = 27° 30'$.

Then, $$P B = \frac{150 \sin 82° 45'}{\sin 27° 30'} = 322.2 \text{ ft.}$$

Since $A P = 275$ ft., $A B = 275 + 322.2 = 597.2$ ft. Ans.

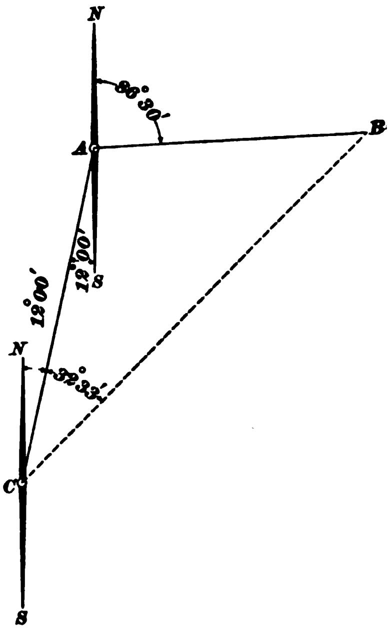


FIG. 6

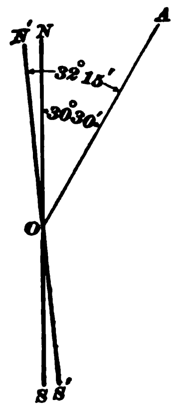


FIG. 7

(6) (*a*) The angle between the lines $A C$ and $A B$, Fig. 6, is equal to $180° - 86° 30' + 12° 00' = 105° 30'$. The length of $B C$ can be found from the relation

$$B C = \sqrt{\overline{A C}^2 + \overline{A B}^2 - 2 \times A C \times A B \cos A}$$

or, substituting the values of $A B$, $A C$, and A,

$$B C = \sqrt{728.5^2 + 586.5^2 - 2 \times 728.5 \times 586.5 \cos 105° 30'}$$
$$= 1{,}050.3 \text{ ft. Ans.}$$

(*b*) In the triangle ABC,

$$\sin C = \frac{AB \sin 105° 30'}{BC} = \frac{586.5 \sin 105° 30'}{1{,}050.3}$$

whence, $C = 32° 33'$

The angle NCB is equal to 12° 00′ + 32° 33′ = 44° 33′ and the bearing of the line CB is N 44° 33′ E. The bearing of BC is the reverse of that of CB or S 44° 33′ W. Ans.

(7) The number of the station 569 ft. from the starting point is Station or Substation 5 + 69. See Art. **34**.

(8) Station 9+89 is 989 ft. from the starting point, and Station 14+04 is 1,404 ft. from the starting point; therefore, the distance between the stations is 1,404 − 989 = 415 ft. Ans.

(9) See Art. **24**.

(10) The angle NON', Fig. 7, between the true meridian and the magnetic meridian, is equal to the difference between the true and the magnetic bearing of the given line. NON' = 32° 15′ − 30° 30′ = 1° 45′. Ans.

COMPASS SURVEYING

(PART 2)

(1) To solve this example, formulas **1** and **2**, Art. **7**, are applied. Since the bearing is southwest, both ranges are negative.

$$t = -19.43 \cos 87° 45' = -.76 \text{ ch. Ans.}$$

$$g = -19.43 \sin 87° 45' = -19.42 \text{ ch. Ans.}$$

The logarithmic work is arranged as follows:

$$\log g = 1.28814;\ g = 19.42 \text{ ch.}$$

$$\log \sin 87° 45' = \bar{1}.99967$$

$$\log 19.43 = 1.28847$$

$$\log \cos 87° 45' = \bar{2}.59395$$

$$\log t = \bar{1}.88242;\ t = .76 \text{ ch.}$$

(2) The tangent of the bearing is found by applying formula **3**, Art. **7**,

$$\tan G = \frac{11.97}{23.16}, \text{ whence } G = 27° 20'$$

As t is negative and g is positive, the bearing is southeast. The bearing given to the nearest quarter degree is S 27° 15′ E. Ans.

The length of the bearing is found by applying formula **4**, Art. **7**,

$$l = \frac{11.97}{\sin 27° 20'} = 26.07 \text{ ch. Ans.}$$

The logarithmic work is arranged as suggested in Art. **7**.

$$\log l = 1.41612;\ l = 26.07$$

$$\log \sin 27° 20' = \bar{1}.66197$$

$$\log 11.97 = 1.07809$$

$$\log 23.16 = 1.36474$$

$$\log \tan G = \bar{1}.71335;\ G = 27° 20'$$

(3) (*a*) The total error of closure is equal to the length of the hypotenuse of a right triangle whose legs are the algebraic sums of the latitude and longitude ranges. See Art. **11**.

$$\sqrt{.07^2 + .13^2} = .15 \text{ ch. Ans.}$$

(*b*) The relative error of closure is equal to the total error of closure divided by the sum of the lengths of all the courses.

$$e = \frac{.15}{41.73} = .004. \text{ Ans.}$$

(4) The latitude and longitude ranges are determined by applying formulas **1** and **2**, respectively, Art. **7**.

For AB, $l = -740.5 \cos 67° 15' = -286.4$
$g = -740.5 \sin 67° 15' = -682.9$
For BC, $l = -369.5 \cos 13° 45' = -358.9$
$g = -369.5 \sin 13° 45' = -87.8$
For CD, $l = 1{,}064 \cos 72° 30' = 320.0$
$g = 1{,}064 \sin 72° 30' = 1{,}014.8$
For DA, $l = 400.9 \cos 37° 00' = 320.2$
$g = -400.9 \sin 37° 00' = -241.3$

The algebraic sum of the latitude ranges is -5.1 and that of the longitude ranges is 2.8. The sum of the lengths of the courses is 2,574.9.

$$\frac{S_t}{S_l} = -\frac{5.1}{2{,}574.9} = -.002, \text{ nearly.}$$

$$\frac{S_g}{S_l} = \frac{2.8}{2{,}574.9} = .001, \text{ nearly.}$$

The correction in the latitude range for AB is $740.5 \times -.002 = -1.5$, nearly; that for BC is $369.5 \times -.002 = -.7$; that for CD is $+1{,}064 \times -.002 = -2.1$; that for DA is $+400.9 \times -.002 = -.8$.

The corrected latitude ranges are as follows:

$-286.4 - (-1.5) = -284.9$ $\quad +320.0 - (-2.1) = +322.1$
$-358.9 - (-\ .7) = -358.2$ $\quad +320.2 - (-\ .8) = +321$

The correction in longitude range for AB is $740.5 \times .001 = .7$; that for BC is $369.5 \times .001 = .4$; that for CD is $1{,}064.0 \times .001 = 1.1$; that for DA is $400.9 \times .001 = .4$. Since the sum of the corrections is 2.6 and the total corrections should be 2.8, the additional correction of .2 is made in the Long. R. of the third course. The corrected longitude ranges are, then,

$-682.9 - .7 = -683.6$ $\quad +1{,}014.8 - 1.3 = +1{,}013.5$
$-87.8 - .4 = -88.2$ $\quad -241.3 - .4 = -241.7$

The corrections for the lengths are determined by formula of Art. **14**.

For AB, $c_l = -286.4 \times (-.002) + (-682.9 \times .001) = -.1$
For BC, $c_l = -358.9 \times (-.002) + (-87.8 \times .001) = +.6$
For CD, $c_l = +320.0 \times (-.002) + 1{,}014.8 \times .001 = +.4$
For DA, $c_l = +320.2 \times (-.002) + (-241.3 \times .001) = -.9$

The corrected lengths are, then,

$740.5 - (-.1) = 740.6$ $\quad 1{,}064.0 - .4 = 1{,}063.6$
$369.5 - .6 = 368.9$ $\quad 400.9 - (-.9) = 401.8$

(5) The relative error of closure is, by formula of Art. **11**, equal to $\sqrt{.002^2 + .001^2} = .002$.

(6) The most westerly corner of the field is determined as explained in Art. **17**. Starting with AB as the first course, the operations are arranged as follows:

Long. R. of AB = + 9.76 = Long. of B
Long. R. of BC = −13.41
− 3.65 = Long. of C
Long. R. of CD = −16.83
−20.48 = Long. of D
Long. R. of DE = +21.78
+ 1.30 = Long. of E

As the longitude of D is the (arithmetically) greatest westing, D is the most westerly corner. Ans.

(7)

Courses	Longitude Ranges	Double Longitudes	Latitude Ranges	Double Areas +	Double Areas −
AB	+255.0	+ 255.0	+570.5	145,477.5	
BC	−558.5	− 48.5	+ 99.0		4,801.5
CD	−449.0	−1,056.0	−352.5	372,240.0	
DE	+217.5	−1,287.5	−449.0	578,087.5	
EA	+535.0	− 535.0	+132.0		70,620.0

1,095,805.0   75,421.5
75,421.5
2)1,020,383.5
S = 510,192 sq. ft., nearly
= 11.71 A.

The double longitudes are calculated as explained in Art. **19.** The work is arranged as follows:

Long. R. of AB = + 255.0 = D. Long. of AB
+ 255.0
Long. R. of BC = − 558.5
− 48.5 = D. Long. of BC
− 558.5
Long. R. of CD = − 449.0
− 1056.0 = D. Long. of CD
− 449.0
Long. R. of DE = + 217.5
− 1287.5 = D. Long. of DE
+ 217.5
Long. R. of EA = + 535.0
− 535.0 = D. Long. of EA

The work is arranged in the table in a manner similar to that shown in Art. **21.** Each double longitude is then multiplied by the latitude range on the right and the product is placed in the double-area column under the proper sign. The difference between the sum of the positive products and the sum of the negative products is the double area. Dividing by 2, the area of the field is found to be 510,192 sq. ft., or, dividing by 43,560, 11.71 A. Ans.

(8) The amount of daily variation in the needle at 11 A. M. on a day in September in a northern locality is found, from the table in Art. **30**, to be 2′. In the morning, the needle is east of the meridian and the bearing of the line is increased. The correct bearing is, then, S (76° 25′ − 2′) E, or S 76° 23′ E. Ans.

(9) The latitudes of the corners are determined as explained in Art. **23**. The latitude of A is equal to the latitude range of the course EA. The latitude of B is equal to the latitude of A plus the latitude range of AB, and so on. The longitudes of the courses are determined in a similar manner. The work is given below:

LATITUDES

	+132.0	= Lat. of A
Lat. R. of AB =	+570.5	
	+702.5	= Lat. of B
Lat. R. of BC =	+99.0	
	+801.5	= Lat. of C
Lat. R. of CD =	−352.5	
	+449.0	= Lat. of D
Lat. R. of DE =	−449.0	
	000.0	= Lat. of E

LONGITUDES

	+535.0	= Long. of A
Long. R. of AB =	+255.0	
	+790.0	= Long. of B
Long. R. of BC =	−558.5	
	+231.5	= Long. of C
Long. R. of CD =	−449.0	
	−217.5	= Long. of D
Long. R. of DE =	+217.5	
	000.0	= Long. of E

(10) The magnetic bearing of the first course, which is represented in Fig. 1 by $O\,1$, is N 15° 45′ E, and since the declination of the needle is 3° 15′ east, it is evident from the figure that the true bearing is equal to the sum of the magnetic bearing and the declination, or 15° 45′ + 3° 15′ = 19° 00′; therefore, the true bearing of AB is N 19° 00′ E. Ans.

For the same reason the true bearing of the second course, which is represented in the figure by $O\,2$, is equal to the sum of the magnetic bearing and the declination, or 88° 30′ + 3° 15′ = 91° 45′. As this is greater than 90°, the true bearing is southeast and is equal to the angle $S\,O\,2$, which is equal to 180° − 91° 45′ = 88° 15′. The true bearing is, therefore, S 88° 15′ E. Ans.

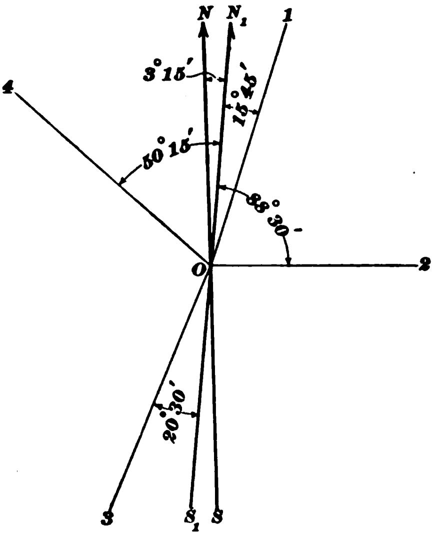


FIG. 1

The magnetic bearing of the third course, which is represented in the figure by $O\,3$, is S 20° 30′ W; it is evident from the figure that the true bearing is equal to the angle $S\,O\,3$, which is equal to the sum of $S\,O\,S_1$ and $S_1\,O\,3$ = 3° 15′ + 20° 30′ = 23° 45′. The true bearing of $O\,3$ is, therefore, S 23° 45′ W. Ans.

The magnetic bearing of the fourth course, which is represented in the figure by $O\,4$, is N 50° 15′ W. It is evident from the figure that the true bearing is equal to the angle $N\,O\,4$, which is equal to $N_1\,O\,4 - N_1\,O\,N$ = 50° 15′ − 3° 15′ = 47° 00′. The true bearing is, therefore, N 47° 00′ W. Ans.

(11) The positions of the true and the magnetic meridian are represented in Fig. 2 by $N\,S$ and $N_1\,S_1$, respectively. The angles representing the magnetic bearings of the four courses are also indicated in the figure. The true bearing of the first course, $O\,1$, is equal to the sum of the angles $1\,O\,N_1$ and $N_1\,O\,N$,

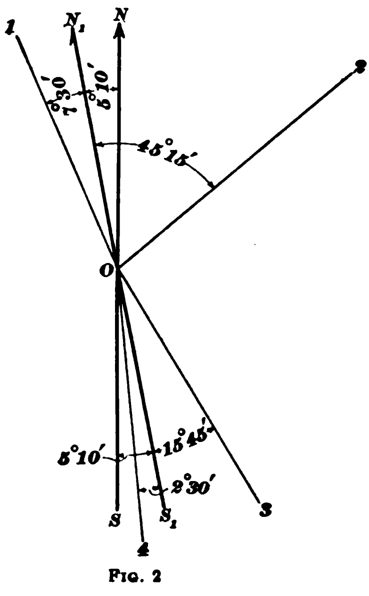


FIG. 2

or 7° 30′ + 5° 10′ = 12° 40′. The true bearing of this course is, therefore, N 12° 40′ W. Ans.

The true bearing of the second course $O\,2$ is equal to the angle $N O 2$, which is equal to $N_1 O 2 - N_1 O N$ = 45° 15′ − 5° 10′ = 40° 5′. The true bearing of this course is, therefore, N 40° 5′ E. Ans.

The true bearing of the third course $O\,3$ is equal to the angle $S O 3$, which is equal to the sum of the angles $S O S_1$ and $S_1 O 3$ = 5° 10′ + 15° 45′ = 20° 55′. The true bearing of this course is, therefore, S 20° 55′ E. Ans.

The true bearing of the fourth course $O\,4$ is equal to the angle $S O 4$, which is equal to $S O S_1 - S_1 O 4$ = 5° 10′ − 2° 30′ = 2° 40′. The true bearing of this course is, therefore, S 2° 40′ E. Ans.

(12) (*a*) The original bearing of the line was N 40° 00′ W and the present bearing is N 38° 45′ W; therefore, the needle has moved west

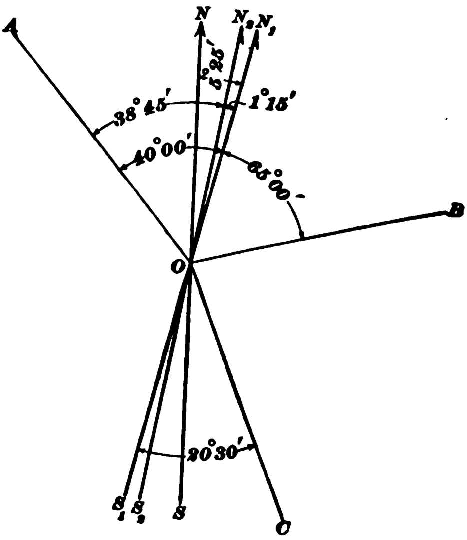


FIG. 3

an amount equal to 40° 00′ − 38° 45′ = 1° 15′. In Fig. 3, the original position of the needle is represented by $N_1 S_1$, the present position, by $N_2 S_2$, and the position of the true meridian by $N S$. The change in position in 20 years is 1° 15′, or 75′; therefore, the yearly change is $\frac{75}{20}$ = 3.7′, say 4′. Ans.

(*b*) The present bearing of the line $O B$ is equal to the angle $B O N_2 = B O N_1 + N_1 O N_2$ = 65° 00′ + 1° 15′ = 66° 15′. The present bearing of the line is, therefore, N 66° 15′ E. Ans.

(*c*) The present bearing of the line OC is equal to the angle $COS_2 = COS_1 - S_2OS_1 = 20° 30' - 1° 15' = 19° 15'$. The present bearing is, therefore, S 19° 15′ E. Ans.

(*d*) The declination of the needle at the time of the original survey was 5° 25′ east, that is, the angle $NON_1 = 5° 25'$. The present declination is equal to the angle $NON_2 = 5° 25' - 1° 15' = 4° 10'$ east. Ans.

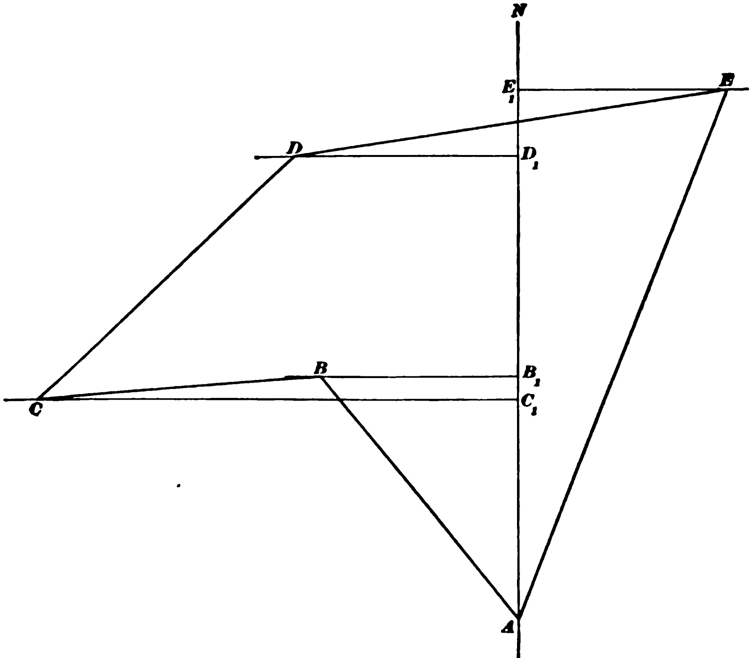


FIG. 4

(13) The latitudes and longitudes of the corners are calculated from the corner A as follows:

LATITUDES

	$+3.50$	= Lat. of B
Lat. R. of BC =	$-\ .34$	
	$+3.16$	= Lat. of C
Lat. R. of CD =	$+3.49$	
	$+6.65$	= Lat. of D
Lat. R. of DE =	$+\ .95$	
	$+7.60$	= Lat. of E
Lat. R. of EA =	-7.60	
	0.00	= Lat. of A

LONGITUDES

$$
\begin{array}{rl}
 & -2.82 = \text{Long. of } B \\
\text{Long. R. of } BC = & \underline{-4.06} \\
 & -6.88 = \text{Long. of } C \\
\text{Long. R. of } CD = & \underline{+3.68} \\
 & -3.20 = \text{Long. of } D \\
\text{Long. R. of } DE = & \underline{+6.21} \\
 & +3.01 = \text{Long. of } E \\
\text{Long. R. of } EA = & \underline{-3.01} \\
 & 0.00 = \text{Long. of } A
\end{array}
$$

The plat of the field, Fig. 4, is then constructed by laying off upwards (since here all latitudes are positive), from A on SN, the latitudes of the corners, thus locating the points B_1, C_1, D_1, and E_1. Through the points B_1, C_1, D_1, and E_1 are drawn indefinite lines perpendicular to SN, and on these lines are laid off the longitudes of the corners B, C, D, and E, respectively. The longitudes of B, C, and D being negative, they are laid off to the left; the longitude of E being positive, it is laid off to the right. The points thus located are joined by straight lines, and the resulting polygon $ABCDE$ is the required plat.

TRANSIT SURVEYING

(PART 1)

---

(1) (*a*) Formula **1**, Art. **17**, solved for n, gives $n = \frac{s}{r}$. In this case, $s = \frac{1}{8}$, and $r = \frac{1}{64}$; hence, $n = \frac{\frac{1}{8}}{\frac{1}{64}} = 8$. Therefore, the vernier must be divided into 8 equal parts. Ans.

(*b*) The vernier, being divided into 8 equal parts, must cover 7 divisions of the scale; therefore, the length of the vernier is $\frac{7}{8}$ in. Ans.

(2) (*a*) Applying formula **1**, Art. **17**,

$$r = \frac{.05}{10} = .005 \text{ ft. Ans.}$$

(*b*) Applying formula **2**, Art. **17**,

$$R = .005 \times 8 = .04 \text{ ft. Ans.}$$

(3) Applying formula **1**, Art. **17**,

$$r = \frac{60'}{10} = 6'. \text{ Ans.}$$

(4) Applying formula **2**, Art. **17**,

$$R = 6 \times 9 = 54'. \text{ Ans.}$$

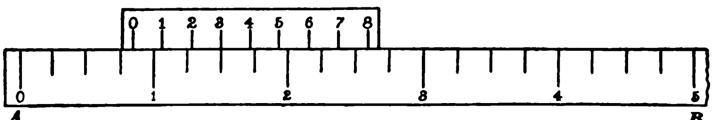


FIG. 1

(5) In Fig. 1, *A B* represents part of a scale divided into quarter-inches. Since the vernier is to read to $\frac{1}{32}$ in., it must be divided into $\frac{\frac{1}{4}}{\frac{1}{32}} = 8$ equal parts, and cover 7 divisions of the scale. The part *0-8* of the vernier is, therefore, $\frac{7}{4} = 1\frac{3}{4}$ in. long. With the vernier in the position shown in Fig. 1, the *3* of the vernier coincides with a division of the scale. The reading of the vernier is, then, $\frac{1}{32} \times 3 = \frac{3}{32}$ in. The reading of the scale is $\frac{3}{4}$ in.; therefore, the whole reading is $\frac{3}{4} + \frac{3}{32} = \frac{27}{32}$ in. Ans. See Art. **16.**

(6) (*a*) The azimuth of the line, counted from the north, is greater than 270°; therefore, the bearing will be northwest and equal to 360° − 276° 57′ = 83° 03′. Therefore, the true bearing of the line is N 83° 03′ W. Ans.

(*b*) The back azimuth of the line is equal to 276° 57′ + 180° = 456° 57′, or 456° 57′ − 360° = 96° 57′. Ans. See Art. **35.**

(7) See Art. **42.**

(8) See Art. **41.**

(9) (*a*) The ranges are determined by formulas **1** and **2**, Art. **39**,

$$t = 569 \cos 343° 41' = 569 \cos (360° - 343° 41')$$
$$= 569 \cos 16° 19' = 546.1 \text{ ft. Ans.}$$
$$g = 569 \sin 343° 41' = 569 [- \sin (360° - 343° 41')]$$
$$= 569 \times - (\sin 16° 19') = - 159.9 \text{ ft. Ans.}$$

(*b*)
$$t = 1{,}026 \cos 233° 04' = 1{,}026 [- \cos (233° 04' - 180°)]$$
$$= 1{,}026 \times - (\cos 53° 04') = -616.5 \text{ ft. Ans.}$$
$$g = 1{,}026 \sin 233° 04' = 1{,}026 [- \sin (233° 04' - 180°)]$$
$$= 1{,}026 \times - (\sin 53° 04') = -820.1 \text{ ft. Ans.}$$

(10) The azimuth of the line whose bearing is S 83° 18′ W is 263° 18′. The angle *B A C*, Fig. 2, is evidently equal to the difference between the azimuth of the line *A B* and that of *A C*, or

354° 29′ − 263° 18′ = 91° 11′. Ans.

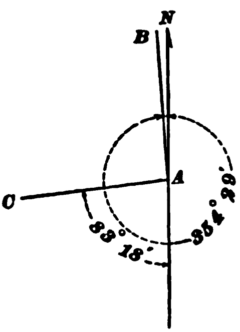


FIG. 2

(11) The calculated bearing of the second line is equal to the sum of the bearing (N 55° 15′ E) of the given line and the deflection angle that is turned to the right. 55° 15′ + 15° 17′ = 70° 32′. Therefore, the calculated bearing of the second line is N 70° 32′ E. Ans.

(12) The calculated bearing of the second line is equal to the sum of the bearing (N 13° 15′ W) of the given line and the deflection angle that is turned to the left. 13° 15′ + 40° 20′ = 53° 35′. Therefore, the calculated bearing of the second line is N 53° 35′ W. Ans.

(13) See Arts. **37**, **38**, and **40.**

(14) Since the longitude range is negative, it is west, and since the latitude range is positive, it is north; therefore, the bearing is northwest; $\tan G = \frac{1,215.8}{1,427.8}$, whence $G = 40° 25'$. The bearing being northwest, the azimuth is equal to $360° - 40° 25' = 319° 35'$. Ans.

TRANSIT SURVEYING

(PART 2)

(1) From Table I,

Upper culmination, February 1, 1905	$4^h\ 39.0^m$
Difference for 1 day, 3.95.	
From Table II, correction under 3.95, opposite 12	43.4
	$3^h\ 55.6^m$
Adding (Art. 29)	11 58.0
Time of lower culmination	15^h 53.6

As this is astronomical time, it shows the interval elapsed since noon of February 12, and corresponds, therefore, to the civil date February 13, $3^h\ 53.6^m$ A. M. We must, therefore, take the time of the previous culmination corresponding to the astronomical date February 11.

Upper culmination, February 1, 1905	$4^h\ 39.0^m$
Correction from Table II	39.5
	$3^h\ 59.5^m$
Adding	11 58.0
Time of lower culmination	$15^h\ 57.5^m$

The civil date corresponding to this date is February 12, $3^h\ 57.5^m$, local time. The difference in longitude between the standard 6-hour meridian (90°) and the given place is 5° 15′, or 21^m. Since the given place is west of the standard-time meridian, standard time is later than local time. The standard time corresponding to the local time, $3^h\ 57.5^m$, is therefore $3^h\ 57.5^m + 21^m = 4^h\ 18.5^m$. Ans.

(2) The difference in longitude between the given places is 44° 25′ 55″, or, in time, $2^h\ 57^m\ 44^s$. The reduction is shown below. See Art. **20**.

```
15 ) 44 ( 2
     30
     --
     14
      60
     ---
     840
      25
     ---
15 ) 865 ( 57
     75
     ---
     115
     105
     ---
      10
       60
      ---
      600
       55
      ---
15 )  655 ( 43.7 say 44
      60
      --
       55
       45
       ---
       100
```

Substituting in formula of Art. **19**,

$$T_a = 9^h\ 41^m + 2^h\ 57^m\ 44^s = 12^h\ 38^m\ 44^s. \quad \text{Ans.}$$

(3)

Upper culmination of Polaris, December 15, 1908	$7^h\ 50.4^m$
Correction from Table II	63.2
	$6^h\ 47.2^m$
Subtracting (Art. **33**)	5 55.0
Time of eastern elongation	$0^h\ 52.2^m$ Ans.

Time of western elongation $= 6^h\ 47.2^m + 5^h\ 55^m = 12^h\ 42.2^m$. Ans.

(4) (*a*) In the triangle $B\,C\,E$, the angle $B\,C\,E = 180° - (25° + 30°\ 13' + 63°\ 27') = 61°\ 20'$.

$$B\,C = \frac{500 \sin 63°\ 27'}{\sin 61°\ 20'} = 509.75 \text{ ft.}$$

In the triangle $B\,E\,D$, the angle $B\,D\,E = 180° - (30°\ 13' + 63°\ 27' + 21°\ 36') = 64°\ 44'$, and the angle $B\,E\,D = 63°\ 27' + 21°\ 36' = 85°\ 03'$.

$$B\,D = \frac{500 \sin 85°\ 03'}{\sin 64°\ 44'} = 550.83 \text{ ft.}$$

In the triangle BCD, let the angle BCD be represented by X, BDC by Y, and CBD by K. Then,

$$\tan\frac{1}{2}(X - Y) = \frac{(BD - BC)}{(BD + BC)}\cot\frac{1}{2}K$$

$$\tan\frac{1}{2}(X - Y) = \frac{550.83 - 509.75}{550.83 + 509.75}\cot 12° 30'$$

whence, $$\frac{1}{2}(X - Y) = 9° 55'$$

$$CD = \frac{550.83 - 509.75}{\sin 9° 55'}\cos 12° 30' = 232.88. \quad \text{Ans.} \quad \text{See Art. 37.}$$

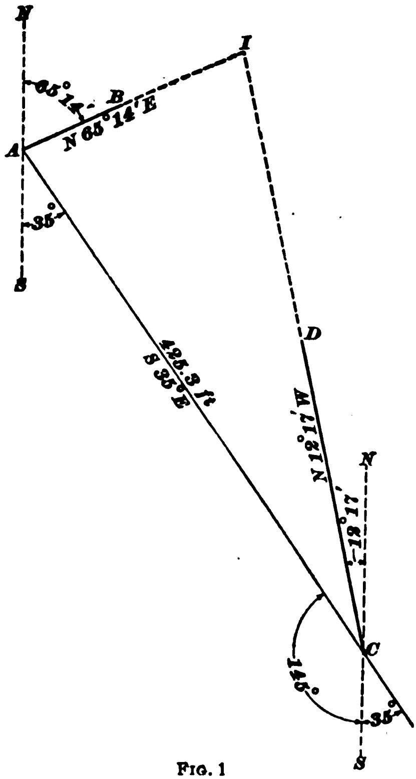


FIG. 1

(*b*) In the triangle EDI, the angle $DEI = 180° - (63° 27' + 21° 36') = 94° 57'$, and the angle $EDI = 180° - (37° 22' + 94° 57') = 47° 41'$.

$$ED = \frac{500 \sin 30° 13'}{\sin 64° 44'} = 278.25 \text{ ft.}$$

$$EI = \frac{278.25 \sin 47° 41'}{\sin 37° 22'} = 339.01 \text{ ft.} \quad \text{Ans.}$$

$$ID = \frac{278.25 \sin 94° 57'}{\sin 37° 22'} = 456.76 \text{ ft.} \quad \text{Ans.}$$

(c) $\frac{1}{2}(X - Y) = 9° 55', \frac{1}{2}(X + Y) = \frac{1}{2}(180° - 25°) = 77° 30';$

whence, $X = \frac{1}{2}(X + Y) + \frac{1}{2}(X - Y) = 87° 25'$

Then, $I = 180° - (87° 25' + 25° + 30° 13') = 37° 22'$. Ans.

(5) In Fig. 1, the angle $IAC = 180° - (65° 14' + 35°) = 79° 46'$, and the angle $ICA = 180° - (12° 17' + 145°) = 22° 43'$. Then, the angle $AIC = 180° - (79° 46' + 22° 43') = 77° 31'$.

$$AI = \frac{425.3 \sin 22° 43'}{\sin 77° 31'} = 168.2 \text{ ft.} \quad \text{Ans.}$$

$$CI = \frac{425.3 \sin 79° 46'}{\sin 77° 31'} = 428.7 \text{ ft.} \quad \text{Ans.}$$

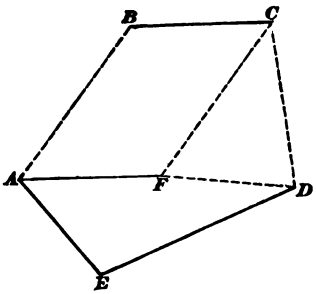


FIG. 2

(6) In the triangle BCG, Fig. II,

$$\tan \frac{1}{2}(CBG - BCG) = \frac{CG - BG}{CG + BG} \cot \tfrac{1}{2} BGC$$

$$\frac{1}{2} BGC = 69° 15'$$

$CG - BG = 417 - 376 = 41$ ft., and $CG + BG = 417 + 376 = 793$ ft.

Then,

$$\tan \frac{1}{2}(CBG - BCG) = \frac{41 \times \cot 69° 15'}{793}$$

whence, $\frac{1}{2}(CBG - BCG) = 1° 07'$

$$\frac{1}{2}(CBG + BCG) = \tfrac{1}{2}(180° - 138° 29') = 20° 46'$$

$$CBG = 21° 53'$$

Then, $BC = \frac{417 \sin 138° 29'}{\sin 21° 53'} = 741.6$ ft. Ans.

The angle $BCG = 180° - (138° 29' + 21° 53') = 19° 38'$

$EBC = 20° 12' + 21° 53' = 42° 05'$

$BCE = 19° 38' + 15° 27' = 35° 05'$

$FEC = 42° 05' + 35° 05' = 77° 10'$. Ans.

$BEC = 180° - 77° 10' = 102° 50'$

$$BE = \frac{741.6 \sin 35° 05'}{\sin 102° 50'} = 437.1 \text{ ft.} \quad \text{Ans.}$$

$$EC = \frac{741.6 \sin 42° 05'}{\sin 102° 50'} = 509.7 \text{ ft.} \quad \text{Ans.}$$

(7) See Art. **48.** In Fig. 2, the side BC is shifted parallel to itself, to the position AF, and AB is shifted parallel to itself to the position FC. The length and azimuth of FD in the polygon $AFDEA$ are determined as explained in Art. **47.** The ranges of the given courses are as follows:

$$\begin{aligned} t_1 &= 300 \cos 86° 23' = +\ 18.9 & g_1 &= 300 \sin 86° 23' = +299.4 \\ t_2 &= 450 \cos 243° 54' = -198.0 & g_2 &= 450 \sin 243° 54' = -404.1 \\ t_3 &= 268 \cos 317° 30' = +197.6 & g_3 &= 268 \sin 317° 30' = -181.1 \\ & \qquad\qquad\qquad\quad\ +\ 18.5 & & \qquad\qquad\qquad\quad\ -285.8 \end{aligned}$$

The ranges of FD are, then, $t = -18.5$, and $g = +285.8$.

$$\tan Z = \frac{+285.8}{-18.5} = -15.4487$$

whence $Z = 93° 42'$, or $180° + 93° 42'$. The sine of Z must have the

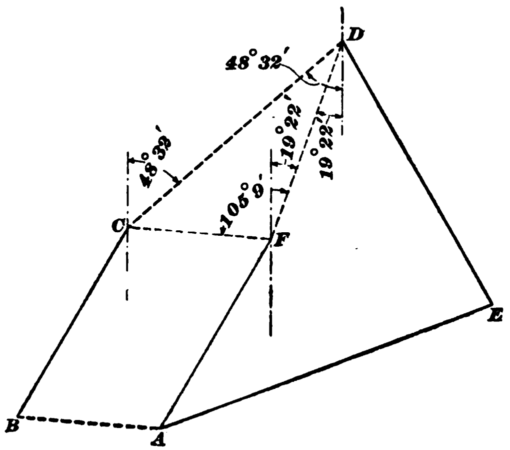


FIG. 3

same sign as g; therefore, since g is positive and $\sin 93° 42'$ is positive, the azimuth of FD is $93° 42'$.

$$FD = \frac{285.8}{\sin 93° 42'} = 286.4 \text{ ft.}$$

In the triangle CFD, the angle CFD is equal to the difference between the azimuths of AB ($= FC$) and FD, or $93° 42' - 33° 45' = 59° 57'$; the angle FDC is equal to the difference between the back azimuths of CD and FD, or $349° 36' - 273° 42' = 75° 54'$; and the angle $FCD = 180° - (75° 54' + 59° 57') = 44° 09'$. Then,

$$FC = AB = \frac{286.4 \sin 75° 54'}{\sin 44° 09'} = 398.8 \text{ ft.} \quad \text{Ans.}$$

$$CD = \frac{286.4 \sin 59° 57'}{\sin 44° 09'} = 355.9 \text{ ft.} \quad \text{Ans.}$$

(8) See Art. **49.** In Fig. 3, the side AB is shifted parallel to itself to the position FC, and BC is shifted parallel to itself to the position AF. In the polygon $AFDEA$, the length and azimuth of the side FD are determined as explained in Art. **47.** The ranges of the given sides are as follows:

$$\begin{array}{llll} t_1 = 458.7 \cos 29° 48' = +398.0 & g_1 = 458.7 \sin 29° 48' = +228.0 \\ t_2 = 632.4 \cos 150° 46' = -551.9 & g_2 = 632.4 \sin 150° 46' = +308.8 \\ t_3 = 729.4 \cos 249° 12' = -259.0 & g_3 = 729.4 \sin 249° 12' = -681.9 \\ \hline \qquad\qquad -412.9 & \qquad\qquad -145.1 \end{array}$$

The ranges of the side FD are, then, $t = +412.9$ and $g = +145.1$.

$$\tan Z = \frac{+145.1}{+412.9}$$

whence $Z = 19° 22'$, $FD = \frac{145.1}{\sin 19° 22'} = 437.6$. In the triangle CFD, $FC = 298$ ft., $FD = 437.6$ ft., and the angle $CDF = 48° 32' - 19° 22' = 29° 10'$; $\sin DCF = \frac{437.6 \sin 29° 10'}{298}$; whence, $DCF = 45° 41'$. $CFD = 180° - (29° 10' + 45° 41') = 105° 9'$. The angle that FC makes with the meridian, measured toward the west, is, then, $105° 9' - 19° 22' = 85° 47'$, and the azimuth of FC, and therefore of AB, is $360° - 85° 47' = 274° 13'$. Ans.

In the triangle CFD,

$$CD = \frac{298 \sin 105° 9'}{\sin 29° 10'} = 590.2 \text{ ft. Ans.}$$

(9) See Art. **50.** The ranges of A_0A_1, A_1A_2, and A_2A_3 are as follows:

$$\begin{array}{llll} t_1 = 600 \cos 340° = +563.8 & g_1 = 600 \sin 340° = -205.2 \\ t_2 = 425 \cos 15° = +410.5 & g_2 = 425 \sin 15° = +110.0 \\ t_3 = 350 \cos 318° = +260.1 & g_3 = 350 \sin 318° = -234.2 \\ \hline \qquad\qquad +1234.4 & \qquad\qquad -329.4 \end{array}$$

The ranges of the closing line A_3A_0, Fig. 4, are, then, $t = -1{,}234.4$ and $g = +329.4$. For the azimuth of this line, we have,

$$\tan Z = \frac{+329.4}{-1{,}234.4}$$

The value of Z is $165° 04'$, since the sine of Z must be positive, and the sine of $165° 04'$ is positive. The angle Z' is equal to $360°$ minus the back azimuth of A_3A_0, or $360° - (180° + 165° 04') = 14° 56'$. The angle $A_0A_3A_4 = Z - 20° = 165° 04' - 20° = 145° 04'$, and the angle $A_3A_4A_0$ is equal to $20°$.

Then,
$$A_3A_4 = \frac{A_3A_0 \sin Z'}{\sin 20°}$$

$$A_0A_4 = \frac{A_3A_0 \sin A_0A_3A_4}{\sin 20°}$$

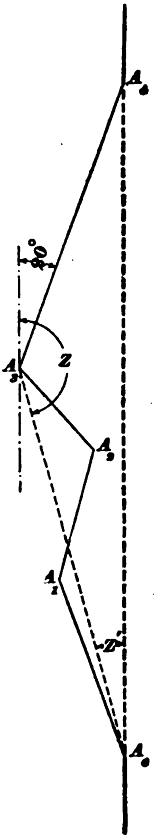
But,
$$A_3A_0 = \frac{g}{\sin Z} = \frac{329.4}{\sin Z'}$$

Then, $A_3A_4 = \frac{329.4 \sin Z'}{\sin Z' \sin 20°} = \frac{329.4}{\sin 20°} = 963.1$ ft.

and $A_0A_4 = \frac{329.4 \sin 145° 04'}{\sin Z' \sin 20°} = 2{,}140.1$ ft. Ans.

(10) The azimuth of BA_1, Fig. 5, is 90°; the azimuth of A_1A_2 $= 115° - 90° = 25°$; the azimuth of $A_2D = 180° - (130° - 25°) = 75°$. The ranges of these courses are as follows:

$t_1 = 500 \cos 90° =$	0 0 0.0	$g_1 = 500 \sin 90° =$	+ 5 0 0
$t_2 = 650 \cos 25° =$	+5 8 9.1	$g_2 = 650 \sin 25° =$	+ 2 7 4.7
$t_3 = 475.2 \cos 75° =$	+1 2 3.0	$g_3 = 475.2 \sin 75° =$	+ 4 5 9.0
	+7 1 2.1		+1 2 3 3.7



The ranges of the closing line DB are, then, $t = -712.1$ and $g = -1{,}233.7$. Therefore, $\tan Z = \frac{-1{,}233.7}{-712.1}$. The sine of the angle Z must be negative. Therefore, $Z = 240°$. The azimuth of DB is, then, 240°.

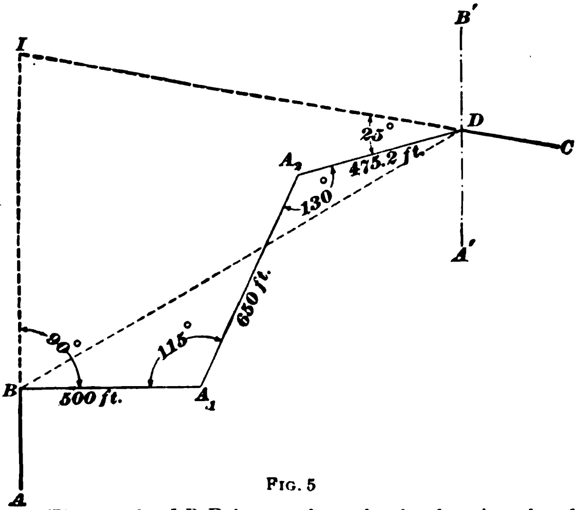


FIG. 5

The angle IBD is equal to the back azimuth of DB, or $(240° + 180°) - 360° = 60°$. The azimuth of A_2D is 75°; therefore, the angle $A_2DB' = 180° - 75°$ $= 105°$, and the angle $IDB' = 105° - 25° = 80°$ = the angle I. Ans.

FIG. 4

The angle $BDI = 180° - (60° + 80°) = 40°$.

$$IB = \frac{BD \sin 40°}{\sin 80°}$$

$$[illegible] = \frac{B[illegible] \sin 40°}{\sin 40°}$$

But,
$$BD = \frac{[illegible]}{\sin 240°} = \frac{[illegible]}{\sin 40°}$$

Then,
$$IB = \frac{[illegible] \sin 40°}{\sin 40° \sin 40°} = [illegible] \pm \quad \text{Ans.}$$

$$[illegible] I = \frac{[illegible] \sin 40°}{\sin 40° \sin 40°} = \frac{[illegible]}{\sin 40°} = [illegible] \pm \quad \text{Ans.}$$

[illegible] The length of [illegible] is equal to
$$\frac{[illegible]}{\sin 40°} = [illegible] \pm \quad \text{Ans.}$$

The azimuth of [illegible] was found in the preceding example to be 240° Ans.

[illegible] *a* Substituting the given values in formula 3, Art. 58,
$$EF = \sqrt{\frac{[illegible]}{[illegible]}} = [illegible] \text{ ch.} \quad \text{Ans.}$$

b Substituting the given values in formula 4, Art. 58,
$$DE = \frac{[illegible]}{[illegible]} = [illegible] \text{ ch.} \quad \text{Ans.}$$

[illegible] The length of EF is determined by applying formula 3, Art. 58. In this case, [illegible] = [illegible], [illegible] = [illegible], m = [illegible], and n = [illegible] − [illegible] = [illegible]. Substituting these values in the formula,
$$x = \sqrt{\frac{[illegible]}{[illegible]}} = [illegible] \text{ ch.} \quad \text{Ans.}$$

Applying formula 5, Art. 58,
$$AE = \frac{[illegible]}{[illegible]} = [illegible] \text{ ch.} \quad \text{Ans.}$$

14. This example is solved by applying the formula of Art. 59. In this case, S = [illegible] = [illegible]. A = [illegible] sq. ch. FC corresponds to GD in the formula, and measures [illegible] ch., and the angle C corresponds to the angle D in the formula, and is equal to 75°. Substituting in the formula,
$$CG = \frac{2 \times [illegible]}{[illegible] \times \sin 75°} = [illegible] \text{ ch.} \quad \text{Ans.}$$

NATURAL TRIGONOMETRIC FUNCTIONS

′	0° Sine	0° Cosine	1° Sine	1° Cosine	2° Sine	2° Cosine	3° Sine	3° Cosine	4° Sine	4° Cosine	′
0	.00000	1.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	1.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	1.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	1.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	1.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	1.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	1.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	1.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	1.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	1.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	1.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	79°		78°		77°		76°		75°		

′	15°		16°		17°		18°		19°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	74°		73°		72°		71°		70°		

′	20° Sine	20° Cosine	21° Sine	21° Cosine	22° Sine	22° Cosine	23° Sine	23° Cosine	24° Sine	24° Cosine	′
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	69°		68°		67°		66°		65°		

′	25° Sine	25° Cosine	26° Sine	26° Cosine	27° Sine	27° Cosine	28° Sine	28° Cosine	29° Sine	29° Cosine	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	64°		63°		62°		61°		60°		

′	30°		31°		32°		33°		34°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	49°		48°		47°		46°		45°		

′	0° Tang	0° Cotang	1° Tang	1° Cotang	2° Tang	2° Cotang	3° Tang	3° Cotang	4° Tang	4° Cotang	′
0	.00000	Infinite	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	.07022	14.2411	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	.07051	14.1821	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	.07080	14.1235	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	.07110	14.0655	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	.07139	14.0079	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	.07168	13.9507	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	.07197	13.8940	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	.07227	13.8378	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	.07256	13.7821	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	.07285	13.7267	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	.07314	13.6719	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	.07344	13.6174	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	.07373	13.5634	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	.07402	13.5098	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	.07431	13.4566	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	.07461	13.4039	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	.07490	13.3515	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	.07519	13.2996	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	.07548	13.2480	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	.07578	13.1969	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	.07607	13.1461	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	.07636	13.0958	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	.07665	13.0458	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	.07695	12.9962	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	.07724	12.9469	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	.07753	12.8981	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	.07782	12.8496	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	.07812	12.8014	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	.07841	12.7536	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	.07870	12.7062	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	.07899	12.6591	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	.07929	12.6124	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	.07958	12.5660	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	.07987	12.5199	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	.08017	12.4742	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	.08046	12.4288	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	.08075	12.3838	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	.08104	12.3390	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	.08134	12.2946	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	.08163	12.2505	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	.08192	12.2067	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	.08221	12.1632	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	.08251	12.1201	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	.08280	12.0772	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	.08309	12.0346	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	.08339	11.9923	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	.08368	11.9504	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	.08397	11.9087	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	.08427	11.8673	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	.08456	11.8262	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	.08485	11.7853	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	.08514	11.7448	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	.08544	11.7045	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	.08573	11.6645	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	.08602	11.6248	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	.08632	11.5853	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	.08661	11.5461	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	.08690	11.5072	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	.08720	11.4685	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	.08749	11.4301	0
′	Cotang	Tang 89°	Cotang	Tang 88°	Cotang	Tang 87°	Cotang	Tang 86°	Cotang	Tang 85°	′

′	5°		6°		7°		8°		9°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	60
1	.08778	11.3919	.10540	9.48781	.12308	8.12481	.14084	7.10038	.15868	6.30189	59
2	.08807	11.3540	.10569	9.46141	.12338	8.10536	.14113	7.08546	.15898	6.29007	58
3	.08837	11.3163	.10599	9.43515	.12367	8.08600	.14143	7.07059	.15928	6.27829	57
4	.08866	11.2789	.10628	9.40904	.12397	8.06674	.14173	7.05579	.15958	6.26655	56
5	.08895	11.2417	.10657	9.38307	.12426	8.04756	.14202	7.04105	.15988	6.25486	55
6	.08925	11.2048	.10687	9.35724	.12456	8.02848	.14232	7.02637	.16017	6.24321	54
7	.08954	11.1681	.10716	9.33155	.12485	8.00948	.14262	7.01174	.16047	6.23160	53
8	.08983	11.1316	.10746	9.30599	.12515	7.99058	.14291	6.99718	.16077	6.22003	52
9	.09013	11.0954	.10775	9.28058	.12544	7.97176	.14321	6.98268	.16107	6.20851	51
10	.09042	11.0594	.10805	9.25530	.12574	7.95302	.14351	6.96823	.16137	6.19703	50
11	.09071	11.0237	.10834	9.23016	.12603	7.93438	.14381	6.95385	.16167	6.18559	49
12	.09101	10.9882	.10863	9.20516	.12633	7.91582	.14410	6.93952	.16196	6.17419	48
13	.09130	10.9529	.10893	9.18028	.12662	7.89734	.14440	6.92525	.16226	6.16283	47
14	.09159	10.9178	.10922	9.15554	.12692	7.87895	.14470	6.91104	.16256	6.15151	46
15	.09189	10.8829	.10952	9.13093	.12722	7.86064	.14499	6.89688	.16286	6.14023	45
16	.09218	10.8483	.10981	9.10646	.12751	7.84242	.14529	6.88278	.16316	6.12899	44
17	.09247	10.8139	.11011	9.08211	.12781	7.82428	.14559	6.86874	.16346	6.11779	43
18	.09277	10.7797	.11040	9.05789	.12810	7.80622	.14588	6.85475	.16376	6.10664	42
19	.09306	10.7457	.11070	9.03379	.12840	7.78825	.14618	6.84082	.16405	6.09552	41
20	.09335	10.7119	.11099	9.00983	.12869	7.77035	.14648	6.82694	.16435	6.08444	40
21	.09365	10.6783	.11128	8.98598	.12899	7.75254	.14678	6.81312	.16465	6.07340	39
22	.09394	10.6450	.11158	8.96227	.12929	7.73480	.14707	6.79936	.16495	6.06240	38
23	.09423	10.6118	.11187	8.93867	.12958	7.71715	.14737	6.78564	.16525	6.05143	37
24	.09453	10.5789	.11217	8.91520	.12988	7.69957	.14767	6.77199	.16555	6.04051	36
25	.09482	10.5462	.11246	8.89185	.13017	7.68208	.14796	6.75838	.16585	6.02962	35
26	.09511	10.5136	.11276	8.86862	.13047	7.66466	.14826	6.74483	.16615	6.01878	34
27	.09541	10.4813	.11305	8.84551	.13076	7.64732	.14856	6.73133	.16645	6.00797	33
28	.09570	10.4491	.11335	8.82252	.13106	7.63005	.14886	6.71789	.16674	5.99720	32
29	.09600	10.4172	.11364	8.79964	.13136	7.61287	.14915	6.70450	.16704	5.98646	31
30	.09629	10.3854	.11394	8.77689	.13165	7.59575	.14945	6.69116	.16734	5.97576	30
31	.09658	10.3538	.11423	8.75425	.13195	7.57872	.14975	6.67787	.16764	5.96510	29
32	.09688	10.3224	.11452	8.73172	.13224	7.56176	.15005	6.66463	.16794	5.95448	28
33	.09717	10.2913	.11482	8.70931	.13254	7.54487	.15034	6.65144	.16824	5.94390	27
34	.09746	10.2602	.11511	8.68701	.13284	7.52806	.15064	6.63831	.16854	5.93335	26
35	.09776	10.2294	.11541	8.66482	.13313	7.51132	.15094	6.62523	.16884	5.92283	25
36	.09805	10.1988	.11570	8.64275	.13343	7.49465	.15124	6.61219	.16914	5.91236	24
37	.09834	10.1683	.11600	8.62078	.13372	7.47806	.15153	6.59921	.16944	5.90191	23
38	.09864	10.1381	.11629	8.59893	.13402	7.46154	.15183	6.58627	.16974	5.89151	22
39	.09893	10.1080	.11659	8.57718	.13432	7.44509	.15213	6.57339	.17004	5.88114	21
40	.09923	10.0780	.11688	8.55555	.13461	7.42871	.15243	6.56055	.17033	5.87080	20
41	.09952	10.0483	.11718	8.53402	.13491	7.41240	.15272	6.54777	.17063	5.86051	19
42	.09981	10.0187	.11747	8.51259	.13521	7.39616	.15302	6.53503	.17093	5.85024	18
43	.10011	9.98931	.11777	8.49128	.13550	7.37999	.15332	6.52234	.17123	5.84001	17
44	.10040	9.96007	.11806	8.47007	.13580	7.36389	.15362	6.50970	.17153	5.82982	16
45	.10069	9.93101	.11836	8.44896	.13609	7.34786	.15391	6.49710	.17183	5.81966	15
46	.10099	9.90211	.11865	8.42795	.13639	7.33190	.15421	6.48456	.17213	5.80953	14
47	.10128	9.87338	.11895	8.40705	.13669	7.31600	.15451	6.47206	.17243	5.79944	13
48	.10158	9.84482	.11924	8.38625	.13698	7.30018	.15481	6.45961	.17273	5.78938	12
49	.10187	9.81641	.11954	8.36555	.13728	7.28442	.15511	6.44720	.17303	5.77936	11
50	.10216	9.78817	.11983	8.34496	.13758	7.26873	.15540	6.43484	.17333	5.76937	10
51	.10246	9.76009	.12013	8.32446	.13787	7.25310	.15570	6.42253	.17363	5.75941	9
52	.10275	9.73217	.12042	8.30406	.13817	7.23754	.15600	6.41026	.17393	5.74949	8
53	.10305	9.70441	.12072	8.28376	.13846	7.22204	.15630	6.39804	.17423	5.73960	7
54	.10334	9.67680	.12101	8.26355	.13876	7.20661	.15660	6.38587	.17453	5.72974	6
55	.10363	9.64935	.12131	8.24345	.13906	7.19125	.15689	6.37374	.17483	5.71992	5
56	.10393	0.62205	.12160	8.22344	.13935	7.17594	.15719	6.36165	.17513	5.71013	4
57	.10422	9.59490	.12190	8.20352	.13965	7.16071	.15749	6.34961	.17543	5.70037	3
58	.10452	9.56791	.12219	8.18370	.13995	7.14553	.15779	6.33761	.17573	5.69064	2
59	.10481	9.54106	.12249	8.16398	.14024	7.13042	.15809	6.32566	.17603	5.68094	1
60	.10510	9.51436	.12278	8.14435	.14054	7.11537	.15838	6.31375	.17633	5.67128	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	84°		83°		82°		81°		80°		

′	10° Tang	10° Cotang	11° Tang	11° Cotang	12° Tang	12° Cotang	13° Tang	13° Cotang	14° Tang	14° Cotang	′
0	.17633	5.67128	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	60
1	.17663	5.66165	.19468	5.13658	.21286	4.69791	.23117	4.32573	.24964	4.00582	59
2	.17693	5.65205	.19498	5.12862	.21316	4.69121	.23148	4.32001	.24995	4.00086	58
3	.17723	5.64248	.19529	5.12069	.21347	4.68452	.23179	4.31430	.25026	3.99592	57
4	.17753	5.63295	.19559	5.11279	.21377	4.67786	.23209	4.30860	.25056	3.99099	56
5	.17783	5.62344	.19589	5.10490	.21408	4.67121	.23240	4.30291	.25087	3.98607	55
6	.17813	5.61397	.19619	5.09704	.21438	4.66458	.23271	4.29724	.25118	3.98117	54
7	.17843	5.60452	.19649	5.08921	.21469	4.65797	.23301	4.29159	.25149	3.97627	53
8	.17873	5.59511	.19680	5.08139	.21499	4.65138	.23332	4.28595	.25180	3.97139	52
9	.17903	5.58573	.19710	5.07360	.21529	4.64480	.23363	4.28032	.25211	3.96651	51
10	.17933	5.57638	.19740	5.06584	.21560	4.63825	.23393	4.27471	.25242	3.96165	50
11	.17963	5.56706	.19770	5.05809	.21590	4.63171	.23424	4.26911	.25273	3.95680	49
12	.17993	5.55777	.19801	5.05037	.21621	4.62518	.23455	4.26352	.25304	3.95196	48
13	.18023	5.54851	.19831	5.04267	.21651	4.61868	.23485	4.25795	.25335	3.94713	47
14	.18053	5.53927	.19861	5.03499	.21682	4.61219	.23516	4.25239	.25366	3.94232	46
15	.18083	5.53007	.19891	5.02734	.21712	4.60572	.23547	4.24685	.25397	3.93751	45
16	.18113	5.52090	.19921	5.01971	.21743	4.59927	.23578	4.24132	.25428	3.93271	44
17	.18143	5.51176	.19952	5.01210	.21773	4.59283	.23608	4.23580	.25459	3.92793	43
18	.18173	5.50264	.19982	5.00451	.21804	4.58641	.23639	4.23030	.25490	3.92316	42
19	.18203	5.49356	.20012	4.99695	.21834	4.58001	.23670	4.22481	.25521	3.91839	41
20	.18233	5.48451	.20042	4.98940	.21864	4.57363	.23700	4.21933	.25552	3.91364	40
21	.18263	5.47548	.20073	4.98188	.21895	4.56726	.23731	4.21387	.25583	3.90890	39
22	.18293	5.46648	.20103	4.97438	.21925	4.56091	.23762	4.20842	.25614	3.90417	38
23	.18323	5.45751	.20133	4.96690	.21956	4.55458	.23793	4.20298	.25645	3.89945	37
24	.18353	5.44857	.20164	4.95945	.21986	4.54826	.23823	4.19756	.25676	3.89474	36
25	.18384	5.43966	.20194	4.95201	.22017	4.54196	.23854	4.19215	.25707	3.89004	35
26	.18414	5.43077	.20224	4.94460	.22047	4.53568	.23885	4.18675	.25738	3.88536	34
27	.18444	5.42192	.20254	4.93721	.22078	4.52941	.23916	4.18137	.25769	3.88068	33
28	.18474	5.41309	.20285	4.92984	.22108	4.52316	.23946	4.17600	.25800	3.87601	32
29	.18504	5.40429	.20315	4.92249	.22139	4.51693	.23977	4.17064	.25831	3.87136	31
30	.18534	5.39552	.20345	4.91516	.22169	4.51071	.24008	4.16530	.25862	3.86671	30
31	.18564	5.38677	.20376	4.90785	.22200	4.50451	.24039	4.15997	.25893	3.86208	29
32	.18594	5.37805	.20406	4.90056	.22231	4.49832	.24069	4.15465	.25924	3.85745	28
33	.18624	5.36936	.20436	4.89330	.22261	4.49215	.24100	4.14934	.25955	3.85284	27
34	.18654	5.36070	.20466	4.88605	.22292	4.48600	.24131	4.14405	.25986	3.84824	26
35	.18684	5.35206	.20497	4.87882	.22322	4.47986	.24162	4.13877	.26017	3.84364	25
36	.18714	5.34345	.20527	4.87162	.22353	4.47374	.24193	4.13350	.26048	3.83906	24
37	.18745	5.33487	.20557	4.86444	.22383	4.46764	.24223	4.12825	.26079	3.83449	23
38	.18775	5.32631	.20588	4.85727	.22414	4.46155	.24254	4.12301	.26110	3.82992	22
39	.18805	5.31778	.20618	4.85013	.22444	4.45548	.24285	4.11778	.26141	3.82537	21
40	.18835	5.30928	.20648	4.84300	.22475	4.44942	.24316	4.11256	.26172	3.82083	20
41	.18865	5.30080	.20679	4.83590	.22505	4.44338	.24347	4.10736	.26203	3.81630	19
42	.18895	5.29235	.20709	4.82882	.22536	4.43735	.24377	4.10216	.26235	3.81177	18
43	.18925	5.28393	.20739	4.82175	.22567	4.43134	.24408	4.09699	.26266	3.80726	17
44	.18955	5.27553	.20770	4.81471	.22597	4.42534	.24439	4.09182	.26297	3.80276	16
45	.18986	5.26715	.20800	4.80769	.22628	4.41936	.24470	4.08666	.26328	3.79827	15
46	.19016	5.25880	.20830	4.80068	.22658	4.41340	.24501	4.08152	.26359	3.79378	14
47	.19046	5.25048	.20861	4.79370	.22689	4.40745	.24532	4.07639	.26390	3.78931	13
48	.19076	5.24218	.20891	4.78673	.22719	4.40152	.24562	4.07127	.26421	3.78485	12
49	.19106	5.23391	.20921	4.77978	.22750	4.39560	.24593	4.06616	.26452	3.78040	11
50	.19136	5.22566	.20952	4.77286	.22781	4.38969	.24624	4.06107	.26483	3.77595	10
51	.19166	5.21744	.20982	4.76595	.22811	4.38381	.24655	4.05599	.26515	3.77152	9
52	.19197	5.20925	.21013	4.75906	.22842	4.37793	.24686	4.05092	.26546	3.76709	8
53	.19227	5.20107	.21043	4.75219	.22872	4.37207	.24717	4.04586	.26577	3.76268	7
54	.19257	5.19293	.21073	4.74534	.22903	4.36623	.24747	4.04081	.26608	3.75828	6
55	.19287	5.18480	.21104	4.73851	.22934	4.36040	.24778	4.03578	.26639	3.75388	5
56	.19317	5.17671	.21134	4.73170	.22964	4.35459	.24809	4.03076	.26670	3.74950	4
57	.19347	5.16863	.21164	4.72490	.22995	4.34879	.24840	4.02574	.26701	3.74512	3
58	.19378	5.16058	.21195	4.71813	.23026	4.34300	.24871	4.02074	.26733	3.74075	2
59	.19408	5.15256	.21225	4.71137	.23056	4.33723	.24902	4.01576	.26764	3.73640	1
60	.19438	5.14455	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	79°		78°		77°		76°		75°		

′	15° Tang	15° Cotang	16° Tang	16° Cotang	17° Tang	17° Cotang	18° Tang	18° Cotang	19° Tang	19° Cotang	′
0	.26795	3.73205	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.26826	3.72771	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.26857	3.72338	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.26888	3.71907	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.26920	3.71476	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.26951	3.71046	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.26982	3.70616	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.27013	3.70188	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.27044	3.69761	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.27076	3.69335	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.27107	3.68909	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.27138	3.68485	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.27169	3.68061	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.27201	3.67638	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.27232	3.67217	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.27263	3.66796	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.27294	3.66376	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.27326	3.65957	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.27357	3.65538	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.27388	3.65121	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.27419	3.64705	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.27451	3.64289	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.27482	3.63874	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.27513	3.63461	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.27545	3.63048	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.27576	3.62636	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.27607	3.62224	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.27638	3.61814	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.27670	3.61405	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.27701	3.60996	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.27732	3.60588	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.27764	3.60181	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.27795	3.59775	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.27826	3.59370	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.27858	3.58966	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.27889	3.58562	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.27921	3.58160	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.27952	3.57758	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.27983	3.57357	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.28015	3.56957	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.28046	3.56557	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.28077	3.56159	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.28109	3.55761	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.28140	3.55364	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.28172	3.54968	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.28203	3.54573	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.28234	3.54179	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.28266	3.53785	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.28297	3.53393	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.28329	3.53001	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.28360	3.52609	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.28391	3.52219	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.28423	3.51829	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.28454	3.51441	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.28486	3.51053	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.28517	3.50666	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.28549	3.50279	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.28580	3.49894	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.28612	3.49509	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.28643	3.49125	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	74°		73°		72°		71°		70°		

′	20° Tang	20° Cotang	21° Tang	21° Cotang	22° Tang	22° Cotang	23° Tang	23° Cotang	24° Tang	24° Cotang	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	.44558	2.24428	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	.44593	2.24252	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	.44627	2.24077	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	.44662	2.23902	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	.44697	2.23727	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	.44732	2.23553	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	.44767	2.23378	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	.44802	2.23204	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	.44837	2.23030	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	.44872	2.22857	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	.44907	2.22683	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	.44942	2.22510	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	.44977	2.22337	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	.45012	2.22164	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	.45047	2.21992	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	.45082	2.21819	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	.45117	2.21647	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	.45152	2.21475	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	.45187	2.21304	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	.45222	2.21132	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	.45257	2.20961	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	.45292	2.20790	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43230	2.31271	.45327	2.20619	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	.45362	2.20449	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	.45397	2.20278	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	.45432	2.20108	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	.45467	2.19938	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	.45502	2.19769	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	.45538	2.19599	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	.45573	2.19430	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	.45608	2.19261	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	.45643	2.19092	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	.45678	2.18923	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	.45713	2.18755	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	.45748	2.18587	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	.45784	2.18419	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	.45819	2.18251	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	.45854	2.18084	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	.45889	2.17916	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	.45924	2.17749	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	.45960	2.17582	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	.45995	2.17416	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	.46030	2.17249	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	.46065	2.17083	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	.46101	2.16917	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	.46136	2.16751	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	.46171	2.16585	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	.46206	2.16420	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	.46242	2.16255	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	.46277	2.16090	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	.46312	2.15925	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	.46348	2.15760	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	.46383	2.15596	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	.46418	2.15432	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	.46454	2.15268	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	.46489	2.15104	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	.46525	2.14940	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	.46560	2.14777	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	.46595	2.14614	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	.46631	2.14451	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	69°		68°		67°		66°		65°		

′	25° Tang	25° Cotang	26° Tang	26° Cotang	27° Tang	27° Cotang	28° Tang	28° Cotang	29° Tang	29° Cotang	′
0	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	60
1	.46666	2.14288	.48809	2.04879	.50989	1.96120	.53208	1.87941	.55469	1.80281	59
2	.46702	2.14125	.48845	2.04728	.51026	1.95979	.53246	1.87809	.55507	1.80158	58
3	.46737	2.13963	.48881	2.04577	.51063	1.95838	.53283	1.87677	.55545	1.80034	57
4	.46772	2.13801	.48917	2.04426	.51099	1.95698	.53320	1.87546	.55583	1.79911	56
5	.46808	2.13639	.48953	2.04276	.51136	1.95557	.53358	1.87415	.55621	1.79788	55
6	.46843	2.13477	.48989	2.04125	.51173	1.95417	.53395	1.87283	.55659	1.79665	54
7	.46879	2.13316	.49026	2.03975	.51209	1.95277	.53432	1.87152	.55697	1.79542	53
8	.46914	2.13154	.49062	2.03825	.51246	1.95137	.53470	1.87021	.55736	1.79419	52
9	.46950	2.12993	.49098	2.03675	.51283	1.94997	.53507	1.86891	.55774	1.79296	51
10	.46985	2.12832	.49134	2.03526	.51319	1.94858	.53545	1.86760	.55812	1.79174	50
11	.47021	2.12671	.49170	2.03376	.51356	1.94718	.53582	1.86630	.55850	1.79051	49
12	.47056	2.12511	.49206	2.03227	.51393	1.94579	.53620	1.86499	.55888	1.78929	48
13	.47092	2.12350	.49242	2.03078	.51430	1.94440	.53657	1.86369	.55926	1.78807	47
14	.47128	2.12190	.49278	2.02929	.51467	1.94301	.53694	1.86239	.55964	1.78685	46
15	.47163	2.12030	.49315	2.02780	.51503	1.94162	.53732	1.86109	.56003	1.78563	45
16	.47199	2.11871	.49351	2.02631	.51540	1.94023	.53769	1.85979	.56041	1.78441	44
17	.47234	2.11711	.49387	2.02483	.51577	1.93885	.53807	1.85850	.56079	1.78319	43
18	.47270	2.11552	.49423	2.02335	.51614	1.93746	.53844	1.85720	.56117	1.78198	42
19	.47305	2.11392	.49459	2.02187	.51651	1.93608	.53882	1.85591	.56156	1.78077	41
20	.47341	2.11233	.49495	2.02039	.51688	1.93470	.53920	1.85462	.56194	1.77955	40
21	.47377	2.11075	.49532	2.01891	.51724	1.93332	.53957	1.85333	.56232	1.77834	39
22	.47412	2.10916	.49568	2.01743	.51761	1.93195	.53995	1.85204	.56270	1.77713	38
23	.47448	2.10758	.49604	2.01596	.51798	1.93057	.54032	1.85075	.56309	1.77592	37
24	.47483	2.10600	.49640	2.01449	.51835	1.92920	.54070	1.84946	.56347	1.77471	36
25	.47519	2.10442	.49677	2.01302	.51872	1.92782	.54107	1.84818	.56385	1.77351	35
26	.47555	2.10284	.49713	2.01155	.51909	1.92645	.54145	1.84689	.56424	1.77230	34
27	.47590	2.10126	.49749	2.01008	.51946	1.92508	.54183	1.84561	.56462	1.77110	33
28	.47626	2.09969	.49786	2.00862	.51983	1.92371	.54220	1.84433	.56501	1.76990	32
29	.47662	2.09811	.49822	2.00715	.52020	1.92235	.54258	1.84305	.56539	1.76869	31
30	.47698	2.09654	.49858	2.00569	.52057	1.92098	.54296	1.84177	.56577	1.76749	30
31	.47733	2.09498	.49894	2.00423	.52094	1.91962	.54333	1.84049	.56616	1.76629	29
32	.47769	2.09341	.49931	2.00277	.52131	1.91826	.54371	1.83922	.56654	1.76510	28
33	.47805	2.09184	.49967	2.00131	.52168	1.91690	.54409	1.83794	.56693	1.76390	27
34	.47840	2.09028	.50004	1.99986	.52205	1.91554	.54446	1.83667	.56731	1.76271	26
35	.47876	2.08872	.50040	1.99841	.52242	1.91418	.54484	1.83540	.56769	1.76151	25
36	.47912	2.08716	.50076	1.99695	.52279	1.91282	.54522	1.83413	.56808	1.76032	24
37	.47948	2.08560	.50113	1.99550	.52316	1.91147	.54560	1.83286	.56846	1.75913	23
38	.47984	2.08405	.50149	1.99406	.52353	1.91012	.54597	1.83159	.56885	1.75794	22
39	.48019	2.08250	.50185	1.99261	.52390	1.90876	.54635	1.83033	.56923	1.75675	21
40	.48055	2.08094	.50222	1.99116	.52427	1.90741	.54673	1.82906	.56962	1.75556	20
41	.48091	2.07939	.50258	1.98972	.52464	1.90607	.54711	1.82780	.57000	1.75437	19
42	.48127	2.07785	.50295	1.98828	.52501	1.90472	.54748	1.82654	.57039	1.75319	18
43	.48163	2.07630	.50331	1.98684	.52538	1.90337	.54786	1.82528	.57078	1.75200	17
44	.48198	2.07476	.50368	1.98540	.52575	1.90203	.54824	1.82402	.57116	1.75082	16
45	.48234	2.07321	.50404	1.98396	.52613	1.90069	.54862	1.82276	.57155	1.74964	15
46	.48270	2.07167	.50441	1.98253	.52650	1.89935	.54900	1.82150	.57193	1.74846	14
47	.48306	2.07014	.50477	1.98110	.52687	1.89801	.54938	1.82025	.57232	1.74728	13
48	.48342	2.06860	.50514	1.97966	.52724	1.89667	.54975	1.81899	.57271	1.74610	12
49	.48378	2.06706	.50550	1.97823	.52761	1.89533	.55013	1.81774	.57309	1.74492	11
50	.48414	2.06553	.50587	1.97681	.52798	1.89400	.55051	1.81649	.57348	1.74375	10
51	.48450	2.06400	.50623	1.97538	.52836	1.89266	.55089	1.81524	.57386	1.74257	9
52	.48486	2.06247	.50660	1.97395	.52873	1.89133	.55127	1.81399	.57425	1.74140	8
53	.48521	2.06094	.50696	1.97253	.52910	1.89000	.55165	1.81274	.57464	1.74022	7
54	.48557	2.05942	.50733	1.97111	.52947	1.88867	.55203	1.81150	.57503	1.73905	6
55	.48593	2.05790	.50769	1.96969	.52985	1.88734	.55241	1.81025	.57541	1.73788	5
56	.48629	2.05637	.50806	1.96827	.53022	1.88602	.55279	1.80901	.57580	1.73671	4
57	.48665	2.05485	.50843	1.96685	.53059	1.88469	.55317	1.80777	.57619	1.73555	3
58	.48701	2.05333	.50879	1.96544	.53096	1.88337	.55355	1.80653	.57657	1.73438	2
59	.48737	2.05182	.50916	1.96402	.53134	1.88205	.55393	1.80529	.57696	1.73321	1
60	.48773	2.05030	.50953	1.96261	.53171	1.88073	.55431	1.80405	.57735	1.73205	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	64°		63°		62°		61°		60°		

′	30°		31°		32°		33°		34°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.57735	1.73205	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	60
1	.57774	1.73089	.60126	1.66318	.62527	1.59930	.64982	1.53888	.67493	1.48163	59
2	.57813	1.72973	.60165	1.66209	.62568	1.59826	.65024	1.53791	.67536	1.48070	58
3	.57851	1.72857	.60205	1.66099	.62608	1.59723	.65065	1.53693	.67578	1.47977	57
4	.57890	1.72741	.60245	1.65990	.62649	1.59620	.65106	1.53595	.67620	1.47885	56
5	.57929	1.72625	.60284	1.65881	.62689	1.59517	.65148	1.53497	.67663	1.47792	55
6	.57968	1.72509	.60324	1.65772	.62730	1.59414	.65189	1.53400	.67705	1.47699	54
7	.58007	1.72393	.60364	1.65663	.62770	1.59311	.65231	1.53302	.67748	1.47607	53
8	.58046	1.72278	.60403	1.65554	.62811	1.59208	.65272	1.53205	.67790	1.47514	52
9	.58085	1.72163	.60443	1.65445	.62852	1.59105	.65314	1.53107	.67832	1.47422	51
10	.58124	1.72047	.60483	1.65337	.62892	1.59002	.65355	1.53010	.67875	1.47330	50
11	.58162	1.71932	.60522	1.65228	.62933	1.58900	.65397	1.52913	.67917	1.47238	49
12	.58201	1.71817	.60562	1.65120	.62973	1.58797	.65438	1.52816	.67960	1.47146	48
13	.58240	1.71702	.60602	1.65011	.63014	1.58695	.65480	1.52719	.68002	1.47053	47
14	.58279	1.71588	.60642	1.64903	.63055	1.58593	.65521	1.52622	.68045	1.46962	46
15	.58318	1.71473	.60681	1.64795	.63095	1.58490	.65563	1.52525	.68088	1.46870	45
16	.58357	1.71358	.60721	1.64687	.63136	1.58388	.65604	1.52429	.68130	1.46778	44
17	.58396	1.71244	.60761	1.64579	.63177	1.58286	.65646	1.52332	.68173	1.46686	43
18	.58435	1.71129	.60801	1.64471	.63217	1.58184	.65688	1.52235	.68215	1.46595	42
19	.58474	1.71015	.60841	1.64363	.63258	1.58083	.65729	1.52139	.68258	1.46503	41
20	.58513	1.70901	.60881	1.64256	.63299	1.57981	.65771	1.52043	.68301	1.46411	40
21	.58552	1.70787	.60921	1.64148	.63340	1.57879	.65813	1.51946	.68343	1.46320	39
22	.58591	1.70673	.60960	1.64041	.63380	1.57778	.65854	1.51850	.68386	1.46229	38
23	.58631	1.70560	.61000	1.63934	.63421	1.57676	.65896	1.51754	.68429	1.46137	37
24	.58670	1.70446	.61040	1.63826	.63462	1.57575	.65938	1.51658	.68471	1.46046	36
25	.58709	1.70332	.61080	1.63719	.63503	1.57474	.65980	1.51562	.68514	1.45955	35
26	.58748	1.70219	.61120	1.63612	.63544	1.57372	.66021	1.51466	.68557	1.45864	34
27	.58787	1.70106	.61160	1.63505	.63584	1.57271	.66063	1.51370	.68600	1.45773	33
28	.58826	1.69992	.61200	1.63398	.63625	1.57170	.66105	1.51275	.68642	1.45682	32
29	.58865	1.69879	.61240	1.63292	.63666	1.57069	.66147	1.51179	.68685	1.45592	31
30	.58905	1.69766	.61280	1.63185	.63707	1.56969	.66189	1.51084	.68728	1.45501	30
31	.58944	1.69653	.61320	1.63079	.63748	1.56868	.66230	1.50988	.68771	1.45410	29
32	.58983	1.69541	.61360	1.62972	.63789	1.56767	.66272	1.50893	.68814	1.45320	28
33	.59022	1.69428	.61400	1.62866	.63830	1.56667	.66314	1.50797	.68857	1.45229	27
34	.59061	1.69316	.61440	1.62760	.63871	1.56566	.66356	1.50702	.68900	1.45139	26
35	.59101	1.69203	.61480	1.62654	.63912	1.56466	.66398	1.50607	.68942	1.45049	25
36	.59140	1.69091	.61520	1.62548	.63953	1.56366	.66440	1.50512	.68985	1.44958	24
37	.59179	1.68979	.61561	1.62442	.63994	1.56265	.66482	1.50417	.69028	1.44868	23
38	.59218	1.68866	.61601	1.62336	.64035	1.56165	.66524	1.50322	.69071	1.44778	22
39	.59258	1.68754	.61641	1.62230	.64076	1.56065	.66566	1.50228	.69114	1.44688	21
40	.59297	1.68643	.61681	1.62125	.64117	1.55966	.66608	1.50133	.69157	1.44598	20
41	.59336	1.68531	.61721	1.62019	.64158	1.55866	.66650	1.50038	.69200	1.44508	19
42	.59376	1.68419	.61761	1.61914	.64199	1.55766	.66692	1.49944	.69243	1.44418	18
43	.59415	1.68308	.61801	1.61808	.64240	1.55666	.66734	1.49849	.69286	1.44329	17
44	.59454	1.68196	.61842	1.61703	.64281	1.55567	.66776	1.49755	.69329	1.44239	16
45	.59494	1.68085	.61882	1.61598	.64322	1.55467	.66818	1.49661	.69372	1.44149	15
46	.59533	1.67974	.61922	1.61493	.64363	1.55368	.66860	1.49566	.69416	1.44060	14
47	.59573	1.67863	.61962	1.61388	.64404	1.55269	.66902	1.49472	.69459	1.43970	13
48	.59612	1.67752	.62003	1.61283	.64446	1.55170	.66944	1.49378	.69502	1.43881	12
49	.59651	1.67641	.62043	1.61179	.64487	1.55071	.66986	1.49284	.69545	1.43792	11
50	.59691	1.67530	.62083	1.61074	.64528	1.54972	.67028	1.49190	.69588	1.43703	10
51	.59730	1.67419	.62124	1.60970	.64569	1.54873	.67071	1.49097	.69631	1.43614	9
52	.59770	1.67309	.62164	1.60865	.64610	1.54774	.67113	1.49003	.69675	1.43525	8
53	.59809	1.67198	.62204	1.60761	.64652	1.54675	.67155	1.48909	.69718	1.43436	7
54	.59849	1.67088	.62245	1.60657	.64693	1.54576	.67197	1.48816	.69761	1.43347	6
55	.59888	1.66978	.62285	1.60553	.64734	1.54478	.67239	1.48722	.69804	1.43258	5
56	.59928	1.66867	.62325	1.60449	.64775	1.54379	.67282	1.48629	.69847	1.43169	4
57	.59967	1.66757	.62366	1.60345	.64817	1.54281	.67324	1.48536	.69891	1.43080	3
58	.60007	1.66647	.62406	1.60241	.64858	1.54183	.67366	1.48442	.69934	1.42992	2
59	.60046	1.66538	.62446	1.60137	.64899	1.54085	.67409	1.48349	.69977	1.42903	1
60	.60086	1.66428	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	
′	59°		58°		57°		56°		55°		′

′	35°		36°		37°		38°		39°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.70021	1.42815	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.70064	1.42726	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.70107	1.42638	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.70151	1.42550	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.70194	1.42462	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.70238	1.42374	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.70281	1.42286	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.70325	1.42198	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.70368	1.42110	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.70412	1.42022	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.70455	1.41934	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.70499	1.41847	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.70542	1.41759	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.70586	1.41672	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.70629	1.41584	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.70673	1.41497	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.70717	1.41409	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.70760	1.41322	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.70804	1.41235	.73457	1.36134	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.70848	1.41148	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.70891	1.41061	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.70935	1.40974	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.70979	1.40887	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.71023	1.40800	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.71066	1.40714	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.71110	1.40627	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.71154	1.40540	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.71198	1.40454	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.71242	1.40367	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.71285	1.40281	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.71329	1.40195	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.71373	1.40109	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.71417	1.40022	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.71461	1.39936	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.71505	1.39850	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.71549	1.39764	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.71593	1.39679	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.71637	1.39593	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.71681	1.39507	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.71725	1.39421	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.71769	1.39336	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.71813	1.39250	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.71857	1.39165	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.71901	1.39079	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.71946	1.38994	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.71990	1.38909	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.72034	1.38824	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.72078	1.38738	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.72122	1.38653	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.72167	1.38568	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.72211	1.38484	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.72255	1.38399	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.72299	1.38314	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.72344	1.38229	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.72388	1.38145	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.72432	1.38060	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.72477	1.37976	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.72521	1.37891	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.72565	1.37807	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.72610	1.37722	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	.96625	1.03493	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	.96681	1.03433	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	.96738	1.03372	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	.96794	1.03312	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	.96850	1.03252	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	.96907	1.03192	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	.96963	1.03132	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	.97020	1.03072	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	.97076	1.03012	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	.97133	1.02952	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	.97189	1.02892	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	.97246	1.02832	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	.97302	1.02772	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	.97359	1.02713	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	.97416	1.02653	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	.97472	1.02593	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	.97529	1.02533	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	.97586	1.02474	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	.97643	1.02414	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	.97700	1.02355	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	.97756	1.02295	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	.97813	1.02236	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	.97870	1.02176	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	.97927	1.02117	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	.97984	1.02057	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	.98041	1.01998	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	.98098	1.01939	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	.98155	1.01879	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	.98213	1.01820	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	.98270	1.01761	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	.98327	1.01702	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	.98384	1.01642	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	.98441	1.01583	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	.98499	1.01524	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	.98556	1.01465	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	.98613	1.01406	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	.98671	1.01347	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	.98728	1.01288	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	.98786	1.01229	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	.98843	1.01170	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	.98901	1.01112	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	.98958	1.01053	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	.99016	1.00994	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	.99073	1.00935	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	.99131	1.00876	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	.99189	1.00818	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	.99247	1.00759	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	.99304	1.00701	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	.99362	1.00642	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	.99420	1.00583	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	.99478	1.00525	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	.99536	1.00467	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	.99594	1.00408	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	.99652	1.00350	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	.99710	1.00291	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	.99768	1.00233	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	.99826	1.00175	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	.99884	1.00116	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	.99942	1.00058	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	1.00000	1.00000	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		45°		

LOGARITHMIC TRIGONOMETRIC FUNCTIONS

0°

"	'	log sin	d	S	T	log tan	c. d.	log cot	C	log cos	
				4.685					5.314		
0	0									0.00000	60
60	1	4̄.46373		57	57	4̄.46373		3.53627	42	0.00000	59
120	2	4̄.76476	30103	57	57	4̄.76476	30103	3.23524	42	0.00000	58
180	3	4̄.94085	17609	57	57	4̄.94085	17609	3.05915	42	0.00000	57
240	4	3̄.06579	12494	57	58	3̄.06579	12494	2.93421	42	0.00000	56
300	5	3̄.16270	9691	57	58	3̄.16270	9691	2.83730	42	0.00000	55
360	6	3̄.24188	7918	57	58	3̄.24188	7918	2.75812	42	0.00000	54
420	7	3̄.30882	6694	57	58	3̄.30882	6694	2.69118	42	0.00000	53
480	8	3̄.36682	5800	57	58	3̄.36682	5800	2.63318	42	0.00000	52
540	9	3̄.41797	5115	57	58	3̄.41797	5115	2.58203	42	0.00000	51
600	10	3̄.46373	4576	57	58	3̄.46373	4576	2.53627	42	0.00000	50
660	11	3̄.50512	4139	57	58	3̄.50512	4139	2.49488	42	0.00000	49
720	12	3̄.54291	3779	57	58	3̄.54291	3779	2.45709	42	0.00000	48
780	13	3̄.57767	3476	57	58	3̄.57767	3476	2.42233	42	0.00000	47
840	14	3̄.60985	3218	57	58	3̄.60986	3219	2.39014	42	0.00000	46
900	15	3̄.63982	2997	57	58	3̄.63982	2996	2.36018	42	0.00000	45
960	16	3̄.66784	2802	57	58	3̄.66785	2803	2.33215	42	0.00000	44
1020	17	3̄.69417	2633	57	58	3̄.69418	2633	2.30582	42	1̄.99999	43
1080	18	3̄.71900	2483	57	58	3̄.71900	2482	2.28100	42	1̄.99999	42
1140	19	3̄.74248	2348	57	58	3̄.74248	2348	2.25752	42	1̄.99999	41
1200	20	3̄.76475	2227	57	58	3̄.76476	2228	2.23524	42	1̄.99999	40
1260	21	3̄.78594	2119	57	58	3̄.78595	2119	2.21405	42	1̄.99999	39
1320	22	3̄.80615	2021	57	58	3̄.80615	2020	2.19385	42	1̄.99999	38
1380	23	3̄.82545	1930	57	58	3̄.82546	1931	2.17454	42	1̄.99999	37
1440	24	3̄.84393	1848	57	58	3̄.84394	1848	2.15606	42	1̄.99999	36
1500	25	3̄.86166	1773	57	58	3̄.86167	1773	2.13833	42	1̄.99999	35
1560	26	3̄.87870	1704	57	58	3̄.87870	1703	2.12130	42	1̄.99999	34
1620	27	3̄.89509	1639	57	58	3̄.89509	1639	2.10491	42	1̄.99999	33
1680	28	3̄.91088	1579	57	58	3̄.91089	1580	2.08911	41	1̄.99999	32
1740	29	3̄.92612	1524	57	59	3̄.92613	1524	2.07387	41	1̄.99998	31
1800	30	3̄.94084	1472	57	59	3̄.94086	1473	2.05914	41	1̄.99998	30
1860	31	3̄.95508	1424	57	59	3̄.95510	1424	2.04490	41	1̄.99998	29
1920	32	3̄.96887	1379	57	59	3̄.96889	1379	2.03111	41	1̄.99998	28
1980	33	3̄.98223	1336	57	59	3̄.98225	1336	2.01775	41	1̄.99998	27
2040	34	3̄.99520	1297	57	59	3̄.99522	1297	2.00478	41	1̄.99998	26
2100	35	2̄.00779	1259	57	59	2̄.00781	1259	1.99219	41	1̄.99998	25
2160	36	2̄.02002	1223	57	59	2̄.02004	1223	1.97996	41	1̄.99998	24
2220	37	2̄.03192	1190	57	59	2̄.03194	1190	1.96806	41	1̄.99997	23
2280	38	2̄.04350	1158	57	59	2̄.04353	1159	1.95647	41	1̄.99997	22
2340	39	2̄.05478	1128	57	59	2̄.05481	1128	1.94519	41	1̄.99997	21
2400	40	2̄.06578	1100	57	59	2̄.06581	1100	1.93419	41	1̄.99997	20
2460	41	2̄.07650	1072	56	60	2̄.07653	1072	1.92347	40	1̄.99997	19
2520	42	2̄.08696	1046	56	60	2̄.08700	1047	1.91300	40	1̄.99997	18
2580	43	2̄.09718	1022	56	60	2̄.09722	1022	1.90278	40	1̄.99997	17
2640	44	2̄.10717	999	56	60	2̄.10720	998	1.89280	40	1̄.99996	16
2700	45	2̄.11693	976	56	60	2̄.11696	976	1.88304	40	1̄.99996	15
2760	46	2̄.12647	954	56	60	2̄.12651	955	1.87349	40	1̄.99996	14
2820	47	2̄.13581	934	56	60	2̄.13585	934	1.86415	40	1̄.99996	13
2880	48	2̄.14495	914	56	60	2̄.14500	915	1.85500	40	1̄.99996	12
2940	49	2̄.15391	896	56	60	2̄.15395	895	1.84605	40	1̄.99996	11
3000	50	2̄.16268	877	56	61	2̄.16273	878	1.83727	39	1̄.99995	10
3060	51	2̄.17128	860	56	61	2̄.17133	860	1.82867	39	1̄.99995	9
3120	52	2̄.17971	843	56	61	2̄.17976	843	1.82024	39	1̄.99995	8
3180	53	2̄.18798	827	56	61	2̄.18804	828	1.81196	39	1̄.99995	7
3240	54	2̄.19610	812	56	61	2̄.19616	812	1.80384	39	1̄.99994	6
3300	55	2̄.20407	797	56	61	2̄.20413	797	1.79587	39	1̄.99994	5
3360	56	2̄.21189	782	56	61	2̄.21195	782	1.78805	39	1̄.99994	4
3420	57	2̄.21958	769	55	61	2̄.21964	769	1.78036	39	1̄.99994	3
3480	58	2̄.22713	755	55	62	2̄.22720	756	1.77280	38	1̄.99994	2
3540	59	2̄.23456	743	55	62	2̄.23462	742	1.76538	38	1̄.99994	1
3600	60	2̄.24186	730	55	62	2̄.24192	730	1.75808	38	1̄.99993	0
				4.685					5.314		
		log cos	d	S	T	log cot	c. d.	log tan	C	log sin	'

89°

″	′	log sin	d	S	T	log tan	c. d.	log cot	C	log cos	
				6̄.685					5.314		
3600	0	2̄.24186	717	55	62	2̄.24192	718	1.75808	38	1̄.99993	60
3660	1	2̄.24903	706	55	62	2̄.24910	706	1.75090	38	1̄.99993	59
3720	2	2̄.25609	695	55	62	2̄.25616	696	1.74384	38	1̄.99993	58
3780	3	2̄.26304	684	55	62	2̄.26312	684	1.73688	38	1̄.99993	57
3840	4	2̄.26988	673	55	63	2̄.26996	673	1.73004	37	1̄.99992	56
3900	5	2̄.27661	663	55	63	2̄.27669	663	1.72331	37	1̄.99992	55
3960	6	2̄.28324	653	55	63	2̄.28332	654	1.71668	37	1̄.99992	54
4020	7	2̄.28977	644	55	63	2̄.28986	643	1.71014	37	1̄.99992	53
4080	8	2̄.29621	634	55	63	2̄.29629	634	1.70371	37	1̄.99992	52
4140	9	2̄.30255	624	55	63	2̄.30263	625	1.69737	37	1̄.99991	51
4200	10	2̄.30879	616	54	63	2̄.30888	617	1.69112	37	1̄.99991	50
4260	11	2̄.31495	608	54	64	2̄.31505	607	1.68495	36	1̄.99991	49
4320	12	2̄.32103	599	54	64	2̄.32112	599	1.67888	36	1̄.99990	48
4380	13	2̄.32702	590	54	64	2̄.32711	591	1.67289	36	1̄.99990	47
4440	14	2̄.33292	583	54	64	2̄.33302	584	1.66698	36	1̄.99990	46
4500	15	2̄.33875	575	54	64	2̄.33886	575	1.66114	36	1̄.99990	45
4560	16	2̄.34450	568	54	65	2̄.34461	568	1.65539	35	1̄.99989	44
4620	17	2̄.35018	560	54	65	2̄.35029	561	1.64971	35	1̄.99989	43
4680	18	2̄.35578	553	54	65	2̄.35590	553	1.64410	35	1̄.99989	42
4740	19	2̄.36131	547	54	65	2̄.36143	546	1.63857	35	1̄.99989	41
4800	20	2̄.36678	539	54	65	2̄.36689	540	1.63311	35	1̄.99988	40
4860	21	2̄.37217	533	53	66	2̄.37229	533	1.62771	34	1̄.99988	39
4920	22	2̄.37750	526	53	66	2̄.37762	527	1.62238	34	1̄.99988	38
4980	23	2̄.38276	520	53	66	2̄.38289	520	1.61711	34	1̄.99987	37
5040	24	2̄.38796	514	53	66	2̄.38809	514	1.61191	34	1̄.99987	36
5100	25	2̄.39310	508	53	66	2̄.39323	509	1.60677	34	1̄.99987	35
5160	26	2̄.39818	502	53	67	2̄.39832	502	1.60168	33	1̄.99986	34
5220	27	2̄.40320	496	53	67	2̄.40334	496	1.59666	33	1̄.99986	33
5280	28	2̄.40816	491	53	67	2̄.40830	491	1.59170	33	1̄.99986	32
5340	29	2̄.41307	485	53	67	2̄.41321	486	1.58679	33	1̄.99985	31
5400	30	2̄.41792	480	53	67	2̄.41807	480	1.58193	33	1̄.99985	30
5460	31	2̄.42272	474	52	68	2̄.42287	475	1.57713	32	1̄.99985	29
5520	32	2̄.42746	470	52	68	2̄.42762	470	1.57238	32	1̄.99984	28
5580	33	2̄.43216	464	52	68	2̄.43232	464	1.56768	32	1̄.99984	27
5640	34	2̄.43680	459	52	68	2̄.43696	460	1.56304	32	1̄.99984	26
5700	35	2̄.44139	455	52	69	2̄.44156	455	1.55844	31	1̄.99983	25
5760	36	2̄.44594	450	52	69	2̄.44611	450	1.55389	31	1̄.99983	24
5820	37	2̄.45044	445	52	69	2̄.45061	446	1.54939	31	1̄.99983	23
5880	38	2̄.45489	441	52	69	2̄.45507	441	1.54493	31	1̄.99982	22
5940	39	2̄.45930	436	51	69	2̄.45948	437	1.54052	31	1̄.99982	21
6000	40	2̄.46366	433	51	70	2̄.46385	432	1.53615	30	1̄.99982	20
6060	41	2̄.46799	427	51	70	2̄.46817	428	1.53183	30	1̄.99981	19
6120	42	2̄.47226	424	51	70	2̄.47245	424	1.52755	30	1̄.99981	18
6180	43	2̄.47650	419	51	70	2̄.47669	420	1.52331	30	1̄.99981	17
6240	44	2̄.48069	416	51	71	2̄.48089	416	1.51911	29	1̄.99980	16
6300	45	2̄.48485	411	51	71	2̄.48505	412	1.51495	29	1̄.99980	15
6360	46	2̄.48896	408	51	71	2̄.48917	408	1.51083	29	1̄.99979	14
6420	47	2̄.49304	404	50	72	2̄.49325	404	1.50675	28	1̄.99979	13
6480	48	2̄.49708	400	50	72	2̄.49729	401	1.50271	28	1̄.99979	12
6540	49	2̄.50108	396	50	72	2̄.50130	397	1.49870	28	1̄.99978	11
6600	50	2̄.50504	393	50	72	2̄.50527	393	1.49473	28	1̄.99978	10
6660	51	2̄.50897	390	50	73	2̄.50920	390	1.49080	27	1̄.99977	9
6720	52	2̄.51287	386	50	73	2̄.51310	386	1.48690	27	1̄.99977	8
6780	53	2̄.51673	382	50	73	2̄.51696	383	1.48304	27	1̄.99977	7
6840	54	2̄.52055	379	50	73	2̄.52079	380	1.47921	27	1̄.99976	6
6900	55	2̄.52434	376	49	74	2̄.52459	376	1.47541	26	1̄.99976	5
6960	56	2̄.52810	373	49	74	2̄.52835	373	1.47165	26	1̄.99975	4
7020	57	2̄.53183	369	49	74	2̄.53208	370	1.46792	26	1̄.99975	3
7080	58	2̄.53552	367	49	75	2̄.53578	367	1.46422	25	1̄.99974	2
7140	59	2̄.53919	363	49	75	2̄.53945	363	1.46055	25	1̄.99974	1
7200	60	2̄.54282		49	75	2̄.54308		1.45692	25	1̄.99974	0
				6̄.685					5.314		
		log cos	d	S	T	log cot	c. d.	log tan	C	log sin	′

2°

	'	log sin	d	S	T	log tan	c. d.	log cot	C	log cos	'
				6.085					5.314		
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	60
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	59
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	58
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	56
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	55
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	54
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	53
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	52
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	51
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	50
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	49
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	48
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	47
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	46
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	45
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	44
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	43
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	42
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	41
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	40
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18	1.99963	39
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18	1.99963	38
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17	1.99962	37
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17	1.99962	36
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17	1.99961	35
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	[illegible]	16	1.99961	34
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	[illegible]	16	1.99960	33
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	1.36574	16	1.99960	32
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	[illegible]	15	1.99959	31
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	[illegible]	15	1.99959	30
[illegible]	[illegible]	[illegible]	[illegible]	44	[illegible]	[illegible]	[illegible]	1.35792	15	1.99958	29
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14	1.99958	28
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14	1.99957	27
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13	1.99956	26
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13	1.99956	25
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13	1.99955	24
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	276	1.34007	12	1.99955	23
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	274	1.33731	12	1.99954	22
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	273	1.33457	12	1.99954	21
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	271	1.33184	11	1.99953	20
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	269	1.32913	11	1.99952	19
[illegible]	[illegible]	[illegible]	[illegible]	41	90	[illegible]	268	1.32644	10	1.99952	18
[illegible]	[illegible]	[illegible]	[illegible]	41	90	[illegible]	266	1.32376	10	1.99951	17
[illegible]	[illegible]	[illegible]	[illegible]	41	90	[illegible]	264	1.32110	10	1.99951	16
[illegible]	[illegible]	[illegible]	[illegible]	41	91	[illegible]	263	1.31846	09	1.99950	15
[illegible]	[illegible]	[illegible]	[illegible]	41	91	[illegible]	261	1.31583	09	1.99949	14
[illegible]	[illegible]	[illegible]	[illegible]	40	92	[illegible]	260	1.31322	08	1.99949	13
[illegible]	[illegible]	[illegible]	[illegible]	40	92	[illegible]	258	1.31062	08	1.99948	12
[illegible]	[illegible]	[illegible]	[illegible]	40	92	[illegible]	257	1.30804	08	1.99948	11
[illegible]	[illegible]	[illegible]	[illegible]	40	93	[illegible]	255	1.30547	07	1.99947	10
[illegible]	[illegible]	[illegible]	[illegible]	40	93	[illegible]	254	1.30292	07	1.99946	9
[illegible]	[illegible]	[illegible]	[illegible]	39	94	[illegible]	252	1.30038	06	1.99946	8
[illegible]	[illegible]	[illegible]	[illegible]	39	94	2.70214	251	1.29786	06	1.99945	7
[illegible]	[illegible]	[illegible]	[illegible]	39	95	[illegible]	249	1.29535	05	1.99944	6
[illegible]	[illegible]	[illegible]	[illegible]	39	95	2.70714	248	1.29286	05	1.99944	5
[illegible]	[illegible]	[illegible]	[illegible]	39	95	[illegible]	246	1.29038	05	1.99943	4
[illegible]	[illegible]	[illegible]	[illegible]	38	96	[illegible]	245	1.28792	04	1.99942	3
[illegible]	[illegible]	[illegible]	[illegible]	38	96	[illegible]	244	1.28547	04	1.99942	2
[illegible]	[illegible]	[illegible]	[illegible]	38	97	[illegible]	243	1.28303	03	1.99941	1
[illegible]	[illegible]	2.71880		38	97	2.71940		1.28060	03	1.99940	0
				6.085					5.314		
		log cos	d	S	T	log cot	c. d.	log tan	C	log sin	'

′	log sin	d	log tan	c. d.	log cot	log cos	
0	$\bar{3}.71880$		$\bar{3}.71940$		1.28060	$\bar{1}.99940$	60
1	$\bar{3}.72120$	240	$\bar{3}.72181$	241	1.27819	$\bar{1}.99940$	59
2	$\bar{3}.72359$	239	$\bar{3}.72420$	239	1.27580	$\bar{1}.99939$	58
3	$\bar{3}.72597$	238	$\bar{3}.72659$	239	1.27341	$\bar{1}.99938$	57
4	$\bar{3}.72834$	237	$\bar{3}.72896$	237	1.27104	$\bar{1}.99938$	56
5	$\bar{3}.73069$	235	$\bar{3}.73132$	236	1.26868	$\bar{1}.99937$	55
6	$\bar{3}.73303$	234	$\bar{3}.73366$	234	1.26634	$\bar{1}.99936$	54
7	$\bar{3}.73535$	232	$\bar{3}.73600$	234	1.26400	$\bar{1}.99936$	53
8	$\bar{3}.73767$	232	$\bar{3}.73832$	232	1.26168	$\bar{1}.99935$	52
9	$\bar{3}.73997$	230	$\bar{3}.74063$	231	1.25937	$\bar{1}.99934$	51
10	$\bar{3}.74226$	229	$\bar{3}.74292$	229	1.25708	$\bar{1}.99934$	50
		228		229			
11	$\bar{3}.74454$		$\bar{3}.74521$		1.25479	$\bar{1}.99933$	49
12	$\bar{3}.74680$	226	$\bar{3}.74748$	227	1.25252	$\bar{1}.99932$	48
13	$\bar{3}.74906$	226	$\bar{3}.74974$	226	1.25026	$\bar{1}.99932$	47
14	$\bar{3}.75130$	224	$\bar{3}.75199$	225	1.24801	$\bar{1}.99931$	46
15	$\bar{3}.75353$	223	$\bar{3}.75423$	224	1.24577	$\bar{1}.99930$	45
16	$\bar{3}.75575$	222	$\bar{3}.75645$	222	1.24355	$\bar{1}.99929$	44
17	$\bar{3}.75795$	220	$\bar{3}.75867$	222	1.24133	$\bar{1}.99929$	43
18	$\bar{3}.76015$	220	$\bar{3}.76087$	220	1.23913	$\bar{1}.99928$	42
19	$\bar{3}.76234$	219	$\bar{3}.76306$	219	1.23694	$\bar{1}.99927$	41
20	$\bar{3}.76451$	217	$\bar{3}.76525$	219	1.23475	$\bar{1}.99926$	40
		216		217			
21	$\bar{3}.76667$		$\bar{3}.76742$		1.23258	$\bar{1}.99926$	39
22	$\bar{3}.76883$	216	$\bar{3}.76958$	216	1.23042	$\bar{1}.99925$	38
23	$\bar{3}.77097$	214	$\bar{3}.77173$	215	1.22827	$\bar{1}.99924$	37
24	$\bar{3}.77310$	213	$\bar{3}.77387$	214	1.22613	$\bar{1}.99923$	36
25	$\bar{3}.77522$	212	$\bar{3}.77600$	213	1.22400	$\bar{1}.99923$	35
26	$\bar{3}.77733$	211	$\bar{3}.77811$	211	1.22189	$\bar{1}.99922$	34
27	$\bar{3}.77943$	210	$\bar{3}.78022$	211	1.21978	$\bar{1}.99921$	33
28	$\bar{3}.78152$	209	$\bar{3}.78232$	210	1.21768	$\bar{1}.99920$	32
29	$\bar{3}.78360$	208	$\bar{3}.78441$	209	1.21559	$\bar{1}.99920$	31
30	$\bar{3}.78568$	208	$\bar{3}.78649$	208	1.21351	$\bar{1}.99919$	30
		206		206			
31	$\bar{3}.78774$		$\bar{3}.78855$		1.21145	$\bar{1}.99918$	29
32	$\bar{3}.78979$	205	$\bar{3}.79061$	206	1.20939	$\bar{1}.99917$	28
33	$\bar{3}.79183$	204	$\bar{3}.79266$	205	1.20734	$\bar{1}.99917$	27
34	$\bar{3}.79386$	203	$\bar{3}.79470$	204	1.20530	$\bar{1}.99916$	26
35	$\bar{3}.79588$	202	$\bar{3}.79673$	203	1.20327	$\bar{1}.99915$	25
36	$\bar{3}.79789$	201	$\bar{3}.79875$	202	1.20125	$\bar{1}.99914$	24
37	$\bar{3}.79990$	201	$\bar{3}.80076$	201	1.19924	$\bar{1}.99913$	23
38	$\bar{3}.80189$	199	$\bar{3}.80277$	201	1.19723	$\bar{1}.99913$	22
39	$\bar{3}.80388$	199	$\bar{3}.80476$	199	1.19524	$\bar{1}.99912$	21
40	$\bar{3}.80585$	197	$\bar{3}.80674$	198	1.19326	$\bar{1}.99911$	20
		197		198			
41	$\bar{3}.80782$		$\bar{3}.80872$		1.19128	$\bar{1}.99910$	19
42	$\bar{3}.80978$	196	$\bar{3}.81068$	196	1.18932	$\bar{1}.99909$	18
43	$\bar{3}.81173$	195	$\bar{3}.81264$	196	1.18736	$\bar{1}.99909$	17
44	$\bar{3}.81367$	194	$\bar{3}.81459$	195	1.18541	$\bar{1}.99908$	16
45	$\bar{3}.81560$	193	$\bar{3}.81653$	194	1.18347	$\bar{1}.99907$	15
46	$\bar{3}.81752$	192	$\bar{3}.81846$	193	1.18154	$\bar{1}.99906$	14
47	$\bar{3}.81944$	192	$\bar{3}.82038$	192	1.17962	$\bar{1}.99905$	13
48	$\bar{3}.82134$	190	$\bar{3}.82230$	192	1.17770	$\bar{1}.99904$	12
49	$\bar{3}.82324$	190	$\bar{3}.82420$	190	1.17580	$\bar{1}.99904$	11
50	$\bar{3}.82513$	189	$\bar{3}.82610$	190	1.17390	$\bar{1}.99903$	10
		188		189			
51	$\bar{3}.82701$		$\bar{3}.82799$		1.17201	$\bar{1}.99902$	9
52	$\bar{3}.82888$	187	$\bar{3}.82987$	188	1.17013	$\bar{1}.99901$	8
53	$\bar{3}.83075$	187	$\bar{3}.83175$	188	1.16825	$\bar{1}.99900$	7
54	$\bar{3}.83261$	186	$\bar{3}.83361$	186	1.16639	$\bar{1}.99899$	6
55	$\bar{3}.83446$	185	$\bar{3}.83547$	186	1.16453	$\bar{1}.99898$	5
56	$\bar{3}.83630$	184	$\bar{3}.83732$	185	1.16268	$\bar{1}.99898$	4
57	$\bar{3}.83813$	183	$\bar{3}.83916$	184	1.16084	$\bar{1}.99897$	3
58	$\bar{3}.83996$	183	$\bar{3}.84100$	184	1.15900	$\bar{1}.99896$	2
59	$\bar{3}.84177$	181	$\bar{3}.84282$	182	1.15718	$\bar{1}.99895$	1
60	$\bar{3}.84358$	181	$\bar{3}.84464$	182	1.15536	$\bar{1}.99894$	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	238	234	229	
6	23.8	23.4	22.9	
7	27.8	27.3	26.7	
8	31.7	31.2	30.5	
9	35.7	35.1	34.4	
10	39.7	39.0	38.2	
20	79.3	78.0	76.3	
30	119.0	117.0	114.5	
40	158.7	156.0	152.7	
50	198.3	195.0	190.8	
	225	220	216	
6	22.5	22.0	21.6	
7	26.3	25.7	25.2	
8	30.0	29.3	28.8	
9	33.8	33.0	32.4	
10	37.5	36.7	36.0	
20	75.0	73.3	72.0	
30	112.5	110.0	108.0	
40	150.0	146.7	144.0	
50	187.5	183.3	180.0	
	212	208	204	
6	21.2	20.8	20.4	
7	24.7	24.3	23.8	
8	28.3	27.7	27.2	
9	31.8	31.2	30.6	
10	35.3	34.7	34.0	
20	70.7	69.3	68.0	
30	106.0	104.0	102.0	
40	141.3	138.7	136.0	
50	176.7	173.3	170.0	
	201	197	193	
6	20.1	19.7	19.3	
7	23.5	23.0	22.5	
8	26.8	26.3	25.7	
9	30.2	29.6	29.0	
10	33.5	32.8	32.2	
20	67.0	65.7	64.3	
30	100.5	98.5	96.5	
40	134.0	131.3	128.7	
50	167.5	164.2	160.8	
	189	185	181	
6	18.9	18.5	18.1	
7	22.1	21.6	21.1	
8	25.2	24.7	24.1	
9	28.4	27.8	27.2	
10	31.5	30.8	30.2	
20	63.0	61.7	60.3	
30	94.5	92.5	90.5	
40	126.0	123.3	120.7	
50	157.5	154.2	150.8	
	4	3	2	1
6	0.4	0.3	0.2	0.1
7	0.5	0.4	0.2	0.1
8	0.5	0.4	0.3	0.1
9	0.6	0.5	0.3	0.2
10	0.7	0.5	0.3	0.2
20	1.3	1.0	0.7	0.3
30	2.0	1.5	1.0	0.5
40	2.7	2.0	1.3	0.7
50	3.3	2.5	1.7	0.8

p. p.

	155	153	151
[illegible]	[illegible]	[illegible]	15.1
[illegible]	[illegible]	17.5	17.6
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	22.7
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
	148	147	1
[illegible]	14.6	14.7	0.1
[illegible]	17.4	17.2	0.1
[illegible]	19.5	19.6	0.1
[illegible]	22.4	22.1	0.2
[illegible]	24.5	24.5	0.2
[illegible]	[illegible]	[illegible]	0.3
[illegible]	74.5	73.5	0.5
[illegible]	99.3	[illegible]	0.7
50	124.2	122.5	0.8
	D. D.		

′	log sin	d	log tan	c. d.	log cot	log cos	
0	$\bar{2}.94030$	144	$\bar{2}.94195$	145	1.05805	$\bar{1}.99834$	60
1	$\bar{2}.94174$	143	$\bar{2}.94340$	145	1.05660	$\bar{1}.99833$	59
2	$\bar{2}.94317$	144	$\bar{2}.94485$	145	1.05515	$\bar{1}.99832$	58
3	$\bar{2}.94461$	142	$\bar{2}.94630$	143	1.05370	$\bar{1}.99831$	57
4	$\bar{2}.94603$	143	$\bar{2}.94773$	144	1.05227	$\bar{1}.99830$	56
5	$\bar{2}.94746$	141	$\bar{2}.94917$	143	1.05083	$\bar{1}.99829$	55
6	$\bar{2}.94887$	142	$\bar{2}.95060$	142	1.04940	$\bar{1}.99828$	54
7	$\bar{2}.95029$	141	$\bar{2}.95202$	142	1.04798	$\bar{1}.99827$	53
8	$\bar{2}.95170$	140	$\bar{2}.95344$	142	1.04656	$\bar{1}.99825$	52
9	$\bar{2}.95310$	140	$\bar{2}.95486$	141	1.04514	$\bar{1}.99824$	51
10	$\bar{2}.95450$		$\bar{2}.95627$		1.04373	$\bar{1}.99823$	50
		139		140			
11	$\bar{2}.95589$	139	$\bar{2}.95767$	141	1.04233	$\bar{1}.99822$	49
12	$\bar{2}.95728$	139	$\bar{2}.95908$	139	1.04092	$\bar{1}.99821$	48
13	$\bar{2}.95867$	138	$\bar{2}.96047$	140	1.03953	$\bar{1}.99820$	47
14	$\bar{2}.96005$	138	$\bar{2}.96187$	138	1.03813	$\bar{1}.99819$	46
15	$\bar{2}.96143$	137	$\bar{2}.96325$	139	1.03675	$\bar{1}.99817$	45
16	$\bar{2}.96280$	137	$\bar{2}.96464$	138	1.03536	$\bar{1}.99816$	44
17	$\bar{2}.96417$	136	$\bar{2}.96602$	137	1.03398	$\bar{1}.99815$	43
18	$\bar{2}.96553$	136	$\bar{2}.96739$	138	1.03261	$\bar{1}.99814$	42
19	$\bar{2}.96689$	136	$\bar{2}.96877$	136	1.03123	$\bar{1}.99813$	41
20	$\bar{2}.96825$		$\bar{2}.97013$		1.02987	$\bar{1}.99812$	40
		135		137			
21	$\bar{2}.96960$	135	$\bar{2}.97150$	135	1.02850	$\bar{1}.99810$	39
22	$\bar{2}.97095$	134	$\bar{2}.97285$	136	1.02715	$\bar{1}.99809$	38
23	$\bar{2}.97229$	134	$\bar{2}.97421$	135	1.02579	$\bar{1}.99808$	37
24	$\bar{2}.97363$	133	$\bar{2}.97556$	135	1.02444	$\bar{1}.99807$	36
25	$\bar{2}.97496$	133	$\bar{2}.97691$	134	1.02309	$\bar{1}.99806$	35
26	$\bar{2}.97629$	133	$\bar{2}.97825$	134	1.02175	$\bar{1}.99804$	34
27	$\bar{2}.97762$	132	$\bar{2}.97959$	133	1.02041	$\bar{1}.99803$	33
28	$\bar{2}.97894$	132	$\bar{2}.98092$	133	1.01908	$\bar{1}.99802$	32
29	$\bar{2}.98026$	131	$\bar{2}.98225$	133	1.01775	$\bar{1}.99801$	31
30	$\bar{2}.98157$		$\bar{2}.98358$		1.01642	$\bar{1}.99800$	30
		131		132			
31	$\bar{2}.98288$	131	$\bar{2}.98490$	132	1.01510	$\bar{1}.99798$	29
32	$\bar{2}.98419$	130	$\bar{2}.98622$	131	1.01378	$\bar{1}.99797$	28
33	$\bar{2}.98549$	130	$\bar{2}.98753$	131	1.01247	$\bar{1}.99796$	27
34	$\bar{2}.98679$	129	$\bar{2}.98884$	131	1.01116	$\bar{1}.99795$	26
35	$\bar{2}.98808$	129	$\bar{2}.99015$	130	1.00985	$\bar{1}.99793$	25
36	$\bar{2}.98937$	129	$\bar{2}.99145$	130	1.00855	$\bar{1}.99792$	24
37	$\bar{2}.99066$	128	$\bar{2}.99275$	130	1.00725	$\bar{1}.99791$	23
38	$\bar{2}.99194$	128	$\bar{2}.99405$	129	1.00595	$\bar{1}.99790$	22
39	$\bar{2}.99322$	128	$\bar{2}.99534$	128	1.00466	$\bar{1}.99788$	21
40	$\bar{2}.99450$		$\bar{2}.99662$		1.00338	$\bar{1}.99787$	20
		127		129			
41	$\bar{2}.99577$	127	$\bar{2}.99791$	128	1.00209	$\bar{1}.99786$	19
42	$\bar{2}.99704$	126	$\bar{2}.99919$	127	1.00081	$\bar{1}.99785$	18
43	$\bar{2}.99830$	126	$\bar{1}.00046$	128	0.99954	$\bar{1}.99783$	17
44	$\bar{2}.99956$	126	$\bar{1}.00174$	127	0.99826	$\bar{1}.99782$	16
45	$\bar{1}.00082$	125	$\bar{1}.00301$	126	0.99699	$\bar{1}.99781$	15
46	$\bar{1}.00207$	125	$\bar{1}.00427$	126	0.99573	$\bar{1}.99780$	14
47	$\bar{1}.00332$	124	$\bar{1}.00553$	126	0.99447	$\bar{1}.99778$	13
48	$\bar{1}.00456$	125	$\bar{1}.00679$	126	0.99321	$\bar{1}.99777$	12
49	$\bar{1}.00581$	123	$\bar{1}.00805$	125	0.99195	$\bar{1}.99776$	11
50	$\bar{1}.00704$		$\bar{1}.00930$		0.99070	$\bar{1}.99775$	10
		124		125			
51	$\bar{1}.00828$	123	$\bar{1}.01055$	124	0.98945	$\bar{1}.99773$	9
52	$\bar{1}.00951$	123	$\bar{1}.01179$	124	0.98821	$\bar{1}.99772$	8
53	$\bar{1}.01074$	122	$\bar{1}.01303$	124	0.98697	$\bar{1}.99771$	7
54	$\bar{1}.01196$	122	$\bar{1}.01427$	123	0.98573	$\bar{1}.99769$	6
55	$\bar{1}.01318$	122	$\bar{1}.01550$	123	0.98450	$\bar{1}.99768$	5
56	$\bar{1}.01440$	121	$\bar{1}.01673$	123	0.98327	$\bar{1}.99767$	4
57	$\bar{1}.01561$	121	$\bar{1}.01796$	122	0.98204	$\bar{1}.99765$	3
58	$\bar{1}.01682$	121	$\bar{1}.01918$	122	0.98082	$\bar{1}.99764$	2
59	$\bar{1}.01803$	120	$\bar{1}.02040$	122	0.97960	$\bar{1}.99763$	1
60	$\bar{1}.01923$		$\bar{1}.02162$		0.97838	$\bar{1}.99761$	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	145	143	141
6	14.5	14.3	14.1
7	16.9	16.7	16.5
8	19.3	19.1	18.8
9	21.8	21.5	21.2
10	24.2	23.8	23.5
20	48.3	47.7	47.0
30	72.5	71.5	70.5
40	96.7	95.3	94.0
50	120.8	119.2	117.5

	139	138	136
6	13.9	13.8	13.6
7	16.2	16.1	15.9
8	18.5	18.4	18.1
9	20.9	20.7	20.4
10	23.2	23.0	22.7
20	46.3	46.0	45.3
30	69.5	69.0	68.0
40	92.7	92.0	90.7
50	115.8	115.0	113.3

	135	133	131
6	13.5	13.3	13.1
7	15.8	15.5	15.3
8	18.0	17.7	17.5
9	20.3	20.0	19.7
10	22.5	22.2	21.8
20	45.0	44.3	43.7
30	67.5	66.5	65.5
40	90.0	88.7	87.3
50	112.5	110.8	109.2

	129	128	126
6	12.9	12.8	12.6
7	15.1	14.9	14.7
8	17.2	17.1	16.8
9	19.4	19.2	18.9
10	21.5	21.3	21.0
20	43.0	42.7	42.0
30	64.5	64.0	63.0
40	86.0	85.3	84.0
50	107.5	106.7	105.0

	125	123	122
6	12.5	12.3	12.2
7	14.6	14.4	14.2
8	16.7	16.4	16.3
9	18.8	18.5	18.3
10	20.8	20.5	20.3
20	41.7	41.0	40.7
30	62.5	61.5	61.0
40	83.3	82.0	81.3
50	104.2	102.5	101.7

	121	120	1
6	12.1	12.0	0.1
7	14.1	14.0	0.1
8	16.1	16.0	0.1
9	18.2	18.0	0.2
10	20.2	20.0	0.2
20	40.3	40.0	0.3
30	60.5	60.0	0.5
40	80.7	80.0	0.7
50	100.8	100.0	0.8

p. p.

′	log sin	d	log tan	c. d	log cot	log cos	′
0	[illegible]		[illegible]		[illegible]	[illegible]	60
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	59
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	58
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	56
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	55
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	54
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	53
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	52
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	51
10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	50
11	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	49
12	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	48
13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	47
14	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	46
15	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	45
16	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	44
17	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	43
18	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	42
19	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	41
20	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	40
21	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	39
22	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	38
23	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	37
24	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	36
25	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	35
26	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	34
27	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	33
28	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	32
29	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	31
30	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	30
31	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	29
32	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28
33	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	27
34	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	26
35	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	25
36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	24
37	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	23
38	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	22
39	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21
40	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99715	20
41	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99712	19
42	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18
43	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17
44	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99705	16
45	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	15
46	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99698	14
47	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99695	13
48	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.99693	12
49	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11
50	[illegible]	106	[illegible]	107	[illegible]	1̄.99690	10
51	[illegible]	105	[illegible]	[illegible]	[illegible]	[illegible]	9
52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8
53	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7
54	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6
55	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5
56	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4
57	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3
58	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2
59	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1
60	1̄.08589	103	1̄.08914	104	0.91086	1̄.99675	0
	log cos	d	log cot	c. d	log tan	log sin	′

p. p.

	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]
20	[illegible]	[illegible]	[illegible]
30	[illegible]	[illegible]	[illegible]
40	[illegible]	[illegible]	[illegible]
50	[illegible]	[illegible]	[illegible]
	118	117	116
6	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]
20	[illegible]	[illegible]	[illegible]
30	[illegible]	[illegible]	[illegible]
40	[illegible]	[illegible]	[illegible]
50	[illegible]	[illegible]	[illegible]
	115	114	113
6	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	[illegible]
20	[illegible]	[illegible]	[illegible]
30	[illegible]	[illegible]	[illegible]
40	[illegible]	[illegible]	[illegible]
50	[illegible]	[illegible]	[illegible]
	112	111	110
6	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]
8	[illegible]	[illegible]	[illegible]
9	[illegible]	[illegible]	[illegible]
10	[illegible]	[illegible]	18.3
20	[illegible]	37.0	36.7
30	[illegible]	55.5	55.0
40	[illegible]	74.0	[illegible]
50	93.3	92.5	91.7
	109	108	107
6	[illegible]	10.8	10.7
7	[illegible]	12.6	12.5
8	[illegible]	14.4	14.3
9	[illegible]	[illegible]	16.1
10	[illegible]	18.0	17.8
20	36.3	36.0	35.7
30	54.5	54.0	53.5
40	[illegible]	72.0	71.3
50	90.8	90.0	89.2
	106	105	104
6	10.6	10.5	10.4
7	12.4	12.3	12.1
8	14.1	14.0	13.9
9	15.9	15.8	15.6
10	17.7	17.5	17.3
20	35.3	35.0	34.7
30	53.0	52.5	52.0
40	70.7	70.0	69.3
50	88.3	87.5	86.7

′	log sin	d	log tan	c. d.	log cot	log cos	
0	1̄.08589	103	1̄.08914	105	0.91086	1̄.99675	60
1	1̄.08692	103	1̄.09019	104	0.90981	1̄.99674	59
2	1̄.08795	102	1̄.09123	104	0.90877	1̄.99672	58
3	1̄.08897	102	1̄.09227	103	0.90773	1̄.99670	57
4	1̄.08999	102	1̄.09330	104	0.90670	1̄.99669	56
5	1̄.09101	101	1̄.09434	103	0 90566	1̄.99667	55
6	1̄.09202	102	1̄.09537	103	0.90463	1̄.99666	54
7	1̄.09304	101	1̄.09640	102	0.90360	1̄.99664	53
8	1̄.09405	101	1̄.09742	103	0.90258	1̄.99663	52
9	1̄.09506	100	1̄.09845	102	0.90155	1̄.99661	51
10	1̄.09606	101	1̄.09947	102	0.90053	1̄.99659	50
11	1̄.09707	100	1̄.10049	101	0.89951	1̄.99658	49
12	1̄.09807	100	1̄.10150	102	0.89850	1̄.99656	48
13	1̄.09907	99	1̄.10252	101	0.89748	1̄.99655	47
14	1̄.10006	100	1̄.10353	101	0.89647	1̄.99653	46
15	1̄.10106	99	1̄.10454	101	0.89546	1̄.99651	45
16	1̄.10205	99	1̄.10555	101	0.89445	1̄.99650	44
17	1̄.10304	98	1̄.10656	100	0.89344	1̄.99648	43
18	1̄.10402	99	1̄.10756	100	0.89244	1̄.99647	42
19	1̄.10501	98	1̄.10856	100	0.89144	1̄.99645	41
20	1̄.10599	98	1̄.10956	100	0.89044	1̄.99643	40
21	1̄.10697	98	1̄.11056	99	0.88944	1̄.99642	39
22	1̄.10795	98	1̄.11155	99	0.88845	1̄.99640	38
23	1̄.10893	97	1̄.11254	99	0.88746	1̄.99638	37
24	1̄.10990	97	1̄.11353	99	0.88647	1̄.99637	36
25	1̄.11087	97	1̄.11452	99	0.88548	1̄.99635	35
26	1̄.11184	97	1̄.11551	98	0.88449	1̄.99633	34
27	1̄.11281	96	1̄.11649	98	0.88351	1̄.99632	33
28	1̄.11377	97	1̄.11747	98	0.88253	1̄.99630	32
29	1̄.11474	96	1̄.11845	98	0.88155	1̄.99629	31
30	1̄.11570	96	1̄.11943	97	0.88057	1̄.99627	30
31	1̄.11666	95	1̄.12040	98	0.87960	1̄.99625	29
32	1̄.11761	96	1̄.12138	97	0.87862	1̄.99624	28
33	1̄.11857	95	1̄.12235	97	0.87765	1̄.99622	27
34	1̄.11952	95	1̄.12332	96	0.87668	1̄.99620	26
35	1̄.12047	95	1̄.12428	97	0.87572	1̄.99618	25
36	1̄.12142	94	1̄.12525	96	0.87475	1̄.99617	24
37	1̄.12236	95	1̄.12621	96	0.87379	1̄.99615	23
38	1̄.12331	94	1̄.12717	96	0.87283	1̄.99613	22
39	1̄.12425	94	1̄.12813	96	0.87187	1̄.99612	21
40	1̄.12519	93	1̄.12909	95	0.87091	1̄.99610	20
41	1̄.12612	94	1̄.13004	95	0.86996	1̄.99608	19
42	1̄.12706	93	1̄.13099	95	0.86901	1̄.99607	18
43	1̄.12799	93	1̄.13194	95	0.86806	1̄.99605	17
44	1̄.12892	93	1̄.13289	95	0.86711	1̄.99603	16
45	1̄.12985	93	1̄.13384	94	0.86616	1̄.99601	15
46	1̄.13078	93	1̄.13478	95	0.86522	1̄.99600	14
47	1̄.13171	92	1̄.13573	94	0.86427	1̄.99598	13
48	1̄.13263	92	1̄.13667	94	0.86333	1̄.99596	12
49	1̄.13355	92	1̄.13761	93	0.86239	1̄.99595	11
50	1̄.13447	92	1̄.13854	94	0.86146	1̄.99593	10
51	1̄.13539	91	1̄.13948	93	0.86052	1̄.99591	9
52	1̄.13630	92	1̄.14041	93	0.85959	1̄.99589	8
53	1̄.13722	91	1̄.14134	93	0.85866	1̄.99588	7
54	1̄.13813	91	1̄.14227	93	0.85773	1̄.99586	6
55	1̄.13904	90	1̄.14320	92	0.85680	1̄.99584	5
56	1̄.13994	91	1̄.14412	92	0.85588	1̄.99582	4
57	1̄.14085	90	1̄.14504	93	0.85496	1̄.99581	3
58	1̄.14175	91	1̄.14597	91	0.85403	1̄.99579	2
59	1̄.14266	90	1̄.14688	92	0.85312	1̄.99577	1
60	1̄.14356		1̄.14780		0.85220	1̄.99575	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	105	104	103
6	10.5	10.4	10.3
7	12.3	12.1	12.0
8	14.0	13.9	13.7
9	15.8	15.6	15.5
10	17.5	17.3	17.2
20	35.0	34.7	34.3
30	52.5	52.0	51.5
40	70.0	69.3	68.7
50	87.5	86.7	85.8

	102	101	100
6	10.2	10.1	10.0
7	11.9	11.8	11.7
8	13.6	13.5	13.3
9	15.3	15.2	15.0
10	17.0	16.8	16.7
20	34.0	33.7	33.3
30	51.0	50.5	50.0
40	68.0	67.3	66.7
50	85.0	84.2	83.3

	99	98
6	9.9	9.8
7	11.6	11.4
8	13.2	13.1
9	14.9	14.7
10	16.5	16.3
20	33.0	32.7
30	49.5	49.0
40	66.0	65.3
50	82.5	81.7

	97	96	95
6	9.7	9.6	9.5
7	11.3	11.2	11.1
8	12.9	12.8	12.7
9	14.6	14.4	14.3
10	16.2	16.0	15.8
20	32.3	32.0	31.7
30	48.5	48.0	47.5
40	64.7	64.0	63.3
50	80.8	80.0	79.2

	94	93	92
6	9.4	9.3	9.2
7	11.0	10.9	10.7
8	12.5	12.4	12.3
9	14.1	14.0	13.8
10	15.7	15.5	15.3
20	31.3	31.0	30.7
30	47.0	46.5	46.0
40	62.7	62.0	61.3
50	78.3	77.5	76.7

	91	90	2
6	9.1	9.0	0.2
7	10.6	10.5	0.2
8	12.1	12.0	0.3
9	13.7	13.5	0.3
10	15.2	15.0	0.3
20	30.3	30.0	0.7
30	45.5	45.0	1.0
40	60.7	60.0	1.3
50	75.8	75.0	1.7

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	
0	$\bar{1}$.19433	80	$\bar{1}$.19971	82	0.80029	$\bar{1}$.99462	60
1	$\bar{1}$.19513	79	$\bar{1}$.20053	81	0.79947	$\bar{1}$.99460	59
2	$\bar{1}$.19592	80	$\bar{1}$.20134	82	0.79866	$\bar{1}$.99458	58
3	$\bar{1}$.19672	79	$\bar{1}$.20216	81	0.79784	$\bar{1}$.99456	57
4	$\bar{1}$.19751	79	$\bar{1}$.20297	81	0.79703	$\bar{1}$.99454	56
5	$\bar{1}$.19830	79	$\bar{1}$.20378	81	0.79622	$\bar{1}$.99452	55
6	$\bar{1}$.19909	79	$\bar{1}$.20459	81	0.79541	$\bar{1}$.99450	54
7	$\bar{1}$.19988	79	$\bar{1}$.20540	81	0.79460	$\bar{1}$.99448	53
8	$\bar{1}$.20067	78	$\bar{1}$.20621	80	0.79379	$\bar{1}$.99446	52
9	$\bar{1}$.20145	78	$\bar{1}$.20701	81	0.79299	$\bar{1}$.99444	51
10	$\bar{1}$.20223	79	$\bar{1}$.20782	80	0.79218	$\bar{1}$.99442	50
11	$\bar{1}$.20302	78	$\bar{1}$.20862	80	0.79138	$\bar{1}$.99440	49
12	$\bar{1}$.20380	78	$\bar{1}$.20942	80	0.79058	$\bar{1}$.99438	48
13	$\bar{1}$.20458	77	$\bar{1}$.21022	80	0.78978	$\bar{1}$.99436	47
14	$\bar{1}$.20535	78	$\bar{1}$.21102	80	0.78898	$\bar{1}$.99434	46
15	$\bar{1}$.20613	78	$\bar{1}$.21182	79	0.78818	$\bar{1}$.99432	45
16	$\bar{1}$.20691	77	$\bar{1}$.21261	80	0.78739	$\bar{1}$.99429	44
17	$\bar{1}$.20768	77	$\bar{1}$.21341	79	0.78659	$\bar{1}$.99427	43
18	$\bar{1}$.20845	77	$\bar{1}$.21420	79	0.78580	$\bar{1}$.99425	42
19	$\bar{1}$.20922	77	$\bar{1}$.21499	79	0.78501	$\bar{1}$.99423	41
20	$\bar{1}$.20999	77	$\bar{1}$.21578	79	0.78422	$\bar{1}$.99421	40
21	$\bar{1}$.21076	77	$\bar{1}$.21657	79	0.78343	$\bar{1}$.99419	39
22	$\bar{1}$.21153	76	$\bar{1}$.21736	78	0.78264	$\bar{1}$.99417	38
23	$\bar{1}$.21229	77	$\bar{1}$.21814	79	0.78186	$\bar{1}$.99415	37
24	$\bar{1}$.21306	76	$\bar{1}$.21893	78	0.78107	$\bar{1}$.99413	36
25	$\bar{1}$.21382	76	$\bar{1}$.21971	78	0.78029	$\bar{1}$.99411	35
26	$\bar{1}$.21458	76	$\bar{1}$.22049	78	0.77951	$\bar{1}$.99409	34
27	$\bar{1}$.21534	76	$\bar{1}$.22127	78	0.77873	$\bar{1}$.99407	33
28	$\bar{1}$.21610	75	$\bar{1}$.22205	78	0.77795	$\bar{1}$.99404	32
29	$\bar{1}$.21685	76	$\bar{1}$.22283	78	0.77717	$\bar{1}$.99402	31
30	$\bar{1}$.21761	75	$\bar{1}$.22361	77	0.77639	$\bar{1}$.99400	30
31	$\bar{1}$.21836	76	$\bar{1}$.22438	78	0.77562	$\bar{1}$.99398	29
32	$\bar{1}$.21912	75	$\bar{1}$.22516	77	0.77484	$\bar{1}$.99396	28
33	$\bar{1}$.21987	75	$\bar{1}$.22593	77	0.77407	$\bar{1}$.99394	27
34	$\bar{1}$.22062	75	$\bar{1}$.22670	77	0.77330	$\bar{1}$.99392	26
35	$\bar{1}$.22137	74	$\bar{1}$.22747	77	0.77253	$\bar{1}$.99390	25
36	$\bar{1}$.22211	75	$\bar{1}$.22824	77	0.77176	$\bar{1}$.99388	24
37	$\bar{1}$.22286	75	$\bar{1}$.22901	76	0.77099	$\bar{1}$.99385	23
38	$\bar{1}$.22361	74	$\bar{1}$.22977	77	0.77023	$\bar{1}$.99383	22
39	$\bar{1}$.22435	74	$\bar{1}$.23054	76	0.76946	$\bar{1}$.99381	21
40	$\bar{1}$.22509	74	$\bar{1}$.23130	76	0.76870	$\bar{1}$.99379	20
41	$\bar{1}$.22583	74	$\bar{1}$.23206	77	0.76794	$\bar{1}$.99377	19
42	$\bar{1}$.22657	74	$\bar{1}$.23283	76	0.76717	$\bar{1}$.99375	18
43	$\bar{1}$.22731	74	$\bar{1}$.23359	76	0.76641	$\bar{1}$.99372	17
44	$\bar{1}$.22805	73	$\bar{1}$.23435	75	0.76565	$\bar{1}$.99370	16
45	$\bar{1}$.22878	74	$\bar{1}$.23510	76	0.76490	$\bar{1}$.99368	15
46	$\bar{1}$.22952	73	$\bar{1}$.23586	75	0.76414	$\bar{1}$.99366	14
47	$\bar{1}$.23025	73	$\bar{1}$.23661	76	0.76339	$\bar{1}$.99364	13
48	$\bar{1}$.23098	73	$\bar{1}$.23737	75	0.76263	$\bar{1}$.99362	12
49	$\bar{1}$.23171	73	$\bar{1}$.23812	75	0.76188	$\bar{1}$.99359	11
50	$\bar{1}$.23244	73	$\bar{1}$.23887	75	0.76113	$\bar{1}$.99357	10
51	$\bar{1}$.23317	73	$\bar{1}$.23962	75	0.76038	$\bar{1}$.99355	9
52	$\bar{1}$.23390	72	$\bar{1}$.24037	75	0.75963	$\bar{1}$.99353	8
53	$\bar{1}$.23462	73	$\bar{1}$.24112	74	0.75888	$\bar{1}$.99351	7
54	$\bar{1}$.23535	72	$\bar{1}$.24186	75	0.75814	$\bar{1}$.99348	6
55	$\bar{1}$.23607	72	$\bar{1}$.24261	74	0.75739	$\bar{1}$.99346	5
56	$\bar{1}$.23679	73	$\bar{1}$.24335	75	0.75665	$\bar{1}$.99344	4
57	$\bar{1}$.23752	71	$\bar{1}$.24410	74	0.75590	$\bar{1}$.99342	3
58	$\bar{1}$.23823	72	$\bar{1}$.24484	74	0.75516	$\bar{1}$.99340	2
59	$\bar{1}$.23895	72	$\bar{1}$.24558	74	0.75442	$\bar{1}$.99337	1
60	$\bar{1}$.23967		$\bar{1}$.24632		0.75368	$\bar{1}$.99335	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	82	81	80
6	8.2	8.1	8.0
7	9.6	9.5	9.3
8	10.9	10.8	10.7
9	12.3	12.2	12.0
10	13.7	13.5	13.3
20	27.3	27.0	26.7
30	41.0	40.5	40.0
40	54.7	54.0	53.3
50	68.3	67.5	66.7

	79	78
6	7.9	7.8
7	9.2	9.1
8	10.5	10.4
9	11.9	11.7
10	13.2	13.0
20	26.3	26.0
30	39.5	39.0
40	52.7	52.0
50	65.8	65.0

	77	76
6	7.7	7.6
7	9.0	8.9
8	10.3	10.1
9	11.6	11.4
10	12.8	12.7
20	25.7	25.3
30	38.5	38.0
40	51.3	50.7
50	64.2	63.3

	75	74
6	7.5	7.4
7	8.8	8.6
8	10.0	9.9
9	11.3	11.1
10	12.5	12.3
20	25.0	24.7
30	37.5	37.0
40	50.0	49.3
50	62.5	61.7

	73	72
6	7.3	7.2
7	8.5	8.4
8	9.7	9.6
9	11.0	10.8
10	12.2	12.0
20	24.3	24.0
30	36.5	36.0
40	48.7	48.0
50	60.8	60.0

	71	3	2
6	7.1	0.3	0.2
7	8.3	0.4	0.2
8	9.5	0.4	0.3
9	10.7	0.5	0.3
10	11.8	0.5	0.3
20	23.7	1.0	0.7
30	35.5	1.5	1.0
40	47.3	2.0	1.3
50	59.2	2.5	1.7

p. p.

'	D. D.	
	66	65
6	6.6	6.5
7	[illegible]	[illegible]
8	[illegible]	[illegible]
9	[illegible]	9.8
10	[illegible]	10.8
20	22.0	[illegible]
30	33.0	32.5
40	44.0	43.3
50	55.0	54.2
	3	2
6	0.3	0.2
7	0.4	0.2
8	0.4	0.3
9	0.5	0.3
10	0.5	0.3
20	1.0	0.7
30	1.5	1.0
40	2.0	1.3
50	2.5	1.7

	log cos	d	log cot	c d	log tan	log sin	'	D. D.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

′	log sin	d	log tan	c. d.	log cot	log cos	
0	$\bar{1}$.28060	65	$\bar{1}$.28865	68	0.71135	$\bar{1}$.99195	60
1	$\bar{1}$.28125	65	$\bar{1}$.28933	67	0.71067	$\bar{1}$.99192	59
2	$\bar{1}$.28190	64	$\bar{1}$.29000	67	0.71000	$\bar{1}$.99190	58
3	$\bar{1}$.28254	65	$\bar{1}$.29067	67	0.70933	$\bar{1}$.99187	57
4	$\bar{1}$.28319	65	$\bar{1}$.29134	67	0.70866	$\bar{1}$.99185	56
5	$\bar{1}$.28384	64	$\bar{1}$.29201	67	0.70799	$\bar{1}$.99182	55
6	$\bar{1}$.28448	64	$\bar{1}$.29268	67	0.70732	$\bar{1}$.99180	54
7	$\bar{1}$.28512	65	$\bar{1}$.29335	67	0.70665	$\bar{1}$.99177	53
8	$\bar{1}$.28577	64	$\bar{1}$.29402	66	0.70598	$\bar{1}$.99175	52
9	$\bar{1}$.28641	64	$\bar{1}$.29468	67	0.70532	$\bar{1}$.99172	51
10	$\bar{1}$.28705	64	$\bar{1}$.29535	66	0.70465	$\bar{1}$.99170	50
11	$\bar{1}$.28769	64	$\bar{1}$.29601	67	0.70399	$\bar{1}$.99167	49
12	$\bar{1}$.28833	63	$\bar{1}$.29668	66	0.70332	$\bar{1}$.99165	48
13	$\bar{1}$.28896	64	$\bar{1}$.29734	66	0.70266	$\bar{1}$.99162	47
14	$\bar{1}$.28960	64	$\bar{1}$.29800	66	0.70200	$\bar{1}$.99160	46
15	$\bar{1}$.29024	63	$\bar{1}$.29866	66	0.70134	$\bar{1}$.99157	45
16	$\bar{1}$.29087	63	$\bar{1}$.29932	66	0.70068	$\bar{1}$.99155	44
17	$\bar{1}$.29150	64	$\bar{1}$.29998	66	0.70002	$\bar{1}$.99152	43
18	$\bar{1}$.29214	63	$\bar{1}$.30064	66	0.69936	$\bar{1}$.99150	42
19	$\bar{1}$.29277	63	$\bar{1}$.30130	65	0.69870	$\bar{1}$.99147	41
20	$\bar{1}$.29340	63	$\bar{1}$.30195	66	0.69805	$\bar{1}$.99145	40
21	$\bar{1}$.29403	63	$\bar{1}$.30261	65	0.69739	$\bar{1}$.99142	39
22	$\bar{1}$.29466	63	$\bar{1}$.30326	65	0.69674	$\bar{1}$.99140	38
23	$\bar{1}$.29529	62	$\bar{1}$.30391	66	0.69609	$\bar{1}$.99137	37
24	$\bar{1}$.29591	63	$\bar{1}$.30457	65	0.69543	$\bar{1}$.99135	36
25	$\bar{1}$.29654	62	$\bar{1}$.30522	65	0.69478	$\bar{1}$.99132	35
26	$\bar{1}$.29716	63	$\bar{1}$.30587	65	0.69413	$\bar{1}$.99130	34
27	$\bar{1}$.29779	62	$\bar{1}$.30652	65	0.69348	$\bar{1}$.99127	33
28	$\bar{1}$.29841	62	$\bar{1}$.30717	65	0.69283	$\bar{1}$.99124	32
29	$\bar{1}$.29903	63	$\bar{1}$.30782	64	0.69218	$\bar{1}$.99122	31
30	$\bar{1}$.29966	62	$\bar{1}$.30846	65	0.69154	$\bar{1}$.99119	30
31	$\bar{1}$.30028	62	$\bar{1}$.30911	64	0.69089	$\bar{1}$.99117	29
32	$\bar{1}$.30090	61	$\bar{1}$.30975	65	0.69025	$\bar{1}$.99114	28
33	$\bar{1}$.30151	62	$\bar{1}$.31040	64	0.68960	$\bar{1}$.99112	27
34	$\bar{1}$.30213	62	$\bar{1}$.31104	64	0.68896	$\bar{1}$.99109	26
35	$\bar{1}$.30275	61	$\bar{1}$.31168	65	0.68832	$\bar{1}$.99106	25
36	$\bar{1}$.30336	62	$\bar{1}$.31233	64	0.68767	$\bar{1}$.99104	24
37	$\bar{1}$.30398	61	$\bar{1}$.31297	64	0.68703	$\bar{1}$.99101	23
38	$\bar{1}$.30459	62	$\bar{1}$.31361	64	0.68639	$\bar{1}$.99099	22
39	$\bar{1}$.30521	61	$\bar{1}$.31425	64	0.68575	$\bar{1}$.99096	21
40	$\bar{1}$.30582	61	$\bar{1}$.31489	63	0.68511	$\bar{1}$.99093	20
41	$\bar{1}$.30643	61	$\bar{1}$.31552	64	0.68448	$\bar{1}$.99091	19
42	$\bar{1}$.30704	61	$\bar{1}$.31616	63	0.68384	$\bar{1}$.99088	18
43	$\bar{1}$.30765	61	$\bar{1}$.31679	64	0.68321	$\bar{1}$.99086	17
44	$\bar{1}$.30826	61	$\bar{1}$.31743	63	0.68257	$\bar{1}$.99083	16
45	$\bar{1}$.30887	60	$\bar{1}$.31806	64	0.68194	$\bar{1}$.99080	15
46	$\bar{1}$.30947	61	$\bar{1}$.31870	63	0.68130	$\bar{1}$.99078	14
47	$\bar{1}$.31008	60	$\bar{1}$.31933	63	0.68067	$\bar{1}$.99075	13
48	$\bar{1}$.31068	61	$\bar{1}$.31996	63	0.68004	$\bar{1}$.99072	12
49	$\bar{1}$.31129	60	$\bar{1}$.32059	63	0.67941	$\bar{1}$.99070	11
50	$\bar{1}$.31189	61	$\bar{1}$.32122	63	0.67878	$\bar{1}$.99067	10
51	$\bar{1}$.31250	60	$\bar{1}$.32185	63	0.67815	$\bar{1}$.99064	9
52	$\bar{1}$.31310	60	$\bar{1}$.32248	63	0.67752	$\bar{1}$.99062	8
53	$\bar{1}$.31370	60	$\bar{1}$.32311	62	0.67689	$\bar{1}$.99059	7
54	$\bar{1}$.31430	60	$\bar{1}$.32373	63	0.67627	$\bar{1}$.99056	6
55	$\bar{1}$.31490	59	$\bar{1}$.32436	62	0.67564	$\bar{1}$.99054	5
56	$\bar{1}$.31549	60	$\bar{1}$.32498	63	0.67502	$\bar{1}$.99051	4
57	$\bar{1}$.31609	60	$\bar{1}$.32561	62	0.67439	$\bar{1}$.99048	3
58	$\bar{1}$.31669	59	$\bar{1}$.32623	62	0.67377	$\bar{1}$.99046	2
59	$\bar{1}$.31728	60	$\bar{1}$.32685	62	0.67315	$\bar{1}$.99043	1
60	$\bar{1}$.31788		$\bar{1}$.32747		0.67253	$\bar{1}$.99040	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	68	67		66	65		64	63
6	6.8	6.7	6	6.6	6.5	6	6.4	6.3
7	7.9	7.8	7	7.7	7.6	7	7.5	7.4
8	9.1	8.9	8	8.8	8.7	8	8.5	8.4
9	10.2	10.1	9	9.9	9.8	9	9.6	9.5
10	11.3	11.2	10	11.0	10.8	10	10.7	10.5
20	22.7	22.3	20	22.0	21.7	20	21.3	21.0
30	34.0	33.5	30	33.0	32.5	30	32.0	31.5
40	45.3	44.7	40	44.0	43.3	40	42.7	42.0
50	56.7	55.8	50	55.0	54.2	50	53.3	52.5

	62	61		60	59		8	2
6	6.2	6.1	6	6.0	5.9	6	0.3	0.2
7	7.2	7.1	7	7.0	6.9	7	0.4	0.2
8	8.3	8.1	8	8.0	7.9	8	0.4	0.3
9	9.3	9.2	9	9.0	8.9	9	0.5	0.3
10	10.3	10.2	10	10.0	9.8	10	0.5	0.3
20	20.7	20.3	20	20.0	19.7	20	1.0	0.7
30	31.0	30.5	30	30.0	29.5	30	1.5	1.0
40	41.3	40.7	40	40.0	39.3	40	2.0	1.3
50	51.7	50.8	50	50.0	49.2	50	2.5	1.7

p. p.

[illegible]	log cos	d	log cot	c d	log tan	log sin	[illegible]	P. P.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

′	log sin	d	log tan	c. d.	log cot	log cos	
0	1̄.35209	54	1̄.36336	58	0.63664	1̄.98872	60
1	1̄.35263	55	1̄.36394	58	0.63606	1̄.98869	59
2	1̄.35318	55	1̄.36452	57	0.63548	1̄.98867	58
3	1̄.35373	54	1̄.36509	57	0.63491	1̄.98864	57
4	1̄.35427	54	1̄.36566	58	0.63434	1̄.98861	56
5	1̄.35481	55	1̄.36624	57	0.63376	1̄.98858	55
6	1̄.35536	54	1̄.36681	57	0.63319	1̄.98855	54
7	1̄.35590	54	1̄.36738	57	0.63262	1̄.98852	53
8	1̄.35644	54	1̄.36795	57	0.63205	1̄.98849	52
9	1̄.35698	54	1̄.36852	57	0.63148	1̄.98846	51
10	1̄.35752	54	1̄.36909	57	0.63091	1̄.98843	50
11	1̄.35806	54	1̄.36966	57	0.63034	1̄.98840	49
12	1̄.35860	54	1̄.37023	57	0.62977	1̄.98837	48
13	1̄.35914	54	1̄.37080	57	0.62920	1̄.98834	47
14	1̄.35968	54	1̄.37137	56	0.62863	1̄.98831	46
15	1̄.36022	53	1̄.37193	57	0.62807	1̄.98828	45
16	1̄.36075	54	1̄.37250	56	0.62750	1̄.98825	44
17	1̄.36129	53	1̄.37306	57	0.62694	1̄.98822	43
18	1̄.36182	54	1̄.37363	56	0.62637	1̄.98819	42
19	1̄.36236	53	1̄.37419	57	0.62581	1̄.98816	41
20	1̄.36289	53	1̄.37476	56	0.62524	1̄.98813	40
21	1̄.36342	53	1̄.37532	56	0.62468	1̄.98810	39
22	1̄.36395	54	1̄.37588	56	0.62412	1̄.98807	38
23	1̄.36449	53	1̄.37644	56	0.62356	1̄.98804	37
24	1̄.36502	53	1̄.37700	56	0.62300	1̄.98801	36
25	1̄.36555	53	1̄.37756	56	0.62244	1̄.98798	35
26	1̄.36608	52	1̄.37812	56	0.62188	1̄.98795	34
27	1̄.36660	53	1̄.37868	56	0.62132	1̄.98792	33
28	1̄.36713	53	1̄.37924	56	0.62076	1̄.98789	32
29	1̄.36766	53	1̄.37980	55	0.62020	1̄.98786	31
30	1̄.36819	52	1̄.38035	56	0.61965	1̄.98783	30
31	1̄.36871	53	1̄.38091	56	0.61909	1̄.98780	29
32	1̄.36924	52	1̄.38147	55	0.61853	1̄.98777	28
33	1̄.36976	52	1̄.38202	55	0.61798	1̄.98774	27
34	1̄.37028	53	1̄.38257	56	0.61743	1̄.98771	26
35	1̄.37081	52	1̄.38313	55	0.61687	1̄.98768	25
36	1̄.37133	52	1̄.38368	55	0.61632	1̄.98765	24
37	1̄.37185	52	1̄.38423	56	0.61577	1̄.98762	23
38	1̄.37237	52	1̄.38479	55	0.61521	1̄.98759	22
39	1̄.37289	52	1̄.38534	55	0.61466	1̄.98756	21
40	1̄.37341	52	1̄.38589	55	0.61411	1̄.98753	20
41	1̄.37393	52	1̄.38644	55	0.61356	1̄.98750	19
42	1̄.37445	52	1̄.38699	55	0.61301	1̄.98746	18
43	1̄.37497	52	1̄.38754	54	0.61246	1̄.98743	17
44	1̄.37549	51	1̄.38808	55	0.61192	1̄.98740	16
45	1̄.37600	52	1̄.38863	55	0.61137	1̄.98737	15
46	1̄.37652	51	1̄.38918	54	0.61082	1̄.98734	14
47	1̄.37703	52	1̄.38972	55	0.61028	1̄.98731	13
48	1̄.37755	51	1̄.39027	55	0.60973	1̄.98728	12
49	1̄.37806	52	1̄.39082	54	0.60918	1̄.98725	11
50	1̄.37858	51	1̄.39136	54	0.60864	1̄.98722	10
51	1̄.37909	51	1̄.39190	55	0.60810	1̄.98719	9
52	1̄.37960	51	1̄.39245	54	0.60755	1̄.98715	8
53	1̄.38011	51	1̄.39299	54	0.60701	1̄.98712	7
54	1̄.38062	51	1̄.39353	54	0.60647	1̄.98709	6
55	1̄.38113	51	1̄.39407	54	0.60593	1̄.98706	5
56	1̄.38164	51	1̄.39461	54	0.60539	1̄.98703	4
57	1̄.38215	51	1̄.39515	54	0.60485	1̄.98700	3
58	1̄.38266	51	1̄.39569	54	0.60431	1̄.98697	2
59	1̄.38317	51	1̄.39623	54	0.60377	1̄.98694	1
60	1̄.38368		1̄.39677		0.60323	1̄.98690	0
	log cos	d	log cot	c. d.	log tan	log sin	′

p. p.

	58	57
6	5.8	5.7
7	6.8	6.7
8	7.7	7.6
9	8.7	8.6
10	9.7	9.5
20	19.3	19.0
30	29.0	28.5
40	38.7	38.0
50	48.3	47.5

	56	55
6	5.6	5.5
7	6.5	6.4
8	7.5	7.3
9	8.4	8.3
10	9.3	9.2
20	18.7	18.3
30	28.0	27.5
40	37.3	36.7
50	46.7	45.8

	54
6	5.4
7	6.3
8	7.2
9	8.1
10	9.0
20	18.0
30	27.0
40	36.0
50	45.0

	53	52
6	5.3	5.2
7	6.2	6.1
8	7.1	6.9
9	8.0	7.8
10	8.8	8.7
20	17.7	17.3
30	26.5	26.0
40	35.3	34.7
50	44.2	43.3

	51	4
6	5.1	0.4
7	6.0	0.5
8	6.8	0.5
9	7.7	0.6
10	8.5	0.7
20	17.0	1.3
30	25.5	2.0
40	34.0	2.7
50	42.5	3.3

	3	2
6	0.3	0.2
7	0.4	0.2
8	0.4	0.3
9	0.5	0.3
10	0.5	0.3
20	1.0	0.7
30	1.5	1.0
40	2.0	1.3
50	2.5	1.7

p. p.

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′	log sin	d	log tan	c. d	log cot	log cos	d	′	p. p.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
	log cos	d	log cot	c. d	log tan	log sin	d	′	p. p.

p. p.	54	53
6	5.4	5.3
[illegible]	[illegible]	[illegible]

p. p.	52	51
6	5.2	5.1
[illegible]	[illegible]	[illegible]

p. p.	50	49
6	5.0	4.9
7	5.8	5.7
8	6.7	6.5
[illegible]	[illegible]	[illegible]

p. p.	48	47
6	4.8	4.7
7	5.6	5.5
8	6.4	6.3
[illegible]	[illegible]	[illegible]

p. p.	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

'	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.41300	47	$\bar{1}$.42805	51	0.57195	$\bar{1}$.98494	3	60
1	$\bar{1}$.41347	47	$\bar{1}$.42856	50	0.57144	$\bar{1}$.98491	3	59
2	$\bar{1}$.41394	47	$\bar{1}$.42906	51	0.57094	$\bar{1}$.98488	4	58
3	$\bar{1}$.41441	47	$\bar{1}$.42957	50	0.57043	$\bar{1}$.98484	3	57
4	$\bar{1}$.41488	47	$\bar{1}$.43007	50	0.56993	$\bar{1}$.98481	4	56
5	$\bar{1}$.41535	47	$\bar{1}$.43057	51	0.56943	$\bar{1}$.98477	3	55
6	$\bar{1}$.41582	46	$\bar{1}$.43108	50	0.56892	$\bar{1}$.98474	3	54
7	$\bar{1}$.41628	47	$\bar{1}$.43158	50	0.56842	$\bar{1}$.98471	4	53
8	$\bar{1}$.41675	47	$\bar{1}$.43208	50	0.56792	$\bar{1}$.98467	3	52
9	$\bar{1}$.41722	46	$\bar{1}$.43258	50	0.56742	$\bar{1}$.98464	4	51
10	$\bar{1}$.41768	47	$\bar{1}$.43308	50	0.56692	$\bar{1}$.98460	3	50
11	$\bar{1}$.41815	46	$\bar{1}$.43358	50	0.56642	$\bar{1}$.98457	4	49
12	$\bar{1}$.41861	47	$\bar{1}$.43408	50	0.56592	$\bar{1}$.98453	3	48
13	$\bar{1}$.41908	46	$\bar{1}$.43458	50	0.56542	$\bar{1}$.98450	3	47
14	$\bar{1}$.41954	47	$\bar{1}$.43508	50	0.56492	$\bar{1}$.98447	4	46
15	$\bar{1}$.42001	46	$\bar{1}$.43558	49	0.56442	$\bar{1}$.98443	3	45
16	$\bar{1}$.42047	46	$\bar{1}$.43607	50	0.56393	$\bar{1}$.98440	4	44
17	$\bar{1}$.42093	47	$\bar{1}$.43657	50	0.56343	$\bar{1}$.98436	3	43
18	$\bar{1}$.42140	46	$\bar{1}$.43707	49	0.56293	$\bar{1}$.98433	4	42
19	$\bar{1}$.42186	46	$\bar{1}$.43756	50	0.56244	$\bar{1}$.98429	3	41
20	$\bar{1}$.42232	46	$\bar{1}$.43806	49	0.56194	$\bar{1}$.98426	4	40
21	$\bar{1}$.42278	46	$\bar{1}$.43855	50	0.56145	$\bar{1}$.98422	3	39
22	$\bar{1}$.42324	46	$\bar{1}$.43905	49	0.56095	$\bar{1}$.98419	4	38
23	$\bar{1}$.42370	46	$\bar{1}$.43954	50	0.56046	$\bar{1}$.98415	3	37
24	$\bar{1}$.42416	45	$\bar{1}$.44004	49	0.55996	$\bar{1}$.98412	3	36
25	$\bar{1}$.42461	46	$\bar{1}$.44053	49	0.55947	$\bar{1}$.98409	4	35
26	$\bar{1}$.42507	46	$\bar{1}$.44102	49	0.55898	$\bar{1}$.98405	3	34
27	$\bar{1}$.42553	46	$\bar{1}$.44151	50	0.55849	$\bar{1}$.98402	4	33
28	$\bar{1}$.42599	45	$\bar{1}$.44201	49	0.55799	$\bar{1}$.98398	3	32
29	$\bar{1}$.42644	46	$\bar{1}$.44250	49	0.55750	$\bar{1}$.98395	4	31
30	$\bar{1}$.42690	45	$\bar{1}$.44299	49	0.55701	$\bar{1}$.98391	3	30
31	$\bar{1}$.42735	46	$\bar{1}$.44348	49	0.55652	$\bar{1}$.98388	4	29
32	$\bar{1}$.42781	45	$\bar{1}$.44397	49	0.55603	$\bar{1}$.98384	3	28
33	$\bar{1}$.42826	46	$\bar{1}$.44446	49	0.55554	$\bar{1}$.98381	4	27
34	$\bar{1}$.42872	45	$\bar{1}$.44495	49	0.55505	$\bar{1}$.98377	4	26
35	$\bar{1}$.42917	45	$\bar{1}$.44544	48	0.55456	$\bar{1}$.98373	3	25
36	$\bar{1}$.42962	46	$\bar{1}$.44592	49	0.55408	$\bar{1}$.98370	4	24
37	$\bar{1}$.43008	45	$\bar{1}$.44641	49	0.55359	$\bar{1}$.98366	3	23
38	$\bar{1}$.43053	45	$\bar{1}$.44690	48	0.55310	$\bar{1}$.98363	4	22
39	$\bar{1}$.43098	45	$\bar{1}$.44738	49	0.55262	$\bar{1}$.98359	3	21
40	$\bar{1}$.43143	45	$\bar{1}$.44787	49	0.55213	$\bar{1}$.98356	4	20
41	$\bar{1}$.43188	45	$\bar{1}$.44836	48	0.55164	$\bar{1}$.98352	3	19
42	$\bar{1}$.43233	45	$\bar{1}$.44884	49	0.55116	$\bar{1}$.98349	4	18
43	$\bar{1}$.43278	45	$\bar{1}$.44933	48	0.55067	$\bar{1}$.98345	3	17
44	$\bar{1}$.43323	44	$\bar{1}$.44981	48	0.55019	$\bar{1}$.98342	4	16
45	$\bar{1}$.43367	45	$\bar{1}$.45029	49	0.54971	$\bar{1}$.98338	4	15
46	$\bar{1}$.43412	45	$\bar{1}$.45078	48	0.54922	$\bar{1}$.98334	3	14
47	$\bar{1}$.43457	45	$\bar{1}$.45126	48	0.54874	$\bar{1}$.98331	4	13
48	$\bar{1}$.43502	44	$\bar{1}$.45174	48	0.54826	$\bar{1}$.98327	3	12
49	$\bar{1}$.43546	45	$\bar{1}$.45222	49	0.54778	$\bar{1}$.98324	4	11
50	$\bar{1}$.43591	44	$\bar{1}$.45271	48	0.54729	$\bar{1}$.98320	3	10
51	$\bar{1}$.43635	45	$\bar{1}$.45319	48	0.54681	$\bar{1}$.98317	4	9
52	$\bar{1}$.43680	44	$\bar{1}$.45367	48	0.54633	$\bar{1}$.98313	4	8
53	$\bar{1}$.43724	45	$\bar{1}$.45415	48	0.54585	$\bar{1}$.98309	3	7
54	$\bar{1}$.43769	44	$\bar{1}$.45463	48	0.54537	$\bar{1}$.98306	4	6
55	$\bar{1}$.43813	44	$\bar{1}$.45511	48	0.54489	$\bar{1}$.98302	3	5
56	$\bar{1}$.43857	44	$\bar{1}$.45559	47	0.54441	$\bar{1}$.98299	4	4
57	$\bar{1}$.43901	45	$\bar{1}$.45606	48	0.54394	$\bar{1}$.98295	4	3
58	$\bar{1}$.43946	44	$\bar{1}$.45654	48	0.54346	$\bar{1}$.98291	3	2
59	$\bar{1}$.43990	44	$\bar{1}$.45702	48	0.54298	$\bar{1}$.98288	4	1
60	$\bar{1}$.44034		$\bar{1}$.45750		0.54250	$\bar{1}$.98284		0
	log cos	d	log cot	c. d.	log tan	log sin	d	'

p. p.

	51	50
6	5.1	5.0
7	6.0	5.8
8	6.8	6.7
9	7.7	7.5
10	8.5	8.3
20	17.0	16.7
30	25.5	25.0
40	34.0	33.3
50	42.5	41.7

	49	48
6	4.9	4.8
7	5.7	5.6
8	6.5	6.4
9	7.4	7.2
10	8.2	8.0
20	16.3	16.0
30	24.5	24.0
40	32.7	32.0
50	40.8	40.0

	47	46
6	4.7	4.6
7	5.5	5.4
8	6.3	6.1
9	7.1	6.9
10	7.8	7.7
20	15.7	15.3
30	23.5	23.0
40	31.3	30.7
50	39.2	38.3

	45	44
6	4.5	4.4
7	5.3	5.1
8	6.0	5.9
9	6.8	6.6
10	7.5	7.3
20	15.0	14.7
30	22.5	22.0
40	30.0	29.3
50	37.5	36.7

	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

p. p.

′	log sin	d	log tan	c.d.	log cot	log cos	d	′
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	log cos	d	log cot	c.d.	log tan	log sin	d	′

P. P.	48	47
6	4.8	4.7
7	5.6	5.5
8	6.4	6.3
9	7.2	7.1
10	8.0	7.8
20	16.0	15.7
30	24.0	23.5
40	32.0	31.3
50	40.0	39.2

P. P.	46	45
6	4.6	4.5
7	5.4	5.2
8	6.1	6.0
9	6.9	6.8
10	7.7	7.5
20	15.3	15.0
30	23.0	22.5
40	30.7	30.0
50	38.3	37.5

P. P.	44	43
6	4.4	4.3
7	5.1	5.0
8	5.9	5.7
9	6.6	6.5
10	7.3	7.2
20	14.7	14.3
30	22.0	21.5
40	29.3	28.7
50	36.7	35.8

P. P.	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

P. P.	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}.46594$	41	$\bar{1}.48534$	45	0.51466	$\bar{1}.98060$	4	60
1	$\bar{1}.46635$	41	$\bar{1}.48579$	45	0.51421	$\bar{1}.98056$	4	59
2	$\bar{1}.46676$	41	$\bar{1}.48624$	45	0.51376	$\bar{1}.98052$	4	58
3	$\bar{1}.46717$	41	$\bar{1}.48669$	45	0.51331	$\bar{1}.98048$	4	57
4	$\bar{1}.46758$	42	$\bar{1}.48714$	45	0.51286	$\bar{1}.98044$	4	56
5	$\bar{1}.46800$	41	$\bar{1}.48759$	45	0.51241	$\bar{1}.98040$	4	55
6	$\bar{1}.46841$	41	$\bar{1}.48804$	45	0.51196	$\bar{1}.98036$	4	54
7	$\bar{1}.46882$	41	$\bar{1}.48849$	45	0.51151	$\bar{1}.98032$	3	53
8	$\bar{1}.46923$	41	$\bar{1}.48894$	45	0.51106	$\bar{1}.98029$	4	52
9	$\bar{1}.46964$	41	$\bar{1}.48939$	45	0.51061	$\bar{1}.98025$	4	51
10	$\bar{1}.47005$	40	$\bar{1}.48984$	45	0.51016	$\bar{1}.98021$	4	50
11	$\bar{1}.47045$	41	$\bar{1}.49029$	44	0.50971	$\bar{1}.98017$	4	49
12	$\bar{1}.47086$	41	$\bar{1}.49073$	45	0.50927	$\bar{1}.98013$	4	48
13	$\bar{1}.47127$	41	$\bar{1}.49118$	45	0.50882	$\bar{1}.98009$	4	47
14	$\bar{1}.47168$	41	$\bar{1}.49163$	44	0.50837	$\bar{1}.98005$	4	46
15	$\bar{1}.47209$	40	$\bar{1}.49207$	45	0.50793	$\bar{1}.98001$	4	45
16	$\bar{1}.47249$	41	$\bar{1}.49252$	44	0.50748	$\bar{1}.97997$	4	44
17	$\bar{1}.47290$	40	$\bar{1}.49296$	45	0.50704	$\bar{1}.97993$	4	43
18	$\bar{1}.47330$	41	$\bar{1}.49341$	44	0.50659	$\bar{1}.97989$	3	42
19	$\bar{1}.47371$	40	$\bar{1}.49385$	45	0.50615	$\bar{1}.97986$	4	41
20	$\bar{1}.47411$	41	$\bar{1}.49430$	44	0.50570	$\bar{1}.97982$	4	40
21	$\bar{1}.47452$	40	$\bar{1}.49474$	45	0.50526	$\bar{1}.97978$	4	39
22	$\bar{1}.47492$	41	$\bar{1}.49519$	44	0.50481	$\bar{1}.97974$	4	38
23	$\bar{1}.47533$	40	$\bar{1}.49563$	44	0.50437	$\bar{1}.97970$	4	37
24	$\bar{1}.47573$	40	$\bar{1}.49607$	45	0.50393	$\bar{1}.97966$	4	36
25	$\bar{1}.47613$	41	$\bar{1}.49652$	44	0.50348	$\bar{1}.97962$	4	35
26	$\bar{1}.47654$	40	$\bar{1}.49696$	44	0.50304	$\bar{1}.97958$	4	34
27	$\bar{1}.47694$	40	$\bar{1}.49740$	44	0.50260	$\bar{1}.97954$	4	33
28	$\bar{1}.47734$	40	$\bar{1}.49784$	44	0.50216	$\bar{1}.97950$	4	32
29	$\bar{1}.47774$	40	$\bar{1}.49828$	44	0.50172	$\bar{1}.97946$	4	31
30	$\bar{1}.47814$	40	$\bar{1}.49872$	44	0.50128	$\bar{1}.97942$	4	30
31	$\bar{1}.47854$	40	$\bar{1}.49916$	44	0.50084	$\bar{1}.97938$	4	29
32	$\bar{1}.47894$	40	$\bar{1}.49960$	44	0.50040	$\bar{1}.97934$	4	28
33	$\bar{1}.47934$	40	$\bar{1}.50004$	44	0.49996	$\bar{1}.97930$	4	27
34	$\bar{1}.47974$	40	$\bar{1}.50048$	44	0.49952	$\bar{1}.97926$	4	26
35	$\bar{1}.48014$	40	$\bar{1}.50092$	44	0.49908	$\bar{1}.97922$	4	25
36	$\bar{1}.48054$	40	$\bar{1}.50136$	44	0.49864	$\bar{1}.97918$	4	24
37	$\bar{1}.48094$	39	$\bar{1}.50180$	43	0.49820	$\bar{1}.97914$	4	23
38	$\bar{1}.48133$	40	$\bar{1}.50223$	44	0.49777	$\bar{1}.97910$	4	22
39	$\bar{1}.48173$	40	$\bar{1}.50267$	44	0.49733	$\bar{1}.97906$	4	21
40	$\bar{1}.48213$	39	$\bar{1}.50311$	44	0.49689	$\bar{1}.97902$	4	20
41	$\bar{1}.48252$	40	$\bar{1}.50355$	43	0.49645	$\bar{1}.97898$	4	19
42	$\bar{1}.48292$	40	$\bar{1}.50398$	44	0.49602	$\bar{1}.97894$	4	18
43	$\bar{1}.48332$	39	$\bar{1}.50442$	43	0.49558	$\bar{1}.97890$	4	17
44	$\bar{1}.48371$	40	$\bar{1}.50485$	44	0.49515	$\bar{1}.97886$	4	16
45	$\bar{1}.48411$	39	$\bar{1}.50529$	43	0.49471	$\bar{1}.97882$	4	15
46	$\bar{1}.48450$	40	$\bar{1}.50572$	44	0.49428	$\bar{1}.97878$	4	14
47	$\bar{1}.48490$	39	$\bar{1}.50616$	43	0.49384	$\bar{1}.97874$	4	13
48	$\bar{1}.48529$	39	$\bar{1}.50659$	44	0.49341	$\bar{1}.97870$	4	12
49	$\bar{1}.48568$	39	$\bar{1}.50703$	43	0.49297	$\bar{1}.97866$	5	11
50	$\bar{1}.48607$	40	$\bar{1}.50746$	43	0.49254	$\bar{1}.97861$	4	10
51	$\bar{1}.48647$	39	$\bar{1}.50789$	44	0.49211	$\bar{1}.97857$	4	9
52	$\bar{1}.48686$	39	$\bar{1}.50833$	43	0.49167	$\bar{1}.97853$	4	8
53	$\bar{1}.48725$	39	$\bar{1}.50876$	43	0.49124	$\bar{1}.97849$	4	7
54	$\bar{1}.48764$	39	$\bar{1}.50919$	43	0.49081	$\bar{1}.97845$	4	6
55	$\bar{1}.48803$	39	$\bar{1}.50962$	43	0.49038	$\bar{1}.97841$	4	5
56	$\bar{1}.48842$	39	$\bar{1}.51005$	43	0.48995	$\bar{1}.97837$	4	4
57	$\bar{1}.48881$	39	$\bar{1}.51048$	44	0.48952	$\bar{1}.97833$	4	3
58	$\bar{1}.48920$	39	$\bar{1}.51092$	43	0.48908	$\bar{1}.97829$	4	2
59	$\bar{1}.48959$	39	$\bar{1}.51135$	43	0.48865	$\bar{1}.97825$	4	1
60	$\bar{1}.48998$		$\bar{1}.51178$		0.48822	$\bar{1}.97821$		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	45	44
6	4.5	4.4
7	5.3	5.1
8	6.0	5.9
9	6.8	6.6
10	7.5	7.3
20	15.0	14.7
30	22.5	22.0
40	30.0	29.3
50	37.5	36.7

	43
6	4.3
7	5.0
8	5.7
9	6.5
10	7.2
20	14.3
30	21.5
40	28.7
50	35.8

	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

	5	4	3
6	0.5	0.4	0.3
7	0.6	0.5	0.4
8	0.7	0.5	0.4
9	0.8	0.6	0.5
10	0.8	0.7	0.5
20	1.7	1.3	1.0
30	2.5	2.0	1.5
40	3.3	2.7	2.0
50	4.2	3.3	2.5

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.48998	39	1̄.51178	43	0.48822	1̄.97821	4	60
1	1̄.49037	39	1̄.51221	43	0.48779	1̄.97817	5	59
2	1̄.49076	39	1̄.51264	42	0.48736	1̄.97812	4	58
3	1̄.49115	38	1̄.51306	43	0.48694	1̄.97808	4	57
4	1̄.49153	39	1̄.51349	43	0.48651	1̄.97804	4	56
5	1̄.49192	39	1̄.51392	43	0.48608	1̄.97800	4	55
6	1̄.49231	38	1̄.51435	43	0.48565	1̄.97796	4	54
7	1̄.49269	39	1̄.51478	42	0.48522	1̄.97792	4	53
8	1̄.49308	39	1̄.51520	43	0.48480	1̄.97788	4	52
9	1̄.49347	38	1̄.51563	43	0.48437	1̄.97784	5	51
10	1̄.49385	39	1̄.51606	42	0.48394	1̄.97779	4	50
11	1̄.49424	38	1̄.51648	43	0.48352	1̄.97775	4	49
12	1̄.49462	38	1̄.51691	43	0.48309	1̄.97771	4	48
13	1̄.49500	39	1̄.51734	42	0.48266	1̄.97767	4	47
14	1̄.49539	38	1̄.51776	43	0.48224	1̄.97763	4	46
15	1̄.49577	38	1̄.51819	42	0.48181	1̄.97759	5	45
16	1̄.49615	39	1̄.51861	42	0.48139	1̄.97754	4	44
17	1̄.49654	38	1̄.51903	43	0.48097	1̄.97750	4	43
18	1̄.49692	38	1̄.51946	42	0.48054	1̄.97746	4	42
19	1̄.49730	38	1̄.51988	43	0.48012	1̄.97742	4	41
20	1̄.49768	38	1̄.52031	42	0.47969	1̄.97738	4	40
21	1̄.49806	38	1̄.52073	42	0.47927	1̄.97734	5	39
22	1̄.49844	38	1̄.52115	42	0.47885	1̄.97729	4	38
23	1̄.49882	38	1̄.52157	43	0.47843	1̄.97725	4	37
24	1̄.49920	38	1̄.52200	42	0.47800	1̄.97721	4	36
25	1̄.49958	38	1̄.52242	42	0.47758	1̄.97717	4	35
26	1̄.49996	38	1̄.52284	42	0.47716	1̄.97713	5	34
27	1̄.50034	38	1̄.52326	42	0.47674	1̄.97708	4	33
28	1̄.50072	38	1̄.52368	42	0.47632	1̄.97704	4	32
29	1̄.50110	38	1̄.52410	42	0.47590	1̄.97700	4	31
30	1̄.50148	37	1̄.52452	42	0.47548	1̄.97696	5	30
31	1̄.50185	38	1̄.52494	42	0.47506	1̄.97691	4	29
32	1̄.50223	38	1̄.52536	42	0.47464	1̄.97687	4	28
33	1̄.50261	37	1̄.52578	42	0.47422	1̄.97683	4	27
34	1̄.50298	38	1̄.52620	41	0.47380	1̄.97679	5	26
35	1̄.50336	38	1̄.52661	42	0.47339	1̄.97674	4	25
36	1̄.50374	37	1̄.52703	42	0.47297	1̄.97670	4	24
37	1̄.50411	38	1̄.52745	42	0.47255	1̄.97666	4	23
38	1̄.50449	37	1̄.52787	42	0.47213	1̄.97662	5	22
39	1̄.50486	37	1̄.52829	41	0.47171	1̄.97657	4	21
40	1̄.50523	38	1̄.52870	42	0.47130	1̄.97653	4	20
41	1̄.50561	37	1̄.52912	41	0.47088	1̄.97649	4	19
42	1̄.50598	37	1̄.52953	42	0.47047	1̄.97645	5	18
43	1̄.50635	38	1̄.52995	42	0.47005	1̄.97640	4	17
44	1̄.50673	37	1̄.53037	41	0.46963	1̄.97636	4	16
45	1̄.50710	37	1̄.53078	42	0.46922	1̄.97632	4	15
46	1̄.50747	37	1̄.53120	41	0.46880	1̄.97628	5	14
47	1̄.50784	37	1̄.53161	41	0.46839	1̄.97623	4	13
48	1̄.50821	37	1̄.53202	42	0.46798	1̄.97619	4	12
49	1̄.50858	38	1̄.53244	41	0.46756	1̄.97615	5	11
50	1̄.50896	37	1̄.53285	42	0.46715	1̄.97610	4	10
51	1̄.50933	37	1̄.53327	41	0.46673	1̄.97606	4	9
52	1̄.50970	37	1̄.53368	41	0.46632	1̄.97602	5	8
53	1̄.51007	36	1̄.53409	41	0.46591	1̄.97597	4	7
54	1̄.51043	37	1̄.53450	42	0.46550	1̄.97593	4	6
55	1̄.51080	37	1̄.53492	41	0.46508	1̄.97589	5	5
56	1̄.51117	37	1̄.53533	41	0.46467	1̄.97584	4	4
57	1̄.51154	37	1̄.53574	41	0.46426	1̄.97580	4	3
58	1̄.51191	36	1̄.53615	41	0.46385	1̄.97576	5	2
59	1̄.51227	37	1̄.53656	41	0.46344	1̄.97571	4	1
60	1̄.51264		1̄.53697		0.46303	1̄.97567		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	43	42
6	4.3	4.2
7	5.0	4.9
8	5.7	5.6
9	6.5	6.3
10	7.2	7.0
20	14.3	14.0
30	21.5	21.0
40	28.7	28.0
50	35.8	35.0

	41
6	4.1
7	4.8
8	5.5
9	6.2
10	6.8
20	13.7
30	20.5
40	27.3
50	34.2

	39	38
6	3.9	3.8
7	4.6	4.4
8	5.2	5.1
9	5.9	5.7
10	6.5	6.3
20	13.0	12.7
30	19.5	19.0
40	26.0	25.3
50	32.5	31.7

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.51264	37	1̄.53697	41	0.46303	1̄.97567	4	60
1	1̄.51301	37	1̄.53738	41	0.46262	1̄.97563	5	59
2	1̄.51338	36	1̄.53779	41	0.46221	1̄.97558	4	58
3	1̄.51374	37	1̄.53820	41	0.46180	1̄.97554	4	57
4	1̄.51411	36	1̄.53861	41	0.46139	1̄.97550	5	56
5	1̄.51447	37	1̄.53902	41	0.46098	1̄.97545	4	55
6	1̄.51484	36	1̄.53943	41	0.46057	1̄.97541	5	54
7	1̄.51520	37	1̄.53984	41	0.46016	1̄.97536	4	53
8	1̄.51557	36	1̄.54025	40	0.45975	1̄.97532	4	52
9	1̄.51593	36	1̄.54065	41	0.45935	1̄.97528	5	51
10	1̄.51629	37	1̄.54106	41	0.45894	1̄.97523	4	50
11	1̄.51666	36	1̄.54147	40	0.45853	1̄.97519	4	49
12	1̄.51702	36	1̄.54187	41	0.45813	1̄.97515	5	48
13	1̄.51738	36	1̄.54228	41	0.45772	1̄.97510	4	47
14	1̄.51774	37	1̄.54269	40	0.45731	1̄.97506	5	46
15	1̄.51811	36	1̄.54309	41	0.45691	1̄.97501	4	45
16	1̄.51847	36	1̄.54350	40	0.45650	1̄.97497	5	44
17	1̄.51883	36	1̄.54390	41	0.45610	1̄.97492	4	43
18	1̄.51919	36	1̄.54431	40	0.45569	1̄.97488	4	42
19	1̄.51955	36	1̄.54471	41	0.45529	1̄.97484	5	41
20	1̄.51991	36	1̄.54512	40	0.45488	1̄.97479	4	40
21	1̄.52027	36	1̄.54552	41	0.45448	1̄.97475	5	39
22	1̄.52063	36	1̄.54593	40	0.45407	1̄.97470	4	38
23	1̄.52099	36	1̄.54633	40	0.45367	1̄.97466	5	37
24	1̄.52135	36	1̄.54673	41	0.45327	1̄.97461	4	36
25	1̄.52171	36	1̄.54714	40	0.45286	1̄.97457	4	35
26	1̄.52207	35	1̄.54754	40	0.45246	1̄.97453	5	34
27	1̄.52242	36	1̄.54794	41	0.45206	1̄.97448	4	33
28	1̄.52278	36	1̄.54835	40	0.45165	1̄.97444	5	32
29	1̄.52314	36	1̄.54875	40	0.45125	1̄.97439	4	31
30	1̄.52350	35	1̄.54915	40	0.45085	1̄.97435	5	30
31	1̄.52385	36	1̄.54955	40	0.45045	1̄.97430	4	29
32	1̄.52421	35	1̄.54995	40	0.45005	1̄.97426	5	28
33	1̄.52456	36	1̄.55035	40	0.44965	1̄.97421	4	27
34	1̄.52492	35	1̄.55075	40	0.44925	1̄.97417	5	26
35	1̄.52527	36	1̄.55115	40	0.44885	1̄.97412	4	25
36	1̄.52563	35	1̄.55155	40	0.44845	1̄.97408	5	24
37	1̄.52598	36	1̄.55195	40	0.44805	1̄.97403	4	23
38	1̄.52634	35	1̄.55235	40	0.44765	1̄.97399	5	22
39	1̄.52669	36	1̄.55275	40	0.44725	1̄.97394	4	21
40	1̄.52705	35	1̄.55315	40	0.44685	1̄.97390	5	20
41	1̄.52740	35	1̄.55355	40	0.44645	1̄.97385	4	19
42	1̄.52775	36	1̄.55395	39	0.44605	1̄.97381	5	18
43	1̄.52811	35	1̄.55434	40	0.44566	1̄.97376	4	17
44	1̄.52846	35	1̄.55474	40	0.44526	1̄.97372	5	16
45	1̄.52881	35	1̄.55514	40	0.44486	1̄.97367	4	15
46	1̄.52916	35	1̄.55554	39	0.44446	1̄.97363	5	14
47	1̄.52951	35	1̄.55593	40	0.44407	1̄.97358	5	13
48	1̄.52986	35	1̄.55633	40	0.44367	1̄.97353	4	12
49	1̄.53021	35	1̄.55673	39	0.44327	1̄.97349	5	11
50	1̄.53056	36	1̄.55712	40	0.44288	1̄.97344	4	10
51	1̄.53092	34	1̄.55752	39	0.44248	1̄.97340	5	9
52	1̄.53126	35	1̄.55791	40	0.44209	1̄.97335	4	8
53	1̄.53161	35	1̄.55831	39	0.44169	1̄.97331	5	7
54	1̄.53196	35	1̄.55870	40	0.44130	1̄.97326	4	6
55	1̄.53231	35	1̄.55910	39	0.44090	1̄.97322	5	5
56	1̄.53266	35	1̄.55949	40	0.44051	1̄.97317	5	4
57	1̄.53301	35	1̄.55989	39	0.44011	1̄.97312	4	3
58	1̄.53336	34	1̄.56028	39	0.43972	1̄.97308	5	2
59	1̄.53370	35	1̄.56067	40	0.43933	1̄.97303	4	1
60	1̄.53405		1̄.56107		0.43893	1̄.97299		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	41	40
6	4.1	4.0
7	4.8	4.7
8	5.5	5.3
9	6.2	6.0
10	6.8	6.7
20	13.7	13.3
30	20.5	20.0
40	27.3	26.7
50	34.2	33.3

	39
6	3.9
7	4.6
8	5.2
9	5.9
10	6.5
20	13.0
30	19.5
40	26.0
50	32.5

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	35	34
6	3.5	3.4
7	4.1	4.0
8	4.7	4.5
9	5.3	5.1
10	5.8	5.7
20	11.7	11.3
30	17.5	17.0
40	23.3	22.7
50	29.2	28.3

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

20°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.53405	35	1̄.56107	39	0.43893	1̄.97299	5	60
1	1̄.53440	35	1̄.56146	39	0.43854	1̄.97294	5	59
2	1̄.53475	34	1̄.56185	39	0.43815	1̄.97289	4	58
3	1̄.53509	35	1̄.56224	40	0.43776	1̄.97285	5	57
4	1̄.53544	34	1̄.56264	39	0.43736	1̄.97280	4	56
5	1̄.53578	35	1̄.56303	39	0.43697	1̄.97276	5	55
6	1̄.53613	34	1̄.56342	39	0.43658	1̄.97271	5	54
7	1̄.53647	35	1̄.56381	39	0.43619	1̄.97266	4	53
8	1̄.53682	34	1̄.56420	39	0.43580	1̄.97262	5	52
9	1̄.53716	35	1̄.56459	39	0.43541	1̄.97257	5	51
10	1̄.53751	34	1̄.56498	39	0.43502	1̄.97252	4	50
11	1̄.53785	34	1̄.56537	39	0.43463	1̄.97248	5	49
12	1̄.53819	35	1̄.56576	39	0.43424	1̄.97243	5	48
13	1̄.53854	34	1̄.56615	39	0.43385	1̄.97238	4	47
14	1̄.53888	34	1̄.56654	39	0.43346	1̄.97234	5	46
15	1̄.53922	35	1̄.56693	39	0.43307	1̄.97229	5	45
16	1̄.53957	34	1̄.56732	39	0.43268	1̄.97224	4	44
17	1̄.53991	34	1̄.56771	39	0.43229	1̄.97220	5	43
18	1̄.54025	34	1̄.56810	39	0.43190	1̄.97215	5	42
19	1̄.54059	34	1̄.56849	38	0.43151	1̄.97210	4	41
20	1̄.54093	34	1̄.56887	39	0.43113	1̄.97206	5	40
21	1̄.54127	34	1̄.56926	39	0.43074	1̄.97201	5	39
22	1̄.54161	34	1̄.56965	39	0.43035	1̄.97196	4	38
23	1̄.54195	34	1̄.57004	38	0.42996	1̄.97192	5	37
24	1̄.54229	34	1̄.57042	39	0.42958	1̄.97187	5	36
25	1̄.54263	34	1̄.57081	39	0.42919	1̄.97182	4	35
26	1̄.54297	34	1̄.57120	38	0.42880	1̄.97178	5	34
27	1̄.54331	34	1̄.57158	39	0.42842	1̄.97173	5	33
28	1̄.54365	34	1̄.57197	38	0.42803	1̄.97168	5	32
29	1̄.54399	34	1̄.57235	39	0.42765	1̄.97163	4	31
30	1̄.54433	33	1̄.57274	38	0.42726	1̄.97159	5	30
31	1̄.54466	34	1̄.57312	39	0.42688	1̄.97154	5	29
32	1̄.54500	34	1̄.57351	38	0.42649	1̄.97149	4	28
33	1̄.54534	33	1̄.57389	39	0.42611	1̄.97145	5	27
34	1̄.54567	34	1̄.57428	38	0.42572	1̄.97140	5	26
35	1̄.54601	34	1̄.57466	38	0.42534	1̄.97135	5	25
36	1̄.54635	33	1̄.57504	39	0.42496	1̄.97130	4	24
37	1̄.54668	34	1̄.57543	38	0.42457	1̄.97126	5	23
38	1̄.54702	33	1̄.57581	38	0.42419	1̄.97121	5	22
39	1̄.54735	34	1̄.57619	39	0.42381	1̄.97116	5	21
40	1̄.54769	33	1̄.57658	38	0.42342	1̄.97111	4	20
41	1̄.54802	34	1̄.57696	38	0.42304	1̄.97107	5	19
42	1̄.54836	33	1̄.57734	38	0.42266	1̄.97102	5	18
43	1̄.54869	34	1̄.57772	38	0.42228	1̄.97097	5	17
44	1̄.54903	33	1̄.57810	39	0.42190	1̄.97092	5	16
45	1̄.54936	33	1̄.57849	38	0.42151	1̄.97087	4	15
46	1̄.54969	34	1̄.57887	38	0.42113	1̄.97083	5	14
47	1̄.55003	33	1̄.57925	38	0.42075	1̄.97078	5	13
48	1̄.55036	33	1̄.57963	38	0.42037	1̄.97073	5	12
49	1̄.55069	33	1̄.58001	38	0.41999	1̄.97068	5	11
50	1̄.55102	34	1̄.58039	38	0.41961	1̄.97063	4	10
51	1̄.55136	33	1̄.58077	38	0.41923	1̄.97059	5	9
52	1̄.55169	33	1̄.58115	38	0.41885	1̄.97054	5	8
53	1̄.55202	33	1̄.58153	38	0.41847	1̄.97049	5	7
54	1̄.55235	33	1̄.58191	38	0.41809	1̄.97044	5	6
55	1̄.55268	33	1̄.58229	38	0.41771	1̄.97039	4	5
56	1̄.55301	33	1̄.58267	37	0.41733	1̄.97035	5	4
57	1̄.55334	33	1̄.58304	38	0.41696	1̄.97030	5	3
58	1̄.55367	33	1̄.58342	38	0.41658	1̄.97025	5	2
59	1̄.55400	33	1̄.58380	38	0.41620	1̄.97020	5	1
60	1̄.55433		1̄.58418		0.41582	1̄.97015		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

	38	37
6	3.8	3.7
7	4.4	4.3
8	5.1	4.9
9	5.7	5.6
10	6.3	6.2
20	12.7	12.3
30	19.0	18.5
40	25.3	24.7
50	31.7	30.8

	35
6	3.5
7	4.1
8	4.7
9	5.3
10	5.8
20	11.7
30	17.5
40	23.3
50	29.2

	34	33
6	3.4	3.3
7	4.0	3.9
8	4.5	4.4
9	5.1	5.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.55433	33	$\bar{1}$.58418	37	0.41582	$\bar{1}$.97015	5	60
1	$\bar{1}$.55466	33	$\bar{1}$.58455	38	0.41545	$\bar{1}$.97010	5	59
2	$\bar{1}$.55499	33	$\bar{1}$.58493	38	0.41507	$\bar{1}$.97005	4	58
3	$\bar{1}$.55532	32	$\bar{1}$.58531	38	0.41469	$\bar{1}$.97001	5	57
4	$\bar{1}$.55564	33	$\bar{1}$.58569	37	0.41431	$\bar{1}$.96996	5	56
5	$\bar{1}$.55597	33	$\bar{1}$.58606	38	0.41394	$\bar{1}$.96991	5	55
6	$\bar{1}$.55630	33	$\bar{1}$.58644	37	0.41356	$\bar{1}$.96986	5	54
7	$\bar{1}$.55663	32	$\bar{1}$.58681	38	0.41319	$\bar{1}$.96981	5	53
8	$\bar{1}$.55695	33	$\bar{1}$.58719	38	0.41281	$\bar{1}$.96976	5	52
9	$\bar{1}$.55728	33	$\bar{1}$.58757	37	0.41243	$\bar{1}$.96971	5	51
10	$\bar{1}$.55761	32	$\bar{1}$.58794	38	0.41206	$\bar{1}$.96966	4	50
11	$\bar{1}$.55793	33	$\bar{1}$.58832	37	0.41168	$\bar{1}$.96962	5	49
12	$\bar{1}$.55826	32	$\bar{1}$.58869	38	0.41131	$\bar{1}$.96957	5	48
13	$\bar{1}$.55858	33	$\bar{1}$.58907	37	0.41093	$\bar{1}$.96952	5	47
14	$\bar{1}$.55891	32	$\bar{1}$.58944	37	0.41056	$\bar{1}$.96947	5	46
15	$\bar{1}$.55923	33	$\bar{1}$.58981	38	0.41019	$\bar{1}$.96942	5	45
16	$\bar{1}$.55956	32	$\bar{1}$.59019	37	0.40981	$\bar{1}$.96937	5	44
17	$\bar{1}$.55988	33	$\bar{1}$.59056	38	0.40944	$\bar{1}$.96932	5	43
18	$\bar{1}$.56021	32	$\bar{1}$.59094	37	0.40906	$\bar{1}$.96927	5	42
19	$\bar{1}$.56053	32	$\bar{1}$.59131	37	0.40869	$\bar{1}$.96922	5	41
20	$\bar{1}$.56085	33	$\bar{1}$.59168	37	0.40832	$\bar{1}$.96917	5	40
21	$\bar{1}$.56118	32	$\bar{1}$.59205	38	0.40795	$\bar{1}$.96912	5	39
22	$\bar{1}$.56150	32	$\bar{1}$.59243	37	0.40757	$\bar{1}$.96907	4	38
23	$\bar{1}$.56182	33	$\bar{1}$.59280	37	0.40720	$\bar{1}$.96903	5	37
24	$\bar{1}$.56215	32	$\bar{1}$.59317	37	0.40683	$\bar{1}$.96898	5	36
25	$\bar{1}$.56247	32	$\bar{1}$.59354	37	0.40646	$\bar{1}$.96893	5	35
26	$\bar{1}$.56279	32	$\bar{1}$.59391	38	0.40609	$\bar{1}$.96888	5	34
27	$\bar{1}$.56311	32	$\bar{1}$.59429	37	0.40571	$\bar{1}$.96883	5	33
28	$\bar{1}$.56343	32	$\bar{1}$.59466	37	0.40534	$\bar{1}$.96878	5	32
29	$\bar{1}$.56375	33	$\bar{1}$.59503	37	0.40497	$\bar{1}$.96873	5	31
30	$\bar{1}$.56408	32	$\bar{1}$.59540	37	0.40460	$\bar{1}$.96868	5	30
31	$\bar{1}$.56440	32	$\bar{1}$.59577	37	0.40423	$\bar{1}$.96863	5	29
32	$\bar{1}$.56472	32	$\bar{1}$.59614	37	0.40386	$\bar{1}$.96858	5	28
33	$\bar{1}$.56504	32	$\bar{1}$.59651	37	0.40349	$\bar{1}$.96853	5	27
34	$\bar{1}$.56536	32	$\bar{1}$.59688	37	0.40312	$\bar{1}$.96848	5	26
35	$\bar{1}$.56568	31	$\bar{1}$.59725	37	0.40275	$\bar{1}$.96843	5	25
36	$\bar{1}$.56599	32	$\bar{1}$.59762	37	0.40238	$\bar{1}$.96838	5	24
37	$\bar{1}$.56631	32	$\bar{1}$.59799	36	0.40201	$\bar{1}$.96833	5	23
38	$\bar{1}$.56663	32	$\bar{1}$.59835	37	0.40165	$\bar{1}$.96828	5	22
39	$\bar{1}$.56695	32	$\bar{1}$.59872	37	0.40128	$\bar{1}$.96823	5	21
40	$\bar{1}$.56727	32	$\bar{1}$.59909	37	0.40091	$\bar{1}$.96818	5	20
41	$\bar{1}$.56759	31	$\bar{1}$.59946	37	0.40054	$\bar{1}$.96813	5	19
42	$\bar{1}$.56790	32	$\bar{1}$.59983	36	0.40017	$\bar{1}$.96808	5	18
43	$\bar{1}$.56822	32	$\bar{1}$.60019	37	0.39981	$\bar{1}$.96803	5	17
44	$\bar{1}$.56854	32	$\bar{1}$.60056	37	0.39944	$\bar{1}$.96798	5	16
45	$\bar{1}$.56886	31	$\bar{1}$.60093	37	0.39907	$\bar{1}$.96793	5	15
46	$\bar{1}$.56917	32	$\bar{1}$.60130	36	0.39870	$\bar{1}$.96788	5	14
47	$\bar{1}$.56949	31	$\bar{1}$.60166	37	0.39834	$\bar{1}$.96783	5	13
48	$\bar{1}$.56980	32	$\bar{1}$.60203	37	0.39797	$\bar{1}$.96778	6	12
49	$\bar{1}$.57012	32	$\bar{1}$.60240	36	0.39760	$\bar{1}$.96772	5	11
50	$\bar{1}$.57044	31	$\bar{1}$.60276	37	0.39724	$\bar{1}$.96767	5	10
51	$\bar{1}$.57075	32	$\bar{1}$.60313	36	0.39687	$\bar{1}$.96762	5	9
52	$\bar{1}$.57107	31	$\bar{1}$.60349	37	0.39651	$\bar{1}$.96757	5	8
53	$\bar{1}$.57138	31	$\bar{1}$.60386	36	0.39614	$\bar{1}$.96752	5	7
54	$\bar{1}$.57169	32	$\bar{1}$.60422	37	0.39578	$\bar{1}$.96747	5	6
55	$\bar{1}$.57201	31	$\bar{1}$.60459	36	0.39541	$\bar{1}$.96742	5	5
56	$\bar{1}$.57232	32	$\bar{1}$.60495	37	0.39505	$\bar{1}$.96737	5	4
57	$\bar{1}$.57264	31	$\bar{1}$.60532	36	0.39468	$\bar{1}$.96732	5	3
58	$\bar{1}$.57295	31	$\bar{1}$.60568	37	0.39432	$\bar{1}$.96727	5	2
59	$\bar{1}$.57326	32	$\bar{1}$.60605	36	0.39395	$\bar{1}$.96722	5	1
60	$\bar{1}$.57358		$\bar{1}$.60641		0.39359	$\bar{1}$.96717		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	38	37
6	3.8	3.7
7	4.4	4.3
8	5.1	4.9
9	5.7	5.6
10	6.3	6.2
20	12.7	12.3
30	19.0	18.5
40	25.3	24.7
50	31.7	30.8

	36	33
6	3.6	3.3
7	4.2	3.9
8	4.8	4.4
9	5.4	5.0
10	6.0	5.5
20	12.0	11.0
30	18.0	16.5
40	24.0	22.0
50	30.0	27.5

	32
6	3.2
7	3.7
8	4.3
9	4.8
10	5.3
20	10.7
30	16.0
40	21.3
50	26.7

	31	6
6	3.1	0.6
7	3.6	0.7
8	4.1	0.8
9	4.7	0.9
10	5.2	1.0
20	10.3	2.0
30	15.5	3.0
40	20.7	4.0
50	25.8	5.0

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

p. p.

22°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.57358		$\bar{1}$.60641		0.39359	$\bar{1}$.96717		60
		31		36			6	
1	$\bar{1}$.57389		$\bar{1}$.60677		0.39323	$\bar{1}$.96711		59
		31		37			5	
2	$\bar{1}$.57420		$\bar{1}$.60714		0.39286	$\bar{1}$.96706		58
		31		36			5	
3	$\bar{1}$.57451		$\bar{1}$.60750		0.39250	$\bar{1}$.96701		57
		31		36			5	
4	$\bar{1}$.57482		$\bar{1}$.60786		0.39214	$\bar{1}$.96696		56
		32		37			5	
5	$\bar{1}$.57514		$\bar{1}$.60823		0.39177	$\bar{1}$.96691		55
		31		36			5	
6	$\bar{1}$.57545		$\bar{1}$.60859		0.39141	$\bar{1}$.96686		54
		31		36			5	
7	$\bar{1}$.57576		$\bar{1}$.60895		0.39105	$\bar{1}$.96681		53
		31		36			5	
8	$\bar{1}$.57607		$\bar{1}$.60931		0.39069	$\bar{1}$.96676		52
		31		36			6	
9	$\bar{1}$.57638		$\bar{1}$.60967		0.39033	$\bar{1}$.96670		51
		31		37			5	
10	$\bar{1}$.57669		$\bar{1}$.61004		0.38996	$\bar{1}$.96665		50
		31		36			5	
11	$\bar{1}$.57700		$\bar{1}$.61040		0.38960	$\bar{1}$.96660		49
		31		36			5	
12	$\bar{1}$.57731		$\bar{1}$.61076		0.38924	$\bar{1}$.96655		48
		31		36			5	
13	$\bar{1}$.57762		$\bar{1}$.61112		0.38888	$\bar{1}$.96650		47
		31		36			5	
14	$\bar{1}$.57793		$\bar{1}$.61148		0.38852	$\bar{1}$.96645		46
		31		36			5	
15	$\bar{1}$.57824		$\bar{1}$.61184		0.38816	$\bar{1}$.96640		45
		31		36			6	
16	$\bar{1}$.57855		$\bar{1}$.61220		0.38780	$\bar{1}$.96634		44
		30		36			5	
17	$\bar{1}$.57885		$\bar{1}$.61256		0.38744	$\bar{1}$.96629		43
		31		36			5	
18	$\bar{1}$.57916		$\bar{1}$.61292		0.38708	$\bar{1}$.96624		42
		31		36			5	
19	$\bar{1}$.57947		$\bar{1}$.61328		0.38672	$\bar{1}$.96619		41
		31		36			5	
20	$\bar{1}$.57978		$\bar{1}$.61364		0.38636	$\bar{1}$.96614		40
		30		36			6	
21	$\bar{1}$.58008		$\bar{1}$.61400		0.38600	$\bar{1}$.96608		39
		31		36			5	
22	$\bar{1}$.58039		$\bar{1}$.61436		0.38564	$\bar{1}$.96603		38
		31		36			5	
23	$\bar{1}$.58070		$\bar{1}$.61472		0.38528	$\bar{1}$.96598		37
		31		36			5	
24	$\bar{1}$.58101		$\bar{1}$.61508		0.38492	$\bar{1}$.96593		36
		30		36			5	
25	$\bar{1}$.58131		$\bar{1}$.61544		0.38456	$\bar{1}$.96588		35
		31		35			6	
26	$\bar{1}$.58162		$\bar{1}$.61579		0.38421	$\bar{1}$.96582		34
		30		36			5	
27	$\bar{1}$.58192		$\bar{1}$.61615		0.38385	$\bar{1}$.96577		33
		31		36			5	
28	$\bar{1}$.58223		$\bar{1}$.61651		0.38349	$\bar{1}$.96572		32
		30		36			5	
29	$\bar{1}$.58253		$\bar{1}$.61687		0.38313	$\bar{1}$.96567		31
		31		35			5	
30	$\bar{1}$.58284		$\bar{1}$.61722		0.38278	$\bar{1}$.96562		30
		30		36			6	
31	$\bar{1}$.58314		$\bar{1}$.61758		0.38242	$\bar{1}$.96556		29
		31		36			5	
32	$\bar{1}$.58345		$\bar{1}$.61794		0.38206	$\bar{1}$.96551		28
		30		36			5	
33	$\bar{1}$.58375		$\bar{1}$.61830		0.38170	$\bar{1}$.96546		27
		31		35			5	
34	$\bar{1}$.58406		$\bar{1}$.61865		0.38135	$\bar{1}$.96541		26
		30		36			6	
35	$\bar{1}$.58436		$\bar{1}$.61901		0.38099	$\bar{1}$.96535		25
		31		35			5	
36	$\bar{1}$.58467		$\bar{1}$.61936		0.38064	$\bar{1}$.96530		24
		30		36			5	
37	$\bar{1}$.58497		$\bar{1}$.61972		0.38028	$\bar{1}$.96525		23
		30		36			5	
38	$\bar{1}$.58527		$\bar{1}$.62008		0.37992	$\bar{1}$.96520		22
		30		35			6	
39	$\bar{1}$.58557		$\bar{1}$.62043		0.37957	$\bar{1}$.96514		21
		31		36			5	
40	$\bar{1}$.58588		$\bar{1}$.62079		0.37921	$\bar{1}$.96509		20
		30		35			5	
41	$\bar{1}$.58618		$\bar{1}$.62114		0.37886	$\bar{1}$.96504		19
		30		36			6	
42	$\bar{1}$.58648		$\bar{1}$.62150		0.37850	$\bar{1}$.96498		18
		30		35			5	
43	$\bar{1}$.58678		$\bar{1}$.62185		0.37815	$\bar{1}$.96493		17
		31		36			5	
44	$\bar{1}$.58709		$\bar{1}$.62221		0.37779	$\bar{1}$.96488		16
		30		35			5	
45	$\bar{1}$.58739		$\bar{1}$.62256		0.37744	$\bar{1}$.96483		15
		30		36			6	
46	$\bar{1}$.58769		$\bar{1}$.62292		0.37708	$\bar{1}$.96477		14
		30		35			5	
47	$\bar{1}$.58799		$\bar{1}$.62327		0.37673	$\bar{1}$.96472		13
		30		35			5	
48	$\bar{1}$.58829		$\bar{1}$.62362		0.37638	$\bar{1}$.96467		12
		30		36			6	
49	$\bar{1}$.58859		$\bar{1}$.62398		0.37602	$\bar{1}$.96461		11
		30		35			5	
50	$\bar{1}$.58889		$\bar{1}$.62433		0.37567	$\bar{1}$.96456		10
		30		35			5	
51	$\bar{1}$.58919		$\bar{1}$.62468		0.37532	$\bar{1}$.96451		9
		30		36			6	
52	$\bar{1}$.58949		$\bar{1}$.62504		0.37496	$\bar{1}$.96445		8
		30		35			5	
53	$\bar{1}$.58979		$\bar{1}$.62539		0.37461	$\bar{1}$.96440		7
		30		35			5	
54	$\bar{1}$.59009		$\bar{1}$.62574		0.37426	$\bar{1}$.96435		6
		30		35			6	
55	$\bar{1}$.59039		$\bar{1}$.62609		0.37391	$\bar{1}$.96429		5
		30		36			5	
56	$\bar{1}$.59069		$\bar{1}$.62645		0.37355	$\bar{1}$.96424		4
		29		35			5	
57	$\bar{1}$.59098		$\bar{1}$.62680		0.37320	$\bar{1}$.96419		3
		30		35			6	
58	$\bar{1}$.59128		$\bar{1}$.62715		0.37285	$\bar{1}$.96413		2
		30		35			5	
59	$\bar{1}$.59158		$\bar{1}$.62750		0.37250	$\bar{1}$.96408		1
		30		35			5	
60	$\bar{1}$.59188		$\bar{1}$.62785		0.37215	$\bar{1}$.96403		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	35
6	3.5
7	4.1
8	4.7
9	5.3
10	5.8
20	11.7
30	17.5
40	23.3
50	29.2

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	30	29
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.59188	30	1̄.62785	35	0.37215	1̄.96403	6	60
1	1̄.59218	29	1̄.62820	35	0.37180	1̄.96397	5	59
2	1̄.59247	30	1̄.62855	35	0.37145	1̄.96392	5	58
3	1̄.59277	30	1̄.62890	36	0.37110	1̄.96387	6	57
4	1̄.59307	29	1̄.62926	35	0.37074	1̄.96381	5	56
5	1̄.59336	30	1̄.62961	35	0.37039	1̄.96376	6	55
6	1̄.59366	30	1̄.62996	35	0.37004	1̄.96370	5	54
7	1̄.59396	29	1̄.63031	35	0.36969	1̄.96365	5	53
8	1̄.59425	30	1̄.63066	35	0.36934	1̄.96360	6	52
9	1̄.59455	29	1̄.63101	34	0.36899	1̄.96354	5	51
10	1̄.59484		1̄.63135		0.36865	1̄.96349		50
		30		35			6	
11	1̄.59514	29	1̄.63170	35	0.36830	1̄.96343	5	49
12	1̄.59543	30	1̄.63205	35	0.36795	1̄.96338	5	48
13	1̄.59573	29	1̄.63240	35	0.36760	1̄.96333	6	47
14	1̄.59602	30	1̄.63275	35	0.36725	1̄.96327	5	46
15	1̄.59632	29	1̄.63310	35	0.36690	1̄.96322	6	45
16	1̄.59661	29	1̄.63345	34	0.36655	1̄.96316	5	44
17	1̄.59690	30	1̄.63379	35	0.36621	1̄.96311	6	43
18	1̄.59720	29	1̄.63414	35	0.36586	1̄.96305	5	42
19	1̄.59749	29	1̄.63449	35	0.36551	1̄.96300	6	41
20	1̄.59778		1̄.63484		0.36516	1̄.96294		40
		30		35			5	
21	1̄.59808	29	1̄.63519	34	0.36481	1̄.96289	5	39
22	1̄.59837	29	1̄.63553	35	0.36447	1̄.96284	6	38
23	1̄.59866	29	1̄.63588	35	0.36412	1̄.96278	5	37
24	1̄.59895	29	1̄.63623	34	0.36377	1̄.96273	6	36
25	1̄.59924	30	1̄.63657	35	0.36343	1̄.96267	5	35
26	1̄.59954	29	1̄.63692	34	0.36308	1̄.96262	6	34
27	1̄.59983	29	1̄.63726	35	0.36274	1̄.96256	5	33
28	1̄.60012	29	1̄.63761	35	0.36239	1̄.96251	6	32
29	1̄.60041	29	1̄.63796	34	0.36204	1̄.96245	5	31
30	1̄.60070		1̄.63830		0.36170	1̄.96240		30
		29		35			6	
31	1̄.60099	29	1̄.63865	34	0.36135	1̄.96234	5	29
32	1̄.60128	29	1̄.63899	35	0.36101	1̄.96229	6	28
33	1̄.60157	29	1̄.63934	34	0.36066	1̄.96223	5	27
34	1̄.60186	29	1̄.63968	35	0.36032	1̄.96218	6	26
35	1̄.60215	29	1̄.64003	34	0.35997	1̄.96212	5	25
36	1̄.60244	29	1̄.64037	35	0.35963	1̄.96207	6	24
37	1̄.60273	29	1̄.64072	34	0.35928	1̄.96201	5	23
38	1̄.60302	29	1̄.64106	34	0.35894	1̄.96196	6	22
39	1̄.60331	28	1̄.64140	35	0.35860	1̄.96190	5	21
40	1̄.60359		1̄.64175		0.35825	1̄.96185		20
		29		34			6	
41	1̄.60388	29	1̄.64209	34	0.35791	1̄.96179	5	19
42	1̄.60417	29	1̄.64243	35	0.35757	1̄.96174	6	18
43	1̄.60446	28	1̄.64278	34	0.35722	1̄.96168	6	17
44	1̄.60474	29	1̄.64312	34	0.35688	1̄.96162	5	16
45	1̄.60503	29	1̄.64346	35	0.35654	1̄.96157	6	15
46	1̄.60532	29	1̄.64381	34	0.35619	1̄.96151	5	14
47	1̄.60561	28	1̄.64415	34	0.35585	1̄.96146	6	13
48	1̄.60589	29	1̄.64449	34	0.35551	1̄.96140	5	12
49	1̄.60618	28	1̄.64483	34	0.35517	1̄.96135	6	11
50	1̄.60646		1̄.64517		0.35483	1̄.96129		10
		29		35			6	
51	1̄.60675	29	1̄.64552	34	0.35448	1̄.96123	5	9
52	1̄.60704	28	1̄.64586	34	0.35414	1̄.96118	6	8
53	1̄.60732	29	1̄.64620	34	0.35380	1̄.96112	5	7
54	1̄.60761	28	1̄.64654	34	0.35346	1̄.96107	6	6
55	1̄.60789	29	1̄.64688	34	0.35312	1̄.96101	6	5
56	1̄.60818	28	1̄.64722	34	0.35278	1̄.96095	5	4
57	1̄.60846	29	1̄.64756	34	0.35244	1̄.96090	6	3
58	1̄.60875	28	1̄.64790	34	0.35210	1̄.96084	5	2
59	1̄.60903	28	1̄.64824	34	0.35176	1̄.96079	6	1
60	1̄.60931		1̄.64858		0.35142	1̄.96073		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	36	35
6	3.6	3.5
7	4.2	4.1
8	4.8	4.7
9	5.4	5.3
10	6.0	5.8
20	12.0	11.7
30	18.0	17.5
40	24.0	23.3
50	30.0	29.2

	34
6	3.4
7	4.0
8	4.5
9	5.1
10	5.7
20	11.3
30	17.0
40	22.7
50	28.3

	30	29
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	28
6	2.8
7	3.3
8	3.7
9	4.2
10	4.7
20	9.3
30	14.0
40	18.7
50	23.3

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.60931	29	1̄.64858	34	0.35142	1̄.96073	6	60
1	1̄.60960	28	1̄.64892	34	0.35108	1̄.96067	5	59
2	1̄.60988	28	1̄.64926	34	0.35074	1̄.96062	6	58
3	1̄.61016	29	1̄.64960	34	0.35040	1̄.96056	6	57
4	1̄.61045	28	1̄.64994	34	0.35006	1̄.96050	5	56
5	1̄.61073	28	1̄.65028	34	0.34972	1̄.96045	6	55
6	1̄.61101	28	1̄.65062	34	0.34938	1̄.96039	5	54
7	1̄.61129	29	1̄.65096	34	0.34904	1̄.96034	6	53
8	1̄.61158	28	1̄.65130	34	0.34870	1̄.96028	6	52
9	1̄.61186	28	1̄.65164	33	0.34836	1̄.96022	5	51
10	1̄.61214	28	1̄.65197	34	0.34803	1̄.96017	6	50
11	1̄.61242	28	1̄.65231	34	0.34769	1̄.96011	6	49
12	1̄.61270	28	1̄.65265	34	0.34735	1̄.96005	5	48
13	1̄.61298	28	1̄.65299	34	0.34701	1̄.96000	6	47
14	1̄.61326	28	1̄.65333	33	0.34667	1̄.95994	6	46
15	1̄.61354	28	1̄.65366	34	0.34634	1̄.95988	6	45
16	1̄.61382	29	1̄.65400	34	0.34600	1̄.95982	5	44
17	1̄.61411	27	1̄.65434	33	0.34566	1̄.95977	6	43
18	1̄.61438	28	1̄.65467	34	0.34533	1̄.95971	6	42
19	1̄.61466	28	1̄.65501	34	0.34499	1̄.95965	5	41
20	1̄.61494	28	1̄.65535	33	0.34465	1̄.95960	6	40
21	1̄.61522	28	1̄.65568	34	0.34432	1̄.95954	6	39
22	1̄.61550	28	1̄.65602	34	0.34398	1̄.95948	6	38
23	1̄.61578	28	1̄.65636	33	0.34364	1̄.95942	5	37
24	1̄.61606	28	1̄.65669	34	0.34331	1̄.95937	6	36
25	1̄.61634	28	1̄.65703	33	0.34297	1̄.95931	6	35
26	1̄.61662	27	1̄.65736	34	0.34264	1̄.95925	5	34
27	1̄.61689	28	1̄.65770	33	0.34230	1̄.95920	6	33
28	1̄.61717	28	1̄.65803	34	0.34197	1̄.95914	6	32
29	1̄.61745	28	1̄.65837	33	0.34163	1̄.95908	6	31
30	1̄.61773	27	1̄.65870	34	0.34130	1̄.95902	5	30
31	1̄.61800	28	1̄.65904	33	0.34096	1̄.95897	6	29
32	1̄.61828	28	1̄.65937	34	0.34063	1̄.95891	6	28
33	1̄.61856	27	1̄.65971	33	0.34029	1̄.95885	6	27
34	1̄.61883	28	1̄.66004	34	0.33996	1̄.95879	6	26
35	1̄.61911	28	1̄.66038	33	0.33962	1̄.95873	5	25
36	1̄.61939	27	1̄.66071	33	0.33929	1̄.95868	6	24
37	1̄.61966	28	1̄.66104	34	0.33896	1̄.95862	6	23
38	1̄.61994	27	1̄.66138	33	0.33862	1̄.95856	6	22
39	1̄.62021	28	1̄.66171	33	0.33829	1̄.95850	6	21
40	1̄.62049	27	1̄.66204	34	0.33796	1̄.95844	5	20
41	1̄.62076	28	1̄.66238	33	0.33762	1̄.95839	6	19
42	1̄.62104	27	1̄.66271	33	0.33729	1̄.95833	6	18
43	1̄.62131	28	1̄.66304	33	0.33696	1̄.95827	6	17
44	1̄.62159	27	1̄.66337	34	0.33663	1̄.95821	6	16
45	1̄.62186	28	1̄.66371	33	0.33629	1̄.95815	5	15
46	1̄.62214	27	1̄.66404	33	0.33596	1̄.95810	6	14
47	1̄.62241	27	1̄.66437	33	0.33563	1̄.95804	6	13
48	1̄.62268	28	1̄.66470	33	0.33530	1̄.95798	6	12
49	1̄.62296	27	1̄.66503	34	0.33497	1̄.95792	6	11
50	1̄.62323	27	1̄.66537	33	0.33463	1̄.95786	6	10
51	1̄.62350	27	1̄.66570	33	0.33430	1̄.95780	5	9
52	1̄.62377	28	1̄.66603	33	0.33397	1̄.95775	6	8
53	1̄.62405	27	1̄.66636	33	0.33364	1̄.95769	6	7
54	1̄.62432	27	1̄.66669	33	0.33331	1̄.95763	6	6
55	1̄.62459	27	1̄.66702	33	0.33298	1̄.95757	6	5
56	1̄.62486	27	1̄.66735	33	0.33265	1̄.95751	6	4
57	1̄.62513	28	1̄.66768	33	0.33232	1̄.95745	6	3
58	1̄.62541	27	1̄.66801	33	0.33199	1̄.95739	6	2
59	1̄.62568	27	1̄.66834	33	0.33166	1̄.95733	5	1
60	1̄.62595		1̄.66867		0.33133	1̄.95728		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

P. P.

	34	33
6	3.4	3.3
7	4.0	3.9
8	4.5	4.4
9	5.1	5.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

	29
6	2.9
7	3.4
8	3.9
9	4.4
10	4.8
20	9.7
30	14.5
40	19.3
50	24.2

	28
6	2.8
7	3.3
8	3.7
9	4.2
10	4.7
20	9.3
30	14.0
40	18.7
50	23.3

	27
6	2.7
7	3.2
8	3.6
9	4.1
10	4.5
20	9.0
30	13.5
40	18.0
50	22.5

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

P. P.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.62595	27	$\bar{1}$.66867	33	0.33133	$\bar{1}$.95728	6	60
1	$\bar{1}$.62622	27	$\bar{1}$.66900	33	0.33100	$\bar{1}$.95722	6	59
2	$\bar{1}$.62649	27	$\bar{1}$.66933	33	0.33067	$\bar{1}$.95716	6	58
3	$\bar{1}$.62676	27	$\bar{1}$.66966	33	0.33034	$\bar{1}$.95710	6	57
4	$\bar{1}$.62703	27	$\bar{1}$.66999	33	0.33001	$\bar{1}$.95704	6	56
5	$\bar{1}$.62730	27	$\bar{1}$.67032	33	0.32968	$\bar{1}$.95698	6	55
6	$\bar{1}$.62757	27	$\bar{1}$.67065	33	0.32935	$\bar{1}$.95692	6	54
7	$\bar{1}$.62784	27	$\bar{1}$.67098	33	0.32902	$\bar{1}$.95686	6	53
8	$\bar{1}$.62811	27	$\bar{1}$.67131	32	0.32869	$\bar{1}$.95680	6	52
9	$\bar{1}$.62838	27	$\bar{1}$.67163	33	0.32837	$\bar{1}$.95674	6	51
10	$\bar{1}$.62865	27	$\bar{1}$.67196	33	0.32804	$\bar{1}$.95668	5	50
11	$\bar{1}$.62892	26	$\bar{1}$.67229	33	0.32771	$\bar{1}$.95663	6	49
12	$\bar{1}$.62918	27	$\bar{1}$.67262	33	0.32738	$\bar{1}$.95657	6	48
13	$\bar{1}$.62945	27	$\bar{1}$.67295	32	0.32705	$\bar{1}$.95651	6	47
14	$\bar{1}$.62972	27	$\bar{1}$.67327	33	0.32673	$\bar{1}$.95645	6	46
15	$\bar{1}$.62999	27	$\bar{1}$.67360	33	0.32640	$\bar{1}$.95639	6	45
16	$\bar{1}$.63026	26	$\bar{1}$.67393	33	0.32607	$\bar{1}$.95633	6	44
17	$\bar{1}$.63052	27	$\bar{1}$.67426	32	0.32574	$\bar{1}$.95627	6	43
18	$\bar{1}$.63079	27	$\bar{1}$.67458	33	0.32542	$\bar{1}$.95621	6	42
19	$\bar{1}$.63106	27	$\bar{1}$.67491	33	0.32509	$\bar{1}$.95615	6	41
20	$\bar{1}$.63133	26	$\bar{1}$.67524	32	0.32476	$\bar{1}$.95609	6	40
21	$\bar{1}$.63159	27	$\bar{1}$.67556	33	0.32444	$\bar{1}$.95603	6	39
22	$\bar{1}$.63186	27	$\bar{1}$.67589	33	0.32411	$\bar{1}$.95597	6	38
23	$\bar{1}$.63213	26	$\bar{1}$.67622	32	0.32378	$\bar{1}$.95591	6	37
24	$\bar{1}$.63239	27	$\bar{1}$.67654	33	0.32346	$\bar{1}$.95585	6	36
25	$\bar{1}$.63266	26	$\bar{1}$.67687	32	0.32313	$\bar{1}$.95579	6	35
26	$\bar{1}$.63292	27	$\bar{1}$.67719	33	0.32281	$\bar{1}$.95573	6	34
27	$\bar{1}$.63319	26	$\bar{1}$.67752	33	0.32248	$\bar{1}$.95567	6	33
28	$\bar{1}$.63345	27	$\bar{1}$.67785	32	0.32215	$\bar{1}$.95561	6	32
29	$\bar{1}$.63372	26	$\bar{1}$.67817	33	0.32183	$\bar{1}$.95555	6	31
30	$\bar{1}$.63398	27	$\bar{1}$.67850	32	0.32150	$\bar{1}$.95549	6	30
31	$\bar{1}$.63425	26	$\bar{1}$.67882	33	0.32118	$\bar{1}$.95543	6	29
32	$\bar{1}$.63451	27	$\bar{1}$.67915	32	0.32085	$\bar{1}$.95537	6	28
33	$\bar{1}$.63478	26	$\bar{1}$.67947	33	0.32053	$\bar{1}$.95531	6	27
34	$\bar{1}$.63504	27	$\bar{1}$.67980	32	0.32020	$\bar{1}$.95525	6	26
35	$\bar{1}$.63531	26	$\bar{1}$.68012	32	0.31988	$\bar{1}$.95519	6	25
36	$\bar{1}$.63557	26	$\bar{1}$.68044	33	0.31956	$\bar{1}$.95513	6	24
37	$\bar{1}$.63583	27	$\bar{1}$.68077	32	0.31923	$\bar{1}$.95507	7	23
38	$\bar{1}$.63610	26	$\bar{1}$.68109	33	0.31891	$\bar{1}$.95500	6	22
39	$\bar{1}$.63636	26	$\bar{1}$.68142	32	0.31858	$\bar{1}$.95494	6	21
40	$\bar{1}$.63662	27	$\bar{1}$.68174	32	0.31826	$\bar{1}$.95488	6	20
41	$\bar{1}$.63689	26	$\bar{1}$.68206	33	0.31794	$\bar{1}$.95482	6	19
42	$\bar{1}$.63715	26	$\bar{1}$.68239	32	0.31761	$\bar{1}$.95476	6	18
43	$\bar{1}$.63741	26	$\bar{1}$.68271	32	0.31729	$\bar{1}$.95470	6	17
44	$\bar{1}$.63767	27	$\bar{1}$.68303	33	0.31697	$\bar{1}$.95464	6	16
45	$\bar{1}$.63794	26	$\bar{1}$.68336	32	0.31664	$\bar{1}$.95458	6	15
46	$\bar{1}$.63820	26	$\bar{1}$.68368	32	0.31632	$\bar{1}$.95452	6	14
47	$\bar{1}$.63846	26	$\bar{1}$.68400	32	0.31600	$\bar{1}$.95446	6	13
48	$\bar{1}$.63872	26	$\bar{1}$.68432	33	0.31568	$\bar{1}$.95440	6	12
49	$\bar{1}$.63898	26	$\bar{1}$.68465	32	0.31535	$\bar{1}$.95434	7	11
50	$\bar{1}$.63924	26	$\bar{1}$.68497	32	0.31503	$\bar{1}$.95427	6	10
51	$\bar{1}$.63950	26	$\bar{1}$.68529	32	0.31471	$\bar{1}$.95421	6	9
52	$\bar{1}$.63976	26	$\bar{1}$.68561	32	0.31439	$\bar{1}$.95415	6	8
53	$\bar{1}$.64002	26	$\bar{1}$.68593	33	0.31407	$\bar{1}$.95409	6	7
54	$\bar{1}$.64028	26	$\bar{1}$.68626	32	0.31374	$\bar{1}$.95403	6	6
55	$\bar{1}$.64054	26	$\bar{1}$.68658	32	0.31342	$\bar{1}$.95397	6	5
56	$\bar{1}$.64080	26	$\bar{1}$.68690	32	0.31310	$\bar{1}$.95391	7	4
57	$\bar{1}$.64106	26	$\bar{1}$.68722	32	0.31278	$\bar{1}$.95384	6	3
58	$\bar{1}$.64132	26	$\bar{1}$.68754	32	0.31246	$\bar{1}$.95378	6	2
59	$\bar{1}$.64158	26	$\bar{1}$.68786	32	0.31214	$\bar{1}$.95372	6	1
60	$\bar{1}$.64184		$\bar{1}$.68818		0.31182	$\bar{1}$.95366		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	33	32
6	3.3	3.2
7	3.9	3.7
8	4.4	4.3
9	5.0	4.8
10	5.5	5.3
20	11.0	10.7
30	16.5	16.0
40	22.0	21.3
50	27.5	26.7

	27
6	2.7
7	3.2
8	3.6
9	4.1
10	4.5
20	9.0
30	13.5
40	18.0
50	22.5

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	7
6	0.7
7	0.8
8	0.9
9	1.1
10	1.2
20	2.3
30	3.5
40	4.7
50	5.8

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

p. p.

26°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.64184	26	1̄.68818	32	0.31182	1̄.95366	6	60
1	1̄.64210	26	1̄.68850	32	0.31150	1̄.95360	6	59
2	1̄.64236	26	1̄.68882	32	0.31118	1̄.95354	6	58
3	1̄.64262	26	1̄.68914	32	0.31086	1̄.95348	7	57
4	1̄.64288	25	1̄.68946	32	0.31054	1̄.95341	6	56
5	1̄.64313	26	1̄.68978	32	0.31022	1̄.95335	6	55
6	1̄.64339	26	1̄.69010	32	0.30990	1̄.95329	6	54
7	1̄.64365	26	1̄.69042	32	0.30958	1̄.95323	6	53
8	1̄.64391	26	1̄.69074	32	0.30926	1̄.95317	7	52
9	1̄.64417	25	1̄.69106	32	0.30894	1̄.95310	6	51
10	1̄.64442	26	1̄.69138	32	0.30862	1̄.95304	6	50
11	1̄.64468	26	1̄.69170	32	0.30830	1̄.95298	6	49
12	1̄.64494	25	1̄.69202	32	0.30798	1̄.95292	6	48
13	1̄.64519	26	1̄.69234	32	0.30766	1̄.95286	7	47
14	1̄.64545	26	1̄.69266	32	0.30734	1̄.95279	6	46
15	1̄.64571	25	1̄.69298	31	0.30702	1̄.95273	6	45
16	1̄.64596	26	1̄.69329	32	0.30671	1̄.95267	6	44
17	1̄.64622	25	1̄.69361	32	0.30639	1̄.95261	7	43
18	1̄.64647	26	1̄.69393	32	0.30607	1̄.95254	6	42
19	1̄.64673	25	1̄.69425	32	0.30575	1̄.95248	6	41
20	1̄.64698	26	1̄.69457	31	0.30543	1̄.95242	6	40
21	1̄.64724	25	1̄.69488	32	0.30512	1̄.95236	7	39
22	1̄.64749	26	1̄.69520	32	0.30480	1̄.95229	6	38
23	1̄.64775	25	1̄.69552	32	0.30448	1̄.95223	6	37
24	1̄.64800	26	1̄.69584	31	0.30416	1̄.95217	6	36
25	1̄.64826	25	1̄.69615	32	0.30385	1̄.95211	7	35
26	1̄.64851	26	1̄.69647	32	0.30353	1̄.95204	6	34
27	1̄.64877	25	1̄.69679	31	0.30321	1̄.95198	6	33
28	1̄.64902	25	1̄.69710	32	0.30290	1̄.95192	7	32
29	1̄.64927	26	1̄.69742	32	0.30258	1̄.95185	6	31
30	1̄.64953	25	1̄.69774	31	0.30226	1̄.95179	6	30
31	1̄.64978	25	1̄.69805	32	0.30195	1̄.95173	6	29
32	1̄.65003	26	1̄.69837	31	0.30163	1̄.95167	7	28
33	1̄.65029	25	1̄.69868	32	0.30132	1̄.95160	6	27
34	1̄.65054	25	1̄.69900	32	0.30100	1̄.95154	6	26
35	1̄.65079	25	1̄.69932	31	0.30068	1̄.95148	7	25
36	1̄.65104	26	1̄.69963	32	0.30037	1̄.95141	6	24
37	1̄.65130	25	1̄.69995	31	0.30005	1̄.95135	6	23
38	1̄.65155	25	1̄.70026	32	0.29974	1̄.95129	7	22
39	1̄.65180	25	1̄.70058	31	0.29942	1̄.95122	6	21
40	1̄.65205	25	1̄.70089	32	0.29911	1̄.95116	6	20
41	1̄.65230	25	1̄.70121	31	0.29879	1̄.95110	7	19
42	1̄.65255	26	1̄.70152	32	0.29848	1̄.95103	6	18
43	1̄.65281	25	1̄.70184	31	0.29816	1̄.95097	7	17
44	1̄.65306	25	1̄.70215	32	0.29785	1̄.95090	6	16
45	1̄.65331	25	1̄.70247	31	0.29753	1̄.95084	6	15
46	1̄.65356	25	1̄.70278	31	0.29722	1̄.95078	7	14
47	1̄.65381	25	1̄.70309	32	0.29691	1̄.95071	6	13
48	1̄.65406	25	1̄.70341	31	0.29659	1̄.95065	6	12
49	1̄.65431	25	1̄.70372	32	0.29628	1̄.95059	7	11
50	1̄.65456	25	1̄.70404	31	0.29596	1̄.95052	6	10
51	1̄.65481	25	1̄.70435	31	0.29565	1̄.95046	7	9
52	1̄.65506	25	1̄.70466	32	0.29534	1̄.95039	6	8
53	1̄.65531	25	1̄.70498	31	0.29502	1̄.95033	6	7
54	1̄.65556	24	1̄.70529	31	0.29471	1̄.95027	7	6
55	1̄.65580	25	1̄.70560	32	0.29440	1̄.95020	6	5
56	1̄.65605	25	1̄.70592	31	0.29408	1̄.95014	7	4
57	1̄.65630	25	1̄.70623	31	0.29377	1̄.95007	6	3
58	1̄.65655	25	1̄.70654	31	0.29346	1̄.95001	6	2
59	1̄.65680	25	1̄.70685	32	0.29315	1̄.94995	7	1
60	1̄.65705		1̄.70717		0.29283	1̄.94988		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	24
6	2.4
7	2.8
8	3.2
9	3.6
10	4.0
20	8.0
30	12.0
40	16.0
50	20.0

	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.1	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

p. p.

63°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.65705	24	1̄.70717	31	0.29283	1̄.94988	6	60
1	1̄.65729	25	1̄.70748	31	0.29252	1̄.94982	7	59
2	1̄.65754	25	1̄.70779	31	0.29221	1̄.94975	6	58
3	1̄.65779	25	1̄.70810	31	0.29190	1̄.94969	7	57
4	1̄.65804	24	1̄.70841	32	0.29159	1̄.94962	6	56
5	1̄.65828	25	1̄.70873	31	0.29127	1̄.94956	7	55
6	1̄.65853	25	1̄.70904	31	0.29096	1̄.94949	6	54
7	1̄.65878	24	1̄.70935	31	0.29065	1̄.94943	7	53
8	1̄.65902	25	1̄.70966	31	0.29034	1̄.94936	6	52
9	1̄.65927	25	1̄.70997	31	0.29003	1̄.94930	7	51
10	1̄.65952	24	1̄.71028	31	0.28972	1̄.94923	6	50
11	1̄.65976	25	1̄.71059	31	0.28941	1̄.94917	6	49
12	1̄.66001	24	1̄.71090	31	0.28910	1̄.94911	7	48
13	1̄.66025	25	1̄.71121	32	0.28879	1̄.94904	6	47
14	1̄.66050	25	1̄.71153	31	0.28847	1̄.94898	7	46
15	1̄.66075	24	1̄.71184	31	0.28816	1̄.94891	6	45
16	1̄.66099	25	1̄.71215	31	0.28785	1̄.94885	7	44
17	1̄.66124	24	1̄.71246	31	0.28754	1̄.94878	7	43
18	1̄.66148	25	1̄.71277	31	0.28723	1̄.94871	6	42
19	1̄.66173	24	1̄.71308	31	0.28692	1̄.94865	7	41
20	1̄.66197	24	1̄.71339	31	0.28661	1̄.94858	6	40
21	1̄.66221	25	1̄.71370	31	0.28630	1̄.94852	7	39
22	1̄.66246	24	1̄.71401	30	0.28599	1̄.94845	6	38
23	1̄.66270	25	1̄.71431	31	0.28569	1̄.94839	7	37
24	1̄.66295	24	1̄.71462	31	0.28538	1̄.94832	6	36
25	1̄.66319	24	1̄.71493	31	0.28507	1̄.94826	7	35
26	1̄.66343	25	1̄.71524	31	0.28476	1̄.94819	6	34
27	1̄.66368	24	1̄.71555	31	0.28445	1̄.94813	7	33
28	1̄.66392	24	1̄.71586	31	0.28414	1̄.94806	7	32
29	1̄.66416	25	1̄.71617	31	0.28383	1̄.94799	6	31
30	1̄.66441	24	1̄.71648	31	0.28352	1̄.94793	7	30
31	1̄.66465	24	1̄.71679	30	0.28321	1̄.94786	6	29
32	1̄.66489	24	1̄.71709	31	0.28291	1̄.94780	7	28
33	1̄.66513	24	1̄.71740	31	0.28260	1̄.94773	6	27
34	1̄.66537	25	1̄.71771	31	0.28229	1̄.94767	7	26
35	1̄.66562	24	1̄.71802	31	0.28198	1̄.94760	7	25
36	1̄.66586	24	1̄.71833	30	0.28167	1̄.94753	6	24
37	1̄.66610	24	1̄.71863	31	0.28137	1̄.94747	7	23
38	1̄.66634	24	1̄.71894	31	0.28106	1̄.94740	6	22
39	1̄.66658	24	1̄.71925	30	0.28075	1̄.94734	7	21
40	1̄.66682	24	1̄.71955	31	0.28045	1̄.94727	7	20
41	1̄.66706	25	1̄.71986	31	0.28014	1̄.94720	6	19
42	1̄.66731	24	1̄.72017	31	0.27983	1̄.94714	7	18
43	1̄.66755	24	1̄.72048	30	0.27952	1̄.94707	7	17
44	1̄.66779	24	1̄.72078	31	0.27922	1̄.94700	6	16
45	1̄.66803	24	1̄.72109	31	0.27891	1̄.94694	7	15
46	1̄.66827	24	1̄.72140	30	0.27860	1̄.94687	7	14
47	1̄.66851	24	1̄.72170	31	0.27830	1̄.94680	6	13
48	1̄.66875	24	1̄.72201	30	0.27799	1̄.94674	7	12
49	1̄.66899	23	1̄.72231	31	0.27769	1̄.94667	7	11
50	1̄.66922	24	1̄.72262	31	0.27738	1̄.94660	6	10
51	1̄.66946	24	1̄.72293	30	0.27707	1̄.94654	7	9
52	1̄.66970	24	1̄.72323	31	0.27677	1̄.94647	7	8
53	1̄.66994	24	1̄.72354	30	0.27646	1̄.94640	6	7
54	1̄.67018	24	1̄.72384	31	0.27616	1̄.94634	7	6
55	1̄.67042	24	1̄.72415	30	0.27585	1̄.94627	7	5
56	1̄.67066	24	1̄.72445	31	0.27555	1̄.94620	6	4
57	1̄.67090	23	1̄.72476	30	0.27524	1̄.94614	7	3
58	1̄.67113	24	1̄.72506	31	0.27494	1̄.94607	7	2
59	1̄.67137	24	1̄.72537	30	0.27463	1̄.94600	7	1
60	1̄.67161		1̄.72567		0.27433	1̄.94593		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	30
6	3.0
7	3.5
8	4.0
9	4.5
10	5.0
20	10.0
30	15.0
40	20.0
50	25.0

	25	24
6	2.5	2.4
7	2.9	2.8
8	3.3	3.2
9	3.8	3.6
10	4.2	4.0
20	8.3	8.0
30	12.5	12.0
40	16.7	16.0
50	20.8	20.0

	23
6	2.3
7	2.7
8	3.1
9	3.5
10	3.8
20	7.7
30	11.5
40	15.3
50	19.2

	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.1	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	1̄.68557	23	1̄.74375	30	0.25625	1̄.94182	7	60
1	1̄.68580	23	1̄.74405	30	0.25595	1̄.94175	7	59
2	1̄.68603	22	1̄.74435	30	0.25565	1̄.94168	7	58
3	1̄.68625	23	1̄.74465	29	0.25535	1̄.94161	7	57
4	1̄.68648	23	1̄.74494	30	0.25506	1̄.94154	7	56
5	1̄.68671	23	1̄.74524	30	0.25476	1̄.94147	7	55
6	1̄.68694	22	1̄.74554	29	0.25446	1̄.94140	7	54
7	1̄.68716	23	1̄.74583	30	0.25417	1̄.94133	7	53
8	1̄.68739	23	1̄.74613	30	0.25387	1̄.94126	7	52
9	1̄.68762	22	1̄.74643	30	0.25357	1̄.94119	7	51
10	1̄.68784	23	1̄.74673	29	0.25327	1̄.94112	7	50
11	1̄.68807	22	1̄.74702	30	0.25298	1̄.94105	7	49
12	1̄.68829	23	1̄.74732	30	0.25268	1̄.94098	8	48
13	1̄.68852	23	1̄.74762	29	0.25238	1̄.94090	7	47
14	1̄.68875	22	1̄.74791	30	0.25209	1̄.94083	7	46
15	1̄.68897	23	1̄.74821	30	0.25179	1̄.94076	7	45
16	1̄.68920	22	1̄.74851	29	0.25149	1̄.94069	7	44
17	1̄.68942	23	1̄.74880	30	0.25120	1̄.94062	7	43
18	1̄.68965	22	1̄.74910	29	0.25090	1̄.94055	7	42
19	1̄.68987	23	1̄.74939	30	0.25061	1̄.94048	7	41
20	1̄.69010	22	1̄.74969	29	0.25031	1̄.94041	7	40
21	1̄.69032	23	1̄.74998	30	0.25002	1̄.94034	7	39
22	1̄.69055	22	1̄.75028	30	0.24972	1̄.94027	7	38
23	1̄.69077	23	1̄.75058	29	0.24942	1̄.94020	8	37
24	1̄.69100	22	1̄.75087	30	0.24913	1̄.94012	7	36
25	1̄.69122	22	1̄.75117	29	0.24883	1̄.94005	7	35
26	1̄.69144	23	1̄.75146	30	0.24854	1̄.93998	7	34
27	1̄.69167	22	1̄.75176	29	0.24824	1̄.93991	7	33
28	1̄.69189	23	1̄.75205	30	0.24795	1̄.93984	7	32
29	1̄.69212	22	1̄.75235	29	0.24765	1̄.93977	7	31
30	1̄.69234	22	1̄.75264	30	0.24736	1̄.93970	7	30
31	1̄.69256	23	1̄.75294	29	0.24706	1̄.93963	8	29
32	1̄.69279	22	1̄.75323	30	0.24677	1̄.93955	7	28
33	1̄.69301	22	1̄.75353	29	0.24647	1̄.93948	7	27
34	1̄.69323	22	1̄.75382	29	0.24618	1̄.93941	7	26
35	1̄.69345	23	1̄.75411	30	0.24589	1̄.93934	7	25
36	1̄.69368	22	1̄.75441	29	0.24559	1̄.93927	7	24
37	1̄.69390	22	1̄.75470	30	0.24530	1̄.93920	8	23
38	1̄.69412	22	1̄.75500	29	0.24500	1̄.93912	7	22
39	1̄.69434	22	1̄.75529	29	0.24471	1̄.93905	7	21
40	1̄.69456	23	1̄.75558	30	0.24442	1̄.93898	7	20
41	1̄.69479	22	1̄.75588	29	0.24412	1̄.93891	7	19
42	1̄.69501	22	1̄.75617	30	0.24383	1̄.93884	8	18
43	1̄.69523	22	1̄.75647	29	0.24353	1̄.93876	7	17
44	1̄.69545	22	1̄.75676	29	0.24324	1̄.93869	7	16
45	1̄.69567	22	1̄.75705	30	0.24295	1̄.93862	7	15
46	1̄.69589	22	1̄.75735	29	0.24265	1̄.93855	8	14
47	1̄.69611	22	1̄.75764	29	0.24236	1̄.93847	7	13
48	1̄.69633	22	1̄.75793	29	0.24207	1̄.93840	7	12
49	1̄.69655	22	1̄.75822	30	0.24178	1̄.93833	7	11
50	1̄.69677	22	1̄.75852	29	0.24148	1̄.93826	7	10
51	1̄.69699	22	1̄.75881	29	0.24119	1̄.93819	8	9
52	1̄.69721	22	1̄.75910	29	0.24090	1̄.93811	7	8
53	1̄.69743	22	1̄.75939	30	0.24061	1̄.93804	7	7
54	1̄.69765	22	1̄.75969	29	0.24031	1̄.93797	8	6
55	1̄.69787	22	1̄.75998	29	0.24002	1̄.93789	7	5
56	1̄.69809	22	1̄.76027	29	0.23973	1̄.93782	7	4
57	1̄.69831	22	1̄.76056	30	0.23944	1̄.93775	7	3
58	1̄.69853	22	1̄.76086	29	0.23914	1̄.93768	8	2
59	1̄.69875	22	1̄.76115	29	0.23885	1̄.93760	7	1
60	1̄.69897		1̄.76144		0.23856	1̄.93753		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	30	29	23	22	8	7
6	3.0	2.9	2.3	2.2	0.8	0.7
7	3.5	3.4	2.7	2.6	0.9	0.8
8	4.0	3.9	3.1	2.9	1.1	0.9
9	4.5	4.4	3.5	3.3	1.2	1.1
10	5.0	4.8	3.8	3.7	1.3	1.2
20	10.0	9.7	7.7	7.3	2.7	2.3
30	15.0	14.5	11.5	11.0	4.0	3.5
40	20.0	19.3	15.3	14.7	5.3	4.7
50	25.0	24.2	19.2	18.3	6.7	5.8

30°

'	log sin	d	log tan	c. d	log cot	log cos	d	'
0	[illegible]		[illegible]		[illegible]	[illegible]		60
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	59
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	58
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	56
5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	55
6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	54
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	53
8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	52
9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	51
10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	50
11	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	49
12	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	48
13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	47
14	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	46
15	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	45
16	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	44
17	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	43
18	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	42
19	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	41
20	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	40
21	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	39
22	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	38
23	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	37
24	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	36
25	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	35
26	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	34
27	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	33
28	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	32
29	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	31
30	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	30
31	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	29
32	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28
33	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	27
34	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	26
35	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	25
36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	24
37	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	23
38	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	22
39	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21
40	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	20
41	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	19
42	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18
43	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17
44	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	16
45	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	15
46	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14
47	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13
48	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	12
49	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11
50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10
51	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9
52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8
53	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7
54	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6
55	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5
56	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4
57	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3
58	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2
59	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1
60	$\bar{1}.71184$	21	$\bar{1}.77877$	[illegible]	0.22123	$\bar{1}.93307$	7	0
'	log cos	d	log cot	c. d	log tan	log sin	d	'

P. P.

	23	22
[illegible]	[illegible]	[illegible]

	22
6	2.2
7	2.6
8	2.9
9	3.3
10	3.7
20	7.3
30	11.0
40	14.7
50	18.3

	21
6	2.1
7	2.5
8	2.8
9	3.2
10	3.5
20	7.0
30	10.5
40	14.0
50	17.5

	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.1
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

59°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.71184	21	$\bar{1}$.77877	29	0.22123	$\bar{1}$.93307	8	60
1	$\bar{1}$.71205	21	$\bar{1}$.77906	29	0.22094	$\bar{1}$.93299	8	59
2	$\bar{1}$.71226	21	$\bar{1}$.77935	28	0.22065	$\bar{1}$.93291	7	58
3	$\bar{1}$.71247	21	$\bar{1}$.77963	29	0.22037	$\bar{1}$.93284	8	57
4	$\bar{1}$.71268	21	$\bar{1}$.77992	28	0.22008	$\bar{1}$.93276	7	56
5	$\bar{1}$.71289	21	$\bar{1}$.78020	29	0.21980	$\bar{1}$.93269	8	55
6	$\bar{1}$.71310	21	$\bar{1}$.78049	28	0.21951	$\bar{1}$.93261	8	54
7	$\bar{1}$.71331	21	$\bar{1}$.78077	29	0.21923	$\bar{1}$.93253	7	53
8	$\bar{1}$.71352	21	$\bar{1}$.78106	29	0.21894	$\bar{1}$.93246	8	52
9	$\bar{1}$.71373	20	$\bar{1}$.78135	28	0.21865	$\bar{1}$.93238	8	51
10	$\bar{1}$.71393	21	$\bar{1}$.78163	29	0.21837	$\bar{1}$.93230	7	50
11	$\bar{1}$.71414	21	$\bar{1}$.78192	28	0.21808	$\bar{1}$.93223	8	49
12	$\bar{1}$.71435	21	$\bar{1}$.78220	29	0.21780	$\bar{1}$.93215	8	48
13	$\bar{1}$.71456	21	$\bar{1}$.78249	28	0.21751	$\bar{1}$.93207	7	47
14	$\bar{1}$.71477	21	$\bar{1}$.78277	29	0.21723	$\bar{1}$.93200	8	46
15	$\bar{1}$.71498	21	$\bar{1}$.78306	28	0.21694	$\bar{1}$.93192	8	45
16	$\bar{1}$.71519	20	$\bar{1}$.78334	29	0.21666	$\bar{1}$.93184	7	44
17	$\bar{1}$.71539	21	$\bar{1}$.78363	28	0.21637	$\bar{1}$.93177	8	43
18	$\bar{1}$.71560	21	$\bar{1}$.78391	28	0.21609	$\bar{1}$.93169	8	42
19	$\bar{1}$.71581	21	$\bar{1}$.78419	29	0.21581	$\bar{1}$.93161	7	41
20	$\bar{1}$.71602	20	$\bar{1}$.78448	28	0.21552	$\bar{1}$.93154	8	40
21	$\bar{1}$.71622	21	$\bar{1}$.78476	29	0.21524	$\bar{1}$.93146	8	39
22	$\bar{1}$.71643	21	$\bar{1}$.78505	28	0.21495	$\bar{1}$.93138	7	38
23	$\bar{1}$.71664	21	$\bar{1}$.78533	29	0.21467	$\bar{1}$.93131	8	37
24	$\bar{1}$.71685	20	$\bar{1}$.78562	28	0.21438	$\bar{1}$.93123	8	36
25	$\bar{1}$.71705	21	$\bar{1}$.78590	28	0.21410	$\bar{1}$.93115	7	35
26	$\bar{1}$.71726	21	$\bar{1}$.78618	29	0.21382	$\bar{1}$.93108	8	34
27	$\bar{1}$.71747	20	$\bar{1}$.78647	28	0.21353	$\bar{1}$.93100	8	33
28	$\bar{1}$.71767	21	$\bar{1}$.78675	29	0.21325	$\bar{1}$.93092	8	32
29	$\bar{1}$.71788	21	$\bar{1}$.78704	28	0.21296	$\bar{1}$.93084	7	31
30	$\bar{1}$.71809	20	$\bar{1}$.78732	28	0.21268	$\bar{1}$.93077	8	30
31	$\bar{1}$.71829	21	$\bar{1}$.78760	29	0.21240	$\bar{1}$.93069	8	29
32	$\bar{1}$.71850	20	$\bar{1}$.78789	28	0.21211	$\bar{1}$.93061	8	28
33	$\bar{1}$.71870	21	$\bar{1}$.78817	28	0.21183	$\bar{1}$.93053	7	27
34	$\bar{1}$.71891	20	$\bar{1}$.78845	29	0.21155	$\bar{1}$.93046	8	26
35	$\bar{1}$.71911	21	$\bar{1}$.78874	28	0.21126	$\bar{1}$.93038	8	25
36	$\bar{1}$.71932	20	$\bar{1}$.78902	28	0.21098	$\bar{1}$.93030	8	24
37	$\bar{1}$.71952	21	$\bar{1}$.78930	29	0.21070	$\bar{1}$.93022	8	23
38	$\bar{1}$.71973	21	$\bar{1}$.78959	28	0.21041	$\bar{1}$.93014	7	22
39	$\bar{1}$.71994	20	$\bar{1}$.78987	28	0.21013	$\bar{1}$.93007	8	21
40	$\bar{1}$.72014	20	$\bar{1}$.79015	28	0.20985	$\bar{1}$.92999	8	20
41	$\bar{1}$.72034	21	$\bar{1}$.79043	29	0.20957	$\bar{1}$.92991	8	19
42	$\bar{1}$.72055	20	$\bar{1}$.79072	28	0.20928	$\bar{1}$.92983	7	18
43	$\bar{1}$.72075	21	$\bar{1}$.79100	28	0.20900	$\bar{1}$.92976	8	17
44	$\bar{1}$.72096	20	$\bar{1}$.79128	28	0.20872	$\bar{1}$.92968	8	16
45	$\bar{1}$.72116	21	$\bar{1}$.79156	29	0.20844	$\bar{1}$.92960	8	15
46	$\bar{1}$.72137	20	$\bar{1}$.79185	28	0.20815	$\bar{1}$.92952	8	14
47	$\bar{1}$.72157	20	$\bar{1}$.79213	28	0.20787	$\bar{1}$.92944	8	13
48	$\bar{1}$.72177	21	$\bar{1}$.79241	28	0.20759	$\bar{1}$.92936	7	12
49	$\bar{1}$.72198	20	$\bar{1}$.79269	28	0.20731	$\bar{1}$.92929	8	11
50	$\bar{1}$.72218	20	$\bar{1}$.79297	29	0.20703	$\bar{1}$.92921	8	10
51	$\bar{1}$.72238	21	$\bar{1}$.79326	28	0.20674	$\bar{1}$.92913	8	9
52	$\bar{1}$.72259	20	$\bar{1}$.79354	28	0.20646	$\bar{1}$.92905	8	8
53	$\bar{1}$.72279	20	$\bar{1}$.79382	28	0.20618	$\bar{1}$.92897	8	7
54	$\bar{1}$.72299	21	$\bar{1}$.79410	28	0.20590	$\bar{1}$.92889	8	6
55	$\bar{1}$.72320	20	$\bar{1}$.79438	28	0.20562	$\bar{1}$.92881	7	5
56	$\bar{1}$.72340	20	$\bar{1}$.79466	29	0.20534	$\bar{1}$.92874	8	4
57	$\bar{1}$.72360	21	$\bar{1}$.79495	28	0.20505	$\bar{1}$.92866	8	3
58	$\bar{1}$.72381	20	$\bar{1}$.79523	28	0.20477	$\bar{1}$.92858	8	2
59	$\bar{1}$.72401	20	$\bar{1}$.79551	28	0.20449	$\bar{1}$.92850	8	1
60	$\bar{1}$.72421		$\bar{1}$.79579		0.20421	$\bar{1}$.92842		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	29	28	21	20
6	2.9	2.8	2.1	2.0
7	3.4	3.3	2.5	2.3
8	3.9	3.7	2.8	2.7
9	4.4	4.2	3.2	3.0
10	4.8	4.7	3.5	3.3
20	9.7	9.3	7.0	6.7
30	14.5	14.0	10.5	10.0
40	19.3	18.7	14.0	13.3
50	24.2	23.3	17.5	16.7

	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.1
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

p. p.

'	log cos	d	log cot	c. d.	log tan	log sin	d	'
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10
[illegible]	1̄.72435	[illegible]	[illegible]	[illegible]	[illegible]	1̄.62433	[illegible]	9
[illegible]	1̄.72455	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8
[illegible]	1̄.72474	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1̄.62370	[illegible]	2
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1
[illegible]	[illegible]	[illegible]	1̄.81252	[illegible]	0.18748	1̄.62359	[illegible]	0
	log cos	d	log cot	c. d.	log tan	log sin	d	'

P. P.

	19
6	1.9
7	2.2
8	2.5
9	2.9
10	3.2
20	6.3
30	9.5
40	12.7
50	15.8

	9	8	7
6	0.9	0.8	0.7
7	1.1	0.9	0.8
8	1.2	1.1	0.9
9	1.4	1.2	1.1
10	1.5	1.3	1.2
20	3.0	2.7	2.3
30	4.5	4.0	3.5
40	6.0	5.3	4.7
50	7.5	6.7	5.8

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}.73611$	19	$\bar{1}.81252$	27	0.18748	$\bar{1}.92359$	8	60
1	$\bar{1}.73630$	20	$\bar{1}.81279$	28	0.18721	$\bar{1}.92351$	8	59
2	$\bar{1}.73650$	19	$\bar{1}.81307$	28	0.18693	$\bar{1}.92343$	8	58
3	$\bar{1}.73669$	20	$\bar{1}.81335$	27	0.18665	$\bar{1}.92335$	9	57
4	$\bar{1}.73689$	19	$\bar{1}.81362$	28	0.18638	$\bar{1}.92326$	8	56
5	$\bar{1}.73708$	19	$\bar{1}.81390$	28	0.18610	$\bar{1}.92318$	8	55
6	$\bar{1}.73727$	20	$\bar{1}.81418$	27	0.18582	$\bar{1}.92310$	8	54
7	$\bar{1}.73747$	19	$\bar{1}.81445$	28	0.18555	$\bar{1}.92302$	9	53
8	$\bar{1}.73766$	19	$\bar{1}.81473$	27	0.18527	$\bar{1}.92293$	8	52
9	$\bar{1}.73785$	20	$\bar{1}.81500$	28	0.18500	$\bar{1}.92285$	8	51
10	$\bar{1}.73805$	19	$\bar{1}.81528$	28	0.18472	$\bar{1}.92277$	8	50
11	$\bar{1}.73824$	19	$\bar{1}.81556$	27	0.18444	$\bar{1}.92269$	9	49
12	$\bar{1}.73843$	20	$\bar{1}.81583$	28	0.18417	$\bar{1}.92260$	8	48
13	$\bar{1}.73863$	19	$\bar{1}.81611$	27	0.18389	$\bar{1}.92252$	8	47
14	$\bar{1}.73882$	19	$\bar{1}.81638$	28	0.18362	$\bar{1}.92244$	9	46
15	$\bar{1}.73901$	20	$\bar{1}.81666$	27	0.18334	$\bar{1}.92235$	8	45
16	$\bar{1}.73921$	19	$\bar{1}.81693$	28	0.18307	$\bar{1}.92227$	8	44
17	$\bar{1}.73940$	19	$\bar{1}.81721$	27	0.18279	$\bar{1}.92219$	8	43
18	$\bar{1}.73959$	19	$\bar{1}.81748$	28	0.18252	$\bar{1}.92211$	9	42
19	$\bar{1}.73978$	19	$\bar{1}.81776$	27	0.18224	$\bar{1}.92202$	8	41
20	$\bar{1}.73997$	20	$\bar{1}.81803$	28	0.18197	$\bar{1}.92194$	8	40
21	$\bar{1}.74017$	19	$\bar{1}.81831$	27	0.18169	$\bar{1}.92186$	9	39
22	$\bar{1}.74036$	19	$\bar{1}.81858$	28	0.18142	$\bar{1}.92177$	8	38
23	$\bar{1}.74055$	19	$\bar{1}.81886$	27	0.18114	$\bar{1}.92169$	8	37
24	$\bar{1}.74074$	19	$\bar{1}.81913$	28	0.18087	$\bar{1}.92161$	9	36
25	$\bar{1}.74093$	20	$\bar{1}.81941$	27	0.18059	$\bar{1}.92152$	8	35
26	$\bar{1}.74113$	19	$\bar{1}.81968$	28	0.18032	$\bar{1}.92144$	8	34
27	$\bar{1}.74132$	19	$\bar{1}.81996$	27	0.18004	$\bar{1}.92136$	9	33
28	$\bar{1}.74151$	19	$\bar{1}.82023$	28	0.17977	$\bar{1}.92127$	8	32
29	$\bar{1}.74170$	19	$\bar{1}.82051$	27	0.17949	$\bar{1}.92119$	8	31
30	$\bar{1}.74189$	19	$\bar{1}.82078$	28	0.17922	$\bar{1}.92111$	9	30
31	$\bar{1}.74208$	19	$\bar{1}.82106$	27	0.17894	$\bar{1}.92102$	8	29
32	$\bar{1}.74227$	19	$\bar{1}.82133$	28	0.17867	$\bar{1}.92094$	8	28
33	$\bar{1}.74246$	19	$\bar{1}.82161$	27	0.17839	$\bar{1}.92086$	9	27
34	$\bar{1}.74265$	19	$\bar{1}.82188$	27	0.17812	$\bar{1}.92077$	8	26
35	$\bar{1}.74284$	19	$\bar{1}.82215$	28	0.17785	$\bar{1}.92069$	9	25
36	$\bar{1}.74303$	19	$\bar{1}.82243$	27	0.17757	$\bar{1}.92060$	8	24
37	$\bar{1}.74322$	19	$\bar{1}.82270$	28	0.17730	$\bar{1}.92052$	8	23
38	$\bar{1}.74341$	19	$\bar{1}.82298$	27	0.17702	$\bar{1}.92044$	9	22
39	$\bar{1}.74360$	19	$\bar{1}.82325$	27	0.17675	$\bar{1}.92035$	8	21
40	$\bar{1}.74379$	19	$\bar{1}.82352$	28	0.17648	$\bar{1}.92027$	9	20
41	$\bar{1}.74398$	19	$\bar{1}.82380$	27	0.17620	$\bar{1}.92018$	8	19
42	$\bar{1}.74417$	19	$\bar{1}.82407$	28	0.17593	$\bar{1}.92010$	8	18
43	$\bar{1}.74436$	19	$\bar{1}.82435$	27	0.17565	$\bar{1}.92002$	9	17
44	$\bar{1}.74455$	19	$\bar{1}.82462$	27	0.17538	$\bar{1}.91993$	8	16
45	$\bar{1}.74474$	19	$\bar{1}.82489$	28	0.17511	$\bar{1}.91985$	9	15
46	$\bar{1}.74493$	19	$\bar{1}.82517$	27	0.17483	$\bar{1}.91976$	8	14
47	$\bar{1}.74512$	19	$\bar{1}.82544$	27	0.17456	$\bar{1}.91968$	9	13
48	$\bar{1}.74531$	18	$\bar{1}.82571$	28	0.17429	$\bar{1}.91959$	8	12
49	$\bar{1}.74549$	19	$\bar{1}.82599$	27	0.17401	$\bar{1}.91951$	9	11
50	$\bar{1}.74568$	19	$\bar{1}.82626$	27	0.17374	$\bar{1}.91942$	8	10
51	$\bar{1}.74587$	19	$\bar{1}.82653$	28	0.17347	$\bar{1}.91934$	9	9
52	$\bar{1}.74606$	19	$\bar{1}.82681$	27	0.17319	$\bar{1}.91925$	8	8
53	$\bar{1}.74625$	19	$\bar{1}.82708$	27	0.17292	$\bar{1}.91917$	9	7
54	$\bar{1}.74644$	18	$\bar{1}.82735$	27	0.17265	$\bar{1}.91908$	8	6
55	$\bar{1}.74662$	19	$\bar{1}.82762$	28	0.17238	$\bar{1}.91900$	9	5
56	$\bar{1}.74681$	19	$\bar{1}.82790$	27	0.17210	$\bar{1}.91891$	8	4
57	$\bar{1}.74700$	19	$\bar{1}.82817$	27	0.17183	$\bar{1}.91883$	9	3
58	$\bar{1}.74719$	18	$\bar{1}.82844$	27	0.17156	$\bar{1}.91874$	8	2
59	$\bar{1}.74737$	19	$\bar{1}.82871$	28	0.17129	$\bar{1}.91866$	9	1
60	$\bar{1}.74756$		$\bar{1}.82899$		0.17101	$\bar{1}.91857$		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	28	27
6	2.8	2.7
7	3.3	3.2
8	3.7	3.6
9	4.2	4.1
10	4.7	4.5
20	9.3	9.0
30	14.0	13.5
40	18.7	18.0
50	23.3	22.5

	20
6	2.0
7	2.3
8	2.7
9	3.0
10	3.3
20	6.7
30	10.0
40	13.3
50	16.7

	19
6	1.9
7	2.2
8	2.5
9	2.9
10	3.2
20	6.3
30	9.5
40	12.7
50	15.8

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

p. p.

34°

′	log sin	d	log tan	c. d.	log cot	log cos	d	′
0	1̄.74756		1̄.82899		0.17101	1̄.91857		60
		19		27			8	
1	1̄.74775		1̄.82926		0.17074	1̄.91849		59
		19		27			9	
2	1̄.74794		1̄.82953		0.17047	1̄.91840		58
		18		27			8	
3	1̄.74812		1̄.82980		0.17020	1̄.91832		57
		19		28			9	
4	1̄.74831		1̄.83008		0.16992	1̄.91823		56
		19		27			8	
5	1̄.74850		1̄.83035		0.16965	1̄.91815		55
		18		27			9	
6	1̄.74868		1̄.83062		0.16938	1̄.91806		54
		19		27			8	
7	1̄.74887		1̄.83089		0.16911	1̄.91798		53
		19		28			9	
8	1̄.74906		1̄.83117		0.16883	1̄.91789		52
		18		27			8	
9	1̄.74924		1̄.83144		0.16856	1̄.91781		51
		19		27			9	
10	1̄.74943		1̄.83171		0.16829	1̄.91772		50
		18		27			9	
11	1̄.74961		1̄.83198		0.16802	1̄.91763		49
		19		27			8	
12	1̄.74980		1̄.83225		0.16775	1̄.91755		48
		19		27			9	
13	1̄.74999		1̄.83252		0.16748	1̄.91746		47
		18		28			8	
14	1̄.75017		1̄.83280		0.16720	1̄.91738		46
		19		27			9	
15	1̄.75036		1̄.83307		0.16693	1̄.91729		45
		18		27			9	
16	1̄.75054		1̄.83334		0.16666	1̄.91720		44
		19		27			8	
17	1̄.75073		1̄.83361		0.16639	1̄.91712		43
		18		27			9	
18	1̄.75091		1̄.83388		0.16612	1̄.91703		42
		19		27			8	
19	1̄.75110		1̄.83415		0.16585	1̄.91695		41
		18		27			9	
20	1̄.75128		1̄.83442		0.16558	1̄.91686		40
		19		28			9	
21	1̄.75147		1̄.83470		0.16530	1̄.91677		39
		18		27			8	
22	1̄.75165		1̄.83497		0.16503	1̄.91669		38
		19		27			9	
23	1̄.75184		1̄.83524		0.16476	1̄.91660		37
		18		27			9	
24	1̄.75202		1̄.83551		0.16449	1̄.91651		36
		19		27			8	
25	1̄.75221		1̄.83578		0.16422	1̄.91643		35
		18		27			9	
26	1̄.75239		1̄.83605		0.16395	1̄.91634		34
		19		27			9	
27	1̄.75258		1̄.83632		0.16368	1̄.91625		33
		18		27			8	
28	1̄.75276		1̄.83659		0.16341	1̄.91617		32
		18		27			9	
29	1̄.75294		1̄.83686		0.16314	1̄.91608		31
		19		27			9	
30	1̄.75313		1̄.83713		0.16287	1̄.91599		30
		18		27			8	
31	1̄.75331		1̄.83740		0.16260	1̄.91591		29
		19		28			9	
32	1̄.75350		1̄.83768		0.16232	1̄.91582		28
		18		27			9	
33	1̄.75368		1̄.83795		0.16205	1̄.91573		27
		18		27			8	
34	1̄.75386		1̄.83822		0.16178	1̄.91565		26
		19		27			9	
35	1̄.75405		1̄.83849		0.16151	1̄.91556		25
		18		27			9	
36	1̄.75423		1̄.83876		0.16124	1̄.91547		24
		18		27			9	
37	1̄.75441		1̄.83903		0.16097	1̄.91538		23
		18		27			8	
38	1̄.75459		1̄.83930		0.16070	1̄.91530		22
		19		27			9	
39	1̄.75478		1̄.83957		0.16043	1̄.91521		21
		18		27			9	
40	1̄.75496		1̄.83984		0.16016	1̄.91512		20
		18		27			8	
41	1̄.75514		1̄.84011		0.15989	1̄.91504		19
		19		27			9	
42	1̄.75533		1̄.84038		0.15962	1̄.91495		18
		18		27			9	
43	1̄.75551		1̄.84065		0.15935	1̄.91486		17
		18		27			9	
44	1̄.75569		1̄.84092		0.15908	1̄.91477		16
		18		27			8	
45	1̄.75587		1̄.84119		0.15881	1̄.91469		15
		18		27			9	
46	1̄.75605		1̄.84146		0.15854	1̄.91460		14
		19		27			9	
47	1̄.75624		1̄.84173		0.15827	1̄.91451		13
		18		27			9	
48	1̄.75642		1̄.84200		0.15800	1̄.91442		12
		18		27			9	
49	1̄.75660		1̄.84227		0.15773	1̄.91433		11
		18		27			8	
50	1̄.75678		1̄.84254		0.15746	1̄.91425		10
		18		26			9	
51	1̄.75696		1̄.84280		0.15720	1̄.91416		9
		18		27			9	
52	1̄.75714		1̄.84307		0.15693	1̄.91407		8
		19		27			9	
53	1̄.75733		1̄.84334		0.15666	1̄.91398		7
		18		27			9	
54	1̄.75751		1̄.84361		0.15639	1̄.91389		6
		18		27			8	
55	1̄.75769		1̄.84388		0.15612	1̄.91381		5
		18		27			9	
56	1̄.75787		1̄.84415		0.15585	1̄.91372		4
		18		27			9	
57	1̄.75805		1̄.84442		0.15558	1̄.91363		3
		18		27			9	
58	1̄.75823		1̄.84469		0.15531	1̄.91354		2
		18		27			9	
59	1̄.75841		1̄.84496		0.15504	1̄.91345		1
		18		27			9	
60	1̄.75859		1̄.84523		0.15477	1̄.91336		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	28	27
6	2.8	2.7
7	3.3	3.2
8	3.7	3.6
9	4.2	4.1
10	4.7	4.5
20	9.3	9.0
30	14.0	13.5
40	18.7	18.0
50	23.3	22.5

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	19
6	1.9
7	2.2
8	2.5
9	2.9
10	3.2
20	6.3
30	9.5
40	12.7
50	15.8

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

p. p.

55°

35°

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}.75859$	18	$\bar{1}.84523$	27	0.15477	$\bar{1}.91336$	8	60
1	$\bar{1}.75877$	18	$\bar{1}.84550$	26	0.15450	$\bar{1}.91328$	9	59
2	$\bar{1}.75895$	18	$\bar{1}.84576$	27	0.15424	$\bar{1}.91319$	9	58
3	$\bar{1}.75913$	18	$\bar{1}.84603$	27	0.15397	$\bar{1}.91310$	9	57
4	$\bar{1}.75931$	18	$\bar{1}.84630$	27	0.15370	$\bar{1}.91301$	9	56
5	$\bar{1}.75949$	18	$\bar{1}.84657$	27	0.15343	$\bar{1}.91292$	9	55
6	$\bar{1}.75967$	18	$\bar{1}.84684$	27	0.15316	$\bar{1}.91283$	9	54
7	$\bar{1}.75985$	18	$\bar{1}.84711$	27	0.15289	$\bar{1}.91274$	8	53
8	$\bar{1}.76003$	18	$\bar{1}.84738$	26	0.15262	$\bar{1}.91266$	9	52
9	$\bar{1}.76021$	18	$\bar{1}.84764$	27	0.15236	$\bar{1}.91257$	9	51
10	$\bar{1}.76039$		$\bar{1}.84791$		0.15209	$\bar{1}.91248$		50
		18		27			9	
11	$\bar{1}.76057$	18	$\bar{1}.84818$	27	0.15182	$\bar{1}.91239$	9	49
12	$\bar{1}.76075$	18	$\bar{1}.84845$	27	0.15155	$\bar{1}.91230$	9	48
13	$\bar{1}.76093$	18	$\bar{1}.84872$	27	0.15128	$\bar{1}.91221$	9	47
14	$\bar{1}.76111$	18	$\bar{1}.84899$	26	0.15101	$\bar{1}.91212$	9	46
15	$\bar{1}.76129$	17	$\bar{1}.84925$	27	0.15075	$\bar{1}.91203$	9	45
16	$\bar{1}.76146$	18	$\bar{1}.84952$	27	0.15048	$\bar{1}.91194$	9	44
17	$\bar{1}.76164$	18	$\bar{1}.84979$	27	0.15021	$\bar{1}.91185$	9	43
18	$\bar{1}.76182$	18	$\bar{1}.85006$	27	0.14994	$\bar{1}.91176$	9	42
19	$\bar{1}.76200$	18	$\bar{1}.85033$	26	0.14967	$\bar{1}.91167$	9	41
20	$\bar{1}.76218$		$\bar{1}.85059$		0.14941	$\bar{1}.91158$		40
		18		27			9	
21	$\bar{1}.76236$	17	$\bar{1}.85086$	27	0.14914	$\bar{1}.91149$	8	39
22	$\bar{1}.76253$	18	$\bar{1}.85113$	27	0.14887	$\bar{1}.91141$	9	38
23	$\bar{1}.76271$	18	$\bar{1}.85140$	26	0.14860	$\bar{1}.91132$	9	37
24	$\bar{1}.76289$	18	$\bar{1}.85166$	27	0.14834	$\bar{1}.91123$	9	36
25	$\bar{1}.76307$	17	$\bar{1}.85193$	27	0.14807	$\bar{1}.91114$	9	35
26	$\bar{1}.76324$	18	$\bar{1}.85220$	27	0.14780	$\bar{1}.91105$	9	34
27	$\bar{1}.76342$	18	$\bar{1}.85247$	26	0.14753	$\bar{1}.91096$	9	33
28	$\bar{1}.76360$	18	$\bar{1}.85273$	27	0.14727	$\bar{1}.91087$	9	32
29	$\bar{1}.76378$	17	$\bar{1}.85300$	27	0.14700	$\bar{1}.91078$	9	31
30	$\bar{1}.76395$		$\bar{1}.85327$		0.14673	$\bar{1}.91069$		30
		18		27			9	
31	$\bar{1}.76413$	18	$\bar{1}.85354$	26	0.14646	$\bar{1}.91060$	9	29
32	$\bar{1}.76431$	17	$\bar{1}.85380$	27	0.14620	$\bar{1}.91051$	9	28
33	$\bar{1}.76448$	18	$\bar{1}.85407$	27	0.14593	$\bar{1}.91042$	9	27
34	$\bar{1}.76466$	18	$\bar{1}.85434$	26	0.14566	$\bar{1}.91033$	10	26
35	$\bar{1}.76484$	17	$\bar{1}.85460$	27	0.14540	$\bar{1}.91023$	9	25
36	$\bar{1}.76501$	18	$\bar{1}.85487$	27	0.14513	$\bar{1}.91014$	9	24
37	$\bar{1}.76519$	18	$\bar{1}.85514$	26	0.14486	$\bar{1}.91005$	9	23
38	$\bar{1}.76537$	17	$\bar{1}.85540$	27	0.14460	$\bar{1}.90996$	9	22
39	$\bar{1}.76554$	18	$\bar{1}.85567$	27	0.14433	$\bar{1}.90987$	9	21
40	$\bar{1}.76572$		$\bar{1}.85594$		0.14406	$\bar{1}.90978$		20
		18		26			9	
41	$\bar{1}.76590$	17	$\bar{1}.85620$	27	0.14380	$\bar{1}.90969$	9	19
42	$\bar{1}.76607$	18	$\bar{1}.85647$	27	0.14353	$\bar{1}.90960$	9	18
43	$\bar{1}.76625$	17	$\bar{1}.85674$	26	0.14326	$\bar{1}.90951$	9	17
44	$\bar{1}.76642$	18	$\bar{1}.85700$	27	0.14300	$\bar{1}.90942$	9	16
45	$\bar{1}.76660$	17	$\bar{1}.85727$	27	0.14273	$\bar{1}.90933$	9	15
46	$\bar{1}.76677$	18	$\bar{1}.85754$	26	0.14246	$\bar{1}.90924$	9	14
47	$\bar{1}.76695$	17	$\bar{1}.85780$	27	0.14220	$\bar{1}.90915$	9	13
48	$\bar{1}.76712$	18	$\bar{1}.85807$	27	0.14193	$\bar{1}.90906$	10	12
49	$\bar{1}.76730$	17	$\bar{1}.85834$	26	0.14166	$\bar{1}.90896$	9	11
50	$\bar{1}.76747$		$\bar{1}.85860$		0.14140	$\bar{1}.90887$		10
		18		27			9	
51	$\bar{1}.76765$	17	$\bar{1}.85887$	26	0.14113	$\bar{1}.90878$	9	9
52	$\bar{1}.76782$	18	$\bar{1}.85913$	27	0.14087	$\bar{1}.90869$	9	8
53	$\bar{1}.76800$	17	$\bar{1}.85940$	27	0.14060	$\bar{1}.90860$	9	7
54	$\bar{1}.76817$	18	$\bar{1}.85967$	26	0.14033	$\bar{1}.90851$	9	6
55	$\bar{1}.76835$	17	$\bar{1}.85993$	27	0.14007	$\bar{1}.90842$	10	5
56	$\bar{1}.76852$	18	$\bar{1}.86020$	26	0.13980	$\bar{1}.90832$	9	4
57	$\bar{1}.76870$	17	$\bar{1}.86046$	27	0.13954	$\bar{1}.90823$	9	3
58	$\bar{1}.76887$	17	$\bar{1}.86073$	27	0.13927	$\bar{1}.90814$	9	2
59	$\bar{1}.76904$	18	$\bar{1}.86100$	26	0.13900	$\bar{1}.90805$	9	1
60	$\bar{1}.76922$		$\bar{1}.86126$		0.13874	$\bar{1}.90796$		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	27	26
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5
9	4.1	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	17
6	1.7
7	2.0
8	2.3
9	2.6
10	2.8
20	5.7
30	8.5
40	11.3
50	14.2

	10
6	1.0
7	1.2
8	1.3
9	1.5
10	1.7
20	3.3
30	5.0
40	6.7
50	8.3

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

p. p.

54°

	log cos	d	log cot	c. d.	log tan	log sin	d	′
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	1.77946	16	1.87711	26	0.12289	1.90235	9	0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.	10	9
6	1.0	0.9
7	1.2	1.1
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

′	log sin	d	log tan	c. d.	log cot	log cos	d	′
0	$\bar{1}$.77946	17	$\bar{1}$.87711	27	0.12289	$\bar{1}$.90235	10	60
1	$\bar{1}$.77963	17	$\bar{1}$.87738	26	0.12262	$\bar{1}$.90225	9	59
2	$\bar{1}$.77980	17	$\bar{1}$.87764	26	0.12236	$\bar{1}$.90216	10	58
3	$\bar{1}$.77997	16	$\bar{1}$.87790	27	0.12210	$\bar{1}$.90206	9	57
4	$\bar{1}$.78013	17	$\bar{1}$.87817	26	0.12183	$\bar{1}$.90197	10	56
5	$\bar{1}$.78030	17	$\bar{1}$.87843	26	0.12157	$\bar{1}$.90187	9	55
6	$\bar{1}$.78047	16	$\bar{1}$.87869	26	0.12131	$\bar{1}$.90178	10	54
7	$\bar{1}$.78063	17	$\bar{1}$.87895	27	0.12105	$\bar{1}$.90168	9	53
8	$\bar{1}$.78080	17	$\bar{1}$.87922	26	0.12078	$\bar{1}$.90159	10	52
9	$\bar{1}$.78097	16	$\bar{1}$.87948	26	0.12052	$\bar{1}$.90149	10	51
10	$\bar{1}$.78113	17	$\bar{1}$.87974	26	0.12026	$\bar{1}$.90139	9	50
11	$\bar{1}$.78130	17	$\bar{1}$.88000	27	0.12000	$\bar{1}$.90130	10	49
12	$\bar{1}$.78147	16	$\bar{1}$.88027	26	0.11973	$\bar{1}$.90120	9	48
13	$\bar{1}$.78163	17	$\bar{1}$.88053	26	0.11947	$\bar{1}$.90111	10	47
14	$\bar{1}$.78180	17	$\bar{1}$.88079	26	0.11921	$\bar{1}$.90101	10	46
15	$\bar{1}$.78197	16	$\bar{1}$.88105	26	0.11895	$\bar{1}$.90091	9	45
16	$\bar{1}$.78213	17	$\bar{1}$.88131	27	0.11869	$\bar{1}$.90082	10	44
17	$\bar{1}$.78230	16	$\bar{1}$.88158	26	0.11842	$\bar{1}$.90072	9	43
18	$\bar{1}$.78246	17	$\bar{1}$.88184	26	0.11816	$\bar{1}$.90063	10	42
19	$\bar{1}$.78263	17	$\bar{1}$.88210	26	0.11790	$\bar{1}$.90053	10	41
20	$\bar{1}$.78280	16	$\bar{1}$.88236	26	0.11764	$\bar{1}$.90043	9	40
21	$\bar{1}$.78296	17	$\bar{1}$.88262	27	0.11738	$\bar{1}$.90034	10	39
22	$\bar{1}$.78313	16	$\bar{1}$.88289	26	0.11711	$\bar{1}$.90024	10	38
23	$\bar{1}$.78329	17	$\bar{1}$.88315	26	0.11685	$\bar{1}$.90014	9	37
24	$\bar{1}$.78346	16	$\bar{1}$.88341	26	0.11659	$\bar{1}$.90005	10	36
25	$\bar{1}$.78362	17	$\bar{1}$.88367	26	0.11633	$\bar{1}$.89995	10	35
26	$\bar{1}$.78379	16	$\bar{1}$.88393	27	0.11607	$\bar{1}$.89985	9	34
27	$\bar{1}$.78395	17	$\bar{1}$.88420	26	0.11580	$\bar{1}$.89976	10	33
28	$\bar{1}$.78412	16	$\bar{1}$.88446	26	0.11554	$\bar{1}$.89966	10	32
29	$\bar{1}$.78428	17	$\bar{1}$.88472	26	0.11528	$\bar{1}$.89956	9	31
30	$\bar{1}$.78445	16	$\bar{1}$.88498	26	0.11502	$\bar{1}$.89947	10	30
31	$\bar{1}$.78461	17	$\bar{1}$.88524	26	0.11476	$\bar{1}$.89937	10	29
32	$\bar{1}$.78478	16	$\bar{1}$.88550	27	0.11450	$\bar{1}$.89927	9	28
33	$\bar{1}$.78494	16	$\bar{1}$.88577	26	0.11423	$\bar{1}$.89918	10	27
34	$\bar{1}$.78510	17	$\bar{1}$.88603	26	0.11397	$\bar{1}$.89908	10	26
35	$\bar{1}$.78527	16	$\bar{1}$.88629	26	0.11371	$\bar{1}$.89898	10	25
36	$\bar{1}$.78543	17	$\bar{1}$.88655	26	0.11345	$\bar{1}$.89888	9	24
37	$\bar{1}$.78560	16	$\bar{1}$.88681	26	0.11319	$\bar{1}$.89879	10	23
38	$\bar{1}$.78576	16	$\bar{1}$.88707	26	0.11293	$\bar{1}$.89869	10	22
39	$\bar{1}$.78592	17	$\bar{1}$.88733	26	0.11267	$\bar{1}$.89859	10	21
40	$\bar{1}$.78609	16	$\bar{1}$.88759	27	0.11241	$\bar{1}$.89849	9	20
41	$\bar{1}$.78625	17	$\bar{1}$.88786	26	0.11214	$\bar{1}$.89840	10	19
42	$\bar{1}$.78642	16	$\bar{1}$.88812	26	0.11188	$\bar{1}$.89830	10	18
43	$\bar{1}$.78658	16	$\bar{1}$.88838	26	0.11162	$\bar{1}$.89820	10	17
44	$\bar{1}$.78674	17	$\bar{1}$.88864	26	0.11136	$\bar{1}$.89810	9	16
45	$\bar{1}$.78691	16	$\bar{1}$.88890	26	0.11110	$\bar{1}$.89801	10	15
46	$\bar{1}$.78707	16	$\bar{1}$.88916	26	0.11084	$\bar{1}$.89791	10	14
47	$\bar{1}$.78723	16	$\bar{1}$.88942	26	0.11058	$\bar{1}$.89781	10	13
48	$\bar{1}$.78739	17	$\bar{1}$.88968	26	0.11032	$\bar{1}$.89771	10	12
49	$\bar{1}$.78756	16	$\bar{1}$.88994	26	0.11006	$\bar{1}$.89761	9	11
50	$\bar{1}$.78772	16	$\bar{1}$.89020	26	0.10980	$\bar{1}$.89752	10	10
51	$\bar{1}$.78788	17	$\bar{1}$.89046	27	0.10954	$\bar{1}$.89742	10	9
52	$\bar{1}$.78805	16	$\bar{1}$.89073	26	0.10927	$\bar{1}$.89732	10	8
53	$\bar{1}$.78821	16	$\bar{1}$.89099	26	0.10901	$\bar{1}$.89722	10	7
54	$\bar{1}$.78837	16	$\bar{1}$.89125	26	0.10875	$\bar{1}$.89712	10	6
55	$\bar{1}$.78853	16	$\bar{1}$.89151	26	0.10849	$\bar{1}$.89702	9	5
56	$\bar{1}$.78869	17	$\bar{1}$.89177	26	0.10823	$\bar{1}$.89693	10	4
57	$\bar{1}$.78886	16	$\bar{1}$.89203	26	0.10797	$\bar{1}$.89683	10	3
58	$\bar{1}$.78902	16	$\bar{1}$.89229	26	0.10771	$\bar{1}$.89673	10	2
59	$\bar{1}$.78918	16	$\bar{1}$.89255	26	0.10745	$\bar{1}$.89663	10	1
60	$\bar{1}$.78934		$\bar{1}$.89281		0.10719	$\bar{1}$.89653		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	27	26	17	16	10	9
6	2.7	2.6	1.7	1.6	1.0	0.9
7	3.2	3.0	2.0	1.9	1.2	1.1
8	3.6	3.5	2.3	2.1	1.3	1.2
9	4.1	3.9	2.6	2.4	1.5	1.4
10	4.5	4.3	2.8	2.7	1.7	1.5
20	9.0	8.7	5.7	5.3	3.3	3.0
30	13.5	13.0	8.5	8.0	5.0	4.5
40	18.0	17.3	11.3	10.7	6.7	6.0
50	22.5	21.7	14.2	13.3	8.3	7.5

'	log sin	d	log tan	c. d.	log cot	log cos	d	'
0	$\bar{1}.78934$	16	$\bar{1}.89281$	26	0.10719	$\bar{1}.89653$	10	60
1	$\bar{1}.78950$	17	$\bar{1}.89307$	26	0.10693	$\bar{1}.89643$	10	59
2	$\bar{1}.78967$	16	$\bar{1}.89333$	26	0.10667	$\bar{1}.89633$	9	58
3	$\bar{1}.78983$	16	$\bar{1}.89359$	26	0.10641	$\bar{1}.89624$	10	57
4	$\bar{1}.78999$	16	$\bar{1}.89385$	26	0.10615	$\bar{1}.89614$	10	56
5	$\bar{1}.79015$	16	$\bar{1}.89411$	26	0.10589	$\bar{1}.89604$	10	55
6	$\bar{1}.79031$	16	$\bar{1}.89437$	26	0.10563	$\bar{1}.89594$	10	54
7	$\bar{1}.79047$	16	$\bar{1}.89463$	26	0.10537	$\bar{1}.89584$	10	53
8	$\bar{1}.79063$	16	$\bar{1}.89489$	26	0.10511	$\bar{1}.89574$	10	52
9	$\bar{1}.79079$	16	$\bar{1}.89515$	26	0.10485	$\bar{1}.89564$	10	51
10	$\bar{1}.79095$		$\bar{1}.89541$		0.10459	$\bar{1}.89554$		50
		16		26			10	
11	$\bar{1}.79111$	17	$\bar{1}.89567$	26	0.10433	$\bar{1}.89544$	10	49
12	$\bar{1}.79128$	16	$\bar{1}.89593$	26	0.10407	$\bar{1}.89534$	10	48
13	$\bar{1}.79144$	16	$\bar{1}.89619$	26	0.10381	$\bar{1}.89524$	10	47
14	$\bar{1}.79160$	16	$\bar{1}.89645$	26	0.10355	$\bar{1}.89514$	10	46
15	$\bar{1}.79176$	16	$\bar{1}.89671$	26	0.10329	$\bar{1}.89504$	9	45
16	$\bar{1}.79192$	16	$\bar{1}.89697$	26	0.10303	$\bar{1}.89495$	10	44
17	$\bar{1}.79208$	16	$\bar{1}.89723$	26	0.10277	$\bar{1}.89485$	10	43
18	$\bar{1}.79224$	16	$\bar{1}.89749$	26	0.10251	$\bar{1}.89475$	10	42
19	$\bar{1}.79240$	16	$\bar{1}.89775$	26	0.10225	$\bar{1}.89465$	10	41
20	$\bar{1}.79256$		$\bar{1}.89801$		0.10199	$\bar{1}.89455$		40
		16		26			10	
21	$\bar{1}.79272$	16	$\bar{1}.89827$	26	0.10173	$\bar{1}.89445$	10	39
22	$\bar{1}.79288$	16	$\bar{1}.89853$	26	0.10147	$\bar{1}.89435$	10	38
23	$\bar{1}.79304$	15	$\bar{1}.89879$	26	0.10121	$\bar{1}.89425$	10	37
24	$\bar{1}.79319$	16	$\bar{1}.89905$	26	0.10095	$\bar{1}.89415$	10	36
25	$\bar{1}.79335$	16	$\bar{1}.89931$	26	0.10069	$\bar{1}.89405$	10	35
26	$\bar{1}.79351$	16	$\bar{1}.89957$	26	0.10043	$\bar{1}.89395$	10	34
27	$\bar{1}.79367$	16	$\bar{1}.89983$	26	0.10017	$\bar{1}.89385$	10	33
28	$\bar{1}.79383$	16	$\bar{1}.90009$	26	0.09991	$\bar{1}.89375$	11	32
29	$\bar{1}.79399$	16	$\bar{1}.90035$	26	0.09965	$\bar{1}.89364$	10	31
30	$\bar{1}.79415$		$\bar{1}.90061$		0.09939	$\bar{1}.89354$		30
		16		25			10	
31	$\bar{1}.79431$	16	$\bar{1}.90086$	26	0.09914	$\bar{1}.89344$	10	29
32	$\bar{1}.79447$	16	$\bar{1}.90112$	26	0.09888	$\bar{1}.89334$	10	28
33	$\bar{1}.79463$	15	$\bar{1}.90138$	26	0.09862	$\bar{1}.89324$	10	27
34	$\bar{1}.79478$	16	$\bar{1}.90164$	26	0.09836	$\bar{1}.89314$	10	26
35	$\bar{1}.79494$	16	$\bar{1}.90190$	26	0.09810	$\bar{1}.89304$	10	25
36	$\bar{1}.79510$	16	$\bar{1}.90216$	26	0.09784	$\bar{1}.89294$	10	24
37	$\bar{1}.79526$	16	$\bar{1}.90242$	26	0.09758	$\bar{1}.89284$	10	23
38	$\bar{1}.79542$	16	$\bar{1}.90268$	26	0.09732	$\bar{1}.89274$	10	22
39	$\bar{1}.79558$	15	$\bar{1}.90294$	26	0.09706	$\bar{1}.89264$	10	21
40	$\bar{1}.79573$		$\bar{1}.90320$		0.09680	$\bar{1}.89254$		20
		16		26			10	
41	$\bar{1}.79589$	16	$\bar{1}.90346$	25	0.09654	$\bar{1}.89244$	11	19
42	$\bar{1}.79605$	16	$\bar{1}.90371$	26	0.09629	$\bar{1}.89233$	10	18
43	$\bar{1}.79621$	15	$\bar{1}.90397$	26	0.09603	$\bar{1}.89223$	10	17
44	$\bar{1}.79636$	16	$\bar{1}.90423$	26	0.09577	$\bar{1}.89213$	10	16
45	$\bar{1}.79652$	16	$\bar{1}.90449$	26	0.09551	$\bar{1}.89203$	10	15
46	$\bar{1}.79668$	16	$\bar{1}.90475$	26	0.09525	$\bar{1}.89193$	10	14
47	$\bar{1}.79684$	15	$\bar{1}.90501$	26	0.09499	$\bar{1}.89183$	10	13
48	$\bar{1}.79699$	16	$\bar{1}.90527$	26	0.09473	$\bar{1}.89173$	11	12
49	$\bar{1}.79715$	16	$\bar{1}.90553$	25	0.09447	$\bar{1}.89162$	10	11
50	$\bar{1}.79731$		$\bar{1}.90578$		0.09422	$\bar{1}.89152$		10
		15		26			10	
51	$\bar{1}.79746$	16	$\bar{1}.90604$	26	0.09396	$\bar{1}.89142$	10	9
52	$\bar{1}.79762$	16	$\bar{1}.90630$	26	0.09370	$\bar{1}.89132$	10	8
53	$\bar{1}.79778$	15	$\bar{1}.90656$	26	0.09344	$\bar{1}.89122$	10	7
54	$\bar{1}.79793$	16	$\bar{1}.90682$	26	0.09318	$\bar{1}.89112$	11	6
55	$\bar{1}.79809$	16	$\bar{1}.90708$	26	0.09292	$\bar{1}.89101$	10	5
56	$\bar{1}.79825$	15	$\bar{1}.90734$	25	0.09266	$\bar{1}.89091$	10	4
57	$\bar{1}.79840$	16	$\bar{1}.90759$	26	0.09241	$\bar{1}.89081$	10	3
58	$\bar{1}.79856$	16	$\bar{1}.90785$	26	0.09215	$\bar{1}.89071$	11	2
59	$\bar{1}.79872$	15	$\bar{1}.90811$	26	0.09189	$\bar{1}.89060$	10	1
60	$\bar{1}.79887$		$\bar{1}.90837$		0.09163	$\bar{1}.89050$		0
	log cos	d	log cot	c. d.	log tan	log sin	d	'

p. p.

	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	17
6	1.7
7	2.0
8	2.3
9	2.6
10	2.8
20	5.7
30	8.5
40	11.3
50	14.2

	16	15
6	1.6	1.5
7	1.9	1.8
8	2.1	2.0
9	2.4	2.3
10	2.7	2.5
20	5.3	5.0
30	8.0	7.5
40	10.7	10.0
50	13.3	12.5

	11
6	1.1
7	1.3
8	1.5
9	1.7
10	1.8
20	3.7
30	5.5
40	7.3
50	9.2

	10	9
6	1.0	0.9
7	1.2	1.1
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.79887	16	$\bar{1}$.90837	26	0.09163	$\bar{1}$.89050	10	60
1	$\bar{1}$.79903	15	$\bar{1}$.90863	26	0.09137	$\bar{1}$.89040	10	59
2	$\bar{1}$.79918	16	$\bar{1}$.90889	25	0.09111	$\bar{1}$.89030	10	58
3	$\bar{1}$.79934	16	$\bar{1}$.90914	26	0.09086	$\bar{1}$.89020	11	57
4	$\bar{1}$.79950	15	$\bar{1}$.90940	26	0.09060	$\bar{1}$.89009	10	56
5	$\bar{1}$.79965	16	$\bar{1}$.90966	26	0.09034	$\bar{1}$.88999	10	55
6	$\bar{1}$.79981	15	$\bar{1}$.90992	26	0.09008	$\bar{1}$.88989	11	54
7	$\bar{1}$.79996	16	$\bar{1}$.91018	25	0.08982	$\bar{1}$.88978	10	53
8	$\bar{1}$.80012	15	$\bar{1}$.91043	26	0.08957	$\bar{1}$.88968	10	52
9	$\bar{1}$.80027	16	$\bar{1}$.91069	26	0.08931	$\bar{1}$.88958	10	51
10	$\bar{1}$.80043	15	$\bar{1}$.91095	26	0.08905	$\bar{1}$.88948	11	50
11	$\bar{1}$.80058	16	$\bar{1}$.91121	26	0.08879	$\bar{1}$.88937	10	49
12	$\bar{1}$.80074	15	$\bar{1}$.91147	25	0.08853	$\bar{1}$.88927	10	48
13	$\bar{1}$.80089	16	$\bar{1}$.91172	26	0.08828	$\bar{1}$.88917	11	47
14	$\bar{1}$.80105	15	$\bar{1}$.91198	26	0.08802	$\bar{1}$.88906	10	46
15	$\bar{1}$.80120	16	$\bar{1}$.91224	26	0.08776	$\bar{1}$.88896	10	45
16	$\bar{1}$.80136	15	$\bar{1}$.91250	26	0.08750	$\bar{1}$.88886	11	44
17	$\bar{1}$.80151	15	$\bar{1}$.91276	25	0.08724	$\bar{1}$.88875	10	43
18	$\bar{1}$.80166	16	$\bar{1}$.91301	26	0.08699	$\bar{1}$.88865	10	42
19	$\bar{1}$.80182	15	$\bar{1}$.91327	26	0.08673	$\bar{1}$.88855	11	41
20	$\bar{1}$.80197	16	$\bar{1}$.91353	26	0.08647	$\bar{1}$.88844	10	40
21	$\bar{1}$.80213	15	$\bar{1}$.91379	25	0.08621	$\bar{1}$.88834	10	39
22	$\bar{1}$.80228	16	$\bar{1}$.91404	26	0.08596	$\bar{1}$.88824	11	38
23	$\bar{1}$.80244	15	$\bar{1}$.91430	26	0.08570	$\bar{1}$.88813	10	37
24	$\bar{1}$.80259	15	$\bar{1}$.91456	26	0.08544	$\bar{1}$.88803	10	36
25	$\bar{1}$.80274	16	$\bar{1}$.91482	25	0.08518	$\bar{1}$.88793	11	35
26	$\bar{1}$.80290	15	$\bar{1}$.91507	26	0.08493	$\bar{1}$.88782	10	34
27	$\bar{1}$.80305	15	$\bar{1}$.91533	26	0.08467	$\bar{1}$.88772	11	33
28	$\bar{1}$.80320	16	$\bar{1}$.91559	26	0.08441	$\bar{1}$.88761	10	32
29	$\bar{1}$.80336	15	$\bar{1}$.91585	25	0.08415	$\bar{1}$.88751	10	31
30	$\bar{1}$.80351	15	$\bar{1}$.91610	26	0.08390	$\bar{1}$.88741	11	30
31	$\bar{1}$.80366	16	$\bar{1}$.91636	26	0.08364	$\bar{1}$.88730	10	29
32	$\bar{1}$.80382	15	$\bar{1}$.91662	26	0.08338	$\bar{1}$.88720	11	28
33	$\bar{1}$.80397	15	$\bar{1}$.91688	25	0.08312	$\bar{1}$.88709	10	27
34	$\bar{1}$.80412	16	$\bar{1}$.91713	26	0.08287	$\bar{1}$.88699	11	26
35	$\bar{1}$.80428	15	$\bar{1}$.91739	26	0.08261	$\bar{1}$.88688	10	25
36	$\bar{1}$.80443	15	$\bar{1}$.91765	26	0.08235	$\bar{1}$.88678	10	24
37	$\bar{1}$.80458	15	$\bar{1}$.91791	25	0.08209	$\bar{1}$.88668	11	23
38	$\bar{1}$.80473	16	$\bar{1}$.91816	26	0.08184	$\bar{1}$.88657	10	22
39	$\bar{1}$.80489	15	$\bar{1}$.91842	26	0.08158	$\bar{1}$.88647	11	21
40	$\bar{1}$.80504	15	$\bar{1}$.91868	25	0.08132	$\bar{1}$.88636	10	20
41	$\bar{1}$.80519	15	$\bar{1}$.91893	26	0.08107	$\bar{1}$.88626	11	19
42	$\bar{1}$.80534	16	$\bar{1}$.91919	26	0.08081	$\bar{1}$.88615	10	18
43	$\bar{1}$.80550	15	$\bar{1}$.91945	26	0.08055	$\bar{1}$.88605	11	17
44	$\bar{1}$.80565	15	$\bar{1}$.91971	25	0.08029	$\bar{1}$.88594	10	16
45	$\bar{1}$.80580	15	$\bar{1}$.91996	26	0.08004	$\bar{1}$.88584	11	15
46	$\bar{1}$.80595	15	$\bar{1}$.92022	26	0.07978	$\bar{1}$.88573	10	14
47	$\bar{1}$.80610	15	$\bar{1}$.92048	25	0.07952	$\bar{1}$.88563	11	13
48	$\bar{1}$.80625	16	$\bar{1}$.92073	26	0.07927	$\bar{1}$.88552	10	12
49	$\bar{1}$.80641	15	$\bar{1}$.92099	26	0.07901	$\bar{1}$.88542	11	11
50	$\bar{1}$.80656	15	$\bar{1}$.92125	25	0.07875	$\bar{1}$.88531	10	10
51	$\bar{1}$.80671	15	$\bar{1}$.92150	26	0.07850	$\bar{1}$.88521	11	9
52	$\bar{1}$.80686	15	$\bar{1}$.92176	26	0.07824	$\bar{1}$.88510	11	8
53	$\bar{1}$.80701	15	$\bar{1}$.92202	25	0.07798	$\bar{1}$.88499	10	7
54	$\bar{1}$.80716	15	$\bar{1}$.92227	26	0.07773	$\bar{1}$.88489	11	6
55	$\bar{1}$.80731	15	$\bar{1}$.92253	26	0.07747	$\bar{1}$.88478	10	5
56	$\bar{1}$.80746	16	$\bar{1}$.92279	25	0.07721	$\bar{1}$.88468	11	4
57	$\bar{1}$.80762	15	$\bar{1}$.92304	26	0.07696	$\bar{1}$.88457	10	3
58	$\bar{1}$.80777	15	$\bar{1}$.92330	26	0.07670	$\bar{1}$.88447	11	2
59	$\bar{1}$.80792	15	$\bar{1}$.92356	25	0.07644	$\bar{1}$.88436	11	1
60	$\bar{1}$.80807		$\bar{1}$.92381		0.07619	$\bar{1}$.88425		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	16
6	1.6
7	1.9
8	2.1
9	2.4
10	2.7
20	5.3
30	8.0
40	10.7
50	13.3

	15
6	1.5
7	1.8
8	2.0
9	2.3
10	2.5
20	5.0
30	7.5
40	10.0
50	12.5

	11	10
6	1.1	1.0
7	1.3	1.2
8	1.5	1.3
9	1.7	1.5
10	1.8	1.7
20	3.7	3.3
30	5.5	5.0
40	7.3	6.7
50	9.2	8.3

p. p.

′	log sin	d	log tan	c. d.	log cot	log cos	d	′
0	1.80807	15	1.92381	26	0.07619	1.88425	10	60
1	1.80822	15	1.92407	26	0.07593	1.88415	11	59
2	1.80837	15	1.92433	25	0.07567	1.88404	10	58
3	1.80852	15	1.92458	26	0.07542	1.88394	11	57
4	1.80867	15	1.92484	26	0.07516	1.88383	11	56
5	1.80882	15	1.92510	25	0.07490	1.88372	10	55
6	1.80897	15	1.92535	26	0.07465	1.88362	11	54
7	1.80912	15	1.92561	26	0.07439	1.88351	11	53
8	1.80927	15	1.92587	25	0.07413	1.88340	10	52
9	1.80942	15	1.92612	26	0.07388	1.88330	11	51
10	1.80957	15	1.92638	25	0.07362	1.88319	11	50
11	1.80972	15	1.92663	26	0.07337	1.88308	10	49
12	1.80987	15	1.92689	26	0.07311	1.88298	11	48
13	1.81002	15	1.92715	25	0.07285	1.88287	11	47
14	1.81017	15	1.92740	26	0.07260	1.88276	10	46
15	1.81032	15	1.92766	26	0.07234	1.88266	11	45
16	1.81047	14	1.92792	25	0.07208	1.88255	11	44
17	1.81061	15	1.92817	26	0.07183	1.88244	10	43
18	1.81076	15	1.92843	25	0.07157	1.88234	11	42
19	1.81091	15	1.92868	26	0.07132	1.88223	11	41
20	1.81106	15	1.92894	26	0.07106	1.88212	11	40
21	1.81121	15	1.92920	25	0.07080	1.88201	10	39
22	1.81136	15	1.92945	26	0.07055	1.88191	11	38
23	1.81151	15	1.92971	25	0.07029	1.88180	11	37
24	1.81166	14	1.92996	26	0.07004	1.88169	11	36
25	1.81180	15	1.93022	26	0.06978	1.88158	10	35
26	1.81195	15	1.93048	25	0.06952	1.88148	11	34
27	1.81210	15	1.93073	26	0.06927	1.88137	11	33
28	1.81225	15	1.93099	25	0.06901	1.88126	11	32
29	1.81240	14	1.93124	26	0.06876	1.88115	10	31
30	1.81254	15	1.93150	25	0.06850	1.88105	11	30
31	1.81269	15	1.93175	26	0.06825	1.88094	11	29
32	1.81284	15	1.93201	26	0.06799	1.88083	11	28
33	1.81299	15	1.93227	25	0.06773	1.88072	11	27
34	1.81314	14	1.93252	26	0.06748	1.88061	10	26
35	1.81328	15	1.93278	25	0.06722	1.88051	11	25
36	1.81343	15	1.93303	26	0.06697	1.88040	11	24
37	1.81358	14	1.93329	25	0.06671	1.88029	11	23
38	1.81372	15	1.93354	26	0.06645	1.88018	11	22
39	1.81387	15	1.93380	26	0.06620	1.88007	11	21
40	1.81402	15	1.93406	25	0.06594	1.87996	11	20
41	1.81417	14	1.93431	26	0.06569	1.87985	10	19
42	1.81431	15	1.93457	25	0.06543	1.87975	11	18
43	1.81446	15	1.93482	26	0.06518	1.87964	11	17
44	1.81461	14	1.93508	25	0.06492	1.87953	11	16
45	1.81475	15	1.93533	26	0.06467	1.87942	11	15
46	1.81490	15	1.93559	25	0.06441	1.87931	11	14
47	1.81505	14	1.93584	26	0.06416	1.87920	11	13
48	1.81519	15	1.93610	26	0.06390	1.87909	11	12
49	1.81534	15	1.93636	25	0.06364	1.87898	11	11
50	1.81549	14	1.93661	26	0.06339	1.87887	10	10
51	1.81563	15	1.93687	25	0.06313	1.87877	11	9
52	1.81578	14	1.93712	26	0.06288	1.87866	11	8
53	1.81592	15	1.93738	25	0.06262	1.87855	11	7
54	1.81607	15	1.93763	26	0.06237	1.87844	11	6
55	1.81622	14	1.93789	25	0.06211	1.87833	11	5
56	1.81636	15	1.93814	26	0.06186	1.87822	11	4
57	1.81651	14	1.93840	25	0.06160	1.87811	11	3
58	1.81665	15	1.93865	26	0.06135	1.87800	11	2
59	1.81680	14	1.93891	25	0.06109	1.87789	11	1
60	1.81694		1.93916		0.06084	1.87778		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

P. P.

	26	25	15	14	11	10
6	2.6	2.5	1.5	1.4	1.1	1.0
7	3.0	2.9	1.8	1.6	1.3	1.2
8	3.5	3.3	2.0	1.9	1.5	1.3
9	3.9	3.8	2.3	2.1	1.7	1.5
10	4.3	4.2	2.5	2.3	1.8	1.7
20	8.7	8.3	5.0	4.7	3.7	3.3
30	13.0	12.5	7.5	7.0	5.5	5.0
40	17.3	16.7	10.0	9.3	7.3	6.7
50	21.7	20.8	12.5	11.7	9.2	8.3

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.81694	15	$\bar{1}$.93916	26	0.06084	$\bar{1}$.87778	11	60
1	$\bar{1}$.81709	14	$\bar{1}$.93942	25	0.06058	$\bar{1}$.87767	11	59
2	$\bar{1}$.81723	15	$\bar{1}$.93967	26	0.06033	$\bar{1}$.87756	11	58
3	$\bar{1}$.81738	14	$\bar{1}$.93993	25	0.06007	$\bar{1}$.87745	11	57
4	$\bar{1}$.81752	15	$\bar{1}$.94018	26	0.05982	$\bar{1}$.87734	11	56
5	$\bar{1}$.81767	14	$\bar{1}$.94044	25	0.05956	$\bar{1}$.87723	11	55
6	$\bar{1}$.81781	15	$\bar{1}$.94069	26	0.05931	$\bar{1}$.87712	11	54
7	$\bar{1}$.81796	14	$\bar{1}$.94095	25	0.05905	$\bar{1}$.87701	11	53
8	$\bar{1}$.81810	15	$\bar{1}$.94120	26	0.05880	$\bar{1}$.87690	11	52
9	$\bar{1}$.81825	14	$\bar{1}$.94146	25	0.05854	$\bar{1}$.87679	11	51
10	$\bar{1}$.81839	15	$\bar{1}$.94171	26	0.05829	$\bar{1}$.87668	11	50
11	$\bar{1}$.81854	14	$\bar{1}$.94197	25	0.05803	$\bar{1}$.87657	11	49
12	$\bar{1}$.81868	14	$\bar{1}$.94222	26	0.05778	$\bar{1}$.87646	11	48
13	$\bar{1}$.81882	15	$\bar{1}$.94248	25	0.05752	$\bar{1}$.87635	11	47
14	$\bar{1}$.81897	14	$\bar{1}$.94273	26	0.05727	$\bar{1}$.87624	11	46
15	$\bar{1}$.81911	15	$\bar{1}$.94299	25	0.05701	$\bar{1}$.87613	12	45
16	$\bar{1}$.81926	14	$\bar{1}$.94324	26	0.05676	$\bar{1}$.87601	11	44
17	$\bar{1}$.81940	15	$\bar{1}$.94350	25	0.05650	$\bar{1}$.87590	11	43
18	$\bar{1}$.81955	14	$\bar{1}$.94375	26	0.05625	$\bar{1}$.87579	11	42
19	$\bar{1}$.81969	14	$\bar{1}$.94401	25	0.05599	$\bar{1}$.87568	11	41
20	$\bar{1}$.81983	15	$\bar{1}$.94426	26	0.05574	$\bar{1}$.87557	11	40
21	$\bar{1}$.81998	14	$\bar{1}$.94452	25	0.05548	$\bar{1}$.87546	11	39
22	$\bar{1}$.82012	14	$\bar{1}$.94477	26	0.05523	$\bar{1}$.87535	11	38
23	$\bar{1}$.82026	15	$\bar{1}$.94503	25	0.05497	$\bar{1}$.87524	11	37
24	$\bar{1}$.82041	14	$\bar{1}$.94528	26	0.05472	$\bar{1}$.87513	12	36
25	$\bar{1}$.82055	14	$\bar{1}$.94554	25	0.05446	$\bar{1}$.87501	11	35
26	$\bar{1}$.82069	15	$\bar{1}$.94579	25	0.05421	$\bar{1}$.87490	11	34
27	$\bar{1}$.82084	14	$\bar{1}$.94604	26	0.05396	$\bar{1}$.87479	11	33
28	$\bar{1}$.82098	14	$\bar{1}$.94630	25	0.05370	$\bar{1}$.87468	11	32
29	$\bar{1}$.82112	14	$\bar{1}$.94655	26	0.05345	$\bar{1}$.87457	11	31
30	$\bar{1}$.82126	15	$\bar{1}$.94681	25	0.05319	$\bar{1}$.87446	12	30
31	$\bar{1}$.82141	14	$\bar{1}$.94706	26	0.05294	$\bar{1}$.87434	11	29
32	$\bar{1}$.82155	14	$\bar{1}$.94732	25	0.05268	$\bar{1}$.87423	11	28
33	$\bar{1}$.82169	15	$\bar{1}$.94757	26	0.05243	$\bar{1}$.87412	11	27
34	$\bar{1}$.82184	14	$\bar{1}$.94783	25	0.05217	$\bar{1}$.87401	11	26
35	$\bar{1}$.82198	14	$\bar{1}$.94808	26	0.05192	$\bar{1}$.87390	12	25
36	$\bar{1}$.82212	14	$\bar{1}$.94834	25	0.05166	$\bar{1}$.87378	11	24
37	$\bar{1}$.82226	14	$\bar{1}$.94859	25	0.05141	$\bar{1}$.87367	11	23
38	$\bar{1}$.82240	15	$\bar{1}$.94884	26	0.05116	$\bar{1}$.87356	11	22
39	$\bar{1}$.82255	14	$\bar{1}$.94910	25	0.05090	$\bar{1}$.87345	11	21
40	$\bar{1}$.82269	14	$\bar{1}$.94935	26	0.05065	$\bar{1}$.87334	12	20
41	$\bar{1}$.82283	14	$\bar{1}$.94961	25	0.05039	$\bar{1}$.87322	11	19
42	$\bar{1}$.82297	14	$\bar{1}$.94986	26	0.05014	$\bar{1}$.87311	11	18
43	$\bar{1}$.82311	15	$\bar{1}$.95012	25	0.04988	$\bar{1}$.87300	12	17
44	$\bar{1}$.82326	14	$\bar{1}$.95037	25	0.04963	$\bar{1}$.87288	11	16
45	$\bar{1}$.82340	14	$\bar{1}$.95062	26	0.04938	$\bar{1}$.87277	11	15
46	$\bar{1}$.82354	14	$\bar{1}$.95088	25	0.04912	$\bar{1}$.87266	11	14
47	$\bar{1}$.82368	14	$\bar{1}$.95113	26	0.04887	$\bar{1}$.87255	12	13
48	$\bar{1}$.82382	14	$\bar{1}$.95139	25	0.04861	$\bar{1}$.87243	11	12
49	$\bar{1}$.82396	14	$\bar{1}$.95164	26	0.04836	$\bar{1}$.87232	11	11
50	$\bar{1}$.82410	14	$\bar{1}$.95190	25	0.04810	$\bar{1}$.87221	12	10
51	$\bar{1}$.82424	15	$\bar{1}$.95215	25	0.04785	$\bar{1}$.87209	11	9
52	$\bar{1}$.82439	14	$\bar{1}$.95240	26	0.04760	$\bar{1}$.87198	11	8
53	$\bar{1}$.82453	14	$\bar{1}$.95266	25	0.04734	$\bar{1}$.87187	12	7
54	$\bar{1}$.82467	14	$\bar{1}$.95291	26	0.04709	$\bar{1}$.87175	11	6
55	$\bar{1}$.82481	14	$\bar{1}$.95317	25	0.04683	$\bar{1}$.87164	11	5
56	$\bar{1}$.82495	14	$\bar{1}$.95342	26	0.04658	$\bar{1}$.87153	12	4
57	$\bar{1}$.82509	14	$\bar{1}$.95368	25	0.04632	$\bar{1}$.87141	11	3
58	$\bar{1}$.82523	14	$\bar{1}$.95393	25	0.04607	$\bar{1}$.87130	11	2
59	$\bar{1}$.82537	14	$\bar{1}$.95418	26	0.04582	$\bar{1}$.87119	12	1
60	$\bar{1}$.82551		$\bar{1}$.95444		0.04556	$\bar{1}$.87107		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	15
6	1.5
7	1.8
8	2.0
9	2.3
10	2.5
20	5.0
30	7.5
40	10.0
50	12.5

	14
6	1.4
7	1.6
8	1.9
9	2.1
10	2.3
20	4.7
30	7.0
40	9.3
50	11.7

	12	11
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.7
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

p. p.

′	log sin	d	log tan	c.d	log cot	log cos	d	′
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	60
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	59
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	58
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	56
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	55
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	54
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	53
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	52
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	51
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	50
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	49
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	48
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	47
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	46
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	45
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	44
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	43
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	42
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	41
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	40
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	39
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	38
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	37
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	36
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	35
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	34
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	33
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	32
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	31
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	30
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	29
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	27
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	26
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	25
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	24
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	23
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	22
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	20
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	19
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	16
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	15
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	13
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	12
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0
	log cos	d	log cot	c.d	log tan	log sin	d	′

P. P.

	28	25	14	13	12	11
6	[illegible]	[illegible]	1.4	1.3	1.2	1.1
7	[illegible]	2.9	1.6	1.5	1.4	1.3
8	[illegible]	3.3	1.9	1.7	1.6	1.5
9	[illegible]	3.8	2.1	2.0	1.8	1.7
10	[illegible]	4.2	2.3	2.2	2.0	1.8
20	[illegible]	8.3	4.7	4.3	4.0	3.7
30	[illegible]	[illegible]	7.0	6.5	6.0	5.5
40	[illegible]	[illegible]	9.3	8.7	8.0	7.3
50	[illegible]	20.8	11.7	10.8	10.0	9.2

′	log sin	d	log tan	c. d.	log cot	log cos	d	
0	$\bar{1}$.83378	14	$\bar{1}$.96966	25	0.03034	$\bar{1}$.86413	12	60
1	$\bar{1}$.83392	13	$\bar{1}$.96991	25	0.03009	$\bar{1}$.86401	12	59
2	$\bar{1}$.83405	14	$\bar{1}$.97016	26	0.02984	$\bar{1}$.86389	12	58
3	$\bar{1}$.83419	13	$\bar{1}$.97042	25	0.02958	$\bar{1}$.86377	11	57
4	$\bar{1}$.83432	14	$\bar{1}$.97067	25	0.02933	$\bar{1}$.86366	12	56
5	$\bar{1}$.83446	13	$\bar{1}$.97092	26	0.02908	$\bar{1}$.86354	12	55
6	$\bar{1}$.83459	14	$\bar{1}$.97118	25	0.02882	$\bar{1}$.86342	12	54
7	$\bar{1}$.83473	13	$\bar{1}$.97143	25	0.02857	$\bar{1}$.86330	12	53
8	$\bar{1}$.83486	14	$\bar{1}$.97168	25	0.02832	$\bar{1}$.86318	12	52
9	$\bar{1}$.83500	13	$\bar{1}$.97193	26	0.02807	$\bar{1}$.86306	11	51
10	$\bar{1}$.83513	14	$\bar{1}$.97219	25	0.02781	$\bar{1}$.86295	12	50
11	$\bar{1}$.83527	13	$\bar{1}$.97244	25	0.02756	$\bar{1}$.86283	12	49
12	$\bar{1}$.83540	14	$\bar{1}$.97269	26	0.02731	$\bar{1}$.86271	12	48
13	$\bar{1}$.83554	13	$\bar{1}$.97295	25	0.02705	$\bar{1}$.86259	12	47
14	$\bar{1}$.83567	14	$\bar{1}$.97320	25	0.02680	$\bar{1}$.86247	12	46
15	$\bar{1}$.83581	13	$\bar{1}$.97345	26	0.02655	$\bar{1}$.86235	12	45
16	$\bar{1}$.83594	14	$\bar{1}$.97371	25	0.02629	$\bar{1}$.86223	12	44
17	$\bar{1}$.83608	13	$\bar{1}$.97396	25	0.02604	$\bar{1}$.86211	11	43
18	$\bar{1}$.83621	13	$\bar{1}$.97421	26	0.02579	$\bar{1}$.86200	12	42
19	$\bar{1}$.83634	14	$\bar{1}$.97447	25	0.02553	$\bar{1}$.86188	12	41
20	$\bar{1}$.83648	13	$\bar{1}$.97472	25	0.02528	$\bar{1}$.86176	12	40
21	$\bar{1}$.83661	13	$\bar{1}$.97497	26	0.02503	$\bar{1}$.86164	12	39
22	$\bar{1}$.83674	14	$\bar{1}$.97523	25	0.02477	$\bar{1}$.86152	12	38
23	$\bar{1}$.83688	13	$\bar{1}$.97548	25	0.02452	$\bar{1}$.86140	12	37
24	$\bar{1}$.83701	14	$\bar{1}$.97573	25	0.02427	$\bar{1}$.86128	12	36
25	$\bar{1}$.83715	13	$\bar{1}$.97598	26	0.02402	$\bar{1}$.86116	12	35
26	$\bar{1}$.83728	13	$\bar{1}$.97624	25	0.02376	$\bar{1}$.86104	12	34
27	$\bar{1}$.83741	14	$\bar{1}$.97649	25	0.02351	$\bar{1}$.86092	12	33
28	$\bar{1}$.83755	13	$\bar{1}$.97674	26	0.02326	$\bar{1}$.86080	12	32
29	$\bar{1}$.83768	13	$\bar{1}$.97700	25	0.02300	$\bar{1}$.86068	12	31
30	$\bar{1}$.83781	14	$\bar{1}$.97725	25	0.02275	$\bar{1}$.86056	12	30
31	$\bar{1}$.83795	13	$\bar{1}$.97750	26	0.02250	$\bar{1}$.86044	12	29
32	$\bar{1}$.83808	13	$\bar{1}$.97776	25	0.02224	$\bar{1}$.86032	12	28
33	$\bar{1}$.83821	13	$\bar{1}$.97801	25	0.02199	$\bar{1}$.86020	12	27
34	$\bar{1}$.83834	14	$\bar{1}$.97826	25	0.02174	$\bar{1}$.86008	12	26
35	$\bar{1}$.83848	13	$\bar{1}$.97851	26	0.02149	$\bar{1}$.85996	12	25
36	$\bar{1}$.83861	13	$\bar{1}$.97877	25	0.02123	$\bar{1}$.85984	12	24
37	$\bar{1}$.83874	13	$\bar{1}$.97902	25	0.02098	$\bar{1}$.85972	12	23
38	$\bar{1}$.83887	14	$\bar{1}$.97927	26	0.02073	$\bar{1}$.85960	12	22
39	$\bar{1}$.83901	13	$\bar{1}$.97953	25	0.02047	$\bar{1}$.85948	12	21
40	$\bar{1}$.83914	13	$\bar{1}$.97978	25	0.02022	$\bar{1}$.85936	12	20
41	$\bar{1}$.83927	13	$\bar{1}$.98003	26	0.01997	$\bar{1}$.85924	12	19
42	$\bar{1}$.83940	14	$\bar{1}$.98029	25	0.01971	$\bar{1}$.85912	12	18
43	$\bar{1}$.83954	13	$\bar{1}$.98054	25	0.01946	$\bar{1}$.85900	12	17
44	$\bar{1}$.83967	13	$\bar{1}$.98079	25	0.01921	$\bar{1}$.85888	12	16
45	$\bar{1}$.83980	13	$\bar{1}$.98104	26	0.01896	$\bar{1}$.85876	12	15
46	$\bar{1}$.83993	13	$\bar{1}$.98130	25	0.01870	$\bar{1}$.85864	13	14
47	$\bar{1}$.84006	14	$\bar{1}$.98155	25	0.01845	$\bar{1}$.85851	12	13
48	$\bar{1}$.84020	13	$\bar{1}$.98180	26	0.01820	$\bar{1}$.85839	12	12
49	$\bar{1}$.84033	13	$\bar{1}$.98206	25	0.01794	$\bar{1}$.85827	12	11
50	$\bar{1}$.84046	13	$\bar{1}$.98231	25	0.01769	$\bar{1}$.85815	12	10
51	$\bar{1}$.84059	13	$\bar{1}$.98256	25	0.01744	$\bar{1}$.85803	12	9
52	$\bar{1}$.84072	13	$\bar{1}$.98281	26	0.01719	$\bar{1}$.85791	12	8
53	$\bar{1}$.84085	13	$\bar{1}$.98307	25	0.01693	$\bar{1}$.85779	13	7
54	$\bar{1}$.84098	14	$\bar{1}$.98332	25	0.01668	$\bar{1}$.85766	12	6
55	$\bar{1}$.84112	13	$\bar{1}$.98357	26	0.01643	$\bar{1}$.85754	12	5
56	$\bar{1}$.84125	13	$\bar{1}$.98383	25	0.01617	$\bar{1}$.85742	12	4
57	$\bar{1}$.84138	13	$\bar{1}$.98408	25	0.01592	$\bar{1}$.85730	12	3
58	$\bar{1}$.84151	13	$\bar{1}$.98433	25	0.01567	$\bar{1}$.85718	12	2
59	$\bar{1}$.84164	13	$\bar{1}$.98458	26	0.01542	$\bar{1}$.85706	13	1
60	$\bar{1}$.84177		$\bar{1}$.98484		0.01516	$\bar{1}$.85693		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	26	25	14	13
6	2.6	2.5	1.4	1.3
7	3.0	2.9	1.6	1.5
8	3.5	3.3	1.9	1.7
9	3.9	3.8	2.1	2.0
10	4.3	4.2	2.3	2.2
20	8.7	8.3	4.7	4.3
30	13.0	12.5	7.0	6.5
40	17.3	16.7	9.3	8.7
50	21.7	20.8	11.7	10.8

	12	11
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.7
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

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′	log sin	d	log tan	c. d.	log cot	log cos	d	′
0	$\bar{1}.84177$	13	$\bar{1}.98484$	25	0.01516	$\bar{1}.85693$	12	60
1	$\bar{1}.84190$	13	$\bar{1}.98509$	25	0.01491	$\bar{1}.85681$	12	59
2	$\bar{1}.84203$	13	$\bar{1}.98534$	26	0.01466	$\bar{1}.85669$	12	58
3	$\bar{1}.84216$	13	$\bar{1}.98560$	25	0.01440	$\bar{1}.85657$	12	57
4	$\bar{1}.84229$	13	$\bar{1}.98585$	25	0.01415	$\bar{1}.85645$	13	56
5	$\bar{1}.84242$	13	$\bar{1}.98610$	25	0.01390	$\bar{1}.85632$	12	55
6	$\bar{1}.84255$	14	$\bar{1}.98635$	26	0.01365	$\bar{1}.85620$	12	54
7	$\bar{1}.84269$	13	$\bar{1}.98661$	25	0.01339	$\bar{1}.85608$	12	53
8	$\bar{1}.84282$	13	$\bar{1}.98686$	25	0.01314	$\bar{1}.85596$	13	52
9	$\bar{1}.84295$	13	$\bar{1}.98711$	26	0.01289	$\bar{1}.85583$	12	51
10	$\bar{1}.84308$	13	$\bar{1}.98737$	25	0.01263	$\bar{1}.85571$	12	50
11	$\bar{1}.84321$	13	$\bar{1}.98762$	25	0.01238	$\bar{1}.85559$	12	49
12	$\bar{1}.84334$	13	$\bar{1}.98787$	25	0.01213	$\bar{1}.85547$	13	48
13	$\bar{1}.84347$	13	$\bar{1}.98812$	26	0.01188	$\bar{1}.85534$	12	47
14	$\bar{1}.84360$	13	$\bar{1}.98838$	25	0.01162	$\bar{1}.85522$	12	46
15	$\bar{1}.84373$	12	$\bar{1}.98863$	25	0.01137	$\bar{1}.85510$	13	45
16	$\bar{1}.84385$	13	$\bar{1}.98888$	25	0.01112	$\bar{1}.85497$	12	44
17	$\bar{1}.84398$	13	$\bar{1}.98913$	26	0.01087	$\bar{1}.85485$	12	43
18	$\bar{1}.84411$	13	$\bar{1}.98939$	25	0.01061	$\bar{1}.85473$	13	42
19	$\bar{1}.84424$	13	$\bar{1}.98964$	25	0.01036	$\bar{1}.85460$	12	41
20	$\bar{1}.84437$	13	$\bar{1}.98989$	26	0.01011	$\bar{1}.85448$	12	40
21	$\bar{1}.84450$	13	$\bar{1}.99015$	25	0.00985	$\bar{1}.85436$	13	39
22	$\bar{1}.84463$	13	$\bar{1}.99040$	25	0.00960	$\bar{1}.85423$	12	38
23	$\bar{1}.84476$	13	$\bar{1}.99065$	25	0.00935	$\bar{1}.85411$	12	37
24	$\bar{1}.84489$	13	$\bar{1}.99090$	26	0.00910	$\bar{1}.85399$	13	36
25	$\bar{1}.84502$	13	$\bar{1}.99116$	25	0.00884	$\bar{1}.85386$	12	35
26	$\bar{1}.84515$	13	$\bar{1}.99141$	25	0.00859	$\bar{1}.85374$	13	34
27	$\bar{1}.84528$	12	$\bar{1}.99166$	25	0.00834	$\bar{1}.85361$	12	33
28	$\bar{1}.84540$	13	$\bar{1}.99191$	26	0.00809	$\bar{1}.85349$	12	32
29	$\bar{1}.84553$	13	$\bar{1}.99217$	25	0.00783	$\bar{1}.85337$	13	31
30	$\bar{1}.84566$	13	$\bar{1}.99242$	25	0.00758	$\bar{1}.85324$	12	30
31	$\bar{1}.84579$	13	$\bar{1}.99267$	26	0.00733	$\bar{1}.85312$	13	29
32	$\bar{1}.84592$	13	$\bar{1}.99293$	25	0.00707	$\bar{1}.85299$	12	28
33	$\bar{1}.84605$	13	$\bar{1}.99318$	25	0.00682	$\bar{1}.85287$	13	27
34	$\bar{1}.84618$	12	$\bar{1}.99343$	25	0.00657	$\bar{1}.85274$	12	26
35	$\bar{1}.84630$	13	$\bar{1}.99368$	26	0.00632	$\bar{1}.85262$	12	25
36	$\bar{1}.84643$	13	$\bar{1}.99394$	25	0.00606	$\bar{1}.85250$	13	24
37	$\bar{1}.84656$	13	$\bar{1}.99419$	25	0.00581	$\bar{1}.85237$	12	23
38	$\bar{1}.84669$	13	$\bar{1}.99444$	25	0.00556	$\bar{1}.85225$	13	22
39	$\bar{1}.84682$	12	$\bar{1}.99469$	26	0.00531	$\bar{1}.85212$	12	21
40	$\bar{1}.84694$	13	$\bar{1}.99495$	25	0.00505	$\bar{1}.85200$	13	20
41	$\bar{1}.84707$	13	$\bar{1}.99520$	25	0.00480	$\bar{1}.85187$	12	19
42	$\bar{1}.84720$	13	$\bar{1}.99545$	25	0.00455	$\bar{1}.85175$	13	18
43	$\bar{1}.84733$	12	$\bar{1}.99570$	26	0.00430	$\bar{1}.85162$	12	17
44	$\bar{1}.84745$	13	$\bar{1}.99596$	25	0.00404	$\bar{1}.85150$	13	16
45	$\bar{1}.84758$	13	$\bar{1}.99621$	25	0.00379	$\bar{1}.85137$	12	15
46	$\bar{1}.84771$	13	$\bar{1}.99646$	26	0.00354	$\bar{1}.85125$	13	14
47	$\bar{1}.84784$	12	$\bar{1}.99672$	25	0.00328	$\bar{1}.85112$	12	13
48	$\bar{1}.84796$	13	$\bar{1}.99697$	25	0.00303	$\bar{1}.85100$	13	12
49	$\bar{1}.84809$	13	$\bar{1}.99722$	25	0.00278	$\bar{1}.85087$	13	11
50	$\bar{1}.84822$	13	$\bar{1}.99747$	26	0.00253	$\bar{1}.85074$	12	10
51	$\bar{1}.84835$	12	$\bar{1}.99773$	25	0.00227	$\bar{1}.85062$	13	9
52	$\bar{1}.84847$	13	$\bar{1}.99798$	25	0.00202	$\bar{1}.85049$	12	8
53	$\bar{1}.84860$	13	$\bar{1}.99823$	25	0.00177	$\bar{1}.85037$	13	7
54	$\bar{1}.84873$	12	$\bar{1}.99848$	26	0.00152	$\bar{1}.85024$	12	6
55	$\bar{1}.84885$	13	$\bar{1}.99874$	25	0.00126	$\bar{1}.85012$	13	5
56	$\bar{1}.84898$	13	$\bar{1}.99899$	25	0.00101	$\bar{1}.84999$	13	4
57	$\bar{1}.84911$	12	$\bar{1}.99924$	25	0.00076	$\bar{1}.84986$	12	3
58	$\bar{1}.84923$	13	$\bar{1}.99949$	26	0.00051	$\bar{1}.84974$	13	2
59	$\bar{1}.84936$	13	$\bar{1}.99975$	25	0.00025	$\bar{1}.84961$	12	1
60	$\bar{1}.84949$		0.00000		0.00000	$\bar{1}.84949$		0
	log cos	d	log cot	c. d.	log tan	log sin	d	′

p. p.

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	14
6	1.4
7	1.6
8	1.9
9	2.1
10	2.3
20	4.7
30	7.0
40	9.3
50	11.7

	13
6	1.3
7	1.5
8	1.7
9	2.0
10	2.2
20	4.3
30	6.5
40	8.7
50	10.8

	12
6	1.2
7	1.4
8	1.6
9	1.8
10	2.0
20	4.0
30	6.0
40	8.0
50	10.0

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INDEX

NOTE.—All items in this index refer first to the section (see the Preface) and then to the page of the section. Thus, "Concrete number, §1, p1," means that concrete number will be found on page 1 of section 1.

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