THE CALCULUS

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(REVISED)

AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS, WITH PRACTICAL APPLICATIONS

PREPARED FOR THE USE OF
THE MIDSHIPMEN OF THE UNITED STATES
NAVAL ACADEMY

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THE CALCULUS.

CHAPTER I.

THE DIFFERENTIAL CALCULUS.

1. The treatment of functions in the fourth-class algebra showed how to find the slope of a graph by determining the derivative of the corresponding function, or to find, by the same means, the tangents to certain curves when the equations of the curves were given. These problems exemplify in an elementary manner the important properties of the Differential Calculus; we now have to extend the same principles further, and must adopt a more generally useful style of notation.

In order to introduce this notation as simply as possible, we shall repeat part of last year's work with some variations.

2. Increments.—In finding the slope of the graph of the function $y=x^2$, we supposed x to be increased to the value $(x+h)$, y thereby being increased to the value $(y+k)$. Either h or k might be negative, indicating a decrease. Such a change in a variable is called an *increment,* and two increments like these, one of which results as a consequence of the other, are called *simultaneous increments.*

We shall now indicate any increment by a composite symbol consisting of the letter Δ (the Greek capital *delta,* equivalent to the English D) followed by the symbol for the variable of which it is an increment. Thus the increment h given to x to increase its value to $(x+h)$, will now be written Δx and read "*delta x,*" or "*increment given to x.*" Similarly for k as used here we shall write Δy (*delta y*).

An increment such as Δx or Δy may be positive or negative, indicating an increase or a decrease.

To represent a certain sort of increment, the use of which will be explained in the next article, the small Roman d is used in place of the capital Δ. The resulting symbols dx, dy, are read *d-x, d-y,* or *differential of x, differential of y.*

The following problem illustrates the use of these symbols for increments.

3. Problem of the Tangent.—To draw a tangent at the point $P_0(x_0, y_0)$ of the parabola, $y=\frac{x^2}{a}$, referred to the pair of perpendicular axes XOY. (Fig. 1.)

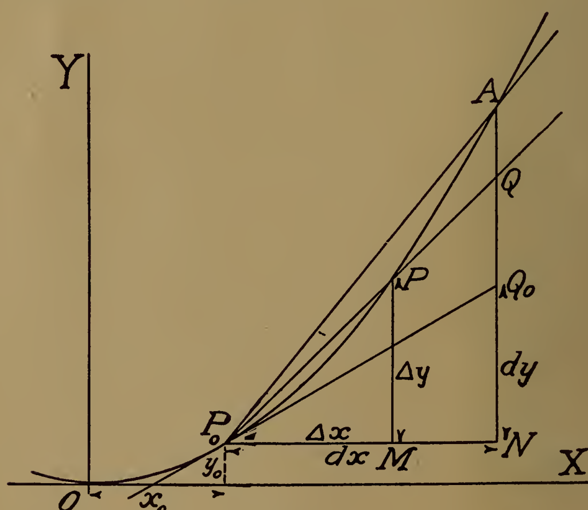


FIG. 1.

As the point P_0 of the tangent is given, it will be sufficient to find the *slope* of the tangent.

Suppose the tangent to be the line P_0Q_0. Through P_0 draw

P_0N parallel to OX of any convenient length dx, and through N draw a parallel to OY, meeting the curve at A and the tangent at Q_0. Call $NQ_0=dy$; then our problem is to find the value of dy, so as to get the ratio, dy/dx, of dy to the assumed length dx. This ratio will be the required slope of the tangent.

Suppose P to be any point of the curve; draw PM parallel to OY, meeting P_0N at M, and the secant P_0P, meeting AN at Q.

Then $P_0M=\Delta x$ and $MP=\Delta y$ are increments which, if given to the coördinates (x_0, y_0) of P_0, will change them to the coördinates $(x_0+\Delta x, y_0+\Delta y)$ of P. If the point P is supposed to slide along the curve through P_0, the secant P_0P will occupy the position of the desired tangent when P passes through the point P_0; during this process the point Q will move along the line AN, passing through Q_0 when P passes through P_0. The desired slope of the tangent, $\dfrac{Q_0N}{P_0N}=\dfrac{dy}{dx}$, is thus the value taken on by the variable ratio $\dfrac{QN}{P_0N}$ when P reaches P_0 and Q reaches Q_0.

But by similar triangles, $\dfrac{QN}{P_0N}=\dfrac{PM}{P_0M}=\dfrac{\Delta y}{\Delta x}$, and when P reaches P_0, $\Delta x=0$. Then the required slope of the tangent is

$$\frac{dy}{dx}=\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x=0}.$$

But now, since P_0 is on the curve,

$$y_0=\frac{x_0^{\,2}}{a},$$

and since P is on the curve,

$$y_0+\Delta y=\frac{(x_0+\Delta x)^2}{a};$$

subtracting, we find

$$\Delta y=\frac{2x_0\Delta x+(\Delta x)^2}{a},$$

$$\frac{\Delta y}{\Delta x}=\frac{2x_0+\Delta x}{a};$$

$$\frac{dy}{dx}=\left[\frac{2x_0+\Delta x}{a}\right]_{\Delta x=0}=\frac{2x_0}{a},$$

$$dy=\frac{2x_0}{a}\cdot dx.$$

Thus, in order to draw the tangent, after laying off P_0N any convenient distance dx to the right, we lay off NQ_0, $\frac{2x_0}{a}$ times as far upward, and draw P_0Q_0, the required tangent.

If the tangent P_0Q_0 makes the angle τ with the axis OX, then evidently

$$\frac{dy}{dx}=\tan\tau=\textit{slope of tangent to curve.}$$

As P_0 may be any point of the curve, the result is expressed in general: The tangent line at any point (x, y) of the parabola $y=\frac{x^2}{a}$ has the slope $\frac{2x}{a}$.

For instance, the slope of the tangent to $y=\frac{x^2}{2}$ at the point $(3, \frac{9}{2})$ is $\frac{2\times 3}{2}=3$; so that, if any distance dx is measured to the right of the point $(3, \frac{9}{2})$ and three times as great a distance upward from the point thus reached, a point of the tangent will be found.

Note the convenience in the discussion above of having two expressions for an increment; the Δ forms have been used for variable increments, and the differential, or d forms, for the fixed increments the ratio of which is to be determined. Moreover, Δy is an increment of the ordinate of the curve, whereas dy is an increment of the ordinate of the tangent.

4. Equation of the Tangent.—For any curve except a straight line, $\frac{dy}{dx} = \tan\tau$, the slope of the tangent, has different values for different points of the curve; its value for any particular point (x_0, y_0) is indicated by

$$\left[\frac{dy}{dx}\right]_{x_0,\, y_0} = \tan\tau_0.$$

Since the equation of the line through (x_0, y_0) having the slope m is $y - y_0 = m(x - x_0)$ (Algebra, Art. 101), the equation of the tangent to a curve at the point (x_0, y_0) of the curve is

$$y - y_0 = (x - x_0)\left[\frac{dy}{dx}\right]_{x_0,\, y_0}.$$

5. *Examples.*

1. Find the general expression for $\frac{dy}{dx} = \left[\frac{\Delta y}{\Delta x}\right]_{\Delta x = 0}$ for the curve $8y = x^3$, and using 1 inch as unit, plot the points of the curve for which $x = -3, -2, -1, 0, 1, 2, 3$ respectively, and at each point construct the tangent, using any convenient value for dx. Sketch the curve.

2. Derive the general expression for $\frac{dy}{dx} = \left[\frac{\Delta y}{\Delta x}\right]_{\Delta x = 0}$ for each of the following curves, and write the equation of the tangent to each at the point indicated:

(a) $x^2 = 8y$ at $(4, 2)$; (b) $x^2 = 27y$ at $(9, 3)$; (c) $4x^3 = y$ at $(\frac{1}{2}, \frac{1}{2})$; (d) $8y = x^3$ at $(2, 1)$; (e) $ay^2 = b^2x$ at (x_0, y_0).

Ans. (a) $x - y = 2$, (b) $2x - 3y = 9$, (c) $3x - y = 1$, (d) $3x - 2y = 4$, (e) $2ay_0y = b^2(x + x_0)$.

6. Variation of Functions.—A variable y is said to be a function of a variable x when it depends upon x for its value, and so changes as x changes. It is often important to measure the relative rate of change of the function y with respect to its variable

x; that is, to tell, when a change is made in x, how many times as much change is made in y. In the case of an algebraic function of the first degree, the relation is evident; for instance, if $y=2x+7$, an increase Δx given to x will change y to $2(x+\Delta x)+7$, increasing it by $\Delta y=2\Delta x$, twice the increase in x. More generally, if $y=mx+b$, and x is increased any amount, y will be increased m times as much. This shows very clearly in the graph of the function—a straight line, where the increment given to x is represented by the horizontal leg of a right triangle, and the corresponding increment of y by the vertical leg. All these triangles are similar for the same graph, and the ratio of the legs in each is $\frac{\Delta y}{\Delta x}=m$. Thus from whatever value and by whatever amount x may be increased, $y=mx+b$ is increased m times as much.

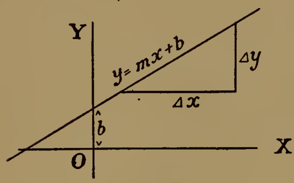


FIG. 2.

When a function has a curved graph, the relation is more complicated; the ratio of the increment of y to the increment of x which produces it depends upon the initial value of x, the amount of the increment given to x, and the *direction* of the increment—whether it is an increase or a decrease. It is, however, of great value to extend the conception of the rate of increase of y relatively to x to mean the limit approached by the ratio of the increments of y and x as these increments approach zero together. This limit, $\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x=0}$, is called the derivative of y with respect to x. Before going more fully into the process of finding derivatives, we shall consider a problem illustrative of the meaning and use of the new idea.

It will be understood, of course, that in this article the letters y and x have been used for function and variable merely for convenience and brevity; all sorts of letters may be expected in various problems. For instance, time is almost invariably indicated by the letter t, and distance (or space) by s.

7. Speed.—The simplest idea of speed is familiar; if a train goes 80 miles in 2 hours, we say it has a speed of 40 miles an hour. The distance moved by any body divided by the time occupied in the motion is called the *mean speed* of the motion.

The following notation for speed is used:

$$\frac{80 \text{ miles}}{2 \text{ hours}} = 40 \text{ m/h} = \frac{80 \times 5280}{2 \times 60 \times 60} = 58\tfrac{2}{3} \text{ f/s (ft. a sec.)}.$$

When the mean speed of a body is the same for all intervals of time during the body's motion, regardless of when they begin or how long they last, the *speed* of the body is defined as being equal to its *mean speed.* When a train is stopped at a station or started from rest, its mean speed changes. If we compute the mean speed for an interval of time immediately following a given instant, then for shorter and shorter intervals also immediately following the given instant, we find that as the interval becomes shorter the mean speed changes, reaching a definite value when the length of the interval is zero; this value is defined as the *speed* at the instant in question.

Suppose, for instance, that a train is slowed down, and that it is known that the number of feet (s) run in t seconds after the power is shut off and the brakes are put on is given by the formula

$$s = 60t - 4t^2,$$

what is the speed four seconds after, or when $t = 4$? When $t = 4$,

$$s = 60t - 4t^2 = 176;$$

in the following table the computation of the mean speed is shown for several intervals immediately following the fourth second since the train began to slow down:

t at end of interval.		s at end of interval.	Distance covered during interval.		Length of interval.		Mean speed during interval.	
5	sec.	200.	24	ft.	1	sec.	24	f/s
4.1	"	178.76	2.76	"	0.1	"	27.6	"
4.01	"	176.2796	0.2796	"	0.01	"	27.96	"
4.001	"	176.027996	0.027996	"	0.001	"	27.996	"
4.000001	"	176.000027999996	0.000027999996	"	0.000001	"	27.999996	"

This mean speed is very evidently approaching the limit 28 f/s. We can demonstrate this rigorously by finding the mean speed for any interval, Δt seconds long, immediately following the instant when $t=4$. Call the distance gone during the interval Δs (it is the increase in s caused by the increase Δt in t). When $t=4+\Delta t$, $s=176+\Delta s$; so

$$176+\Delta s=60(4+\Delta t)-4(4+\Delta t)^2=176+28\Delta t-4(\Delta t)^2 \text{ft.},$$

and

$$\Delta s=28\Delta t-4(\Delta t)^2 \text{ ft.}$$

The mean speed during the interval of Δt seconds is:

$$\text{Mean speed}=\frac{\Delta s}{\Delta t}=28-4\Delta t \text{ f/s.}$$

If we replace Δt in this expression for the mean speed by the values used for the length of the intervals in the table, we obtain the numerical values given in the table for the mean speed; and it is now evident that if we make $\Delta t=0$, we get for the actual speed after four seconds

$$\frac{ds}{dt}=\left[\frac{\Delta s}{\Delta t}\right]_{\Delta t=0}=[28-4\Delta t]_{\Delta t=0} f/s=28 \text{ f/s.}$$

8. We can find the speed when the train has gone any number, t, of seconds since the slowing down began. As before, take any interval Δt seconds just after the instant in question, and let Δs be the distance gone during the interval, s the distance gone at the beginning of the interval:

$$s = 60t - 4t^2,$$

$$s + \Delta s = 60(t + \Delta t) - 4(t + \Delta t)^2;$$

$$\Delta s = 60\Delta t - 8t\Delta t - 4(\Delta t)^2;$$

$$\frac{\Delta s}{\Delta t} = 60 - 8t - 4\Delta t$$

is the mean speed during the interval; the speed is

$$\frac{ds}{dt} = \left[\frac{\Delta s}{\Delta t}\right]_{\Delta t = 0} = 60 - 8t.$$

The speed is zero when $t = 7\frac{1}{2}$ secs.; then $s = 225$ ft. We thus see that after the brakes are put on, the train will go 225 ft. in $7\frac{1}{2}$ secs. and then stop.

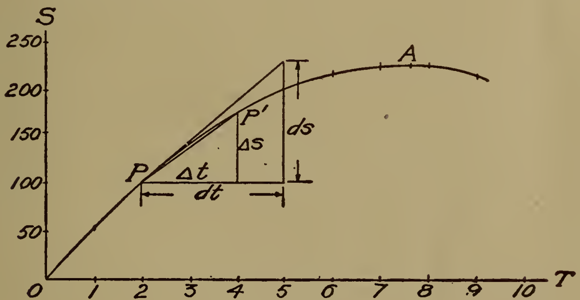


FIG. 3.

9. There is a very important connection between this problem and the preceding one, a relation which becomes apparent when we draw the graph of the function $s = 60t - 4t^2$. In Fig. 3 this graph is drawn so that a unit length for s represents 50 ft., and a unit of t represents 1 sec. Each step in the process of finding the speed at the end of t seconds can be shown graphically: the point P shows a distance of s ft. traversed after t secs.; the point P' shows $(s + \Delta s)$ ft. after $(t + \Delta t)$ secs.; the mean speed in this interval, or $\frac{\Delta s}{\Delta t}$, is the slope of the secant line PP'; the limit approached by the mean speed, or the actual speed at P, is the slope of the tangent line at P.

According to the notation of the preceding article, we should represent the slope of this graph by $\tan \tau = \frac{ds}{dt}$, where dt is any convenient length (drawn to represent 3 secs. in the figure) and ds is so taken that $\frac{ds}{dt}$ shall be equal to the value of $\left[\frac{\Delta s}{\Delta t}\right]_{\Delta t=0}$. For this reason we have used the fraction $\frac{ds}{dt}$ to represent the speed.

Notice that $\tan \tau = 60 - 8t$ becomes zero when $t = 7\frac{1}{2}$, and is negative for larger values of t, as appears either from inspection of the graph or of the algebraic expression $60 - 8t$ (τ may be measured from the axis of abscissas to either part of the tangent, and changes from the third quadrant to the second, or from the first to the fourth, as t passes through the value $7\frac{1}{2}$). This corresponds in the problem to a speed which decreases to nothing and then becomes negative. The problem as stated and the graph representing it end at the value $t = 7\frac{1}{2}$, indicated by the point A; if, however, the train were stopped by reversing the engines, it would be only momentarily stationary and then begin to back. In this case, the backing would be indicated by the negative speed, and represented by the part of the graph beyond A.

Note again the difference in the use of the symbols compounded with Δ and those compounded with d. Δs and Δt represent simultaneous *variable* increments, made to approach the limit zero (and reach it); ds and dt represent any fixed values whose ratio is this limit. Sometimes dt, ds, and Δs are grouped together as simultaneous increments; in this case all three represent constants; dt represents some increment of time, Δs the actual distance traversed in this interval, and ds the distance that would have been traversed if the speed at the beginning of the interval had remained unchanged throughout the interval. On the graph (Fig. 4), dt appears as an increment of the abscissa of the point P, Δs as the corresponding increment of the ordinate of the curve, and ds as the corresponding increment of the ordinate of the tangent.

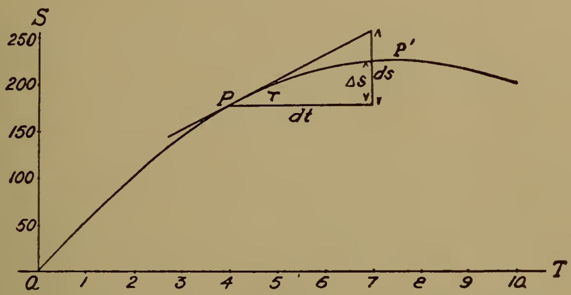


FIG. 4.

10. *Examples.*

1. A body falling freely near the earth describes a distance $s=16t^2$ ft. (nearly), when t is the elapsed time in seconds; find the velocity at the end of 1 sec., at the end of 2 secs., and at the instant of starting.

2. A body projected vertically upward from a height, h, with a velocity, v, will at the end of t secs. have gone to a height $s=h+vt-16t^2$ ft.; in each of the following three cases, find the speed after t secs., and the speed when the body reaches the ground; and in each case draw the graph representing s as a function of t, and draw τ to represent speed $=\tan \tau$.

Case 1: $h=100$, $v=+18$ (body thrown upward at 18 f/s).
Ans. $18-32t$ and -82 f/s.
Case 2: $h=100$, $v=-18$ (body thrown downward at 18 f/s).
Case 3: $h=100$, $v=0$ (body dropped).

11. The General Problem of the Derivative.—In the problem of the speed of the train, we have an instance of the general problem of the differential calculus; the problem, that is, of finding a measure for the relative rate of increase of a function as compared with its independent variable; the distance traversed by the train is the function $60t-4t^2$ of the number of seconds elapsed since the brakes were applied. We may have occasion to consider the rate of increase of any function relatively to its variable; for functions of other variables than the time, this rate will have different meanings, but for any function whatever its measure is the slope of the tangent to the graph of the function, and the method of its determination is the same as for the time-rate or speed.

If a point is moved along the graph of any function from left to right, it will go up or down according as the function increases or decreases with an increase of the argument. The curve shows by its steepness how fast the function changes in value as compared with its argument. The direction of the curve, which changes from point to point, is at any particular point the same as the direction of the tangent at that point, and is determined by the angle τ from the axis of x to the tangent. Thus the value of $\tan\tau$ for the graph is naturally the measure of the rate of change of the function, as compared with its argument.

The rate of change of any function as compared with its argument, or the derivative of the function, is always determined in general form, as a function of the argument; the determination for a particular value of the argument has been made in these early examples merely as a means of simplifying the conception of the subject.

The distinctions already noted are made in general between the uses of the symbols compounded with Δ and those compounded with d; and, in addition, it is customary for the sake of brevity, when the function is indicated by y or by $f(x)$, to indicate the derivative by y' or $f'(x)$ (read "y prime," "function prime x"). We thus have the general definition or notation of the following article.

12. Derivative of a Function.—Given the function $y=f(x)$, its relative rate of increase as compared with its argument x is called its *derivative with respect to the argument,* and is determined as follows:

$$f'(x) = \left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]_{\Delta x=0} = \left[\frac{\Delta y}{\Delta x}\right]_{\Delta x=0} = \frac{dy}{dx} = y'.$$

The law according to which a change in x causes a change in y or $f(x)$ is more explicitly stated by giving the value of the *differential* of the function:

$$dy = f'(x)\cdot dx \text{ or } df(x) = f'(x)\cdot dx.$$

Finding either the derivative or the differential is called *differentiating* the function. As an example, we have already shown that if

$$y = f(x) = x^2,$$

$$y' = f'(x) = \left[\frac{(x+\Delta x)^2 - x^2}{\Delta x}\right]_{\Delta x=0} = 2x = \frac{dy}{dx}.$$

$$dy = 2x\cdot dx; \quad d(x^2) = 2x\cdot dx.$$

That is, the change in x^2 is (momentarily) $2x$ times the change in x.

13. Relation between Derivative and Graph.—If the tangent to the graph of $y=f(x)$ makes the angle τ with the axis of x,

$$y' = f'(x) = \frac{dy}{dx} = \tan\tau.$$

When a function increases as its argument increases, its derivative is positive, and the angle τ for its graph is in the first or third quadrant; when a function decreases as its argument increases, its derivative is negative, and the angle τ is in the second or fourth quadrant; when, as the argument increases, the function is neither increasing nor decreasing, but changing from one state to the other, its derivative is zero and $\tau = 0°$ or $180°$; i. e., the graph is parallel to the axis of abscissas. The function in this last case is said to have an *extreme* value, or an *extremum;* a *maximum* or a *minimum* according as the value is *greater* or *less* than the adjoining values on either side.

The reader should notice, in the examples already expounded and in those that follow, that the determination of the value $\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x=0} = \frac{dy}{dx} = y'$ in every case consists in evaluating an indeterminate fraction, a function of Δx in the form $\frac{0}{0}$. (See Algebra, Art. 45.)

14. *Examples.*

Apply the formal definition of the derivative to the determination of the following:

1. Find the derivative of $ax+b$ with respect to x. (a and b constants.) Ans. a.
2. Find the x-derivative of x^2+a. Ans. $2x$.
3. Find the x-derivative of bx^2. Ans. $2bx$.
4. Find the differentials of x^3 and of at^3. Ans. $3x^2dx$, $3at^2dt$.

15. Rules for Differentiating.—The process of determining a derivative by means of the formal definition is used only to establish a few general rules, from which we can find the derivative of any function or combination of functions. Several of these rules were established in the Alegbra (Arts. 86-87, 113-114). These are here repeated in the new notation, with some additions. The notation $\frac{d}{dx}(f(x))$, for $\frac{df(x)}{dx}$ is merely a matter of convenience when the parenthesis is long.

The derivative of the sum of two functions is the sum of their derivatives. If $f(x)$ and $\phi(x)$ are two functions of x, we have by definition

$$\frac{d}{dx}[f(x)+\phi(x)] = \left[\frac{f(x+\Delta x)+\phi(x+\Delta x)-[f(x)+\phi(x)]}{\Delta x}\right]_{\Delta x=0}$$

$$=\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right]_{\Delta x=0}$$

$$=f'(x)+\phi'(x).$$

$$d[f(x)+\phi(x)]=f'(x)dx+\phi'(x)dx;$$

so that *the same rule holds for the differential of a sum.*

The derivative of the difference of two functions is the difference of their derivatives. The proof is precisely similar. It is evident that these rules may be extended to apply to the derivative or differential of the algebraic sum of any number of functions.

The derivative of a constant is zero. This is obvious, as the change in a constant c, corresponding to the change in any variable, is nothing; or the change in the constant is zero times the change in the variable. Again, the graph of $y=c$ is a straight line, having $\tau=0$ and $\tan\tau=0$ for all values of x.

As a corollary of this and the preceding, the derivative of $[f(x)+c]$, where c is any constant, is $f'(x)$, or

$$\frac{d}{dx}[f(x)+c]=f'(x).$$

Also,

$$d[f(x)+c]=f'(x)dx.$$

The derivative of the product of a constant and a function of x is equal to the product of the constant and the derivative of the function.

For by definition,

$$\frac{d}{dx}[cf(x)]=\left[\frac{c[f(x+\Delta x)]-cf(x)}{\Delta x}\right]_{\Delta x=0}=cf'(x).$$

As a differential,

$$d[cf(x)]=cf'(x)dx.$$

The last two rules are conveniently stated: *In differentiation, a constant term vanishes; a constant factor persists.*

$\frac{d}{dx}(x^n) = nx^{n-1}$, **or** $d(x^n) = nx^{n-1}\,dx$.—By definition,

$$\frac{d}{dx}(x^n) = \left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]_{\Delta x=0}$$

By the binomial formula,

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2$$
$$+\text{terms containing higher powers of } (\Delta x).$$

Hence

$$\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]_{\Delta x=0} = \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + (\text{higher powers of } \Delta x)\right]_{\Delta x=0} = nx^{n-1}.$$

This theorem is here proved on the basis of the binomial theorem, itself proved (Algebra, Art. 54) only for positive integral values of n. The extension of the proof to other values of n is made in a later article of this book (Art. 19.)

Derivative of a Product.—*The derivative* (*or differential*) *of the product of two functions is the sum of the products formed by multiplying each function by the derivative* (*or differential*) *of the other.*

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$, **or** $d(uv) = vdu + udv$, **where** u **and** v **are functions of** x.—Let Δu and Δv be the increments of u and v corresponding to the increment Δx of the independent variable x; then, by definition,

$$\frac{d}{dx}(uv) = \left[\frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x}\right]_{\Delta x=0}$$
$$= \left[\frac{v\cdot\Delta u + u\cdot\Delta v + \Delta u\cdot\Delta v}{\Delta x}\right]_{\Delta x=0}$$

$$=\left[v\frac{\Delta u}{\Delta x}+u\frac{\Delta v}{\Delta x}+\frac{\Delta u}{\Delta x}\cdot\frac{\Delta v}{\Delta x}\cdot\Delta x\right]_{\Delta x=0};$$

$$\left[\frac{\Delta u}{\Delta x}\right]_{\Delta x=0}=\frac{du}{dx};\quad\left[\frac{\Delta v}{\Delta x}\right]_{\Delta x=0}=\frac{dv}{dx};\text{ and}\left[\frac{\Delta u}{\Delta x}\cdot\frac{\Delta v}{\Delta x}\cdot\Delta x\right]_{\Delta x=0}=0.$$

Hence

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx},$$

or

$$d(uv)=v\cdot du+u\cdot dv.$$

This formula enables us to differentiate the product of any number of functions of a single variable; and in combination with other rules to differentiate any polynomial function of two or more variables. For example,

$$\begin{aligned}&d(x^3-3x^2y+y^2-2xy+3x)\\&\quad=3x^2dx-3[x^2dy+yd(x^2)]+2ydy-2(xdy+ydx)+3dx\\&\quad=(3x^2-6xy-2y+3)dx-(3x^2-2y+2x)dy.\end{aligned}$$

16. *Examples.*

1. Find the x-derivatives of the following functions: (a) $2-3x+x^3$, (b) x^2-x^3, (c) $x^4-14x^2+24x+12$.
Ans. (a) $3(x^2-1)$; (b) $x(3x-2)$; (c) $4(x^3-7x+6)$.

2. Find the value of x for which each of the derivatives of Example 1 is zero. Ans. (a) ±1; (b) $0, \frac{2}{3}$; (c) $1, 2, -3$.

3. Trace the curve $y=x^2-x^3$, and find the equation of the tangent at the points where the curve intersects the x-axis.
Ans. $y=0$ and $x+y=1$.

Find the derivatives of the following:

4. x^7, $10x^4$. Ans. $7x^6$, $40x^3$.

5. x^5-3x^3+17x. Ans. $5x^4-9x^2+17$.

6. at^2-bt+c. Ans. $2at-b$.

Find the differentials of the following:

7. $-as^2$, $6s^6$. Ans. $-2asds$, $36s^5ds$.

8. $-\frac{gt^2}{2}+v_0t+s$. Ans. $(-gt+v_0)dt$.

9. $-gt+v_0$. Ans. $-gdt$.

10. In each of the following identities, find the derivative of the left member by the rule for the product, and check by finding the derivative of the right member.

$$x^2 \cdot x^3 \equiv x^5; \quad x^2 \cdot x^2 \equiv x^4; \quad x^n \cdot x^m \equiv x^{m+n}; \quad x^n \cdot x^n \equiv x^{2n}.$$

11. Differentiate $u \cdot u = x$ by the rule for the product, finding $u\frac{du}{dx} = \frac{1}{2}$, and thence show that $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$.

17. Derivative of an Implicit Function (Algebra, Art. 114).— When a function is defined by an implicit relation such as $x^2 + y^2 = a^2$, or, in general, $f(x, y) = 0$, it is understood that x being given any value, y must have such a value as to make the relation true, and is thus confined by the relation to a value or values depending upon the value of x; y is, in other words, a function of x in the ordinary sense. For any pair of values of x and y so related, then, $x^2 + y^2$ or $f(x, y)$ is a constant, and its differential is zero. Differentiating $x^2 + y^2 = a^2$, we thus find that

$$2xdx + 2ydy = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y},$$

thereby determining the derivative of y with respect to x.

Whatever may be the form of $f(x, y)$, the equation $df(x, y) = 0$ involves dx and dy, so that differentiating any implicit relation between two variables results in an equation involving their differentials, which can be solved to give a value of the derivative of one with respect to the other.

18. *Examples.*

Find the derivative of y with respect to x in each of the following:

1. $x^2 - xy + y^2 = 1$. Ans. $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$.

2. $3x^2-3y^2+4x-5y+2=0$. Ans. $\frac{dy}{dx}=\frac{6x+4}{6y+5}$.

3. $x^3+y^3-3xy+x-y=0$. Ans. $\frac{dy}{dx}=-\frac{3x^2-3x+1}{3y^2-3y-1}$.

19. The Differential of x^n when n is Fractional or Negative.— We have proved (Art. 15) that $dx^n=x^{n-1}dx$ for the case in which n is positive and integral. Suppose now that n is a negative integer, and let $n=-m$, m being of course a positive integer. Let $y=x^n=x^{-m}$; then

$$x^m y=1.$$

Then (Arts. 15 and 17):

$$x^m dy+mx^{m-1}y dx=0;$$

$$\frac{dy}{dx}=-mx^{-1}y=nx^{n-1}.$$

Thus the rule is proved for all integral values of n, positive or negative.

Suppose, further, that n is fractional (either positive or negative), and let $n=\frac{p}{q}$, p and q being integral. Let $y=x^n=x^{\frac{p}{q}}$; then

$$y^q=x^p,$$

$$qy^{q-1}dy=px^{p-1}dx,$$

$$\frac{dy}{dx}=\frac{p}{q}\frac{x^{p-1}}{y^{q-1}}=\frac{p}{q}x^{p-1-\frac{p}{q}(q-1)}=\frac{p}{q}x^{\frac{p-q}{q}},$$

$$\frac{dy}{dx}=nx^{n-1}.$$

This proves the rule for any rational value of n, positive or negative, integral or fractional; we are thus enabled to find the derivative of any algebraic expression.

20. *Examples.*

Find the derivative of each of the following:

1. $\dfrac{1}{x^2}=x^{-2}$. Ans. $-2x^{-3}=\dfrac{-2}{x^3}$.

2. $\dfrac{1}{x}$. Ans. $-\dfrac{1}{x^2}$.

3. $\sqrt{x}=x^{\frac{1}{2}}$. Ans. $\frac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{x}}$.

4. $\sqrt{x^3}$. Ans. $\frac{3}{2}\sqrt{x}$.

5. $\sqrt[3]{\dfrac{1}{x^2}}$. Ans. $-\frac{2}{3}x^{-\frac{5}{3}}$.

6. $\dfrac{1}{\sqrt{x^3}}$. Ans. $\dfrac{-3}{2\sqrt{x^5}}$.

7. $t^{\frac{3}{2}}-t^{\frac{1}{2}}$. Ans. $\frac{3}{2}t^{\frac{1}{2}}-\frac{1}{2}t^{-\frac{1}{2}}$.

8. $s^2-\sqrt{s}$. Ans. $2s-\dfrac{1}{2\sqrt{s}}$.

Find the differentials of each of the following:

9. $\sqrt{at^3}$. Ans. $\frac{3}{2}\sqrt{at}\cdot dt$.

10. $\sqrt{\dfrac{x^2}{a}}$. Ans. $\dfrac{dx}{\sqrt{a}}$.

11. $2\sqrt{\dfrac{x}{a}}$. Ans. $\dfrac{dx}{\sqrt{ax}}$.

12. $\sqrt{\dfrac{a}{x}}$. Ans. $-\frac{1}{2}\sqrt{\dfrac{a}{x^3}}$.

Find the value of $\dfrac{dy}{dx}$ from each of the following:

13. $\left(\dfrac{x}{a}\right)^{\frac{1}{2}}+\left(\dfrac{y}{b}\right)^{\frac{1}{2}}=1$. Ans. $\dfrac{dy}{dx}=-\sqrt{\dfrac{by}{ax}}$.

14. $\left(\dfrac{x}{a}\right)^{\frac{2}{3}}+\left(\dfrac{y}{b}\right)^{\frac{2}{3}}=1$. Ans. $\dfrac{dy}{dx}=\sqrt[3]{\dfrac{b^2y}{a^2x}}$.

21. The Differential of a Function of a Function.—If we are told that of three men, A, B, and C, A does twice as much work under given conditions as B, and B three times as much as C, we conclude readily that A does six times as much as C. By the same connection of ideas, if y is a function of x, and z is a function of y; for instance, if $y=x^2$ and $z=y^3$, so that the rate of increase of y is $2x$ times that of x, and the rate of increase of z is $3y^2$ times that of y, the rate of increase of z is clearly $3y^2\times 2x$ or $6xy^2$ times that of x. This is formulated in general:

If $y=f(x)$ and $z=\phi(y)$, so that

$$\frac{dy}{dx}=f'(x),\quad \frac{dz}{dy}=\phi'(y),$$

$$\frac{dz}{dx} = \frac{dy}{dx} \cdot \frac{dz}{dy} = f'(x) \cdot \phi'(y).$$

The identity is sufficiently evident from the fact that dy cancels out from the differential expressions for the two derivatives. The importance of this relation in actual work is very great; there is indeed no single principle upon which so much depends in the practical use of the calculus.

Suppose we wish $d(x^2+a)^5$. We may say:

Let $y=(x^2+a)$; then

$$d(x^2+a)^5 = d(y^5) = 5y^4dy;$$

but $dy=2xdx$; so

$$d(y^5) = 5y^4 \cdot 2xdx = 10x(x^2+a)^4dx.$$

With a little practice, however, most of these steps may be omitted or taken mentally, as follows:

$$\begin{aligned} d(x^2+a)^5 &= 5(x^2+a)^4 d(x^2+a) \\ &= 5(x^2+a)^4 2xdx \\ &= 10x(x^2+a)^4dx. \end{aligned}$$

Again, given $x^2+y^2=a^2$, $z=-\frac{x}{y}$; to find $\frac{dz}{dx}$ as a function of x and y. We have:

$$2xdx+2ydy=0;$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

And again,

$$z = -xy^{-1},$$

$$dz = xy^{-2}dy - y^{-1}dx,$$

$$\frac{dz}{dx} = \frac{x}{y^2}\frac{dy}{dx} - y^{-1} = \frac{x}{y^2}\left(-\frac{x}{y}\right) - \frac{1}{y},$$

$$\frac{dz}{dx} = -\frac{x^2}{y^3} - \frac{1}{y} = -\frac{x^2+y^2}{y^3}.$$

Since $x^2+y^2=a^2$, this can be simplified:

$$\frac{dz}{dx} = -\frac{a^2}{y^3}.$$

22. To most persons the simplest way to use the rules of differentiation is to learn the expressions for fundamental differentials, and to regard the derivative of a $f(x)$ as the result of finding the differential and dividing by dx.

Moreover, as the principle mentioned in Art. 21 is of the greatest importance, it is well to memorize the fundamental rules in such a way that it shall not be readily possible to lose sight of its application. So stated, the rules applying to algebraic functions are as collected below:

If u and v are any functions of the independent variable,

$$d(u \pm v) = du \pm dv,$$
$$d(uv) = udv + vdu,$$
$$d(u^n) = nu^{n-1}du.$$

No other rules for algebraic functions should be memorized except as they force themselves on the attention through persistent occurrence; unless an exception be made in favor of the rule for a quotient, a special case of the rule for a product, which is readily seen to be

$$d\frac{u}{v} = duv^{-1} = \frac{vdu - udv}{v^2}.$$

This rule is useful only when both terms of the fraction are variable. If either term is constant, the rule for u^n is simpler to apply.

23. *Examples.*

1. Prove the rule for differentiating a quotient.

2. Show that $d\,\dfrac{x^2+3}{x-7} = \dfrac{x^2-14x-3}{x^2-14x+49}\,dx.$

Show, without first expanding the expressions to be differentiated, that:

3. $d(a-x)^4 = -4(a-x)^3dx.$

4. $d(3a+2x)^n = 2n(3a+2x)^{n-1}dx.$

5. $d(4-7x^2)^5 = -70x(4-7x^2)^4dx.$

6. $d(3-2x^3)^6 = -36x^2(3-2x^3)^5dx.$

7. $d(a^4+a^2x^2+x^4)^3 = 6x(a^2+2x^2)(a^4+a^2x^2+x^4)^2dx.$

8. $d(a-x)^2(a+x)^3 = (a-x)(a+x)^2(a-5x)\,dx.$

9. $d\,\dfrac{a}{x+a} = \dfrac{-a\,dx}{(x+a)^2},\ d\,\dfrac{a}{a-x} = \dfrac{a\,dx}{(a-x)^2}.$

[Note: Use u^n rule.]

10. $d(2a-3x)^2(3a-2x)^3 = -30(a-x)(2a-3x)(3a-2x)^2dx.$

11. $d\,\dfrac{a^2}{(a^2+x^2)^2} = \dfrac{-4a^2x\,dx}{(a^2+x^2)^3}.$ [Note: Use u^n rule.]

12. $d\sqrt{a^2+x^2} = \dfrac{x\,dx}{\sqrt{a^2+x^2}}.$ [Note: $=d(a^2+x^2)^{\frac{1}{2}}$.]

13. $d\,\dfrac{1}{\sqrt{a^2-x^2}} = \dfrac{x\,dx}{\sqrt{(a^2-x^2)^3}}.$

14. $d\sqrt{\dfrac{a-x}{a+x}} = \dfrac{-a\,dx}{(a+x)\sqrt{a^2-x^2}}.$

15. $d\sqrt{\dfrac{a+x}{a-x}} = \dfrac{a\,dx}{(a-x)\sqrt{a^2-x^2}}.$

16. $d\,\dfrac{(a-x)^2}{(a+x)^3} = \dfrac{-(a-x)(5a-x)}{(a+x)^4}\,dx.$

17. $d\,\dfrac{(3a-2x)^3}{(2a-3x)^2} = \dfrac{6(a+x)(3a-2x)^2}{(2a-3x)^3}\,dx.$

18. $d(a-x)^{\frac{3}{4}}(a+x)^{\frac{1}{4}} = -\frac{1}{2}(a+2x)(a-x)^{-\frac{1}{4}}(a+x)^{-\frac{3}{4}}dx.$

19. $d\,\dfrac{1}{u} = \dfrac{-du}{u^2},\ d\sqrt{u} = \dfrac{du}{2\sqrt{u}},\ d\,\dfrac{1}{\sqrt{u}} = \dfrac{-du}{\sqrt{u^3}},$

$d\,\dfrac{1}{u^2} = \dfrac{-2du}{u^3}.$

20. $d\sqrt{(a^2+3x^2)^3} = 9x\sqrt{a^2+3x^2}\,dx.$

21. $d\,\dfrac{a^2+2abx+b^2x^2}{2(a+b)^3} = \dfrac{b(a+bx)}{(a+b)^3}\,dx.$

22. $d[\sqrt{a-x}+\sqrt{a+x}] = \dfrac{\sqrt{a-x}-\sqrt{a+x}}{2\sqrt{a^2-x^2}}\,dx.$

23. $d\sqrt{\dfrac{2x^3-3a^3}{3a}} = \dfrac{x^2\sqrt{3}\,dx}{\sqrt{a(2x^3-3a^3)}}.$

24. $d\sqrt[3]{\dfrac{x^2-a^2}{x^2+a^2}} = \dfrac{4a^2x\,dx}{3(x^2-a^2)^{\frac{2}{3}}(x^2+a^2)^{\frac{4}{3}}}.$

25. $d\sqrt[4]{\dfrac{a^2-x^2}{a^2+x^2}} = -a^2x(a^2-x^2)^{-\frac{3}{4}}(a^2+x^2)^{-\frac{5}{4}}dx.$

26. $d(u+v)^n = n(u+v)^{n-1}(du+dv).$

27. $d(u^n+v^n) = nu^{n-1}du + nv^{n-1}dv.$

28. $d(a+bt^n)^m = nmb(a+bt^n)^{m-1}t^{n-1}dt.$

29. $d\,\dfrac{a}{bp^2-cp^3} = \dfrac{-a(2bp-3cp^2)}{(bp^2-cp^3)^2}\,dp.$

30. If $x=at^2$, $y=bt$, $d(x^2+y^2) = 2t(2a^2t^2+b^2)\,dt.$

31. If $x=at^2$, $y=bt$, $d\,\dfrac{y}{x} = \dfrac{-b\,dt}{at^2}.$

32. If $x=at^2$, $y=bt$, $d\,\dfrac{b^2}{2ay} = \dfrac{-b\,dt}{2at^2}.$

33. In 30-32, find the corresponding derivatives with respect to x (see Art. 21):

$$\frac{d(x^2+y^2)}{dx} = \frac{2t(2a^2t^2+b^2)\,dt}{2at\,dt} = \frac{2a^2t^2+b^2}{a}.$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{-b}{2a^2t^3}\,;\quad \frac{d}{dx}\left(\frac{b^2}{2ay}\right) = \frac{-b}{4a^2t^3}.$$

34. If $y = \dfrac{b}{a}\sqrt{a^2-x^2}$, $\dfrac{dy}{dx} = \dfrac{-bx}{a\sqrt{a^2-x^2}}.$

35. If $y = \dfrac{b}{a}\sqrt{x^2-a^2}$, $\dfrac{dy}{dx} = \dfrac{bx}{a\sqrt{x^2-a^2}}.$

36. In 34, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = -ab(a^2-x^2)^{-\frac{3}{2}} = -\dfrac{b^4}{a^2y^3}.$

37. In 35, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = -\dfrac{b^4}{a^2y^3}.$

24. Trigonometric Functions. Circular Measure.—When we compare the increase of a function with that of its argument, the units used are of essential importance. For instance, in the speed problem of Art. 7, if the speed had been computed in yards a second rather than feet a second, the rates would have been one-third as great numerically; if in yards an hour, twenty times as great.

Any trigonometric function, such as $\sin\theta$, being the ratio of two lengths, is an abstract number, so that there is no question of its unit; but the angle θ itself may be measured in terms of various units. Of these the most convenient for use in the calculus is the *radian,* or the unit of *circular measure;* and in this subject it is invariably the unit used. So it must be remembered that *for the purposes of the calculus, the argument of any trigonometric function is always understood to be expressed in circular measure.*

As this understanding always exists, it is customary to use merely the name of the argument, without reference to the unit; thus we speak of comparing the changes in $\sin\theta$ and θ, meaning by θ the circular measure of θ, or the value of θ in radians.

When the name of an angle is used to indicate a quantity, the quantity indicated is always the circular measure of the angle.

25. Derivatives of Trigonometric Functions.—In order to find the derivatives of $\sin\theta$ and $\cos\theta$, we shall need to know the values when $\Delta\theta=0$ of certain forms. Construct a figure (Fig. 5) showing graphically the circular measure $\left(\frac{a}{r}\right)$, the sine $\left(\frac{s}{r}\right)$ and the cosine $\left(\frac{c}{r}\right)$ of an angle $\Delta\theta$, and add the tangents as shown. Note that doubling $\Delta\theta$ doubles also s, a, and t. As $\Delta\theta$ approaches zero as a limit, c approaches r as a limit, and s, t, and a each approach zero.

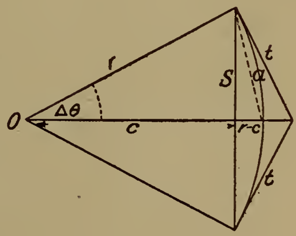


FIG. 5.

By geometry,

$$\left.\begin{aligned} &2s<2a<2t,\\ &\frac{s}{r}<\frac{a}{r}<\frac{t}{r}\,; \end{aligned}\right\}\text{ or: }\left\{\begin{aligned} &\sin\Delta\theta<\Delta\theta<\tan\Delta\theta,\\ &1<\frac{\Delta\theta}{\sin\Delta\theta}<\sec\Delta\theta. \end{aligned}\right.$$

Now

$$\left[\sec \Delta\theta = \frac{r}{c}\right]_{\Delta\theta=0} = 1;$$

hence

$$\left[\frac{\Delta\theta}{\sin \Delta\theta}\right]_{\Delta\theta=0} = 1.$$

By means of this proposition, we can now find the derivatives of $\sin\theta$ and $\cos\theta$.

By definition,

$$\frac{d \sin\theta}{d\theta} = \left[\frac{\sin(\theta+\Delta\theta) - \sin\theta}{\Delta\theta}\right]_{\Delta\theta=0},$$

which may be written

$$\frac{d \sin\theta}{d\theta} = \left[\frac{2\cos\dfrac{2\theta+\Delta\theta}{2} \cdot \sin\dfrac{\Delta\theta}{2}}{\Delta\theta}\right]_{\Delta\theta=0}$$

$$= \left[\cos(\theta+\tfrac{1}{2}\Delta\theta) \cdot \frac{\sin(\frac{1}{2}\Delta\theta)}{\frac{1}{2}\Delta\theta}\right]_{\frac{1}{2}\Delta\theta=0},$$

$$\frac{d \sin\theta}{d\theta} = \cos\theta,$$

since the limit of the fraction is 1, or

$$d(\sin\theta) = \cos\theta \, d\theta.$$

Similarly,

$$\frac{d \cos\theta}{d\theta} = \left[\frac{\cos(\theta+\Delta\theta) - \cos\theta}{\Delta\theta}\right]_{\Delta\theta=0}$$

$$= \left[-\sin(\theta+\tfrac{1}{2}\Delta\theta)\,\frac{\sin(\frac{1}{2}\Delta\theta)}{\frac{1}{2}\Delta\theta}\right]_{\frac{1}{2}\Delta\theta=0},$$

$$\frac{d\cos\theta}{d\theta} = -\sin\theta,$$

or

$$d(\cos\theta) = -\sin\theta \, d\theta.$$

We can find these derivatives much more simply by making use of the expressions for the sine and the cosine as algebraic functions of the circular measure, which are shown to be (see Brown's Trigonometry, Art. 66).

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \ldots \ldots,$$

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots \ldots,$$

whence the derivatives of the sine and cosine of an angle with respect to its circular measure are

$$\frac{d\sin\theta}{d\theta} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots \ldots = \cos\theta,$$

$$\frac{d\cos\theta}{d\theta} = -\theta + \frac{\theta^3}{\lfloor 3} - \frac{\theta^5}{\lfloor 5} + \ldots \ldots = -\sin\theta.$$

We can now differentiate the other trigonometric functions, which are algebraic functions of *sine* and *cosine*.

$$\frac{d\sec\theta}{d\theta} = \frac{d}{d\theta}(\cos\theta)^{-1} = -(\cos\theta)^{-2}(-\sin\theta)$$

$$= \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta.$$

$$\frac{d\csc\theta}{d\theta} = \frac{d}{d\theta}(\sin\theta)^{-1} = -(\sin\theta)^{-2}(\cos\theta)$$

$$= -\frac{\cos\theta}{\sin^2\theta} = -\csc\theta\cot\theta.$$

$$\frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\sin\theta(\cos\theta)^{-1}$$

$$= -\sin\theta(\cos\theta)^{-2}(-\sin\theta) + (\cos\theta)^{-1}\cos\theta$$

$$= 1 + \tan^2\theta = \sec^2\theta.$$

$$\frac{d}{d\theta}\cot\theta = \frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\sin\theta(-\sin\theta) - \cos\theta(\cos\theta)}{\sin^2\theta}$$

$$= \frac{-1}{\sin^2\theta} = -\csc^2\theta.$$

One of the last two has been derived by the rule for the product, the other by the rule for the quotient; each is readily handled in both ways.

26. The principle of differentiating a function of a function by successive application of several rules (Art. 21) occurs constantly in treating trigonometric functions. E. g.,

$$d(\sin^3 2\theta) = 3\sin^2 2\theta\, d(\sin 2\theta)$$
$$= 3\sin^2 2\theta \cos 2\theta\, d(2\theta) = 6\sin^2 2\theta \cos 2\theta\, d\theta.$$

It is likely to be the case that a trigonometric expression can be put in several different forms, one of which may be simplest for one purpose, another for another purpose. For instance, we might write

$$3\sin 4\theta \sin 2\theta\, d\theta \ \text{ for } \ 6\sin^2 2\theta \cos 2\theta\, d\theta,$$

or

$$24\sin^2\theta\cos^2\theta(\cos^2\theta - \sin^2\theta)\,d\theta,$$

or might express this last form in terms of $\sin^2\theta$ only or of $\cos^2\theta$ only. The answers to the examples may need some such reduction before they agree with those in the text.

27. *Examples.*

1. Prove the rule for $d\tan\theta$ from the rule for $d\dfrac{u}{v}$.

2. Prove the rule for $d\cot\theta$ from the rule for duv.

3. Since, if $\sin\theta = y$, and $\cos\theta = x$, we have $x^2+y^2=1$, $\dfrac{dy}{d\theta}=x$, prove from these two relations that $\dfrac{dx}{d\theta}=-y$. Again, prove this from the relation $\cos\theta=\sin(\frac{\pi}{2}-\theta)$.

Prove the following:

4. $d\sin^2 x = \sin 2x\, dx = -d\cos^2 x.$

5. $d\sin(x^2) = 2x\cos(x^2)\,dx.$

6. $d\cos(x^2) = -2x\sin(x^2)\,dx.$

7. $d\sqrt{1-\cos\theta} = \frac{1}{2}\sqrt{2}\cos\frac{\theta}{2}\,d\theta.$

8. $d\sqrt{1+\cos\theta} = -\frac{1}{2}\sqrt{2}\sin\frac{\theta}{2}\,d\theta.$

9. $d\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta.$

10. $d\,\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{d\theta}{1+\cos\theta}.$

11. $d(\csc\theta - \cot\theta) = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta.$

12. $d \cos 5\theta = -5 \sin 5\theta \, d\theta.$

13. $d(\sin 7\theta - \sin 3\theta) = (7 \cos 7\theta - 3 \cos 3\theta)\, d\theta.$

14. $d \cos 5\theta \sin 2\theta = \frac{1}{2}(7 \cos 7\theta - 3 \cos 3\theta)\, d\theta.$

15. $d \tan^2 t = \dfrac{2 \sin t}{\cos^3 t}\, dt.$

16. $d \sin ax^2 = 2ax \cos ax^2 dx.$

17. If $x = a \cos \phi$ and $y = b \sin \phi$, $\dfrac{dy}{dx} = -\dfrac{b}{a} \cot \phi.$

18. If $x = a \sec \phi$ and $y = b \tan \phi$, $\dfrac{dy}{dx} = \dfrac{b}{a} \csc \phi.$

19. In (17), $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = +\dfrac{b}{a} \csc^2 \phi \dfrac{d\phi}{dx} = -\dfrac{b}{a^2} \csc^3 \phi.$

20. In (18), $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = -\dfrac{b}{a^2} \cot^3 \phi.$

Compare Exs. 17-20 with Exs. 34-37, Art. 23.

21. $d(\sec \theta + \tan \theta)^n = n \sec \theta (\sec \theta + \tan \theta)^n.$

22. $d(\cos^4 \theta - \sin^4 \theta) = -2 \sin 2\theta \, d\theta.$

23. For what value of the angle is its sine increasing one-half as fast as the angle? The tangent twice as fast?

Ans. $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

28. The Inverse of the Trigonometric Functions.—The symbol *sin*$^{-1}$ (which is read "inverse sine") stands for "the angle whose sine is . . . . " When it is used in the calculus to represent a quantity, it signifies "*the circular measure of the angle whose sine is* . . . . " So for the other inverse functions. In the familiar graphic representation of Fig. 6, the straight lines s, c, t, and the arc θ represent the sine, cosine, tangent, and circular measure of the angle θ. Thus s is the sine of the arc θ, and θ is the arc of the sine s. Consequently many mathematicians read the symbol *sin*$^{-1}$ "arc-sine"; e. g., $\theta = \sin^{-1} s = \cos^{-1} c = \tan^{-1} t$ is read: "θ equals arc-sine of s, arc-cosine of c, arc-tangent of t."

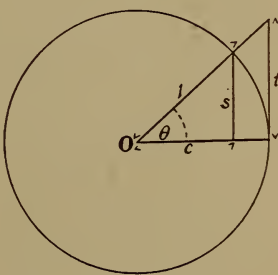


FIG. 6.

29. Derivatives of the Inverse Trigonometric Functions.—The relations $\theta = \sin^{-1}\frac{x}{a}$ and $\sin\theta = \frac{x}{a}$ are equivalent; differentiating the second, we find that

$$\cos\theta\, d\theta = \frac{dx}{a}\,; \quad d\theta = \frac{dx}{a\cos\theta} = \frac{dx}{\sqrt{a^2 - x^2}}\,;$$

hence

$$d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{a^2-x^2}}.$$

By the same method we derive the following list of differentials:

$$d\sin^{-1}\frac{x}{a} = \frac{dx}{\sqrt{a^2-x^2}} = -d\cos^{-1}\frac{x}{a},$$

$$d\tan^{-1}\frac{x}{a} = \frac{a\,dx}{a^2+x^2} = -d\cot^{-1}\frac{x}{a},$$

$$d\sec^{-1}\frac{x}{a} = \frac{a\,dx}{x\sqrt{x^2-a^2}} = -d\csc^{-1}\frac{x}{a}.$$

To these may be added, if $\operatorname{versin}\theta = 1 - \cos\theta$,

$$d\operatorname{versin}^{-1}\frac{x}{a} = \frac{dx}{\sqrt{2ax-x^2}}.$$

It is helpful to notice, in order to remember when the factor a appears in the differential of an inverse function of $\frac{x}{a}$, that such a differential is always of degree zero, a, x, and dx each representing a length, and so being of degree 1.

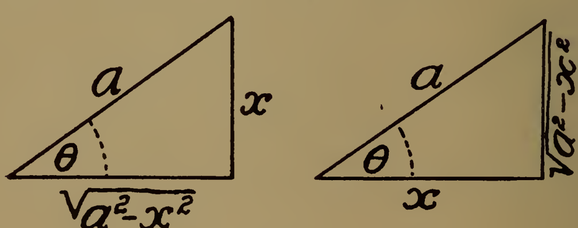


FIG. 7.

It should also be noticed that an angle may be given much more simply as the inverse of one function than as the inverse of another; the best method of transformation is to draw the geometric figure. For instance,

$$\theta = \cos^{-1}\frac{\sqrt{a^2-x^2}}{a} = \sin^{-1}\frac{x}{a},$$

$$\theta = \tan^{-1}\frac{\sqrt{a^2-x^2}}{x} = \cos^{-1}\frac{x}{a}.$$

30. *Examples.*

Prove the following:

1. $d\sin^{-1}\frac{x-2}{3} = \frac{dx}{\sqrt{5+4x-x^2}}$.

2. $d\cos^{-1}\frac{b-x}{a} = \frac{dx}{\sqrt{a^2-b^2+2bx-x^2}}$.

3. $d\tan^{-1}\frac{2x-1}{2} = \frac{4dx}{4x^2-4x+5}$.

4. $d\tan^{-1}\frac{x-b}{2a} = \frac{2adx}{x^2-2bx+b^2+4a^2}$.

5. $\frac{d}{dx}\sin^{-1}(\cos x) = -1$.

6. $\frac{d}{dx}\sin(\cos^{-1}x) = \frac{-x}{\sqrt{1-x^2}}$.

7. $\frac{d}{dx}(x\sin^{-1}x+\sqrt{1-x^2}) = \sin^{-1}x$.

8. $\frac{d}{dx}\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$.

9. $\frac{d}{dx}\csc^{-1}\frac{\sqrt{x^2+a^2}}{x} = \frac{a}{a^2+x^2}$.

10. $\frac{d}{dx}\sec^{-1}\frac{a}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$.

11. $\frac{d}{dx}\tan^{-1}\sqrt{\frac{1-x}{1+x}} = -\frac{-1}{2\sqrt{1-x^2}}$.

(Note: Let $x = \cos\theta$ and write the radical as a function of $\frac{\theta}{2}$.)

12. $\frac{d}{dx}\sin^{-1}\sqrt{\frac{1-x}{2}}=-\frac{1}{2\sqrt{1-x^2}}$.

13. $\frac{d}{dx}\cos^{-1}\sqrt{\frac{1+x}{2}}=-\frac{1}{2\sqrt{1-x^2}}$.

14. $\frac{d}{dx}\sin^{-1}2x\sqrt{1-x^2}=\frac{2}{\sqrt{1-x^2}}$.

15. $\frac{d}{dx}\tan^{-1}\frac{2x}{1-x^2}=\frac{2}{1+x^2}$.

31. Exponential and Logarithmic Functions.—The derivatives of exponential expressions like a^x, e^x, can best be derived from their expansion into infinite series (see Algebra, Brown and Capron, Art. 115). The value of e^x, thus expressed, is

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\ldots,$$

the derivative of which is

$$\frac{d(e^x)}{dx}=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\ldots=e^x.$$

Hence the derivative of e^x is e^x itself.

$$de^x=e^x\cdot dx.$$

Differential formulas of logarithmic expressions are simplest when the base is e; logarithms to the base e (natural logarithms) are therefore used more than any others in the calculus; and so for convenience we write $\log x$ for $\log_e x$, and always understand the base to be e when no base is written.

The value of $d\log x$ may be obtained from the equation $y=\log_e x$, which in exponential form, is

$$e^y=x.$$

Differentiating, we find

$$e^y\,dy=dx,\quad \frac{dy}{dx}=\frac{1}{e^y}=\frac{1}{x};$$

so that

$$d\log x=\frac{dx}{x}.$$

The differential of a^x may be obtained from the series for a^x, or as follows: If $y = a^x$,

$$\log y = (\log a)x,$$

$$\frac{dy}{y} = (\log a)\,dx,$$

$$dy = (\log a)\,y\,dx,$$

$$d(a^x) = a^x \cdot \log a \cdot dx.$$

For $d\log_a x$ we have, if $y = \log_a x$,

$$x = a^y,$$

$$dx = a^y \log a\, dy,$$

$$dy = \frac{dx}{a^y \log a} = \frac{dx}{x \log a}.$$

As $\dfrac{1}{\log_e a} = \log_a e$, the *modulus* of the system having a as its base,

$$d \log_a x = \log_a e \frac{dx}{x}.$$

When $a = 10$, the modulus, $\log_{10} e = .43429+$, is usually denoted by μ, so that

$$d \log_{10} x = \frac{\mu dx}{x} = \frac{.43429 dx}{x}.$$

32. The Logarithms of the Trigonometric Functions.—By combining the principles of Arts. 25 and 31, we readily obtain the differentials of the logarithms of the trigonometric functions. For instance,

$$d \log \sin \theta = \frac{d \sin \theta}{\sin \theta} = \frac{\cos \theta\, d\theta}{\sin \theta} = \cot \theta\, d\theta, \text{ etc.}$$

The full list is:

$$d \log \sin \theta = \cot \theta \,.\, d\theta,$$

$$d \log \cos \theta = -\tan \theta \,.\, d\theta,$$

$$d \log \tan \theta = 2 \csc 2\theta \,.\, d\theta,$$

$$d \log \csc \theta = -\cot \theta \,.\, d\theta,$$

$$d \log \sec \theta = \tan \theta \,.\, d\theta,$$

$$d \log \cot \theta = -2 \csc 2\theta \,.\, d\theta.$$

33. Logarithmic Differentiation.—It often happens that the logarithm of a function is easier to differentiate than the function itself. For instance, finding $d\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$ involves a good deal of algebraic work, which is lessened appreciably by the following process:

Let

$$y=\sqrt{\frac{a^2+x^2}{a^2-x^2}};$$

then

$$\log y=\tfrac{1}{2}\log(a^2+x^2)-\tfrac{1}{2}\log(a^2-x^2),$$

$$\frac{dy}{y}=\frac{xdx}{a^2+x^2}+\frac{xdx}{a^2-x^2}=\frac{2a^2xdx}{a^4-x^4},$$

$$dy=y\frac{2a^2xdx}{a^4-x^4}=\sqrt{\frac{a^2+x^2}{a^2-x^2}}\cdot\frac{2a^2xdx}{a^4-x^4}=\frac{2a^2xdx}{(a^2-x^2)^{\frac{3}{2}}(a^2+x^2)^{\frac{1}{2}}}.$$

Besides being convenient for products, quotients, powers, and roots, logarithmic differentiation is *necessary* for exponential functions in which *both base and exponent are variable.*

For instance, to find dx^x; if $y=x^x$,

$$\log y=x\log x,$$

$$\frac{dy}{y}=dx+\log x dx,$$

$$dy=y(1+\log x)\,dx=x^x(1+\log x)\,dx.$$

34. *Examples.*

1. $\dfrac{d}{dx}\log\dfrac{x-a}{x+a}=\dfrac{2a}{x^2-a^2}.$

2. $\dfrac{d}{dx}\log\dfrac{x-a}{x-b}=\dfrac{a-b}{(x-a)(x-b)}.$

3. $\dfrac{d}{dx}e^{a\sin x}=ae^{a\sin x}\cos x.$

4. $y=\log\dfrac{e^x}{1+e^x}.\quad y'=\dfrac{1}{1+e^x}.$

5. $y = \log(x + \sqrt{1+x^2})$. $y' = \frac{1}{\sqrt{1+x^2}}$.

6. $f(x) = \log(\log x)$. $f'(x) = \frac{1}{x \log x}$.

7. $f(\theta) = \log(\theta + \sin\theta)$. $f'(\theta) = \frac{1+\cos\theta}{\theta + \sin\theta}$.

8. $f(\theta) = \log(\sec\theta + \tan\theta)$. $f'(\theta) = \sec\theta$.

9. $f(\theta) = \log\left(\frac{1-\cos\theta}{1+\cos\theta}\right)$. $f'(\theta) = 2\csc\theta$.

10. $f(\theta) = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$. $f'(\theta) = \sec\theta$.

11. $y = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}}$. $y' = \frac{1}{(1-x)\sqrt{(1-x^2)}}$.

12. $y = \frac{\sqrt{(x+1)(x+3)^9}}{(x+2)^4}$. $y' = \frac{x^2(x+3)^{\frac{7}{2}}}{(x+2)^5(x+1)^{\frac{1}{2}}}$.

35. We shall proceed after this article to some applications of derivatives, and for some time shall develop no further rules for differentiating; it will therefore be convenient to collect at this place the rules we have so far derived. These rules are given as expressions for differentials. As a practical rule for finding derivatives, we have:

To find the derivative of a variable z with respect to a variable w, find the quotient $\frac{dz}{dw}$, *and if any derivative appears in this quotient, determine and substitute its value.*

d constant $= 0$.

*$d(u \pm v) = du \pm dv$.

*$duv = udv + vdu$.

$d\frac{u}{v} = \frac{vdu - udv}{v^2}$.

*$du^n = nu^{n-1}du$.

$d\frac{1}{u} = \frac{-du}{u^2}$.

$d \csc u = -\csc u \cot u du$.

$d \sin^{-1} u = \frac{du}{\sqrt{1-u^2}} = -d \cos^{-1} u$.

$d \tan^{-1} u = \frac{du}{1+u^2} = -d \cot^{-1} u$.

$d \sec^{-1} u = \frac{du}{u\sqrt{u^2-1}} = -d \csc^{-1} u$.

$de^u = e^u du$.

$d\sqrt{u} = \frac{du}{2\sqrt{u}}$.

$*d \sin u = \cos u du$.

$d \tan u = \sec^2 u du$.

$d \sec u = \sec u \tan u du$.

$d \cos u = -\sin u du$.

$d \cot u = -csc^2 u du$.

$*da^u = a^u \log a du$.

$d \log u = \frac{du}{u}$.

$d \log_a u = \log_a e \frac{du}{u}$.

$d \log_{10} u = \frac{\mu du}{u} = \frac{.43429 du}{u}$.

du^v is found by logarithmic differentiation.

From the rules marked with an asterisk, all the others can be derived.

36. *Miscellaneous Examples.*

Prove the following:

1. $\frac{d}{d\theta}(\theta - \sin\theta \cos\theta) = 2\sin^2\theta$.

2. $\frac{d}{dx} \frac{\sin x}{a - b\cos x} = \frac{a\cos x - b}{(a - b\cos x)^2}$.

3. $\frac{d}{dx}(2x \sin x + [2 - x^2]\cos x) = x^2 \sin x$.

4. $\frac{d}{dx} \sin^{-1} \frac{1-x}{1+x} = \frac{-1}{(1+x)\sqrt{x}}$.

5. $\frac{d}{dx} \sin^{-1} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{x\sqrt{a^2 - b^2}}{(x^2 - b^2)\sqrt{x^2 - a^2}}$.

6. $\frac{d}{dx} \tan^{-1} \frac{x\sqrt{3}}{x+2} = \frac{\sqrt{3}}{2(x^2 + x + 1)}$.

7. $\frac{d}{dx} \cos^{-1} \frac{b + a\cos x}{a + b\cos x} = \frac{\sqrt{a^2 - b^2}}{a + b\cos x}$.

8. $\frac{d}{dx}\left[2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)\right] = \frac{\sqrt{a^2 - b^2}}{a + b\cos x}$.

Solve the following by letting $x = a\sin\phi$, finding the derivative with respect to ϕ, and thence the derivative with respect to x.

9. $\frac{d}{dx} \frac{x}{\sqrt{a^2 - x^2}} = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.

10. $\frac{d}{dx}\frac{\sqrt{a^2-x^2}}{a^2x}=-\frac{1}{x^2\sqrt{a^2-x^2}}.$

Make similar substitutions in the following:

11. $\frac{d}{dx}\frac{x^3}{(1-x^2)^3}=\frac{3x^2(1+x^2)}{(1-x^2)^4}.$

12. $\frac{d}{dx}\frac{x^3}{(1-x^2)^{\frac{3}{2}}}=\frac{3x^2}{(1-x^2)^{\frac{5}{2}}}.$

13. $\frac{d}{dx}\frac{1-x}{\sqrt{1+x^2}}=-\frac{1+x}{(1+x^2)^{\frac{3}{2}}}.$

14. $d\log(\sec\theta+\tan\theta)=\sec\theta\, d\theta.$

15. $d[\sec\theta\tan\theta+\log(\sec\theta+\tan\theta)]=2\sec^3\theta\, d\theta.$

16. $de^{-x^2}=-2xe^{-x^2}dx.$

17. $dx^{\sin x}=x^{\sin x-1}\sin x+x^{\sin x}\cos x\log x.$

37. Problems in Speed and Time-Rates.—Speed has already been defined as the relative rate of increase in the distance s traversed from a fixed point, as compared with the time t elapsed since a given instant. It is equal to

$$\frac{ds}{dt}=\left[\frac{\Delta s}{\Delta t}\right]_{\Delta t=0},$$

the derivative of s with respect to t. Speed may thus be called the *time-rate of distance*. Any other variable has in the same sense a *time-rate*. Thus if θ is the angle generated by the spoke of a wheel during any time, t, $\frac{d\theta}{dt}$ is the time-rate of the rotation, called the *angular velocity*. Again if A is the area of an expanding surface, or V the volume of an expanding solid, $\frac{dA}{dt}$ or $\frac{dV}{dt}$ is the *time-rate of expansion*, and so on. All these time-rates are sometimes called *velocities*, or simply *rates*. They are all, of course, *derivatives with respect to time*.

Aside from certain general principles, there is no theory of this application of derivatives; such difficulties as arise in the solution of the problem are chiefly matters of algebra and geometry.

We shall discuss a few examples suggestive of the methods most commonly useful.

Example 1: A rope attached to a boat is being hauled in at the rate of $2\frac{1}{2}$ f/s by a man on a wharf, whose hands are 12 ft. higher up than the point of attachment of the rope. Find the speed of the boat (a) in general, (b) when it is 9 ft. from the wharf.

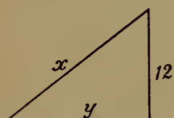


FIG. 8.

Let y be the distance of the boat from the wharf at any time, and x its distance from the man; then we have given

$$\frac{dx}{dt}=\frac{-5}{2}\text{ f/s},\quad y^2+144=x^2,$$

and are to find $\frac{dy}{dt}$. Then

$$y=\sqrt{x^2-144};\quad dy=\frac{xdx}{\sqrt{x^2-144}}.$$

Dividing by dt,

$$\frac{dy}{dt}=\frac{x}{\sqrt{x^2-144}}\frac{dx}{dt}=\frac{-5x}{2\sqrt{x^2-144}}\text{ f/s, in general.}$$

When $y=9$, $x=\sqrt{144+y^2}=15$,

$$\left.\frac{dy}{dt}\right]_{y=9}=\frac{-75}{18}\text{ f/s}=-4\tfrac{1}{6}\text{ f/s}.$$

Note the principle actually employed here: The time-rate of x is given $=\frac{5}{2}$ f/s; we find the x-rate of $y=\frac{x}{\sqrt{x^2-144}}$; then since the increase in y is $\frac{x}{\sqrt{x^2-144}}$ times that in x, and the increase in x is $\frac{-5}{2}$ that in t, the increase in y is

$$\frac{dy}{dt}=\frac{dy}{dx}\,\frac{dx}{dt}=\frac{-5x}{2\sqrt{x^2-144}}\text{ times that in } t.$$

Note that both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are negative, since x and y both decrease.

Note particularly that nothing can be accomplished by taking $y=9$ at the start; the general case must be used to get an *equation* expressing the functional relation between y and x.

Example 2: Two railroad tracks cross at right-angles; on each a train is approaching the crossing; one, 17 mi. off, is going west at 12 m/h; the other, 22 mi. off, is going south at 15 m/h. Find the rate at which they are approaching each other (a) at the end of 40 min., (b) when the west-bound train is $1\frac{1}{2}$ mi. west of the crossing. After t hours, let the train going west be x mi. east of the crossing; the train going south, y mi. north of it; and let the two trains be z mi. apart.

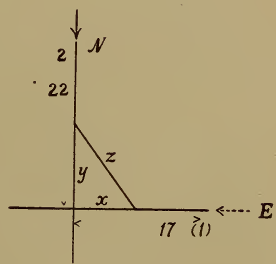


FIG. 9.

We have:

$$\frac{dx}{dt} = -12 \text{ m/h}, \quad \frac{dy}{dt} = -15 \text{ m/h},$$

$$z^2 = x^2 + y^2;$$

so that

$$2z\,dz = 2x\,dx + 2y\,dy,$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right),$$

$$\frac{dz}{dt} = \frac{1}{z}(-12x - 15y) \text{ m/h} = -\frac{3}{z}(4x + 5y) \text{ m/h}.$$

At the end of 40 min., the trains will have gone 8 mi. west and 10 mi. south respectively, so that $x=9$ mi., $y=12$ mi., and therefore $z=15$ mi. Hence,

$$\left[\frac{dz}{dt}\right]_{t=\frac{2}{3}} = -\tfrac{3}{15}(\overline{4\times 9}+\overline{5\times 12})\,\text{m/h} = -\tfrac{96}{5}\,\text{m/h}.$$

At the end of 40 min. the trains are approaching at 19.2 m/h.

When $x=-\frac{3}{2}$, the west-bound train will have gone $\frac{37}{2}$ mi. in $\frac{37}{24}$ hrs., and the south-bound train will be $\frac{9}{8}$ mi. south of the crossing, so that $y=-\frac{9}{8}$; consequently $z=\frac{15}{8}$. Hence,

$$\left[\frac{dz}{dt}\right]_{t=\frac{37}{24}} = \frac{-3\times 8}{15}[-4\times\tfrac{3}{2}-5\times\tfrac{9}{8}]\,\text{m/h} = \tfrac{93}{5}\,\text{m/h},$$

and the trains are receding at 18.6 m/h.

Since $x=17-12t$ and $y=22-15t$, it would be possible to express z directly as a function of t, and thence find $\frac{dz}{dt}$, but the computation would be much more laborious. The general results would be

$$z=\sqrt{773-1068t+369t^2},$$

$$\frac{dz}{dt} = \frac{-534+369t}{\sqrt{773-1068t+369t^2}}.$$

Example 3: A man is trotting around a circular track 4 mi. in diameter at the rate of 6 mi. an hour. Find the rate at which his distance from a fixed point of the track is increasing.

Let the fixed point be A, and the man's position at any time be M; let the central angle AOM be θ, and the straight line $AM=x$; then

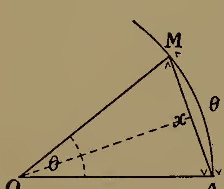


FIG. 10.

$$x=2\sin\tfrac{\theta}{2}\ (2\text{ mi.}) = 4\sin\tfrac{\theta}{2}\text{ mi.}$$

$$dx=2\cos\tfrac{\theta}{2}\,d\theta\text{ mi.}$$

and

$$\frac{dx}{dt}=2\cos\tfrac{\theta}{2}\,\frac{d\theta}{dt}\ \text{m/h};$$

but in this relation θ must be in circular measure. (See Art 24.)

As the radius is 2 mi., 6 mi. = 3 radii, and the speed of 6 m/h gives an angular velocity, $\frac{d\theta}{dt} = 3$; hence

$$\frac{dx}{dt} = 3(2 \cos \tfrac{\theta}{2} \text{ m/h}) = 6 \cos \tfrac{\theta}{2} \text{ m/h}.$$

With θ between 0 and π, this rate is positive, and the man is receding from A; with θ between π and 2π, it is negative, and the man is approaching A; this cycle then repeats.

This solution can also be applied to any case of two objects moving in the same circle, their relative speed taking the place of the speed of the man in this problem.

Examples.

4. One end of a ladder 29 ft. long is against a vertical wall, the other on a horizontal floor. If the lower end slides along the floor at $1\frac{1}{2}$ f/s, how fast is the upper end slipping down the wall when the lower end is 20 ft. from the wall? 25 ft.?

Ans. $1\frac{3}{7}$ ft. a sec.; $\frac{25}{24}\sqrt{6}$ f/s = 2.55 f/s approx.

5. The free end of a ball of string is attached to a wall; 8 ft. lower down, the ball is moving horizontally at the rate of $4\frac{1}{4}$ f/s; how fast is the string unwinding when the ball is 15 ft. from the wall? Ans. $3\frac{3}{4}$ f/s.

6. A stone drops from a height of 100 ft. upon level ground. Given that its speed when it has fallen s ft. is $8\sqrt{s}$ f/s, find the speed of its shadow on the ground, when the stone is at a height of $9\frac{3}{4}$ ft., the altitude of the sun being 30°.

Ans. $76\sqrt{3}$ f/s = 131.6 f/s.

7. If a shadow of the stone in example 6 is cast by a light 20 ft. above the ground and 10 ft. from the path of the stone, how fast will this shadow be moving when the stone is 19 ft. above the ground? $9\frac{3}{4}$ ft.? Ans. 14400 f/s and nearly 144.7 f/s.

8. A stone cast into a pool causes a circular wave, the front of which moves at the rate of $2\frac{1}{2}$ in. a second. Show that the area of the circle increases at the rate of $5\pi r$ sq. in. a second, r being the variable radius of the circle.

9. A gun is fired from a balloon 200 ft. above the ground, producing a spherical wave in the air, which moves at the rate of

1100 ft. a second. Find the rate at which the surface of this wave and the volume enclosed by it are increasing when the wave reaches the ground.

Ans. Surface, at $1{,}760{,}000\pi$ sq. ft./sec.; volume, at $176{,}000{,}000\pi$ cu. ft./sec.

10. Gas is pumped at the rate of 10 cu. in./sec. into a spherical toy balloon; how fast are the surface and radius of the balloon increasing: (a) when the radius is 5 in.? (b) When the balloon holds a cubic foot?

Ans. (a) Surface at 4 sq. in./sec.; radius at $\frac{1}{10\pi}$ in./sec. (b) Surface at $\frac{10}{3}\left(\frac{\pi}{6}\right)^{\frac{1}{3}}$ sq. in./sec.; radius at $\frac{5}{432}\left(\frac{6}{\pi}\right)^{\frac{1}{3}}$ in./sec.

11. Two ships, A and B, are on the same meridian, 117 mi. apart. A is sailing due east at the rate of 10 m/h, B is sailing due north, toward A, at the rate of 15 m/h. At what rate are they approaching each other after they have been sailing 3 hrs.? After they have been sailing 5 hrs. and 12 min.? When are they neither approaching nor receding? What is the closest approach they make to each other?

Ans. $+10$ m/h, $+1$ m/h, after 5 hrs. 24 min., $18\sqrt{13}$ mi.

12. Two trains are 12 mi. and 6 mi. respectively from a crossing where their routes intersect at right angles; the first train is approaching the crossing at the rate of 42 m/h, the second receding from it at the rate of 36 m/h; are the trains approaching or receding, and at what rate: (a) after 10 min.? (b) After 2 hrs.? (c) When will the distance between them be least?

Ans. (a) Receding, $17\frac{1}{13}$ m/h; (b) receding, about 54.94 m/h; (c) at the end of $5\frac{11}{17}$ min.

13. Two railroad tracks make an angle $AOB = 60°$; $AO = 8$ mi., $BO = 5$ mi. A train at A is approaching O at 20 m/h, and a train at B is approaching O at 30 m/h. Find (a) at what rate the trains are approaching each other; at what rate they will be receding or approaching: (b) when B reaches O; (c) when A reaches O; (d) at the end of 30 min.

Ans. (a) 20 m/h; (b) approaching at 5 m/h; (c) receding at 20 m/h; (d) receding at about 22.9 m/h.

14. A solid expands so that the time-rate $\frac{dx}{dt}$ of the expansion of any one of its linear dimensions is kx, k being a constant. Show that the time-rate $\frac{dV}{dt}$ of its volume is $3kV$.

15. The roadway of a bridge is 20 yds. above the roadway below, and the two run perpendicular to each other. One man is going over the bridge at 3 m/h, and another, directly under him, is going at 8 m/h. At what rate will they be receding from each other at the end of 3 min.? Ans. About 8.54 m/h.

16. A city street has a vertical wall on one side, and 75 ft. from it, on the other side, is a light. A man starts 15 ft. directly up the street from this light and crosses straight over at 4 m/h. Find the rate at which his shadow is moving horizontally along the wall (a) when he has gone 10 ft.; (b) when he has gone 50 ft. Ans. (a) 45 m/h; (b) 1.8 m/h.

17. Wine is poured into a conical glass 3 in. deep at a uniform rate, filling the glass in 8 secs. At what rate is the surface rising at the end of 1 sec.? When it reaches the brim?

Ans. $\frac{1}{2}$ in./sec.; $\frac{1}{8}$ in./sec.

18. A beam 20 ft. long rests against a vertical wall and a horizontal floor; a bar is attached by hinges to its middle and against the angle of wall and floor. The beam slides down so that the angle θ between this bar and the wall increases at the rate of 18° a second. Find the rate at which each end of the beam is moving when $\theta = 30°, 45°, 60°$.

Ans. $\theta = 30°, 45°, 60°$; rate of upper end $= \pi$ f/s, $\pi\sqrt{2}$ f/s, $\pi\sqrt{3}$ f/s; rate of lower end $= \pi\sqrt{3}$ f/s, $\pi\sqrt{2}$ f/s, π f/s.

19. The connecting rod, PA, of a stationary engine is 5 ft. long; and the crank AC is 1 ft.; if the crank revolves at the uniform rate of two revolutions a second, find the speed of the piston-rod (or of the point P) when the angle PCA is 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°. Ans. If $PC = x$, when

$$\theta = 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°, \frac{dx}{dt} = 0$$

$= -\frac{16}{7}\pi\sqrt{2},\ -4\pi,\ -\frac{12}{7}\pi\sqrt{2},\ 0,\ \frac{12}{7}\pi\sqrt{2},\ 4\pi,\ \frac{16}{7}\pi\sqrt{2},\ 0.$

20. A wheel 3 ft. in diameter rolls along level ground at the uniform rate of 10 m/h. Find the rate of the *forward* motion of the bottom, top, foremost, and hindmost points of the rim.

Ans. x being the horizontal distance moved by a point fixed on the rim, measured from the point where it touched the road, $x = \frac{3}{2}(\theta - \sin\theta)$ ft., θ being the angle generated by the radius to this point, and when $\theta = 0°\ 90°, 180°, 270°, \frac{dx}{dt} = 0$, 10 m/h, 20 m/h, 10 m/h, the horizontal speeds of the bottom, hindmost, top, and foremost points respectively.

21. A line tangent to a circle of 10-in. radius moves across the circle at the rate of $\frac{1}{2}$ in./sec., keeping always parallel to its first position. Find the rate of increase of the area of the segment next the point of tangency when the line has moved $\frac{1}{4}$ way, $\frac{1}{2}$ way, and $\frac{3}{4}$ way across the circle.

Ans. $5\sqrt{3}$, 10, and $5\sqrt{3}$ sq. in./sec. respectively.

22. A plane moves across a sphere of 10-in. radius as the line moved across the circle in example 21; find the rate of increase of the corresponding volume.

Ans. $37\frac{1}{2}\pi$ cu. in./sec., 50π cu. in./sec., and $37\frac{1}{2}\pi$ cu. in./sec. respectively.

23. An eccentric circular cam of radius a inches revolves about a point O at a distance b from its center C; the point O being in line with a rod which bears upon the rim. Let the rod bear upon the point R of the rim, and call $OR=r$, and the angle $COR=\theta$. Show from the triangle COR that

$$r=b\cos\theta+\sqrt{a^2-b^2\sin^2\theta},$$

so that if the cam makes n revolutions a second, the speed of the rod's motion is $\frac{dr}{dt}=-2\pi nb\left[\sin\theta+\frac{b\sin\theta\cos\theta}{\sqrt{a^2-b^2\sin^2\theta}}\right]$.

CHAPTER II.

Analytic Geometry.

38. Geometrical Applications of Analysis.—The relation between a function and its graph is utilized in two ways: first, the properties of a function may be made evident by reference to the graph; and second, the properties of a geometric locus may be studied by means of its equation. The second of these was expounded to a certain extent in *Chapter VII, Loci of Equations,* of the Algebra. We shall begin here with a recapitulation of the results of that chapter.

39. Relation between a Locus and its Equation.—If the values of the coördinates x and y are varied independently, the point (x, y) will move all over the plane; but if x and y are dependent, owing to the existence of a relation $f(x, y)=0$, the motion of the point (x, y) is restricted to a curve. If $f(x, y)=0$, when solved, gives $y=F(x)$, this curve is the graph of $F(x)$, and:

The curve is the locus of $f(x, y)=0$;

$f(x, y)=0$ *is the equation of the curve;*

The coördinates of any point of the curve satisfy $f(x, y)=0$ *and the coördinates of any other point do not;* or

Any point whose coördinates satisfy $f(x, y)=0$ *lies on the curve, and any point whose coördinates do not satisfy* $f(x, y)=0$ *does not lie on the curve.* (Algebra, Art. 98.)

40. The Linear Equation and the Straight Line.—If the equation $f(x, y)=0$ of a curve is of the *first degree,* or *linear,* that is, of the form $Ax+By+C=0$, its locus is a *straight line.*

The constants of the line being the x-intercept a, the y-intercept b, the inclination to the axis of abscissas τ, and the slope m,

$$m=\tan\tau=-\frac{A}{B}, \quad a=-\frac{C}{A}, \quad b=-\frac{C}{B}.$$

A straight line is traced by determining two of its points, or one point and its slope.

The equation of the straight line through the point (x_1, y_1), *and having the slope* m *or the inclination* τ_1 *to the x-axis, is*

$$y - y_1 = m(x - x_1),$$

or

$$y - y_1 = \tan \tau_1 (x - x_1).$$

The equation of the straight line through any two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

In terms of the constants m, a, and b defined above, *the equation of any straight line may be written*

$$y = mx + b \text{ (slope and intercept equation)},$$

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ (intercept equation)}.$$

(Algebra, Arts. 99-101.)

41. The Angle between Two Straight Lines.—It is evident from Fig. 11 that if two straight lines, (1) $y = m_1 x + b_1$ and (2) $y = m_2 x + b_2$, are inclined to the axis of x at the angles τ_1 and τ_2 respectively, the angle α between them is

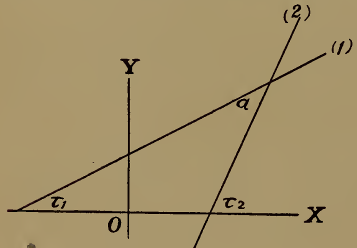


FIG. 11.

$$\alpha = \tau_2 - \tau_1,$$

whence

$$\tan \alpha = \frac{\tan \tau_2 - \tan \tau_1}{1 + \tan \tau_1 \tan \tau_2}.$$

As the slopes of the lines are $m_1 = \tan \tau_1$, $m_2 = \tan \tau_2$,

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

If the lines are parallel, $\alpha = 0$, $\tan \alpha = 0$, and $m_1 = m_2$; or *the slopes are equal.*

If the lines are perpendicular, $\alpha = \frac{\pi}{2}$, $\tan \alpha = \infty$, and $m_1 m_2 = -1$, or *the slopes are negative reciprocals.*

42. The Mid-Point and the Distance between Two Points.—The point half-way between (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}\right).$$

The distance between these points is

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

(Algebra, Arts. 104-105.)

43. The Distance of a Point from a Line.—The distance from (x_1, y_1) to $Ax+By+C=0$ is (Algebra, Art. 106)

$$d = \frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

44. Intersections.—The coördinates of the point or points of intersection of any two curves are the simultaneous solutions of the equations of the curves. (Algebra, Arts. 107-108.)

45. Finding the Equation of a Geometric Locus.—To determine the algebraic equation of the curve traced by a point which moves under given restrictions:

(1) Refer the problem to a pair of axes, letting the variable coördinates (x, y) be the distances from these axes of any point whatever on the curve;

(2) State in the form of an equation in terms of x and y the definition of the curve or the conditions of the problem. (Algebra, Art. 109.)

46. The Equation of the Circle.—The equation of the circle having a radius of length a and the point (b, c) for center is

$$(x-b)^2+(y-c)^2=a^2.$$

If the center is at the origin, the equation is

$$x^2+y^2=a^2.$$

Any equation of the form $ax^2+ay^2+dx+ey+f=0$ represents a circle, the radius being $\sqrt{\frac{d^2+e^2-4af}{4a^2}}$ and the center at $\left(\frac{-d}{2a}, \frac{-e}{2a}\right)$. (Algebra, Art. 110.)

47. Tangents to Circles.—For any point on the circle

$$x^2+y^2=a^2$$

we have

$$2xdx+2ydy=0,$$

$$\frac{dy}{dx}=\frac{-x}{y}.$$

Then, if $m_1=\tan\tau_1$ is the slope of the tangent at the point (x_1, y_1) (see Art. 4),

$$m_1=-\frac{x_1}{y_1},$$

and the equation of the tangent is (Algebra, Arts. 115-116)

$$x_1x+y_1y=a^2.$$

In the same way, the tangent to $(x-b)^2+(y-c)^2=a^2$ is

$$(x-x_1)(x_1-b)+(y-y_1)(y_1-c)=0,$$

or

$$(x_1-b)(x-b)+(y_1-c)(y-c)=a^2.$$

48. Tangent to Circle in Terms of its Slope.—If a line having the slope m is tangent to the circle $x^2+y^2=a^2$, its equation is

$$y=mx\pm a\sqrt{1+m^2}.$$

If this line is tangent to any circle $(x-b)^2+(y-c)^2=a^2$, its equation is

$$y-c=m(x-b)\pm a\sqrt{1+m^2}.$$

These forms of the equation are useful in determining the tangents to a circle from an outside point. (Algebra, Arts. 117-118.)

49. Loci of Quadratic Equations.—Any quadratic equation, of the general form

$$ax^2+bxy+cy^2+dx+ey+f=0$$

represents some one of four types of loci:

(1) Two straight lines (distinct or coincident, parallel, or intersecting).

(2) A closed oval called an ellipse (of which the circle is a special case): condition $b^2-4ac<0$.

(3) A single-branched open curve called a parabola: condition $b^2-4ac=0$.

(4) A two-branched open curve called a hyperbola: condition $b^2-4ac>0$.

In the Algebra, the circle was defined as a geometric locus, and its equation found; but the ellipse, parabola, and hyperbola were treated merely as the loci of certain equations, or as the graphs of certain functions. We shall next proceed to introduce these last curves again as geometric loci; but to facilitate the process we shall need to be able to use different sets of coördinate axes in handling the same curve.

50. Transformation of Coördinates. Shifting the Origin.—Suppose we know the equation $f(x, y)=0$ of a curve referred to a pair of rectangular axes XOY and wish to find its equation $F(x', y')=0$ referred to a pair of rectangular axes $X'O'Y'$ parallel to these. the new origin O' being the point (x_0, y_0) in the old system. Let P be any point of the curve, its coördinates being (x, y) in the original system, and (x', y') in the new system.

Then it is evident from Fig. 12 that the relations between the old and the new coördinates are

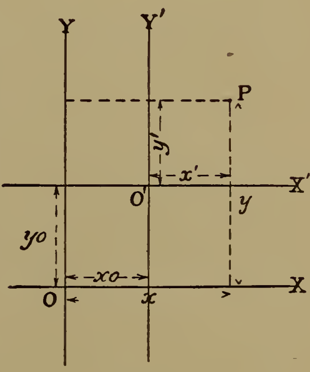


FIG. 12.

$$x = x_0 + x',$$
$$y = y_0 + y'.$$

Substituting these values, we shall have

$$f(x_0 + x', y_0 + y') \equiv F(x', y') = 0,$$

the desired equation.

51. Rotation of Axes.—Suppose now that, having the equation $f(x, y) = 0$ of a curve referred to a pair of rectangular axes XOY, we wish to find the equation $F(x', y') = 0$ of the same curve when the axes of reference have been rotated about the origin through the angle θ to the new position $X'OY'$.

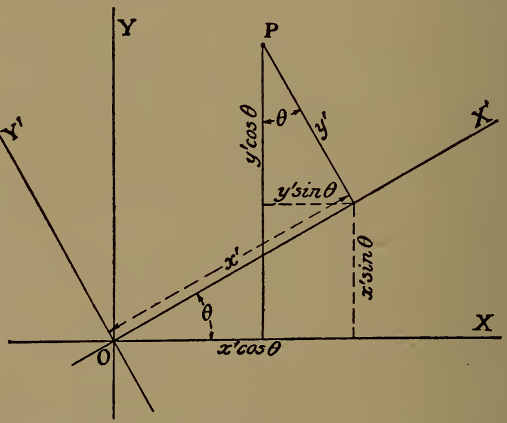


FIG. 13.

Let P be any point of the curve, and let its coördinates be (x, y) in the original system and (x', y') in the new system. Then it is evident from Fig. 13 that

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

The principal object of transformation of axes is to simplify the equation of the curve. It will be found that shifting the origin makes no change in the terms of highest degree.

As an example, suppose that, having the equation

$$x^2-xy+y^2+2x-3y+2=0,$$

we use a new pair of axes parallel to the old axes, having the point $(-\frac{1}{3}, \frac{4}{3})$ of the old system (the center of the ellipse) as new origin. The new equation, when the primes are dropped, is

$$x^2-xy+y^2=\tfrac{1}{3}.$$

If we now turn the axes about the new origin through the angle 45°, the equation (after dropping primes) is

$$x^2+3y^2=\tfrac{2}{3}.$$

52. *Examples.*

1. Shift the origin of the equation

$$8x^2-4xy+5y^2-8x-16y-16=0$$

to the point (1, 2), and transform the equation thus derived by rotating the axes through the angle $\tan^{-1} 2$.

Ans. $4x^2+9y^2=36$.

2. Shift the origin of the equation

$$6x^2-5xy-6y^2-19y-22x+5=0$$

to the point (1, −2), and rotate the axes of the equation thus derived through the angle $\tan^{-1} 5$. Ans. $x^2-y^2=2$.

3. Transfer the origin of the equation

$$11x^2-4xy+8y^2-50x-52y+11=0.$$

to the point (3, 4), and then rotate the axes of the derived equation through the angle $\tan^{-1}\left(-\frac{1}{2}\right)$. Ans. $\frac{x^2}{14}+\frac{y^2}{24}=1$.

4. Rotate the axes of the equation $x^2-y^2=a^2$ through an angle of 45°. Ans. $2xy=-a^2$.

53. Conic Sections.—A *conic section* is the locus of a point which moves so that its distance from a fixed point, the *focus*, has a constant ratio to its distance from a fixed line, the *directrix*.

The constant ratio, which is designated by *e*, is called the *eccentricity* and determines, by its value, whether the curve is a parabola, an ellipse, or a hyperbola.

Half the distance from the focus to the directrix is called the *parameter*, and is designated by p. The parameter determines the size of the conic, the eccentricity its shape.

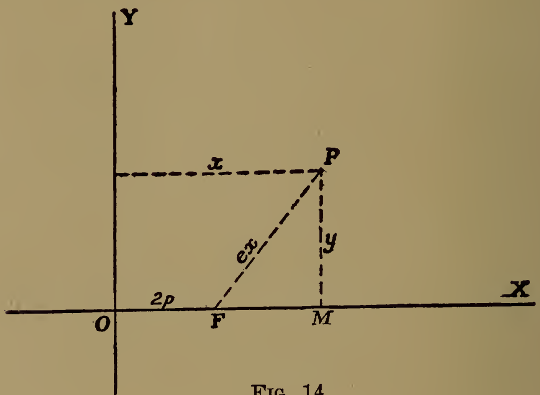


FIG. 14.

If we take as rectangular axes the directrix and a straight line through the focus, as in Fig. 14, then by definition, $FP = ex$. As $2p$ is taken for the fixed distance OF we have, from the right triangle FPM,

$$(x-2p)^2+y^2=e^2x^2, \tag{1}$$

the general equation of the conic.

When equation (1) is rearranged in the form

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

as

$$(1-e^2)x^2+y^2-4px+4p^2=0,$$

the discriminant is seen to be

$$b^2-4ac=4(e^2-1).$$

Hence the curve is:

An ellipse when $e<1$.
A parabola when $e=1$.
A hyperbola when $e>1$.

54. Properties of the Conics.—The line through the focus perpendicular to the directrix is called the *transverse axis* or *principal diameter* of the conic, and the points where it intersects the conic are called the *vertices*. The double ordinate through the focus is called the *latus rectum*.

To locate the vertices we solve equation (1) with the equation, $y=0$, of the principal diameter, getting

$$(x-2p)^2=e^2x^2,$$
$$x-2p=\pm ex,$$
$$x=\frac{2p}{1+e} \text{ or } \frac{2p}{1-e}.$$

If $e<1$, both roots are positive; if $e=1$, one root is p, the other is infinite; and if $e>1$, one root is positive, the other negative. Hence:

Both vertices of an ellipse are on the same side of the directrix as the focus;

The parabola has only one (finite) vertex, midway between the focus and the directrix;

The hyperbola has a vertex on each side of the directrix.

The latus rectum of the general conic $(x-2p)^2+y^2=e^2x^2$ is $4pe$, since the ordinate corresponding to the abscissa $2p$ is $\pm 2pe$. In the parabola, this length is $4p$, twice the distance from the focus to the directrix; in the ellipse, it is less; in the hyperbola, greater.

55. Typical Equations. The Parabola.—When $e=1$, the general equation (1) becomes

$$y^2=4p(x-p). \qquad (2)$$

To simplify equation (2), shift the origin to the vertex (p, o), as shown in Fig. 15. The equation then becomes

$$y^2 = 4px, \tag{3}$$

the typical form of the equation of a parabola, the coördinate axes being the principal diameter and the tangent at the vertex.

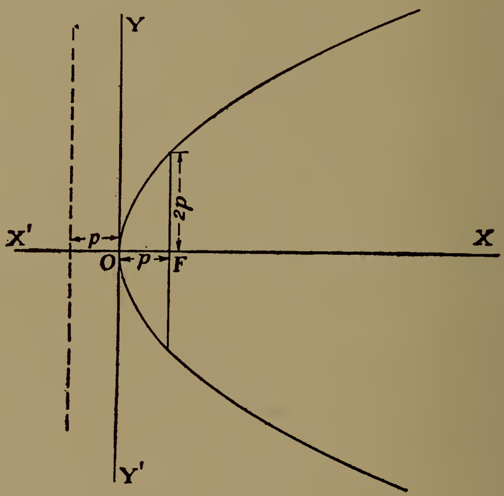


FIG. 15.

56. *Examples.*

1. What is the equation in typical form of the parabola which has the double ordinate $2b$ corresponding to the abscissa a?

Ans. $\dfrac{y^2}{b^2} = \dfrac{x}{a}$.

2. Write the equation of the parabola, vertex at origin, when the directrix is $x = p$; when it is $y = p$; when it is $y = -p$.

Ans. $y^2 = -4px$; $x^2 = -4py$; $x^2 = 4py$.

3. Show that the line $y = mx$ always cuts the parabola $y^2 = 4px$ in two points. What are the points when $m = 0$?

4. Find the equations of the following parabolas:

(*a*) Vertex $(1, 2)$ focus $(-2, 2)$.

Ans. $(y-2)^2 = -12(x-1)$.

(*b*) Vertex (2, 3), focus (4, 3).
Ans. $y^2-6y-8x+25=0.$

(*c*) Vertex (2, 2), focus (2, −4).
Ans. $x^2-4x+24y-44=0.$

[Hint: Write the equation of each referred to its transverse axis and tangent at vertex; then shift to the required axes.]

57. The Central Conics: Ellipse and Hyperbola.—When e is not$=1$, the general conic

$$(x-2p)^2+y^2=e^2x^2 \qquad (1)$$

has, as we have seen, two vertices, at

$$\left(\frac{2p}{1+e},\ 0\right) \text{ and } \left(\frac{2p}{1-e},\ 0\right).$$

The line joining the vertices is called the *major axis* of the conic; its length is denoted by $2a$. The point midway between the vertices will be seen later to bisect every chord through that point; it is therefore called the *center*. (See Figs. 16 and 17.)

58. The Ellipse.—The length of the *semi- major axis* for the *ellipse* is

$$a=\tfrac{1}{2}\left(\frac{2p}{1-e}-\frac{2p}{1+e}\right)=\frac{2pe}{1-e^2},$$

and the distance from the focus to the directrix, in terms of this length, is

$$2p=\frac{a(1-e^2)}{e}=\frac{a}{e}-ae.$$

The abscissa of the center is

$$\tfrac{1}{2}\left(\frac{2p}{1-e}+\frac{2p}{1+e}\right)=\frac{2p}{1-e^2}=\frac{a}{e}.$$

(See Fig. 16.)

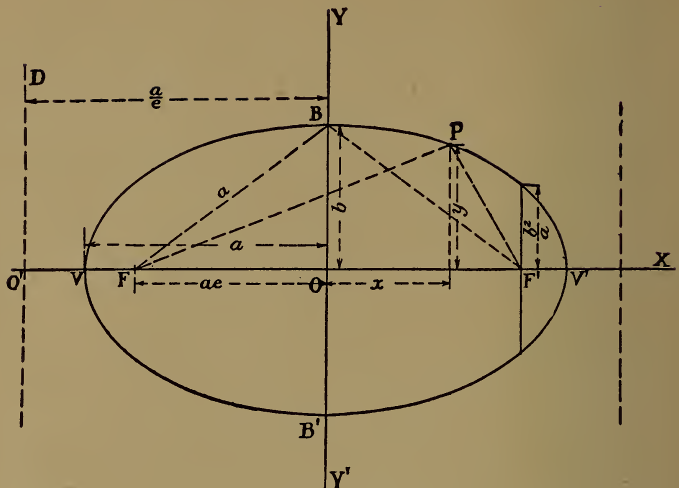


FIG. 16.

Equation (1) is much simplified when the origin is shifted to the center. If we first replace $2p$ in (1) by its value $\frac{a}{e} - ae$, getting

$$\left(x - \frac{a}{e} + ae\right)^2 + y^2 = e^2x^2,$$

and then shift the origin to $\left(\frac{a}{e},\ 0\right)$, we have, after dropping primes,

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,$$

or

$$(1 - e^2)x^2 + y^2 = a^2,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \qquad (4)$$

59. The Hyperbola.—The length of the *semi- major axis* of the *hyperbola* is

$$a=\tfrac{1}{2}\left(\frac{2p}{e+1}+\frac{2p}{e-1}\right)=\frac{2pe}{e^2-1},$$

and the distance from the focus to the directrix, in terms of this length, is

$$2p=\frac{a(e^2-1)}{e}=ae-\frac{a}{e}.$$

The abscissa of the center is $\frac{2p}{1-e^2}=\frac{-a}{e}$. (Fig. 17.) Replac-

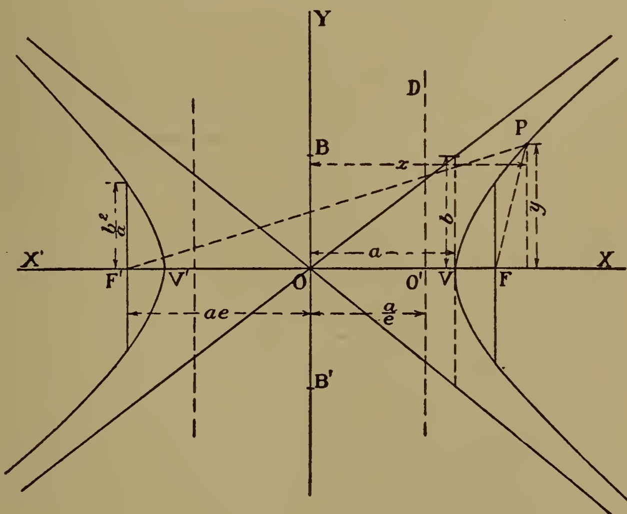


FIG. 17.

ing $2p$ by $ae-\frac{a}{e}$, shifting the origin to $\left(-\frac{a}{e},\ 0\right)$, and dropping primes, we have again

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1. \qquad (4)$$

This one equation thus represents either an ellipse or a hyperbola according as $e<1$ or $e>1$.

60. Symmetry of the Central Conics.—It is now evident from equation (4) why the conics for which e is not $=1$ are called central conics, and why the point taken as our new origin is the center. For, corresponding to any point (x_1, y_1) lying on (4) is another point $(-x_1, -y_1)$, also lying on (4), and as the new origin is half-way between the two points, it bisects any chord drawn through it. Moreover, the points $(x_1, -y_1)$ and $(-x_1, y_1)$ also lie on (4), so that the new axes of coördinates are axes of symmetry for the conic. These axes of symmetry are called the *principal axes* or *principal diameters* of the central conic; the one perpendicular to the directrix being the *major axis*, and the one parallel to the directrix, the *minor axis*, or *conjugate axis.*

61. Typical Equations of Ellipse and Hyperbola.—The intercepts on the minor axis, $x=0$, of a central conic

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

are $\pm a\sqrt{1-e^2}$, real in the case of the ellipse, imaginary for the hyperbola.

In the ellipse, the length of the *semi- minor axis* is denoted by b;

$$b=a\sqrt{1-e^2} \qquad (e<1),$$

so that we have as *the typical equation of the ellipse referred to its principal diameters as coördinate axes:*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{5}$$

In the hyperbola, $a^2(1-e^2)$ is represented by $-b^2$;

$$b=a\sqrt{e^2-1} \qquad (e>1),$$

so that we have as the typical eqnation of the hyperbola referred to its principal diameters as coördinate axes:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (6)$$

In this form, the asymptotes of the hyperbola are

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0,$$

or

$$y = \pm \frac{b}{a} x.$$

In the hyperbola, then, b is the length of an ordinate of either asymptote drawn from either vertex. For convenience in certain statements, b is often called the semi- minor axis of the hyperbola.

62. Dimensions of the Central Conics.—From what has already been said, the values of the following are evident:

For any central conic having a major axis of length $2a$ and eccentricity e, we have the lengths:

	$e<1$. Ellipse.	$e>1$. Hyperbola.
Center to directrix:	$\frac{a}{e}$.	$\frac{a}{e}$.
Focus to directrix:	$\frac{a}{e} - ae$.	$ae - \frac{a}{e}$.
Center to focus:	ae.	ae.
Latus rectum:	$2a(1-e^2)$.	$2a(e^2-1)$.
Minor axis:	$b=a\sqrt{1-e^2}$.	$[b=a\sqrt{e^2-1}]$.

In terms of a and b, the eccentricity is:

$$e = \sqrt{\frac{a^2-b^2}{a^2}} \text{ for the ellipse, } \quad e = \sqrt{\frac{a^2+b^2}{a^2}} \text{ for the hyperbola.}$$

On account of the symmetry of the curves, the ellipse has a directrix $\frac{a}{e}$ to the right of the center, and a focus ae to the right in addition to the directrix and focus at the same distances to the left; the hyperbola has a pair to the left as well as one to the right.

63. *Examples.*

1. The focus of a conic is $\frac{9}{4}$ in. from its directrix, and the eccentricity is $\frac{4}{5}$. Find the equation referred to the directrix and a perpendicular through the focus, and reduce to the typical form. Draw a figure to scale, showing the foci, directrices, center, latus rectum, and minor axis.

2. Treat the same problem as that of example 1, changing the eccentricity to $\frac{5}{4}$ and replacing the minor axis by the asymptotes.

3. Determine the dimensions of the following curves:

(1) $9x^2+25y^2=225$. Ans. $e=\frac{4}{5}$, etc.

(2) $9x^2-16y^2=144$. Ans. $e=\frac{5}{4}$, etc.

(3) $9x^2+25y^2+18x-100y-116=0$. Ans. $e=\frac{4}{5}$, etc.

(4) $y^2=(a-x)(x+a)$.

(5) Write the equations of the circles, radius a, when the center is:

(a) at $(-a, 0)$, (b) at $(a, 0)$, (c) at $(0, -a)$, (d) at $(0, a)$.

4. Prove that a circle having its center at an extremity of the minor axis of an ellipse and its radius equal to the semi- major axis will intersect the major axis at the foci.

5. Prove that a circle concentric with a hyperbola and passing through the intersections of the asymptotes with the tangents at the vertices will also pass through the foci.

6. Show that the circle is a special case of the ellipse, having equal semi-axes, the foci coincident with the center, eccentricity zero, parameter infinite, and directrix at infinity.

7. Suppose a right circular cone cut by a horizontal plane in a circular section. If the plane turns about a diameter of the circle, how will the eccentricity of the section vary? What is the eccentricity of a pair of intersecting straight lines regarded as a conic?

64. Conjugate Diameters.—*The locus of the middle points of a set of parallel chords in a conic section is a diameter.* This is proved as follows: Let the equation of the conic be

$$(x-2p)^2+y^2=e^2x^2, \tag{1}$$

as in Art. 53, and let the common slope of the parallel chords be m, and the general equation of all the chords be

$$y=mx\mp c, \tag{2}$$

m thus being a fixed constant and c a quantity that is constant for any one position of the chord (2), but varies as the chord moves. (A quantity of this sort is called a parameter.)

The extremities (x_1, y_1) and (x_2, y_2) of any one of the chords are the two intersections of (1) and (2), so that x_1 and x_2 are the two roots of

$$(x-2p)^2+(mx+c)^2=e^2x^2,$$

or of

$$x^2(1-e^2+m^2)+2(mc-2p)x+(4p^2+c^2)=0. \qquad (3)$$

If $P'(x', y')$ is the mid-point of this chord,

$$x'=\frac{x_1+x_2}{2},$$

and is half the sum of the roots of (3). By the Theory of Quadratics (Algebra, Art. 11), this is

$$x'=\frac{2p-mc}{1-e^2+m^2},$$

so that

$$c=\frac{x'}{m}(e^2-1-m^2)+\frac{2p}{m}.$$

Since P' lies on the line $y=mx+c$, we have

$$y'=mx'+c,$$

or

$$y'=mx'+\frac{x'}{m}(e^2-1-m^2)+\frac{2p}{m}.$$

This relation between the fixed constants and the coördinates of P', since it is free from the parameter c, expresses the restriction under which the point P' moves, or is the equation of the locus of P'. Simplifying the equation and dropping primes, we have

$$y=\frac{1}{m}[(e^2-1)x+2p], \qquad (4)$$

as the equation of the locus of the mid-point of a set of chords of slope m in the general conic.

If the conic is a parabola, e=1, and the locus is

$$y=\frac{2p}{m}, \tag{5}$$

a line parallel to the transverse axis. Any such line is called a diameter; if we regard the parabola as a conic having one vertex at infinity, and therefore its center also at infinity, we may say that all these parallels pass through its center.

If the conic is an ellipse, its center is at $\left(\frac{a}{e}, 0\right)$ and

$$2p=\frac{a}{e}(1-e^2);$$

equation (4) may therefore be written

$$y=\frac{e^2-1}{m}\left(x-\frac{a}{e}\right),$$

and, referred to central axes, becomes

$$y=\frac{e^2-1}{m}x. \tag{6}$$

If the conic is a hyperbola, its center is at $\left(-\frac{a}{e}, 0\right)$ and $2p=\frac{a}{e}(e^2-1)$; equation (4) may be written

$$y=\frac{e^2-1}{m}\left(x+\frac{a}{e}\right),$$

and, referred to central axes, becomes

$$y=\frac{e^2-1}{m}x. \tag{6}$$

For any central conic, then, *the locus of the middle points of a set of parallel chords, of slope m, is a line of slope* $\frac{e^2-1}{m}$ *through the center, or a diameter.*

The locus of the mid-points of chords parallel to the diameter (6) is $y=mx$, one of the original set of parallel chords. This is seen by putting $\frac{e^2-1}{m}$ for m in the italicised result of the preceding paragraph.

Thus we have the theorem: *If the product of the slopes of two diameters of a central conic is* (e^2-1), *each of the diameters bisects all the chords parallel to the other.*

Two such diameters are called *conjugate diameters.*

65. Focal Radii of Central Conics.—The straight lines joining the point $P(x, y)$ of a central conic, as in Figs. 16 and 17, with the foci F and F_1, are called *focal radii.* In the case of the ellipse, these distances are by definition:

$$FP = e\left(\frac{a}{e} + x\right) = a + ex,$$

and

$$F_1P = e\left(\frac{a}{e} - x\right) = a - ex;$$

their sum is therefore constant, and equal to $2a$.

In the case of the hyperbola, the focal radii are

$$FP = e\left(x - \frac{a}{e}\right) = ex - a$$

and

$$F_1P = e\left(x + \frac{a}{e}\right) = ex + a;$$

their difference is therefore constant, and equal to $2a$.

Consequently the *ellipse* may be defined as *the locus of a point moving so that the sum of its distances from two fixed points is constant;* and the *hyperbola* as *the locus of a point moving so that the difference between its distances from two fixed points is constant.*

66. *Examples.*

1. The principal axes of a central conic divide the plane into four quadrants. Show that two conjugate diameters of an ellipse never lie in the same quadrant, but two conjugate diameters of a hyperbola always do.

2. What is the equation of the diameter conjugate to $y = 2x$ in the conic $\frac{x^2}{9} + \frac{y^2}{4} = 1$? In the conic $\frac{x^2}{9} - \frac{y^2}{4} = 1$?

Ans. $y = -\frac{2}{9}x$; $y = \frac{2x}{9}$.

3. Show that if a diameter $y=mx$ meets a hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

in real points, the conjugate diameter does not.

4. Two pins are stuck in a piece of paper and an endless string is held taut about them by a pencil point. What curve will be drawn when the pencil moves? Find its dimensions if the string is 7 in. long and the pins are 3 in. apart.

Ans. $a=2,\ e=\frac{3}{4},\ b=\frac{1}{2}\sqrt{7}$, etc.

5. Show that if the extremities of a diameter of a central conic $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$ are (x_1, y_1) and $(-x_1, -y_1)$, the extremities of the conjugate diameter are

$$\left(\frac{y_1}{\sqrt{1-e^2}},\ -x_1\sqrt{1-e^2}\right) \text{ and } \left(\frac{-y_1}{\sqrt{1-e^2}},\ x_1\sqrt{1-e^2}\right).$$

67. Tangents, Normals, Subtangents, and Subnormals.—We have already seen that if a curve is given in rectangular coördinates by an equation $f(x, y)=0$, the slope of its tangent at any point is the value of $\frac{dy}{dx}$ at that point. If P (Fig. 18) is any point of the curve, draw any horizontal length from P to represent dx, and a vertical length in proper proportion to represent dy, and complete the triangle by drawing the line marked ds. The resulting triangle is called *the differential triangle for the point P.*

Let PT and PN be the tangent and normal to the curve at P, meeting the axis of x at T and N respectively, and let PM be the ordinate of P. Represent the angle PTM as usual by τ, and note that the angle MPN and the angle from dx to ds are each equal to τ, as all the right triangles in the figure are similar.

We are to find the equations of *the tangent PT* and *the normal PN*, and the lengths PT, PN, TM, NM. These four lengths are called: *PT, the tangent; PN, the normal; TM, the subtangent;* and *NM, the subnormal* of the curve for the point P. Abbreviate the four required lengths as shown in the figure, and let the coördinates of P be (a, b). Determine the general expression for $\frac{dy}{dx}$ for the curve; then

$$\left[\frac{dy}{dx}\right]_{\substack{x=a\\y=b}} = \tan\tau = \frac{b}{st} = \frac{sn}{b}.$$

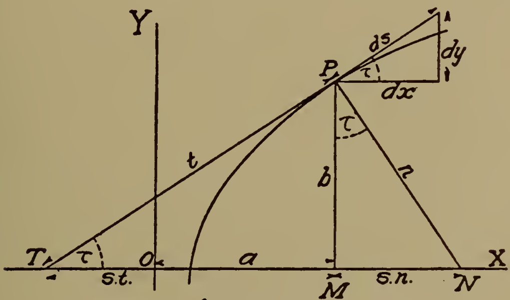


FIG. 18.

Hence

$$st = b\left[\frac{dx}{dy}\right]_{\substack{x=a\\y=b}}; \quad sn = b\left[\frac{dy}{dx}\right]_{\substack{x=a\\y=b}};$$

$$\left[\frac{ds}{dy}\right]_{\substack{x=a\\y=b}} = \operatorname{cosec}\tau = \frac{t}{b}; \quad t = b\left[\frac{ds}{dy}\right]_{\substack{x=a\\y=b}};$$

$$\left[\frac{ds}{dx}\right]_{\substack{x=a\\y=b}} = \sec\tau = \frac{n}{b}; \quad n = b\left[\frac{ds}{dx}\right]_{\substack{x=a\\y=b}}.$$

Since $ds = \sqrt{(dx)^2 + (dy)^2}$,

$$t = b\left[\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right]_{\substack{x=a\\y=b}};$$

$$n = b\left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right]_{\substack{x=a\\y=b}}.$$

The equation of the tangent is $y-b=\tan\tau(x-a)$

or

$$y-b=\left[\frac{dy}{dx}\right]_{\substack{x=a\\y=b}}\cdot(x-a).$$

The equation of the normal is $y-b=-\cot\tau\ (x-a)$

or

$$y-b=-\left[\frac{dx}{dy}\right]_{\substack{x=a\\y=b}}\cdot(x-a).$$

This discussion exhibits the general methods of finding the two equations and the four lengths. In any numerical case, the simplest process is to sketch a figure similar to Fig. 18, mark the values of a, b, dx, dy, and ds for the given point, and derive the results directly from the similar triangles.

68. Application of Art. 67 to the Conics. Parabola.—Given any parabola, of parameter p, refer it to its transverse axis and the tangent at its vertex as coördinate axes. Its equation is then

$$y^2=4px,$$

and its slope at any point is

$$\frac{dy}{dx}=\frac{2p}{y}.$$

If P_1 (x_1, y_1) is any point on the parabola, the tangent at that point is

$$y-y_1=\frac{2p}{y_1}(x-x_1),$$

or

$$y_1y-2px=y_1^2-2px_1=2px_1,$$

since, the point P_1 being on the parabola, $y_1^2=4px_1$.

The tangent to $y^2=4px$ at any point (x_1, y_1) lying on the curve is $y_1y=2p(x+x_1)$.

The normal at P_1 is

$$y-y_1=-\frac{y_1}{2p}(x-x_1),$$

or replacing x_1 by its value in terms of y_1,

$$y_1x+2py=\frac{y_1}{4p}(8p^2+y_1^2).$$

For the differential triangle, if we take $dx=y_1$, we have $dy=2p$, $ds=\sqrt{4p^2+y_1^2}$.

The *subtangent* $=y_1\frac{y_1}{2p}=\frac{y_1^2}{2p}=\frac{4px_1}{2p}=2x_1$.

The *subnormal* $=y_1\frac{2p}{y_1}=2p$.

The *tangent* $=y_1\frac{\sqrt{4p^2+y_1^2}}{2p}=2\sqrt{x_1(x_1+p)}$.

The *normal* $=y_1\frac{\sqrt{4p^2+y_1^2}}{y_1}=2\sqrt{p(x_1+p)}$.

Note that x_1 is the distance of P_1 from the tangent at the vertex, p is the distance of the vertex from either the directrix or the focus, (x_1+p) is the distance of P_1 from the directrix.

The Central Conics.—The typical equation of any central conic (ellipse or hyperbola), in which the coördinates are measured from the principal axes, is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1, \quad \text{or} \quad y^2=(1-e^2)(a^2-x^2).$$

From this equation,

$$\frac{dy}{dx}=(e^2-1)\frac{x}{y}.$$

In the differential triangle at any point P_1 of the conic, if we take $dx=y_1$, then

$$dy=(e^2-1)x_1,$$

and

$$ds=\sqrt{y_1^2+(e^2-1)^2x_1^2}=\sqrt{(1-e^2)(a^2-e^2x_1^2)}.$$

From these, remembering that P_1 is on the conic, we derive the following:

Equation of tangent: $\frac{x_1 x}{a^2}+\frac{y_1 y}{a^2(1-e^2)}=1.$

Equation of normal: $\frac{x}{x_1}+(e^2-1)\frac{y}{y_1}=e^2.$

Subtangent $=\frac{x_1^2-a^2}{x_1}.$

Subnormal $=(e^2-1)x_1.$

Tangent $=\frac{1}{x_1}\sqrt{(a^2-x_1^2)(a^2-e^2x_1^2)}.$

Normal $=\sqrt{(1-e^2)(a^2-e^2x_1^2)}.$

No attention is paid to the signs of the last four, as merely the lengths are of interest. The absolute value of each of them is to be used.

The quantity $(a^2-e^2x_1^2)$ is the product of the focal radii of P_1.

For the typical equation of the ellipse and hyperbola, $a^2(1-e^2)$ is replaced by b^2 and $-b^2$ respectively; so:

The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at any point* (x_1, y_1) *lying on the ellipse is* $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1.$

The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *at any point* (x_1, y_1) *lying on the hyperbola is* $\frac{x_1 x}{a^2}-\frac{y_1 y}{b^2}=1.$

It can be shown that the equation of the tangent to any conic at any point (x_1, y_1) lying on the conic can be written by first writing the equation of the conic with xx in place of x^2, yy for y^2, $\frac{1}{2}(xy+yx)$ for xy, $\frac{1}{2}(x+x)$ for x, and $\frac{1}{2}(y+y)$ for y, and then affixing the subscript 1 to the alternate letters.

69. Tangents and Normals to the Parabola in Terms of the Slope.—The tangent to $y^2=4px$ may be written:

$$y=\frac{2p}{y_1}x+\frac{2px_1}{y_1},$$

or, since $y_1^2=4px_1$,

$$y=\frac{2p}{y_1}x+\frac{y_1}{2}.$$

If we represent the slope of the tangent by m,

$$m=\frac{2p}{y_1},$$

$$\frac{y_1}{2}=\frac{p}{m};$$

so that *the equation of the tangent of slope m to the parabola* $y^2=4px$ is

$$y=mx+\frac{p}{m}.$$

The normal to the parabola $y^2=4px$ at the point P_1 of the parabola may be written:

$$y=-\frac{y_1}{2p}x+y_1+\frac{y_1^3}{8p^2}.$$

If we represent the slope of the normal by m,

$$m=-\frac{y_1}{2p};$$

$$y_1=-2pm;$$

$$\frac{y_1^3}{8p^2}=-pm^3,$$

so that *the equation of the normal of slope m to the parabola* $y^2=4px$ is

$$y=mx-2pm-pm^3.$$

Note the significance of these forms; if we draw a line of any slope m and make the y-intercept $=p/m$, the line will be tangent to the parabola $y^2=4px$; if the y-intercept is made $=-2pm-pm^3$, the line will be perpendicular to the parabola.

These forms furnish a means of determining the tangents or normals to the parabola from any point, whether it lies on the parabola or not.

For instance, to find the tangent from (4, 7) to $y^2=6x$, since any tangent is in general

$$y=mx+\frac{3}{2m},$$

and the desired tangents pass through (4, 7), we have

$$7=4m+\frac{3}{2m};$$

solving this for m we get $m=\frac{3}{2}$ or $\frac{1}{4}$. Hence the tangents are

$$2y-3x=2 \quad \text{and} \quad 4y-x=24.$$

The slopes of the tangents from (x_1, y_1) to $y^2=4px$ are in the same way $\frac{1}{2}(y_1 \pm \sqrt{y_1^2-4px_1})$; both are real for a point outside the parabola, they coincide for a point on the parabola, and are imaginary for a point inside the parabola.

To find the normals from (12, 6) to $y^2=6x$; since they are of the form

$$y=mx-2pm-pm^3$$

and pass through (12, 6), we have

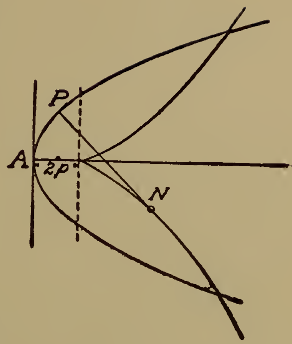
$$6=12m-3m-\tfrac{3}{2}m^3$$

to determine m; that is, the roots of $m^3-6m+4=0$ are the slopes of the required normals. These roots are 2, $\sqrt{3}-1$, and $-\sqrt{3}-1$ or 2, .732 and -2.732.

There are thus three normals from this one point to the parabola. The equation for the slope, m, of a normal from (x_1, y_1) to $y^2=4px$ is similarly seen to be

$$m^3+\frac{2p-x_1}{p}m+\frac{y_1}{p}=0.$$

According as the discriminant of this equation (Algebra, Art. 74) is negative, zero, or positive, there will be three different normals, two coincident and one different, or one real normal and two imaginary.



[The discriminant is

$$\frac{1}{108p^3}(27py_1^2+4[2p-x_1]^3),$$

so that the curve $27py^2=4(x-2p)^3$, called a semi-cubical parabola, is the boundary of the points from each of which three real normals can be drawn to the parabola.]

70. Tangents to the Central Conics in Terms of the Slope.— The tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$ or

$$y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}.$$

If we represent its slope by m,

$$m=-\frac{b^2x_1}{a^2y_1},$$

and since

$$x_1^2=\frac{a^2}{b^2}(b^2-y_1^2),$$

$$m=\pm\frac{b}{ay_1}\sqrt{b^2-y_1^2};$$

whence

$$\frac{b^2}{y_1}=\pm\sqrt{a^2m^2+b^2};$$

so that *the equation of the tangent of slope* m *to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$y=mx\pm\sqrt{a^2m^2+b^2}.$$

In the same way, we find that *the equation of the tangent of slope m to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

For the *circle* $x^2 + y^2 = a^2$, a special case of the ellipse, the similar equation is

$$y = mx \pm a\sqrt{1 + m^2}$$

(Algebra, Art. 117).

These equations are of the same service as those for the parabola; the equations of the normals of the central conics in terms of m are less interesting, and much more complicated.

71. Properties of Tangents to Conics.—*The focal radius and diameter through a point of a parabola make equal angles with the tangent at that point.*

The focal radii to a point on a central conic make equal angles with the tangent at that point.

It is in consequence of the first of these properties that a beam of light parallel to the principal axis of a parabolic mirror converges at the focus, and that rays diverging from the focus are reflected as a beam of parallel rays.

From the second property it follows that rays diverging from one focus of an elliptical mirror are brought to a focus at the other, and rays diverging from the focus of a hyperbolic mirror diverge after reflection as if they had come from the other focus.

The second property is proved as follows:

The slope of the tangent to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

is

$$m_1 = (e^2 - 1)\frac{x_1}{y_1}.$$

The slope of the focal radius joining (x_1, y_1) to $(ae, 0)$ is

$$m_2 = \frac{y_1}{x_1 - ae}.$$

The tangent of the angle between these lines is

$$\frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\dfrac{y_1}{x_1 - ae} + (1 - e^2)\dfrac{x_1}{y_1}}{1 - (1 - e^2)\dfrac{x_1}{y_1} \cdot \dfrac{y_1}{x_1 - ae}} = \frac{y_1^2 + (1 - e^2) x_1 (x_1 - ae)}{y_1 (x_1 - ae) - (1 - e^2) x_1 y_1}.$$

Since $y_1^2 = (1 - e^2)(a^2 - x_1^2)$, this reduces to

$$\frac{a(1 - e^2)(a - ex_1)}{-ey_1(a - ex_1)} = -\frac{a(1 - e^2)}{ey_1}.$$

The slope of the other focal radius is

$$m_3 = \frac{y_1}{x_1 + ae},$$

so that by changing e to $-e$ throughout the preceding discussion and reversing the order of the lines, we find

$$\frac{m_1 - m_3}{1 + m_1 m_3} = -\frac{a(1 - e^2)}{ey_1}.$$

The angles are thus shown to be equal.

72. *Perpendicular tangents of a conic intersect in a circle called the director circle,* which in the parabola is a straight line, the directrix (a circle of infinite radius). This may be shown in two ways. For example, if the equation of the tangent to the ellipse

$$y = mx \pm \sqrt{a^2 m^2 + b^2}$$

be arranged as a quadratic equation in m, it will give

$$m^2 + 2m\frac{xy}{a^2 - x^2} + \frac{b^2 - y^2}{a^2 - x^2} = 0.$$

In order that the two direction ratios m_1 and m_2 derived from this equation shall represent tangents at right angles to each other, they must have the relation

$$m_1 m_2 = -1;$$

that is, the absolute term of the quadratic

$$\frac{b^2 - y^2}{a^2 - x^2} = m_1 m_2 = -1,$$

which gives

$$x^2 + y^2 = a^2 + b^2.$$

For the parabola, we should have, in a similar way,

$$m^2 - m\frac{y}{x} + \frac{p}{x} = 0,$$

or

$$m_1 m_2 = \frac{p}{x} = -1,$$

or

$$x = -p.$$

The same results are readily obtained by finding the locus of the intersection of two perpendicular tangents:

$$y - mx = \sqrt{a^2m^2 + b^2},$$
$$my + x = \sqrt{a^2 + b^2m^2}.$$

If the two equations of these perpendicular tangents are squared and added, there results

$$(x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2);$$

and by eliminating m, we obtain

$$x^2 + y^2 = a^2 + b^2.$$

73. Pedal Curves.—Perpendiculars from the focus to the tangents of a conic meet on a circle.

The equations of the tangent of the ellipse and the perpendicular from the focus upon it are

$$y - mx = \sqrt{a^2m^2 + b^2},$$
$$my + x = ae.$$

Squaring and adding these equations and noting that $b^2 = a^2(1-e^2)$, we get

$$(x^2+y^2)(1+m^2)=a^2(1+m^2),$$

or, eliminating m,

$$x^2+y^2=a^2,$$

as the locus of the intersection of the two lines.

This circle is called the *major auxiliary circle* of the ellipse, for reasons that will appear in Art. 87, Parametric Equations.

In the case of the parabola, m may be directly eliminated; the equations are:

$$y=mx+\frac{p}{m},$$

$$y=-\frac{x}{m}+\frac{p}{m}.$$

Subtracting,

$$0=x\left(m+\frac{1}{m}\right),$$

or

$$x=0.$$

The interpretation of the tangent at the vertex of the parabola as corresponding to the auxiliary circle of the ellipse is evident from the fact that the center of the parabola is infinitely distant from the vertex.

The locus of the intersection of a tangent of a given curve with the perpendicular from a given point is called a *pedal curve*. We have just derived the pedals of the conics when the given point is the focus. The pedal for any other point is derived in a similar way; the equations of the tangent and of the perpendicular from the given point are written in terms of m, and the arbitrary parameter m eliminated.

As an illustration we will derive the pedal curve of the parabola $y^2=4ax$, when the given point is $(-a, 0)$. The equations of the tangent and of its perpendicular from $(-a, 0)$ are:

$$y=mx+\frac{a}{m}, \tag{1}$$

$$y=\frac{x+a}{-m}. \tag{2}$$

The elimination of m and the resulting equation are both simplified by shifting the origin of (1) and (2) to the given point $(-a, 0)$. The lines are then

$$y=m(x-a)+\frac{a}{m}, \tag{1$'$}$$

$$y=-\frac{x}{m}. \tag{2$'$}$$

Putting $\frac{-x}{y}$ for m in (1)$'$ and simplifying, we get

$$x(x^2+y^2)-a(x^2-y^2)=0$$

as the equation of the pedal curve referred to the given point as origin.

74. *Examples.*

1. Find in detail the equations of the tangent and normal and the lengths of the tangent, normal, subtangent, and subnormal for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) lying on the curve.

2. Give the equations and lengths of the tangent and normal and the lengths of the subtangent and subnormal to each of the following at the point indicated.

$y^2=12x$ at the point having the abscissa 3.

$\frac{x^2}{9}+\frac{y^2}{4}=1$ at the point having the abscissa 2.

$\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point having the abscissa 3.

3. Find the tangents from (2, 7) and the normals from (15, 6) to the parabola $y^2=12x$.

Ans. $y=3x+1$, $2y=x+12$, $y=x-9$, $y=-2x+36$.

4. Find the slopes of the two tangents to $\frac{x^2}{9}+\frac{y^2}{4}=1$ from $(h, 2)$; from $(3, k)$; from $(3, 2)$. (See Algebra, Art. 46.)

Ans. $\frac{4h}{h^2-9}$ and 0; $\frac{k^2-4}{6k}$ and ∞; 0 and ∞.

5. Find the equations of tangent and normal and lengths of subtangent and subnormal for the curve $y=e^x$ where it crosses the y-axis. Ans. $y-x=1$, $y+x=1$, 1, 1.

6. Find the lengths of tangent, normal, subtangent, and subnormal for $y=e^x$, where $x=\log\sqrt{3}$. Ans. 2, $2\sqrt{3}$, 1, 3.

7. Show that the lengths of subtangent and subnormal for the curve $y=\sin x$ are in general $\tan x$ and $\frac{1}{2}\sin 2x$.

8. Find the subtangent of $y=\tan x$ when $x=\frac{\pi}{4}$. Ans. $\frac{1}{2}$.

9. Find the angle between $y=\sin x$ and $y=\cos x$.

Ans. $\tan^{-1}2\sqrt{2}$.

10. Show that the tangents at the extremities of the latus rectum of a conic meet on the directrix.

11. Let PM be the ordinate, PN the normal of a point P on the conic $y^2=(1-e^2)(a^2-x^2)$, and call the foci F_1 and F_2. Show that $NF_1:NF_2=PF_1:PF_2$, so that by plane geometry PN bisects the angle F_1PF_2.

12. Prove the property of the parabolic mirror.

13. Derive the pedal of the circle $x^2+y^2=a^2$ with respect to the point $(a, 0)$.

Ans. $(a, 0)$ being taken as a new origin, the pedal is the *cardioid* $(x^2+y^2+ax)^2=a^2(x^2+y^2)$.

14. Derive the pedal of the rectangular hyperbola $x^2-y^2=a^2$, with respect to its center.

Ans. The lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$, the origin being the same as for the hyperbola.

75. The Second Derivative.—If y is a function of x, the derivative of y with respect to x is also a function of x, the derivative of which may be taken with respect to x.

The x-derivative of the x-derivative of a function is called the second x-derivative of the function, or the second derivative of the function with respect to x.

The derivative of the second derivative is called the *third derivative,* and so on.

The notation used in expressing the higher derivatives is as follows:

If $y=f(x)$, then

$$y'=f'(x)=\frac{df(x)}{dx}=\frac{dy}{dx} \text{ represents the first derivative;}$$

$$y''=f''(x)=\frac{df'(x)}{dx}=\frac{dy'}{dx} \text{ represents the second derivative;}$$

$$y'''=f'''(x)=\frac{df''(x)}{dx}=\frac{dy''}{dx} \text{ represents the third derivative;}$$

and so on.

When it is convenient to consider that the value of dx, which is arbitrary, is always the same throughout a discussion, another set of forms can be used to denote the higher derivatives. For if dx is considered constant, and we find the *x-derivative* of $\frac{dy}{dx}$ by differentiating and dividing by dx, we get $y''=\frac{d(dy)}{(dx)^2}$; repeating the process, we get $y'''=\frac{d[d(dy)]}{(dx)^3}$, and so on.

These numerators are cumbersome, so they are abbreviated to d^2y, d^3y, expressions which are read "*d* second *y*" or "second differential of *y*," etc., and which mean "the differential of the differential of *y*," etc.

The notation, when x is the independent variable, is thus:

$$y=f(x),$$

$$y'=f'(x)=\frac{dy}{dx},$$

$$y''=f''(x)=\frac{d^2y}{dx^2},$$

$$y'''=f'''(x)=\frac{d^3y}{dx^3},$$

etc.

It will be noticed that in place of $(dx)^2$ we have written dx^2, which in these forms is always taken to mean "the square of dx." Whenever there is any danger of confusion with the differential of x^2, the parentheses must be retained.

The second derivative is the only one that is susceptible of interpretation by itself, though the others occur very commonly. The simplest interpretation of the second derivative occurs in connection with functions of *time*. If $s=f(t)$ is the distance traversed during the time t,

$$s'=f'(t)=\frac{ds}{dt},$$

is, as we already know, the *speed* of the motion, or the time-rate of increase of the distance. $s''=f''(t)=\frac{d^2s}{dt^2}$ is called the *acceleration* of the motion, and is the time-rate of increase of the speed. Thus when a body falls to the earth, going s ft. in t secs, the law of its falling is (nearly)

$$s=16t^2;$$

from which

$$s'=\frac{ds}{dt}=32t \text{ f/s},$$

or the speed is $32t$ ft. a second; and

$$s''=\frac{d^2s}{dt^2}=32 \text{ f/s}^2,$$

or the *acceleration* is 32 ft. a second in each second; that is, there is an increase in the velocity of 32 ft. a second during each second that the body falls.

In problems of motion, then, the first derivative is the speed, the second derivative is the acceleration. The third derivative occurs in some sorts of motion, but has no special name.

In geometrical problems in which a curve is represented by an equation $y=f(x)$ in connection with a system of axes, the first derivative, $\frac{dy}{dx}$, is the slope of the curve, as we have seen. The second derivative has no special name, but is very significant, both by itself and as an element of an important expression.

76. Inflections.—Let $y=f(x)$ be the equation of a curve in rectangular coördinates; then

$$y'=f'(x)=\frac{dy}{dx}=\tan\tau$$

is the slope of the curve, and

$$y''=f''(x)=\frac{d^2y}{dx^2}$$

is the derivative of the slope, and is consequently positive when the slope is increasing (algebraically), negative when the slope is decreasing, and zero when the slope is changing from one state to the other. A point which appears to be the junction of opposite bends, like the middle point of the letter *S*, evidently has this

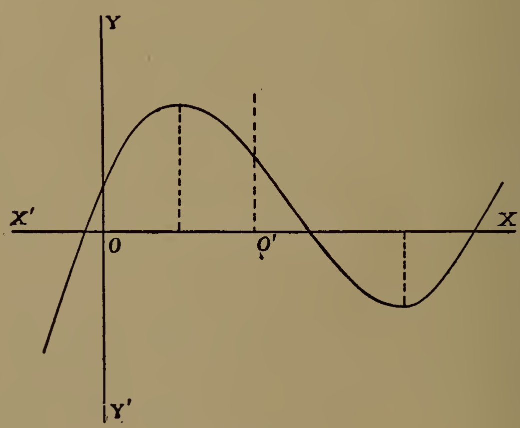


FIG. 19.

last property; such a point is called a *point of inflection,* and *at a point of inflection of a curve,* $y=f(x)$,

$$y''=f''(x)=\frac{d^2y}{dx^2}=0.$$

Any graph of a cubic function, such as we studied in the Algebra, has just one point of inflection. Thus the equation of the curve on page 99 of the Algebra is

$$10y=2x^3-15x^2+24x+6.$$

And for this curve,

$$10y'=f'(x)=6x^2-30x+24,$$
$$10y''=f''(x)=12x-30,$$
$$y''=0 \text{ when } x=\tfrac{5}{2}.$$

As x increases from $-\infty$ to $+\infty$, the slope y' decreases, as is evident from the figure, from $+\infty$, becoming zero when $x=1$, decreasing further to the value $\frac{-27}{20}$ when $x=\frac{5}{2}$ (at the *inflection*), after which it increases to $+\infty$, passing through the value zero when $x=4$.

In this figure the inflection is clearly the junction of the part of the curve convex upward with the part convex downward, so that *the tangent at the inflection crosses the curve.* Inspection of an adjacent secant shows further that *at the inflection the curve and the tangent have three common points.* This appears also analytically from the fact that the equation of the tangent, at the point $(\frac{5}{2}, \frac{7}{20})$,

$$10y=\frac{149-54x}{4},$$

when solved simultaneously with the equation of the curve, gives

$$(2x-5)^3=0$$

to determine the intersections.

77. *Examples.*

1. Show that the tangent to the inflection for $y = x^3 - 3x^2 - 45x + 7$ is $48x + y = 8$.

2. Show that if the abscissas of the highest and lowest points of a cubic parabola $a^2y = x^3 + bx^2 + c^2x + d^3$ are x_1 and x_2, the abscissa of the inflection is $\frac{x_1 + x_2}{2}$.

3. Find the inflection of $y = x^3 + px + q$.

4. Find the tangent to the curve $y(1 + \log x) = x$ at its inflection. Ans. $4y - x = e$.

5. Find the tangents at the inflections of the curve (Witch)

$$y(a^2 + x^2) = a^3.$$

Ans. $2y \pm x = 2a$.

6. Show that if $y = x^{-3 \log x}$, $\frac{y'}{y} = -\frac{6 \log x}{x}$, $\frac{yy'' - y'^2}{y^2} = -\frac{6}{x^2}(1 - \log x)$, $y'' = -\frac{6y}{x^2}(1 + 2\log x)(1 - 3\log x)$, so that the curve $y = x^{-3 \log x}$ has inflections at $(e^{-\frac{1}{2}}, e^{-\frac{3}{4}})$ and $(e^{\frac{1}{3}}, e^{-\frac{1}{3}})$.

78. Curvature. Radius of Curvature.—One of the most important characteristics of a curve, especially in connection with the bending of beams and other structural supports, machine parts, etc., is the sharpness with which the curve bends, or its *curvature.*

Some curves bend more sharply than others, and except for the straight line and the circle, any curve bends more sharply at some points than at others; the curvature of any conic section, for instance, is greatest at the ends of its major axis, and decreases at increasing distances from these points.

To measure the curvature of any given curve at a given point, P_1, let the tangent at P_1 slide along the curve, its point of contact moving through the arc $P_1P_2 = \Delta s$, and the tangent itself turning through the angle $\Delta\alpha$ (Fig. 20). The angle of turn, $\Delta\alpha$, in comparison with the distance, Δs, that the point of contact has to move in order to produce this turn, indicates the sharpness of the curvature at P_1. The quotient, $\frac{\Delta\alpha}{\Delta s}$, is the *mean curvature* of the curve from P_1 to P_2. This mean curvature varies, in general, for the same point P_1, according to the length of Δs. As usual in such cases, its value when $\Delta s = 0$ is taken as the actual curvature at P_1; i. e.,

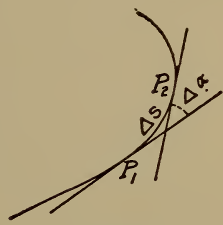


FIG. 20.

$$\left[\frac{\Delta\alpha}{\Delta s}\right]_{\Delta s=0} = \chi,$$

the curvature at P_1.

79. Given a circle of radius a, to find its curvature at a given point P_1 (Fig. 21). Choose any second point, P_2, and call the arc $P_1P_2 = \Delta s$. Call the angle turned through by the tangent as its contact moves from P_1 to P_2, $\Delta\alpha$. Choose any diameter OA and call the angle $P_1OA = \theta$, the angle $P_2OA = \theta + \Delta\theta$. Then by geometry,

$$\Delta\alpha = \Delta\theta,$$

and using circular measure throughout, we have

$$\Delta s = a\Delta\theta;$$

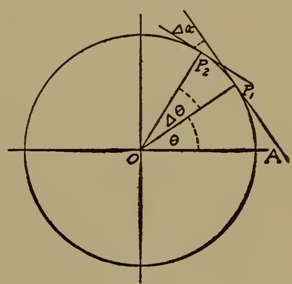


FIG. 21.

$$\frac{\Delta\alpha}{\Delta s} = \frac{\Delta\theta}{a\Delta\theta} = \frac{1}{a}.$$

For the circle, then, the mean curvature is constant for all points and for all lengths of the increment of arc; the curvature of a circle at any point is the reciprocal of the radius.

For any straight line, $\Delta\alpha = 0$ for any value of Δs, and the curvature is constantly zero. (This is consistent with the conception of a straight line as a circle of infinite radius.)

80. To compute the curvature of any curve at a given point, let ϕ be the angle made by a tangent to the curve with any fixed line OA (Fig. 22), and $\Delta\phi$ be the increment ϕ receives when the contact moves through the arc $P_1P_2 = \Delta s$; then $\Delta a = \Delta\phi$, and the curvature is

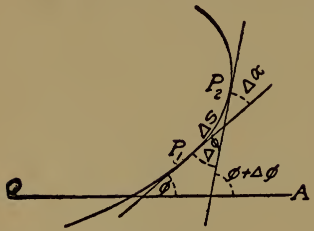


FIG. 22.

$$\chi = \left[\frac{\Delta\phi}{\Delta s}\right]_{\Delta s=0} = \frac{d\phi}{ds}.$$

The value at P_1 of $\frac{d\phi}{ds}$ is the actual curvature at P_1.

The most convenient method of designating the curvature of a given curve is to give the radius of the circle which has the same curvature; this radius is denoted by ρ (rho, the Greek r); then as the curvature of the circle is $\frac{1}{\rho}$,

$$\frac{1}{\rho} = \frac{d\phi}{ds}, \quad \rho = \frac{ds}{d\phi}.$$

This circle is called the *circle of curvature;* its radius ρ is called the *radius of curvature,* and its center the *center of curvature.*

81. The Differential of Arc.—In order to use this formula in connection with curves whose properties are defined by equations, we shall need analytical expressions for ds and $d\phi$. The symbol ds is the same that we used earlier for the hypotenuse of the differential triangle, and is used here because $\frac{ds}{dx}$ is $\sec\tau$ when ds is the differential of arc as well as when $ds = \sqrt{(dx)^2 + (dy)^2}$. That is, *the differential of arc and the hypotenuse of the differential triangle are the same.*

From the familiarity that we have already gained with the intimate relation between a curve and its tangent, especially with their infinitesimal increments, this idea probably will seem an obvious inference; if it is at all vague, the following explanation should make it clear.

82. By the length of a straight line is meant the number of times it will contain some recognized standard or unit; as a definition for the length of a curved line we have: "the limit approached by the length of a broken line formed of chords of the curve as the length of each chord approaches zero, and as the number of chords consequently increases indefinitely." Now suppose we have two points of a curve, $P(x, y)$ and $P'(x+\Delta x, y+\Delta y)$, between which is the arc Δs. Let the slopes at P and P' be $\tan\tau$ and $\tan(\tau+\Delta\tau)$. Suppose a broken line to be inscribed in the arc from P to P', and for convenience let the chords of which it is composed have equal projections along Δx. (See Fig. 23.)

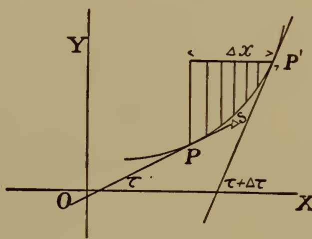


FIG. 23.

The inclination of each chord to the x-axis is between the values τ and $(\tau+\Delta\tau)$; its length is the length of its projection multiplied by the secant of its inclination. If there are n chords, the length of the broken line is $\frac{\Delta x}{n}$ times the sum of all n secants, or is Δx times the average of the secants. This average is certainly between $\sec\tau$ and $\sec(\tau+\Delta\tau)$ in value; so the length of the broken line is between the values $\Delta x \sec\tau$ and $\Delta x \sec(\tau+\Delta\tau)$. This is true independently of the number of the chords composing the broken line, and so holds true when this number is increased indefinitely; hence *the length of the arc Δs is intermediate in value between $\Delta x \sec\tau$ and $\Delta x \sec(\tau+\Delta\tau)$.*

Thus we have:

$$\text{The value of } \frac{\Delta s}{\Delta x} \text{ is between } \sec\tau \text{ and } \sec(\tau+\Delta\tau).$$

As Δx approaches zero, $\sec(\tau+\Delta\tau)$ approaches $\sec\tau$, and so

$$\left[\frac{\Delta s}{\Delta x}\right]_{\Delta x=0}=\frac{ds}{dx}=\sec\tau,$$

where $\frac{ds}{dx}$ is the x-derivative of the arc s, and ds is the differential of arc.

But

$$\sec\tau=\sqrt{1+\tan^2\tau}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\sqrt{(dx)^2+(dy)^2}}{dx};$$

hence

$$ds=\sqrt{(dx)^2+(dy)^2},$$

where ds is the differential of arc.

ds is really $\pm\sqrt{(dx)^2+(dy)^2}$; in any problem in which the sign is of importance, it must be chosen consistently with the relations

$$\frac{ds}{dx}=\sec\tau,\quad \frac{ds}{dy}=\csc\tau,$$

etc., so that the derivatives shall indicate by their signs whether s increases or decreases with x or y and shall be of the same signs as the corresponding functions of τ.

83. Curvature in Rectangular Coördinates.—Given a curve $y=f(x)$ referred to rectangular coördinates, to find its radius of curvature at any point. Take the axis of x as the fixed line OA in the preceding article; then

$$\phi=\tau,$$

$$\chi=\frac{d\phi}{ds}=\frac{d\tau}{ds},\quad \rho=\frac{ds}{d\tau},$$

$$\tau=\tan^{-1}\frac{dy}{dx}=\tan^{-1}y'.$$

Differentiating τ, and dividing $d\tau$ by ds, we have:

$$d\tau=\frac{d(y')}{1+y'^2},$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} \cdot dx,$$

$$\frac{d\tau}{ds} = \frac{d(y')}{(1+y'^2)^{\frac{3}{2}} \cdot dx}.$$

Since

$$\frac{dy'}{dx} = y'',$$

$$\chi = \frac{d\tau}{ds} = \frac{y''}{(1+y'^2)^{\frac{3}{2}}},$$

$$\rho = \frac{ds}{d\tau} = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

If the problem is one in which x is the independent variable throughout, y'' may be replaced by $\frac{d^2y}{dx^2}$; the formula is often written

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

We commonly pay no attention to the sign of ρ; whenever there is any point in considering it, the simplest method is to observe that the sign of $\rho = \frac{ds}{d\tau}$ shows whether s and τ are increasing together or not. The value of ds is $\pm\sqrt{1+y'^2} \cdot dx$ according as the arc s and the abscissa x increase together or do not, and this of course depends partly on the point of the curve from which s is measured, and the direction of the curve that is considered positive.

84. Radius of Curvature of the Central Conics.—As an illustration of the use of the formula, we will find the radius of curvature of any central conic (origin at center).

$$y^2 = (1-e^2)(a^2-x^2). \tag{1}$$

$$y' = (e^2-1)\frac{x}{y}.$$

$$y'' = (e^2-1)\frac{y-xy'}{y^2} = \frac{(e^2-1)}{y^2}\left[y-(e^2-1)\frac{x^2}{y}\right]$$
$$= \frac{e^2-1}{y^3}\,[y^2+(1-e^2)x^2].$$

Substituting the value of y^2 from (1) in the numerator, we have

$$y'' = -(1-e^2)^2\frac{a^2}{y^3}.$$

Now

$$1+y'^2 = \frac{y^2+(e^2-1)^2x^2}{y^2} = \frac{(1-e^2)(a^2-x^2)+(e^2-1)^2x^2}{y^2}$$

by the same substitution, or

$$1+y'^2 = \frac{1-e^2}{y^2}(a^2-e^2x^2).$$

Hence

$$\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{\dfrac{(1-e^2)^{\frac{3}{2}}}{y^3}(a^2-e^2x^2)^{\frac{3}{2}}}{-(1-e^2)^2\dfrac{a^2}{y^3}},$$

or the positive value of ρ is

$$\rho = \frac{(a^2-e^2x^2)^{\frac{3}{2}}}{a^2\sqrt{1-e^2}} = \frac{(e^2x^2-a^2)^{\frac{3}{2}}}{a^2\sqrt{e^2-1}}.$$

For the ellipse, $a\sqrt{1-e^2}=b$, and for the hyperbola, $a\sqrt{e^2-1}=b$; hence:

The radius of curvature of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *at any point* (x, y) is

$$\frac{1}{ab}(a^2-e^2x^2)^{\frac{3}{2}};$$

The radius of curvature of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ *at any point* (x, y) is

$$\frac{1}{ab}(e^2x^2-a^2)^{\frac{3}{2}}.$$

For either curve, ρ is thus the $\frac{3}{2}$ power of the product of the focal radii drawn to the given point, divided by the product of the semi-axes.

85. *Examples.*

1. Derive the value of ρ for the parabola $y^2=4px$ directly, first using $y'=\sqrt{\frac{p}{x}}$, then using $y'=\frac{2p}{y}$.

Ans. $\rho=\frac{2}{\sqrt{p}}(x+p)^{\frac{3}{2}}=\frac{1}{4p^2}(y^2+4p^2)^{\frac{3}{2}}$.

2. Find ρ for the parabola $\frac{y^2}{b^2}=\frac{x}{a}$.

3. What is the radius of curvature for a point of inflection?

4. Find ρ for $y=\cos x$, and show that the least value of ρ is for one of the highest or lowest points, when the center of curvature is on $y=0$.

5. Show that at the vertex, ρ is equal to the semi- latus rectum for any conic, and at an extremity of the minor axis of the ellipse, $\rho=\frac{a^2}{b}$.

6. Derive the value of ρ for the ellipse directly from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Ans. $\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

7. Do the same for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

8. Find the normal and the radius of curvature for

$$y=\frac{c}{2}\left(e^{x/c}+e^{-x/c}\right).$$

Ans. $\rho=n=y^2/c$.

86. Auxiliary Circles of the Ellipse.—Consider the following locus problem. Two concentric circles are given (Fig. 24), of radii a and b $(a>b)$, and two perpendicular diameters, OX and OY; let any radius $Op'p$ cut the smaller circle at p' and the larger at p; to find the locus of a point, P, moving so as always to have the same abscissa as p and the same ordinate as p', with reference to the axes of coördinates XOY.

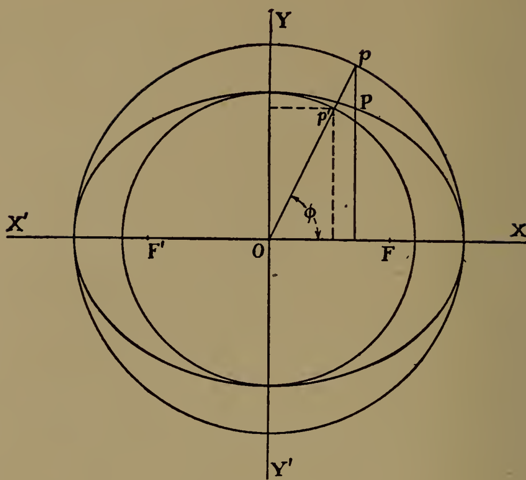


FIG. 24.

Let the angle $pOX=\phi$; then for any value of ϕ,

$$\left.\begin{aligned} x&=a\cos\phi,\\ y&=b\sin\phi.\end{aligned}\right\} \qquad (1)$$

Since $\sin\phi=\frac{y}{b}$, $\cos\phi=\frac{x}{a}$, and $\sin^2\phi+\cos^2\phi=1$, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad (2)$$

a relation true for any position occupied by P under the given conditions. The locus is therefore an ellipse of semi-axes a and b. The larger circle, tangent to the ellipse at the extremities of its major axis, is called the *major auxiliary circle* of the ellipse; and the smaller circle, tangent to the ellipse at the extremities of the minor axis, is called the *minor auxiliary circle* of the ellipse. The points p and p' are spoken of as the points of the auxiliary circles *corresponding* to the point P of the ellipse. ϕ is called the *eccentric angle* of the point P.

This proposition furnishes a simple means of constructing any number of points of the ellipse by the aid of the auxiliary circles, and there are other relations through which certain constructions are made possible. For instance, the subtangent of any ellipse having the major axis $2a$ is $\frac{a^2-x^2}{x}$. Therefore if any number of ellipses are drawn with a common major axis, points on them having the same abscissa will have the same subtangent, and the major auxiliary circle is one of these ellipses. Thus a tangent may be drawn to the ellipse at any point, or from any point of OX, by first drawing the tangent to the major auxiliary circle for the corresponding point.

87. Parametric Equations.—Besides these geometric relations, and much more important, is the fact that the pair of equations (1) represent the same relation between x and y as the single equation (2), for in most analytic work it is easier to use equations (1) than equation (2).

Equations (1) express each of the coördinates of the variable point as a function of an auxiliary variable, ϕ, so that by assuming all possible values of ϕ, all possible pairs of values of x and y can be found, and the points of the locus determined. An auxiliary variable of this sort is called a *parameter,* and a pair of equations such as these are called *parametric equations* of the corresponding curve.

It is often convenient to replace the ordinary equation of a curve by a pair of parametric equations, doing so arbitrarily without any thought of the geometric significance of the parameter so introduced. Any pair will do if, when the parameter is eliminated, the original equation is produced. The parametric equations of the ellipse might have been introduced in this way; for, given

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

since

$$\cos^2\phi + \sin^2\phi = 1$$

we are led at once to assume

$$\frac{x}{a} = \cos\phi \quad \text{and} \quad \frac{y}{b} = \sin\phi,$$

as the *parametric equations of the ellipse.*

In the same way, and for the same reason, we assume

$$\left.\begin{array}{l} x = a\cos\phi \\ y = a\sin\phi \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x = b\cos\phi \\ y = b\sin\phi \end{array}\right\}$$

as *the parametric equations of the major and minor auxiliary circles respectively.* It is evident from the figure that the parameter ϕ is the same in all three of these.

Again, in

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

since

$$\sec^2\phi - \tan^2\phi = 1,$$

if we assume

$$\frac{x^2}{a^2} = \sec^2\phi,$$

then

$$\frac{y^2}{b^2} = \tan^2\phi;$$

so that

$$x = a\sec\phi, \quad y = b\tan\phi$$

are the *parametric equations of the hyperbola.*

It is entirely unnecessary, in using this pair of equations in analytic work, to consider the geometric relations between ϕ and the hyperbola.

88. Derivatives in Connection with Parametric Equations.— All our derivative formulas can be applied directly to parametric equations, but of course *the parameter is the independent variable,* so that the second x-derivative of y cannot be formed by differentiating the differential of y and dividing by $(dx)^2$, but must be formed by taking the x-derivative of $\frac{dy}{dx}$. This is best done by differentiating the first derivative and dividing by dx.

As an illustration, consider the ellipse

$$x = a\cos\phi, \quad y = b\sin\phi.$$

Evidently,

$$\left.\begin{array}{l} dx = -a\sin\phi\, d\phi \\ dy = b\cos\phi\, d\phi \end{array}\right\} ds = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\cdot d\phi;$$

$$\tan\tau = y' = \frac{dy}{dx} = -\frac{b}{a}\cot\phi;$$

$$y'' = \frac{d(y')}{dx} = \frac{\frac{b}{a}\csc^2\phi\cdot d\phi}{-a\sin\phi\cdot d\phi},$$

or

$$y'' = \frac{-b\csc^3\phi}{a^2};$$

$$\rho = \pm\frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{\pm\left(1+\frac{b^2}{a^2}\cot^2\phi\right)^{\frac{3}{2}}}{\frac{-b}{a^2}\csc^3\phi}$$

$$= \mp\frac{1}{ab}\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{\frac{3}{2}}.$$

These results may be used as they stand, for to any given point of the ellipse corresponds a value of ϕ, and substituting this value will give the slope, curvature, etc., at the given point. If desired, the results may be expressed independently of the parameter. Putting $\sin\phi = \frac{y}{b}$, $\cos\phi = \frac{x}{a}$, $b^2 = a^2(1-e^2)$, ρ becomes, for instance,

$$\rho = \frac{\pm 1}{ab}(a^2 - e^2x^2)^{\frac{3}{2}}.$$

(Cf. Art. 84.)

The value of ρ can be derived directly from the definition, $\rho = \frac{ds}{d\tau}$, and for some parametric equations this is much the more convenient way.

Thus, since $\tan\tau = -\frac{b}{a}\cot\phi$,

$$\sec^2\tau\, d\tau = \frac{b}{a}\csc^2\phi\, d\phi.$$

Substituting $1 + \tan^2\tau = 1 + \frac{b^2}{a^2}\cot^2\phi$ for $\sec^2\tau$, we find $d\tau$ and get for $\frac{ds}{d\tau} = \rho$ the same result as before.

89. The Cycloid.—One of the most interesting of the transcendental curves is the *cycloid*, defined as follows: If a circle of fixed size rolls along a straight line, the curve described by a point of its circumference is called a *cycloid*. The line on which the circle rolls is called the *directrix* of the cycloid. The curve consists of an indefinite number of *arches*, touching the directrix in points called *cusps;* the piece of the directrix between two

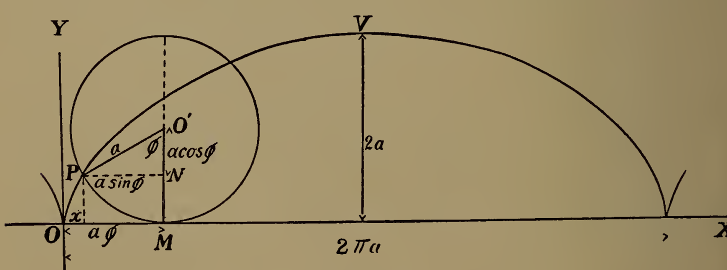


FIG. 25.

cusps is called the *base* of the corresponding arch, and the point of an arch furthest from the base is called a *vertex*. (See Fig. 25.)

If the radius of the rolling circle is a, the base of an arch is evidently $2\pi a$, and the height $2a$.

Take the directrix as axis of x, and one of the points where the generating point touches the directrix as the origin; take the axis of y on the same side of the directrix as the rolling circle. Consider the position $P(x, y)$ occupied by the generating point when the circle touches the directrix at M, its center having moved to O', and the radius drawn to P having turned through the angle ϕ.

Evidently $OM = \text{arc } PM = a\phi$, and $PN = a \sin\phi$;

$$x = OM - PN.$$

Also $O'M = a$, $O'N = a\cos\phi$;

$$y = O'M - O'N.$$

Consequently

$$\left.\begin{aligned} x &= a(\phi - \sin\phi), \\ y &= a(1 - \cos\phi). \end{aligned}\right\}$$

are a pair of parametric equations of the cycloid.

It is possible to eliminate ϕ from these equations and express x as an explicit function of y, but the resulting form is inconvenient. From the parametric equations, however, the curve is readily studied.

We can very easily obtain the coördinates of any number of points; for instance, with

$$\phi = 0;\ \frac{\pi}{3};\ \frac{\pi}{2};\ \frac{2\pi}{3};\ \pi;\ \frac{4\pi}{3};\ \frac{3\pi}{2};\ \frac{5\pi}{3};\ 2\pi;\ \text{etc.},$$

we have

$$\left.\begin{matrix} x = 0; & .18a; & .57a; & 1.23a; & \pi a; & 5.06a; & 5.71a; & 6.10a; & 6.28a \\ y = 0; & .5a; & a; & 1.5a; & 2a; & 1.5a; & a; & .5a; & 0 \end{matrix}\right\}\text{etc.}$$

Since $(\pi-\phi)$ and $(\pi+\phi)$ give the same value to y, and give to x values differing equally from πa, the arch is symmetrical with regard to $x=\pi a$. Since increasing ϕ by 2π increases x by $2\pi a$ and leaves y unchanged, the curve consists of equal arches, stretching indefinitely in both directions.

Differentiating the equations, we have:

$$dx = a(1-\cos\phi)\,d\phi = 2a\sin^2\frac{\phi}{2}\,d\phi,$$

$$dy = a\sin\phi\,d\phi = 2a\sin\frac{\phi}{2}\cos\frac{\phi}{2}\,d\phi,$$

$$ds^2 = dx^2 + dy^2 = 4a^2\sin^2\frac{\phi}{2}\left(\sin^2\frac{\phi}{2}+\cos^2\frac{\phi}{2}\right)(d\phi)^2$$
$$= 4a^2\sin^2\frac{\phi}{2}(d\phi)^2,$$

$$ds = \pm 2a\sin\frac{\phi}{2}\,d\phi.$$

If s is measured from the origin and is positive in the direction taken by the generating point as ϕ increases, then, since $\frac{ds}{d\phi}$ is to be positive always,

$ds = 2a\sin\frac{\phi}{2}\,d\phi$ for the arch from $\phi=0$ to $\phi=2\pi$;

$ds = -2a\sin\frac{\phi}{2}\,d\phi$ for the adjacent arches; and so alternately.

$$\left.\begin{aligned}\frac{dy}{dx} &= \tan\tau = \cot\frac{\phi}{2},\\ \frac{dy}{ds} &= \sin\tau = \cos\frac{\phi}{2},\\ \frac{dx}{ds} &= \cos\tau = \sin\frac{\phi}{2};\end{aligned}\right\}\text{ whence } \tau = \frac{\pi}{2} - \frac{\phi}{2}.$$

Hence, in Fig. 25, the tangent at P is parallel to the bisector of the angle $PO'M$, and consequently passes through the highest point of the rolling circle.

The sub-normal is

$$y \tan \tau = 2a \sin^2 \frac{\phi}{2} \cot \frac{\phi}{2} = a \sin \phi,$$

and so $= PN$; the normal therefore passes through M, the lowest point of the rolling circle.

The length of the normal is

$$n = y \sec \tau = 2a \sin^2 \frac{\phi}{2} \csc \frac{\phi}{2} = 2a \sin \frac{\phi}{2};$$

and the length of the radius of curvature is

$$\rho = \pm \frac{ds}{d\tau} = - \frac{2a \sin \frac{\phi}{2}\, d\phi}{d\left(\frac{\pi}{2} - \frac{\phi}{2}\right)} = 4a \sin \frac{\phi}{2}.$$

(The positive value of ρ is $-\frac{ds}{d\tau}$, since τ decreases as s increases, τ being measured from the positive direction of OX to the positive direction of the tangent.)

Since $\rho = 2n$, the radius of curvature is bisected at the point M.

90. As an example of the arbitrary use of the parameter, suppose we wish to study the curve given by

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

If we let $\left(\frac{x}{a}\right)^{\frac{2}{3}} = \cos^2 \phi$, we find $\left(\frac{y}{b}\right)^{\frac{2}{3}} = \sin^2 \phi$; whence

$$x = a \cos^3 \phi, \quad y = b \sin^3 \phi$$

are parametric equations of the curve.

From these it appears that the x-intercepts of the curve are $\pm a$, the y-intercepts $\pm b$, and that these are the extreme values of

x and y. As $\tan\tau = -\frac{b}{a}\tan\phi$, the curve is tangent to each axis at its intercept, and so has four cusps.

This curve, when $b=a$, is called the *astroid*. In this case, $\tau=\pi-\phi$, $ds=\pm 3a\sin\phi\cos\phi\, d\phi$, + in the first and third quadrants, − in the others, if s is measured positively contra-clockwise around the curve. The length of the tangent is

$$t = y\csc\tau = y\csc\phi = a\sin^2\phi.$$

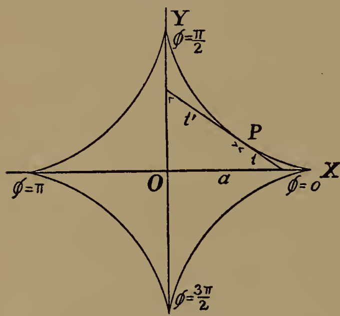


FIG. 26.

The distance along the tangent from its contact to the y-axis is

$$t' = -x\sec\tau = x\sec\phi = a\cos^2\phi;$$

hence $t+t'=a$; and the part of the tangent intercepted between the axes is of the constant length a.

The length of the radius of curvature is

$$\rho = \pm 3a\sin\phi\cos\phi = \pm\sqrt[3]{axy},$$

with signs chosen as for ds. As $\rho = \frac{3a}{2}\sin 2\phi$, its greatest value is $\frac{3a}{2}$, for the points half-way between two cusps. The normals at these points are $\frac{a}{2}$ in length and reach to the origin.

91. *Examples.*

1. From the equations for the hyperbola,

$$x = a\sec\phi, \quad y = b\tan\phi,$$

find $\tan\tau$, the subnormal, and the radius of curvature.

Ans. $\tan\tau = \frac{b}{a}\csc\phi$; s. n. $= \frac{b^2}{a}\sec\phi$; $\rho = \frac{(a^2\sin^2\phi + b^2)^{\frac{3}{2}}}{ab\cos^3\phi}$ or $\frac{(a^2\tan^2\phi + b^2\sec^2\phi)^{\frac{3}{2}}}{ab}$.

What limitations do these equations place on the value of $\frac{y}{x}$?

2. In the case of the ellipse

$$x=a\cos\phi,\quad y=b\sin\phi,$$

given that the value $\phi=\phi_1$, determines the point $P_1(x_1, y_1)$, show that $\phi=\phi_1\pm\frac{\pi}{2}$ determines the extremities of the diameter through the conjugate point.

3. In the ellipse of example 2, show that if ρ is the radius of curvature at P_1; and $2r$ is the length of the diameter through the point conjugate to P_1, then $ab\rho=r^3$.

4. Show that $x=at^2$, $y=bt$ are parametric equations of a parabola; find in terms of t the lengths of the subtangent, the subnormal, and the radius of curvature. Show that the extremities of the latus rectum are given by $t=\pm\frac{b}{2a}$, and find the values of the three lengths for these points and for the vertex.

5. Show that a point at a distance b from the center of the rolling circle of the cycloid generates a curve having for its equations

$$x=a\phi-b\sin\phi,\quad y=a-b\cos\phi.$$

(Any such curve is a *trochoid.*)

Show that $y''=\frac{b(a\cos\phi-b)}{(a-b\cos\phi)^3}$; so that, if $b<a$, the points given by $\phi=\cos^{-1}\frac{b}{a}$ are inflections. (Prolate cycloid.)

Show that, if $a<b$, the values ϕ_1 and $-\phi_1$, for which $\sin\phi_1=\frac{a\phi}{b}$, give coincident points of the curve. (Curtate cycloid.)

Show that for either trochoid, $\rho=\pm\frac{(a^2+b^2-2ab\cos\phi)^{\frac{3}{2}}}{b(a\cos\phi-b)}$, and that for the points where the curtate cycloid cuts the directrix, $\rho=\sqrt{b^2-a^2}$.

6. Transform the equations of the cycloid, using for axes the tangent and normal at the vertex, y-axis upward, getting

$$x'=a(\phi'+\sin\phi'),\quad y'=a(-1+\cos\phi').\quad (\phi'=\phi-\pi.)$$

7. A jointed parallelogram $PAOB$ has one vertex fixed at the origin of coördinates O and moves so that the angle AOX is equal to twice the angle BOX; if the sides are $OA=a$, $OB=b$, show that the locus of P is given by the equations

$$\left.\begin{aligned} x&=a\cos 2\phi+b\cos\phi, \\ y&=a\sin 2\phi+b\sin\phi. \end{aligned}\right\}$$

This curve is called a *limaçon*. If $b=2a$, it is the *cardioid*.

Show that, for the cardioid, $\tau=\frac{1}{2}(\pi+3\phi)$, $\rho=\frac{8a}{3}\cos\frac{\phi}{2}$. Find the values of x, y, τ, and ρ for the points given by $\phi=0$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$. Draw the curve.

8. Find a pair of parametric equations for

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1,$$

and show that they represent the part between (a, o) and (o, b) of the parabola represented by the four irrational equations included in

$$\pm\left(\frac{x}{a}\right)^{\frac{1}{2}}\pm\left(\frac{y}{b}\right)^{\frac{1}{2}}=1,$$

or by the single equation

$$\left(\frac{x}{-a}+\frac{y}{b}-1\right)^2=4\frac{x}{a}.$$

9. Find the parametric equations of the *companion to the cycloid*, the locus of the point N; and show that, if the origin is shifted to the point $(\pi a, a)$, the equations become

$$x'=a\phi',\quad y'=a\cos\phi',$$

where $\phi'=\phi-\pi$; so that the companion is the cosine curve $y'=a\cos\frac{x'}{a}$.

10. Show that

$$x=a(\cos\phi+\phi\sin\phi),\quad y=a(\sin\phi-\phi\cos\phi)$$

represent a spiral for which $\rho=a\phi$.

92. Polar Coördinates.—We are familiar with the method of locating a point in a plane by giving its distances (called rectangular coördinates) from two given perpendicular lines. There are other methods of determining the position of a point in a plane, and of these the most important is by means of *polar coördinates*. If, as in Fig. 27, we choose a point of the plane O, and a line, OA, through O, then the distance $r=OP$ and the angle $\theta=AOP$ are polar coördinates of P. The point O is called the *pole* or *origin*, and the line OA the *initial line* of the system of polar coördinates. The distance r and the angle θ are called the *radius vector* and the *vectorial angle* of the point P. The point is designated by (r, θ).

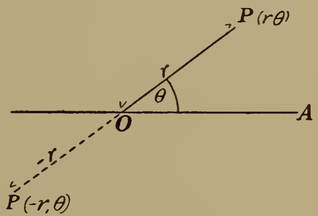


FIG. 27.

The angle θ is measured counter-clockwise from the initial line if positive, clockwise if negative. The distance r is measured in the direction thus determined if r is positive, in the opposite direction if r is negative.

93. Transformation of Coördinates.—The relation between polar and rectangular coördinates is simple if they have the same origin, and the same line as initial line and axis of x. Thus, in Fig. 28 we evidently have

$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \left.\begin{array}{l} x^2+y^2=r^2 \\ \dfrac{y}{x}=\tan\theta. \end{array}\right\}$$

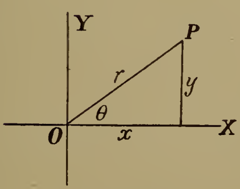


FIG. 28.

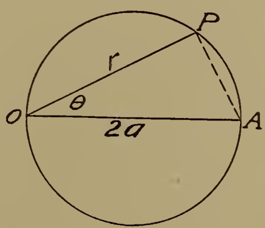


FIG. 29.

The equation of a curve in polar coördinates may be found either by expressing its geometric definition in terms of its polar coördinates or by transforming its equation as given in rectangular coördinates. For instance, in the circle of Fig. 29, if the pole O is on the circumference, and the initial line is the diameter through O, it is evident from the geometry of the figure that

$$r=2a\cos\theta,$$

a being the radius.

Again, the equation of this circle in rectangular coördinates is

$$(x-a)^2+y^2=a^2, \quad \text{or} \quad x^2+y^2=2ax,$$

which, through the formulas of transformation, becomes in polar coördinates

$$r^2=2ar\cos\theta,$$

or

$$r=2a\cos\theta.$$

As another example, consider the parametric equations

$$\left.\begin{aligned} x&=a\cos 2\phi+2a\cos\phi \\ y&=a\sin 2\phi+2a\sin\phi \end{aligned}\right\}$$

which represent (Ex. 7, Art. 91) a *cardioid.* If we first shift the origin a distance a to the left, these become

$$\left.\begin{aligned} x&=a+a\cos 2\phi+2a\cos\phi, \\ y&=a\sin 2\phi+2a\sin\phi. \end{aligned}\right\}$$

These, since $1+\cos 2\phi=2\cos^2\phi$, $\sin 2\phi=2\sin\phi\cos\phi$, may be written

$$\left.\begin{aligned} x&=2a\cos\phi(1+\cos\phi), \\ y&=2a\sin\phi(1+\cos\phi). \end{aligned}\right\}$$

Evidently

$$\tan\theta=\frac{y}{x}=\tan\phi; \quad \phi=\theta.$$

$$r^2=x^2+y^2=[2a(1+\cos\phi)]^2(\sin^2\phi+\cos^2\phi),$$

$$r=2a(1+\cos\phi);$$

and $r=2a(1+\cos\theta)$ is the polar equation of the cardioid, the pole being the cusp, the initial line the axis of symmetry.

From this equation, $r=2a\cos\theta+2a$, it is evident that the cardioid is the locus of points distant $2a$ further from the origin than the points of the circle $r=2a\cos\theta$.

94. *Examples.*

Find the polar equations of the following curves, taking the origin and x-axis of the rectangular system as the pole and initial line of the polar system, except when directed otherwise.

1. $x=a;\ y=b.$
2. $x^2+y^2=a^2.$
3. Ellipse: major axis $2a$, eccentricity e, pole at left focus, initial line the major axis. (Hint: After substituting, solve the equation for r.)

$$Ans.\quad r=\frac{a(1-e^2)}{1-e\cos\theta}\ \text{or}\ r=-\frac{a(1-e^2)}{1+e\cos\theta}.$$

4. Parabola: parameter p, pole at focus.

$$Ans.\quad r=p\csc^2\tfrac{\theta}{2}\quad\text{or}\quad r=-p\sec^2\tfrac{\theta}{2}.$$

5. The *lemniscate* (Art. 74), $(x^2+y^2)^2=a^2(x^2-y^2)$.

$$\text{Ans.}\quad r^2=a^2\cos 2\theta.$$

6. The cardioid (Art. 74), $(x^2+y^2)^2+2ax(x^2+y^2)-a^2y^2=0$.

$$\text{Ans.}\quad r=-a(1+\cos\theta)\ \text{or}\ r=a(1-\cos\theta).$$

In each of the examples with two answers, tracing the curve from the two polar equations will show that the two equations are equivalent. This may also be seen from the fact that the point determined from one equation by $\theta=\alpha$ is the same as the point determined from the other by $\theta=\alpha+\pi$.

95. Derivatives with Polar Coördinates.—Let a curve be given by an equation in polar coördinates, and let it be required to find the angle ψ from the positive direction of the radius vector at any point P to the tangent at the same point. (See Figs. 30 and 31.) This value can be obtained directly, or by transforming to polar coördinates the results already obtained in rectangular coördinates. Following the second method we have:

$$\left.\begin{aligned} x&=r\cos\theta, & dx&=\cos\theta\,dr-r\sin\theta\,d\theta;\\ y&=r\sin\theta, & dy&=\sin\theta\,dr+r\cos\theta\,d\theta;\end{aligned}\right\}$$

$$\tan\tau = \frac{dy}{dx} = \frac{\sin\theta\,dr + \cos\theta \cdot rd\theta}{\cos\theta\,dr - \sin\theta \cdot rd\theta}.$$

Dividing each term of this fraction by $\cos\theta\,dr$, we have

$$\tan\tau = \frac{\tan\theta + \frac{rd\theta}{dr}}{1 - \tan\theta\frac{rd\theta}{dr}}.$$

But since $\tau = \theta + \psi$,

$$\tan\tau = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi}.$$

Hence

$$\tan\psi = \frac{rd\theta}{dr}.$$

As $r = f(\theta)$, $\frac{dr}{d\theta} = f'(\theta)$ is represented by r'; so

$$\tan\psi = \frac{r}{r'}.$$

The differential of arc ds is obtained similarly:

$$(ds)^2 = (dx)^2 + (dy)^2 = (rd\theta)^2 + (dr)^2.$$

The other functions of ψ may thus be expressed:

$$\sec^2\psi = 1 + \tan^2\psi = \frac{(rd\theta)^2 + (dr)^2}{(dr)^2} = \frac{(ds)^2}{(dr)^2}, \quad \text{etc.}$$

$$\cos\psi = \frac{dr}{ds}, \quad \sin\psi = \frac{rd\theta}{ds}, \quad \text{etc.}$$

96. These results are conveniently viewed in the following figure, called the *polar differential triangle.*

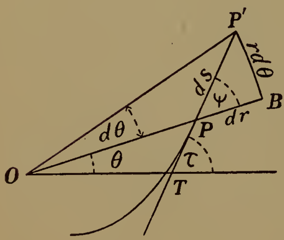


FIG. 30.

In Fig. 30, regarding the arcs PP' and $P'B$ as straight lines of lengths ds and $rd\theta$, PB as of length dr, B as a rectilinear right angle, and $P'PB$ as the angle ψ between the radius vector OP and the curve, the results just obtained are easily remembered:

$$\tan\psi = \frac{rd\theta}{dr}; \quad \cos\psi = \frac{dr}{ds}; \quad \sin\psi = \frac{rd\theta}{ds}.$$

It is usual to regard θ as the independent variable in all treatments by polar coördinates.

97. Applications of the Polar Differential Triangle.—Corresponding to a point $P(r, \theta)$ of a curve $r = f(\theta)$ (Fig. 31), draw the radius vector OP, the tangent PT, the normal PN, and a perpendicular to OP through O meeting PT and PN at T and N.

PT and PN are called the polar tangent and normal, OT and ON the polar subtangent and subnormal. Let p be the perpendicular from O to PT.

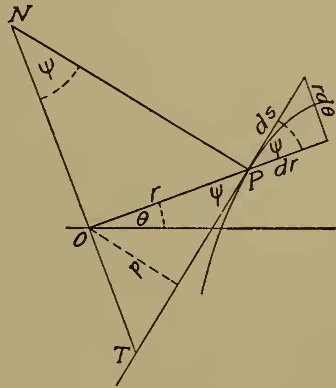


FIG. 31.

$$PT = \text{tangent} \quad = r\sec\psi \doteq \frac{rds}{dr} = r\sqrt{1 + \left(\frac{rd\theta}{dr}\right)^2} = \frac{r}{r'}\sqrt{r^2 + r'^2}.$$

$$PN = \text{normal} \quad = r\csc\psi = \frac{rds}{rd\theta} = \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r'^2}.$$

$$OT = \text{subtangent} = r\tan\psi = \frac{r^2 d\theta}{dr} = \frac{r^2}{r'}.$$

$$ON = \text{subnormal} = r\cot\psi = \frac{rdr}{rd\theta} = \frac{dr}{d\theta} = r'.$$

$$p = r\sin\psi \quad = \frac{r^2 d\theta}{ds} = \frac{r^2 d\theta}{\sqrt{(dr)^2 + (rd\theta)^2}}$$

$$p=\frac{r^2}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}=\frac{r^2}{\sqrt{r^2+r'^2}}.$$

The radius of curvature is found from the definition $\rho=\frac{ds}{d\tau}$,

and the relations $\tau=\psi+\theta$, $\tan\psi=\frac{r}{r'}$, whence

$$\tan\tau=\frac{r+r'\tan\theta}{r'-r\tan\theta}=\frac{r\cos\theta+r'\sin\theta}{r'\cos\theta-r\sin\theta}.$$

98. *Examples.*

1. Find for $r=2a\cos\theta$, the six lengths of Art. 97.

2. Show that for the cardioid $r=2a\sin^2\frac{\theta}{2}$,

$$\tau=\frac{3\theta}{2}\pm n\pi,\quad ds=2a\sin\tfrac{\theta}{2}d\theta,\quad \rho=\tfrac{2}{3}\sqrt{2ar}.$$

3. Show that for the lemniscate $r^2=a^2\cos 2\theta$,

$$\psi=\frac{\pi}{2}+2\theta,\quad p=\frac{r^3}{a^2},\quad \rho=\frac{a^2}{3r}.$$

4. Show that in the parabola, $r=a\sec^2\frac{\theta}{2}$, $\sqrt{r^2+r'^2}=a\sec^3\frac{\theta}{2}$, the polar subtangent $=2a\csc\theta$, and the perpendicular from the focus on a tangent $=a\sec\frac{\theta}{2}$.

5. Find the radius of curvature of the parabola $r=a\sec^2\frac{\theta}{2}$, and its values for the vertex and the extremities of the latus rectum. Ans. $\rho=2a\sec^3\frac{\theta}{2}$; $2a$, and $4a\sqrt{2}$.

6. Find the radius of curvature of the ellipse $r=\frac{a(1-e^2)}{1+e\cos\theta}$, and its values for the points given by $\theta=0$, $\frac{\pi}{2}$, π, $\cos^{-1}(-e)$.

7. Find $\tan\psi$ for the Spiral of Archimedes, $r=a\theta$, and for the Logarithmic or Equiangular Spiral $r=ae^{n\theta}$.

8. Show that the radius of curvature for $r=ae^{n\theta}$ is $r\sqrt{1+n^2}$, and for $r=a\theta$ is $\frac{(1+\theta^2)^{\frac{3}{2}}}{2+\theta^2}a$.

HIGHER DEGREE CURVES.

99. Curve-Tracing.—In order to find the form of a curve from its equation, we can either *plot* the curve or *trace* it. In plotting a curve, we find the coördinates of a large number of points, and by marking these points obtain a dotted line to serve as a guide in sketching the curve. In tracing a curve, we locate a few important points, find approximate forms of the curve at these points, and then sketch the curve. Thus an ellipse can be constructed point by point with the aid of the auxiliary circles; or a circumscribed parallelogram may be used as an approximate form, as in the algebra; or again, the circles of curvature at the ends of the axes will furnish closer approximations.

100. Approximate Forms.—If $f(x, y)=0$ is the given curve, and $\phi(x, y)=0$ is another, meeting $f(x, y)=0$ in two or more points coincident at P, $\phi(x, y)=0$ is a *tangent* or an *approximate form* to $f(x, y)=0$ at P. The greater the number of points coincident at P, the closer is the approximation.

Approximate Forms at the Origin and at Infinity.—When two equations are solved simultaneously, the number of common solutions is in general the product of the degrees of the equations. It is always easy to see how many of these solutions have the value zero or are infinite.

For instance, consider the cubic equation

$$x^3+x^2-y=0. \qquad (1)$$

It should have three solutions in common with the linear equation

$$y=0. \qquad (2)$$

But $x^3+x^2=0$ or $x^2(x+1)=0$ gives $x=0$ twice; so $(0, 0)$ is a double pair of solutions, and the x-axis is tangent to the curve (1).

The curve $y=x^2$ should have *six* intersections with (1); but $x^3=0$ is of the third degree only, and, considered as of the sixth degree, has three zero roots and three infinite roots. Hence the parabola $y=x^2$ meets (1) three times at the origin and three times at infinity. (See Art. 46, Algebra.) The parabola $y=x^2$ is a closer approximation to (1) at the origin than the tangent.

The curve $y=x^3$ should meet the curve nine times; but $x^2=0$, considered as an equation of the ninth degree, has two roots corresponding to the origin, and seven infinite. $y=x^3$ is therefore a closer approximation than $y=x^2$ at infinity, but not so close at the origin.

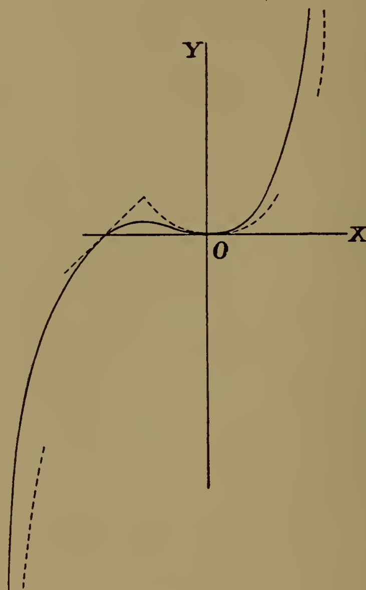


FIG. 32.

We therefore take $y=x^2$ as the form at the origin, and $y=x^3$ as the form at infinity, using the part of $y=x^2$ near the origin and the remoter parts of $y=x^3$. (See Fig. 32.)

From the foregoing, the significance of the following general principles will be clear.

The terms of lowest degree give the form at the origin (if the curve goes through the origin); *the terms of highest degree give the form at infinity.*

The terms of the very lowest degree give tangent lines at the origin; terms not homogeneous, but lower in degree than the rest of the equation, give a curvilinear form at the origin.

The terms of the very highest degree give lines intersecting the curve at infinity, but not necessarily tangent to it; terms not homogeneous, but higher in degree than the rest of the equation, give a curvilinear form always tangent at infinity, but not necessarily of as close tangency as possible.

101. Asymptotes.—Tangents to a curve at infinity are called *asymptotes*. In Art. 128 of the Algebra, a method was given of

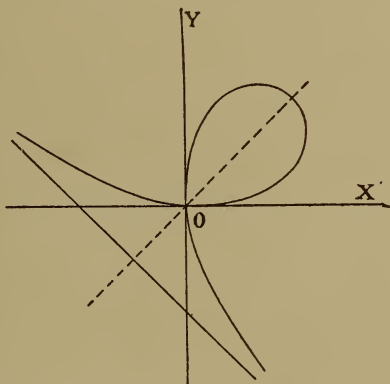


FIG. 33.

determining the asymptotes of a hyperbola. This method applies equally well to curves of higher degree, both in demonstration and in application. Thus *any real factor of the terms of highest degree in the equation may be evaluated to give an asymptote.* If there are no such *real* factors, there are no infinite branches, and the curve is closed. If the evaluation gives an infinite result, the corresponding branch is parabolic.

The curve

$$x^3+y^3-3xy=0$$

has for tangents at the origin

$$xy=0 \quad \text{or} \quad x=0 \quad \text{and} \quad y=0,$$

the coördinate axes. Also,

$$x+y=\left.\frac{3xy}{x^2-xy+y^2}\right]_{y=-x=\infty}=\left.\frac{-3x^2}{3x^2}\right]=-1,$$

so that $x+y=-1$ is an asymptote of the curve. (See Fig. 33.)

Again,

$$2x^2y+y^2+8x=0$$

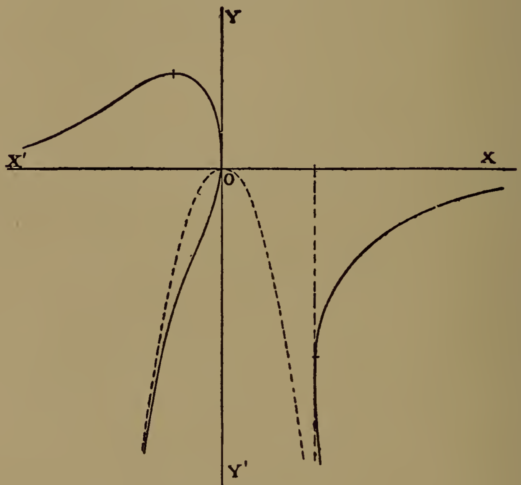


FIG. 34.

has for tangent at the origin $8x=0$, the y-axis, and for a closer approximation at the origin

$$y^2=-8x,$$

a parabola.

Evaluating $y(2x^2+y)=-8x$ for the parabolic factor, we get

$$2x^2+y=\left.\frac{-8x}{y}\right]_{y=-2x^2=\infty}=\left.\frac{4}{x}\right]_{x=\infty}=0,$$

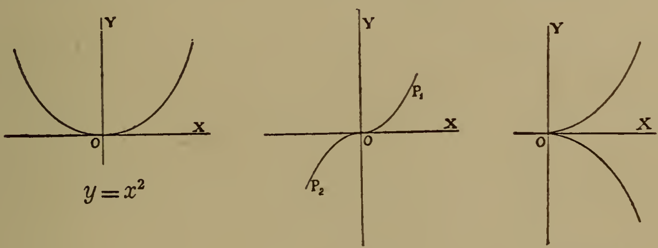
or the parabola $y=-2x^2$ as a curvilinear asymptote.

Evaluating for the linear factor, we get

$$y = \frac{-8x}{y+2x^2}\Bigg]_{\substack{y=0\\x=\infty}} = -\frac{4}{x} = 0,$$

or the x-axis, $y=0$, as a rectilinear asymptote. (See Fig. 34.)

102. Typical Forms at the Origin and at Infinity.—The *parabolic forms* $y=x^2$, $y=x^3$, $y^2=x^3$, etc., of the general form $y^m = \pm x^n$, are of frequent occurrence as approximate forms; each of them is readily traced by using the general principles above. For instance, in $y=x^2$, the term of lowest degree gives $y=0$, the



$y=x^2$ $y=x^3$ $y^2=x^3$

FIG. 35.

x-axis, as tangent at the origin, and y is evidently positive for any real value of x. (Fig. 35.)

$y=x^3$ is also tangent to the x-axis at the origin, and y is of the same sign as x.

$y^2=x^3$ is tangent to the x-axis at the origin, and x is positive for all real values of y.

$x^2=-y^3$ is tangent to $x=0$, the y-axis, and y is always negative. This is the same curve as $y^2=x^3$, but with the cusp pointing upward.

A second-degree parabola of the form $y^2=ax$ has $a/2$ for the radius of curvature at the vertex; this enables us to approximate more closely to the proper curvature at the origin for a curve having $y^2 - ax$ for its lowest terms. For instance, the curve of Fig. 34, having the form $y^2 = -8x$ at the origin, is very closely tangent to a circle of radius 4 with its center on the x-axis at $(-4, 0)$.

The *forms at infinity of the parabolic types*, $y = x^2$, $y = x^3$, $y^2 = x^3$, etc., are from their highest terms similar to $x = 0$, or the y-axis. As a point (x, y) moves out on one of these curves, x and y both increase indefinitely, but the curve becomes more and more closely parallel to the y-axis. In the same way, $x = y^2$, $x = y^3$, $x^2 = y^3$, etc., go further and further from the axes, becoming more and more nearly parallel to the x-axis.

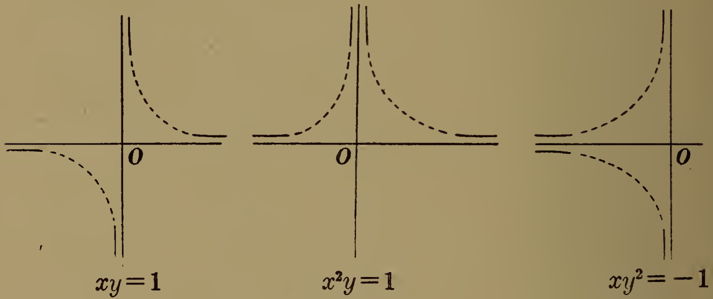


FIG. 36.

The *hyperbolic forms*, $xy = 1$, $x^2y = 1$, $xy^2 = 1$, etc., of the general form $x^n y^m = \pm 1$, evidently have the coördinate axes as asymptotes, since for each $y = \infty$ when $x = 0$ and $x = \infty$ when $y = 0$. For $xy = 1$, x and y are of the same sign; so the two branches are in the first and third quadrants. For $x^2y = 1$, y is positive for all real values of x; so the two branches are in the first and second quadrants; $xy^2 = 1$ is in the first and fourth quadrants, $xy = -1$ is in the second and fourth, and so on.

103. Uses of Derivatives.—Besides the forms at the origin and at infinity, it is often necessary to find the form of the curve at some other important points. Generally no more is done than to find the slope at some convenient points, especially the intersections with the axes, and to locate the points where the curve has some particular slope, especially 0 or ∞, sometimes ± 1.

For instance, the curve of Fig. 32,

$$y = x^2 + x^3,$$

cuts the x-axis at $(-1, 0)$. Its slope is in general

$$\frac{dy}{dx} = 2x + 3x^2,$$

and is $+1$ at this point. Again, its slope becomes 0 when $x = 0$ or $-\frac{2}{3}$; so the curve is parallel to the x-axis at $(-\frac{2}{3}, \frac{4}{27})$. These data aid materially in giving the proper form to the curve.

Again, the curve of Fig. 34,

$$2x^2y + y^2 + 8x = 0, \qquad (1)$$

has for its slope

$$\frac{dy}{dx} = -\frac{4(xy+2)}{2(x^2+y)},$$

and this slope is 0 for the point on (1) for which $xy = -2$.

Putting $y = -2/x$ in (1),

$$-4x + \frac{4}{x^2} + 8x = 0; \quad x = -1, \quad y = -\frac{2}{x} = 2.$$

Thus the curve is parallel to the x-axis at $(-1, 2)$.

The slope of (1) is ∞ for the point on (1) for which $y = -x^2$, or

$$-2x^4 + x^4 + 8x = 0,$$

whence

$$x = 0 \text{ or } 2 \quad \text{and} \quad y = -x^2 = 0 \text{ or } -4.$$

Thus the curve is parallel to the y-axis at the origin and at $(2, -4)$.

The form of the equation (Fig. 33)

$$x^3 + y^3 - 3xy = 0$$

shows plainly enough that the curve is symmetrical with regard to $y=x$, so that the end of the loop is its intersection with $y=x$.

The tangents at the inflections of a curve are of great assistance in drawing the curve, and are sometimes readily found, particularly if the equation can be put in the form $y=f(x)$.

104. Analysis of the Equation.—The general principles given for selecting terms out of an equation to represent a valuable approximation to the curve at the origin or at infinity are very broad in application, but do not cover all cases.

For instance, in $x^3+y^3-3xy=0$,

$$x^3-3xy=x(x^2-3y)=0 \quad \text{and} \quad y^3-3xy=y(y^2-3x)=0$$

are of the same degree as the curve, but give two parabolic forms at the origin, as well as the tangent axes. It is a tiresome process to test all the combinations of terms in an equation in order to find the ones which give close approximations, but the selection of such terms is greatly simplified by the employment of a graphic analyzer called the *analytical triangle*.

105. The Analytical Triangle.—The analytical triangle is formed by plotting each term of a complete equation, in rectangular coördinates, with the *exponents* of x and y for the *coördinates* of the corresponding point. The analytical triangle AOB of the cubic equation is given in Fig. 37; a complete equation of the n-th degree would have $(n+1)$ points on each side of the triangle formed by joining its outside terms.

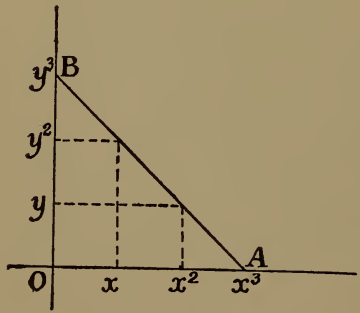


FIG. 37.

The sides OA and OB are called the *analytical axes* of x and y, respectively. The points on the analytical axis of x represent all the terms of the equation which do not contain

y; the points on the analytical axis of y, the terms which do not contain x; while the points on the side AB represent all the highest degree terms of the equation, and, as we have seen, determine the form of the curve at infinity.

The vertices O, A, and B are called the *fundamental points* of the analytical triangle; the vertex O corresponds to the origin of coördinates, the vertex A to the point at infinity in the direction of the axis of x, and the vertex B to the point at infinity in the direction of the axis of y.

106. The equation of a given curve is *placed upon the analytical triangle,* when all the terms which compose it are designated by some distinguishing mark. The cubic equation

$$x^3+y^3-3xy+2x^2+y=0 \quad (1)$$

is placed upon the analytical triangle in Fig. 38, where each term is indicated by drawing a small circle around it; no attention is paid to its coefficient, as the object of the graphic analysis is to indicate the terms which give the equations of approximate forms, and the meaning of the missing terms of the complete equation.

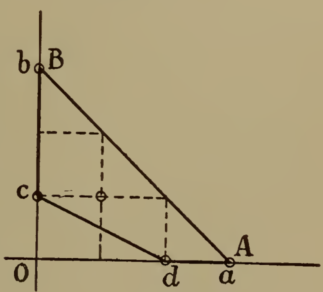


FIG. 38.

107. The Analytical Polygon.—If straight lines are drawn joining marked points of the analytical triangle so as to form a convex polygon, outside of which no marked point lies, the figure thus formed is called the *analytical polygon.* The polygon $abcd$ in Fig. 38 is the analytical polygon for the equation given in Art. 106, the marked point within it having no significance in determining the general characteristics of the curve.

The analytical polygon must have either a side or a vertex on each side of the analytical triangle. An equation formed by setting equal to zero the terms of the original equation upon any side of the polygon, enables us to obtain an approximation to some part of the curve. For brevity's sake, we say that the side of the polygon *gives* the form of this part of the curve. An *outer* side of the analytical polygon is one such that every marked point lies between it and the origin; an *inner* side is one such that no marked point lies between it and the origin; the remaining sides, if any exist, lie on the legs (analytical axes) of the analytical triangle.

(1) *An outer side gives an approximate form of the curve at infinity, indicating always a linear or a curvilinear asymptote.*

(2) *An inner side gives an approximate form at the origin.*

(3) *A side on one of the analytical axes gives, by solution, the intersections of the corresponding coördinate axis with the curve.*

While these three precepts furnish the fundamental analysis of an equation by means of its *analytical polygon* formed by the marked points of the analytical triangle, the unmarked points are of importance in determining the curve, especially if they include one or more of the fundamental points.

If the point O is unmarked, the equation has no absolute term, and the curve passes through the origin. If one of the points A or B is unmarked, it is at the vertex of a small triangle cut off from the analytical triangle by an adjacent side of the analytical polygon, or by such a side produced; *this side is said to cut off the unmarked vertex, and gives the approximate form of the curve only in the direction corresponding to the unmarked vertex.* If the terms forming such a side contain a monomial factor, it is to be disregarded; if such factor should give an asymptote, this may be found more advantageously from some other side of the polygon.

The terms of a side lying on AB, being homogeneous, may always be decomposed into factors of the first degree, either real or imaginary; each of the real linear factors gives a direction in which the curve goes off to infinity, and by evaluating such factors by the method employed in Art. 101, we may determine the asymptotes if any exist; if the evaluation of a linear factor does not lead to a finite value, the curve has no asymptote corresponding to it, but goes off to infinity in a parabolic form. It is not necessary to evaluate the monomial factors, as the infinite branches corresponding to them are given to better advantage by the sides of the analytical polygon cutting off the unmarked vertex A or B.

108. The sides of the analytical polygon *abcd* in Fig. 38 will furnish the following analysis of the given equation:

$$x^3+y^3-3xy+2x^2+y=0.$$

The *outer* side *ab* gives

$$x^3+y^3=0 \tag{1}$$

as the first approximation of the infinite branch, showing the linear asymptote corresponding to the factor $(x+y)$.

The side *bc*, lying in the analytical axis of y, gives the equation

$$y^3+y=0, \tag{2}$$

which determines the intersections of the given curve with the axis of y.

The side *ad*, lying in the analytical axis of x, gives

$$x^3+2x^2=0, \tag{3}$$

which determines the intersections of the curve with the axis of x.

The side *cd*, since the analytical origin is *unmarked*, gives the approximate form of the curve at the origin,

$$2x^2+y=0. \tag{4}$$

Equation (2) shows the curve to intersect the axis of y in but one real point, the origin.

Equation (3) shows the curve to intersect the axis of x *twice* at the origin and at $(-2, 0)$.

The slope, $\frac{dy}{dx} = \frac{-3x^2+3y-4x}{3y^2-3x+1}$, is $\frac{-4}{7}$ at $(-2, 0)$.

Equation (1), when evaluated from the other terms of the equation, gives the asymptote

$$(y+x) = \left.\frac{3xy-2x^2-y}{y^2-xy+x^2}\right]_{y=-x=\infty} = -\left.\frac{5x^2}{3x^2}\right] = -\frac{5}{3}. \quad (5)$$

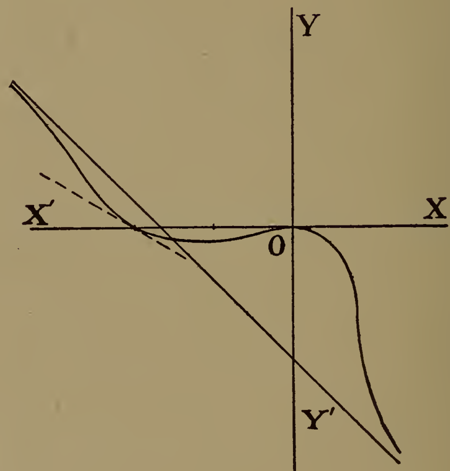


FIG. 39.

The curve is now readily traced, by drawing the asymptote, drawing in the form at the origin, and marking the only other point where the curve crosses the axes. (See Fig. 39.)

109. The three equations which we have already discussed in Arts. 100 and 101 we shall now subject to more complete analysis by reference to their analytical polygons.

For $y = x^2 + x^3$ (Figs. 40 and 32):

$y = 0$ gives one intersection with the y-axis at the origin, two at infinity.

$x^2+x^3=0$ gives two intersections with the x-axis at the origin, and one at $(-1, 0)$.

$y=x^2$ is the form at the origin.

$y=x^3$ is the form at infinity in the direction of the y-axis.

For $x^3+y^3-3xy=0$ (Figs. 41 and 33):

$y^3=0$ gives three intersections with the axis of y at the origin, and $x^3=0$ gives three with the axis of x.

x^3+y^3 must be factored and its real factor evaluated, giving $x+y=-1$ as the rectilinear asymptote.

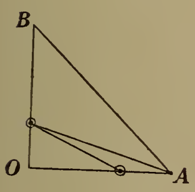


FIG. 40.

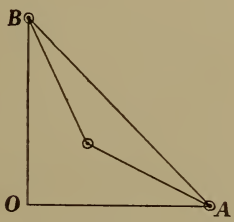


FIG. 41.

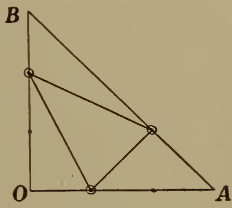


FIG. 42.

The forms at the origin are:

$$x^3-3xy=x(x^2-3y)=0$$

and

$$y^3-3xy=y(y^2-3x)=0,$$

a parabola $x^2=3y$ tangent to the x-axis and above it, and a parabola $y^2=3x$ tangent to the y-axis and to the right of it.

For $2x^2y+y^2+8x=0$ (Figs. 42 and 34):

$y^2=0$ gives two intersections with the y-axis at the origin, one at infinity.

$x=0$ gives one intersection with the x-axis at the origin, two at infinity.

$y^2=-8x$ is the form at the origin.

$2x^2y+y^2=0$ or $y=-2x^2$ is the form at infinity in the direction of the axis of y.

$2x^2y+8x=0$ or $xy=-4$ is the form at infinity in the direction of the axis of x. (The curve bears no resemblance to $xy=-4$ in the other direction.) The curve therefore approaches the x-axis as an asymptote, lying below it on the right, and above it on the left.

110. Parametric Equations $x=f(m)$, $y=mf(m)$.—If a curve has two or more tangents at the origin, that is, if the lowest terms of its equation are of at least the second degree, points of the curve may be determined by a certain pair of parametric equations much more readily than by the single equation. These parametric equations are determined by assuming

$$y=mx,$$

and thence expressing both x and y as functions of the parameter m. For instance, applying this method to the strophoid,

$$x(x^2+y^2)+a(x^2-y^2)=0,$$

we find

$$x=\frac{m^2-1}{m^2+1}a, \quad y=m\frac{m^2-1}{m^2+1}a,$$

from which, by assuming various values of m, any number of points of the curve may be determined.

The parameter m of any point of a curve is of course the slope of the radius vector to the point, or is the same as $\tan\theta$ in polar coördinates.

111. *Examples.*

1. $y^2(2a-x)=x^3$. (Cissoid.)
2. $(a^2-x^2)y^2=a^2x^2$.
3. $(a^2+x^2)y^2=a^2x^2$.
4. $y(a^2+x^2)=a^3$.
5. $a^4y^2=(a^2-x^2)^3$.
6. $a^4y^2=(a^2-x^2)x^4$.
7. $x^2(a^2-y^2)=a^4$.
8. $y^2(a^2-x^2)=x^4$.

9. $axy - x^3 = 2a^3$.
10. $a^2y - 4a^2x + x^3 = 0$.
11. $y^2(x - 2a) + 4a^2x = 0$. (Witch.)
12. $xy^2 = a(x^2 + a^2)$.
13. $a^3y = (a + x)^2(a^2 - x^2)$.
14. $y^2(a - x) = x^2(a + x)$. (Strophoid.)
15. $x^3 - y^3 - x + y^2 = 0$.
16. $y^4 - 3axy^2 + 2ax^3 = 0$.
17. $y^3 + ax^2 - axy = 0$.
18. $x^4 + 2ay^3 - 3axy^2 = 0$.
19. $x^4 + y^4 = a^2xy$.
20. $x^4 + 2a^2x^2 - 7a^2xy + 3a^2y^2 = 0$.

CHAPTER III.

Maxima and Minima.

112. Maxima and Minima, Extrema.—Sometimes the graph of a function has a point which is further from the axis of x than any other point of the graph in the immediate vicinity, as at A or B in Fig. 43. The value of the function at such a point

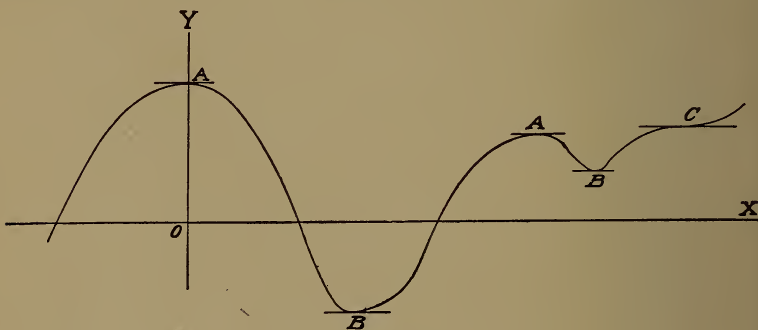


Fig. 43.

is an *extremum;* a *maximum* if it is larger than the values near by, as at A, a *minimum* if it is smaller, as at B.

If the coördinates of A or B are (x_0, y_0), y_0 is an extreme value of the function, x_0 is the value of the independent variable that makes the function a maximum or a minimum, and

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \left[f'(x)\right]_{x=x_0} = f'(x_0) = 0.$$

This principle is not altogether general, for the graph may have cusps, as in Fig. 44, where $f(x)$ is an extremum, although $f'(x)$ is not 0. These cases, however, we shall disregard.

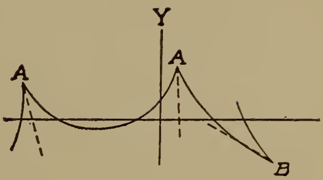


FIG. 44.

With simple functions, the converse of the principle is commonly true (i. e., $f(x)$ has an extremum whenever $f'(x)=0$), but it is evident that if the graph has an inflection where $f'(x)=0$, the function has not an extremum (C, Fig. 43).

On either side of a point where $f(x)$ is a maximum, $\tan\tau=f'(x)$ is decreasing as x increases, and its derivative, $f''(x)$, is negative. On either side of a point where $f(x)$ is a minimum, $\tan\tau=f'(x)$ is increasing as x increases, and its derivative, $f''(x)$, is positive.

We therefore have as tests:

$$f'(x)=0,\quad f''(x)<0,\quad f(x) \textit{ is a maximum.}$$
$$f'(x)=0,\quad f''(x)>0,\quad f(x) \textit{ is a minimum.}$$

The cases in which $f'(x)=0$ and $f''(x)=0$ include all such inflections as C in Fig. 43, and also points at which $f(x)$ has an extremum (e. g., $y=x^4$ when $x=0$).

The test for extrema may be stated as follows:

If, as x increases through the value x_0, $f'(x)$ passes through the value zero and *changes sign*, $f(x_0)$ is an extremum; if $f'(x)$ changes from + to −, a maximum; if from − to +, a minimum.

In geometric problems, it is almost always easy to see whether an extremum exists or not, and of which kind it is.

If $y=f(x)$ is positive, y^2 and y have extreme values of the same sort for the same values of x; for the sign of $dy^2=2ydy$ is the same as that of dy, and both become zero together. Again, y and $\frac{1}{y}$ have extreme values of opposite sorts for the same value of x; for $d\,\dfrac{1}{y} = -\dfrac{dy}{y^2}$ is opposite in sign to dy and becomes zero when $dy=0$. We may thus often simplify the problem of finding the extreme values of a given function, by finding those of the square of the function, or of its reciprocal, and interpreting the results accordingly.

113. *Examples.*

1. Find the dimensions of the open box of greatest capacity which can be made from a piece of tin plate 3 in. square, by cutting a small square from each corner and folding up the edges.

Let x = the side of the small square; i. e., the depth of the box. Then $(3-2x)$ is a side of the bottom, and the volume V is

$$V=x(3-2x)^2.$$

$\dfrac{dV}{dx}=V'=3(3-2x)(1-2x)=0$ when $x=\frac{3}{2}$ or $\frac{1}{2}$. The extreme values of V therefore occur when $x=\frac{3}{2}$ and when $x=\frac{1}{2}$, and are $V=0$, the minimum, and $V=2$ cu. in., the maximum.

2. To find the cylinder of revolution of given volume that shall have the minimum surface (for instance, a closed tin can of given volume, of such dimensions as to require the least amount of material in its construction).

Let y = the height of the cylinder and x = the radius of the base; then $S=2\pi x^2+2\pi xy$, where x and y are subject to the restrictions that the volume shall be constant, or

$$V=\pi x^2 y=C.$$

By elimination, S might be expressed as an explicit function of x or y, and the value of x or y found to make $\dfrac{dS}{dx}=0$, or $\dfrac{dS}{dy}=0$. It is simpler however to proceed as follows:

$$dS=2\pi\{(2x+y)\,dx+x\,dy\},$$

which will be zero when S is a minimum.

Since V is constant, we also have

$$dV=\pi(x^2dy+2xy\,dx)=0,$$

and from this equation

$$dy = -\frac{2y}{x}\,dx.$$

Substituting this value of dy in the value of dS,

$$dS = 2\pi\{(2x+y)\,dx - 2y\,dx\},$$

and

$$\frac{dS}{dx} = 2\pi(2x-y) = 0,$$

if $y=2x$.

We thus have the proper proportions of the cylinder, and its actual dimensions can be computed from the relations

$$V = \pi x^2 y \quad \text{and} \quad y = 2x,$$

from which

$$V = 2\pi x^3 \quad \text{and} \quad x = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}.$$

3. Prove that the maximum area of a rectangle which can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, having its sides parallel to the principal axes, is $2ab$.

4. Prove the results of example 3, using the parametric equations $x = a\cos\phi$, $y = b\sin\phi$.

5. Prove that the maximum volume formed by revolving a rectangle inscribed as in examples 3 and 4 about the x-axis is $\frac{4\pi b^2 a}{3\sqrt{3}}$.

6. A triangle is inscribed in a parabolic segment having a base $2b$ and altitude a, the vertex of the triangle being at the mid-point of the base of the segment and the base of the triangle parallel to the base of the segment; find the maximum value of the area of the triangle. Ans. $A = \frac{2ab}{3\sqrt{3}}$.

7. An open cylindrical can is to contain 231 cu. in. What is the least amount of tin that can be used to make it?

Ans. 165.42 sq. in.

8. Find the most economical proportions for a closed cylindrical tin can, if in making each of the circular ends it is necessary to use up a piece in the shape of a regular hexagon circumscribing the circle.

Ans. h being the height and D the diameter, $\frac{h}{D}=\frac{2\sqrt{3}}{\pi}=\frac{11}{10}$ nearly.

9. Find the most economical proportions for a cylindrical tin cup, making the same allowance for waste as in the preceding example. Ans. $\frac{h}{D}=\frac{\sqrt{3}}{\pi}=\frac{11}{20}$ nearly.

10. A man is in a rowboat, 4 mi. from the nearest point, A, on a straight beach, and is bound for a point B on the beach, 25 mi. beyond A. He can row $2\frac{1}{2}$ m/h, and walk $3\frac{1}{2}$ m/h. Where shall he land to reach his destination as soon as possible?

Ans. $\frac{5}{3}\sqrt{6}$ mi. from A.

11. The altitude of a right circular cone is h, and the radius of its base is a. Find the greatest volume of an inscribed right circular cylinder. Ans. $V=\frac{4}{27}\pi a^2 h$.

12. Find the dimensions of the cylinder in example 11 if its total surface is to be a maximum.

Ans. Radius$=\frac{ha}{2(h-a)}$, height$=\frac{h(h-2a)}{2(h-a)}$.

13. Determine the cone of maximum lateral surface and the one of minimum lateral surface inscribed in a paraboloid of revolution of height a, and diameter of base a.

Ans. The altitudes are $\frac{a}{2}$ and $\frac{a}{3}$ respectively.

14. In a circle of fixed radius a a rectangle of sides $2x$ and $2y$ is inscribed; the figure is revolved about the diameter perpendicular to the side $2x$. Find the dimensions so that (a) the rectangle shall have the maximum area, (b) the cylinder shall have the maximum volume, (c) the cylinder shall have the maximum lateral surface.

Ans. (a) and (c), $x=y=\frac{a}{\sqrt{2}}$; (b) $x=\frac{a}{3}\sqrt{6}$, $y=\frac{a}{\sqrt{3}}$.

15. From a steamer A, going north at 8 m/h, a steamer B is observed going west at 10 m/h. If A turns just as B crosses its path, what straight course must A take in order to cross the course of B as near B as possible?

Ans. About N. 53° 8′ W.

16. The base of a column 9 ft. high is 16 ft. above the eye of an observer. How far off must he stand for the column to subtend the greatest possible angle? Ans. 20 ft.

17. An isosceles triangle is circumscribed about a parabolic segment, the base of which is parallel to the tangent at the vertex. Show that the area of the triangle is least when its altitude is $\frac{4}{3}$ of the altitude of the segment.

18. The expenditure of coal in steaming a ship is proportional to the time and to the cube of the speed; find the most economical speed against a current having the speed a.

Ans. $\frac{a}{2}$ actual speed, $\frac{3a}{2}$ through the water.

19. A circular sector is to have a given perimeter and as large an area as possible; what must be its angle?

Ans. 2 radians.

20. The strength of a beam is proportional to the breadth and the square of the depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius a.

Ans. $\frac{2a}{3}\sqrt{3}$ by $\frac{2a}{3}\sqrt{6}$.

21. A triangle is inscribed in an ellipse, its vertex at a vertex of the ellipse, its base a double ordinate. Find the greatest area it can have. Ans. $\frac{3ab}{4}\sqrt{3}$, a and b being the semi-axes.

22. If the figure of example 21 is revolved to form a cone inscribed in an ellipsoid, what is the greatest volume the cone can have? Ans. $\frac{32}{81}\pi ab^2$.

23. A rectangular strip of copper is to be bent so that its cross-section is a circular arc; show that to give the maximum capacity, the arc must be a semicircle.

24. Two circular plates, each of radius a, are to be cut and bent into conical surfaces and put together to form a can-buoy. What must be the radius of the base of each cone if the buoy is to be as large as possible? Ans. $\frac{a}{3}\sqrt{6}$.

25. A wall 27 ft. high is 64 ft. from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside of the wall. Ans. 125 ft.

26. Find the least volume that can be left between a sphere and a circumscribed cone of revolution. (Hint: Find two expressions for the area of the section of the cone through its altitude, and thus get a relation involving x, y, and a, the radius of the base and the altitude of the cone, and the radius of the sphere.) Ans. $\frac{4}{3}\pi a^3$.

CHAPTER IV.

INTEGRATION.

114. Definition.—*If $f(x)dx$ is the differential of $F(x)$, $F(x)$ is an integral of $f(x)dx$.*

For instance, since $2xdx$ is the differential of x^2, x^2 is an integral of $2xdx$. $d(x^2+c)$, if c is any constant, is also $2xdx$, and in general:

$F(x)$ being an integral of $f(x) \cdot dx$, and c being any constant, $F(x)+c$ is also an integral of $f(x)dx$; that is, any differential expression has innumerable integrals, any two of which differ by a constant.

Thus *integration*, as the process of finding an integral is called, is the *inverse of differentiation*, and like most inverse processes, leads to multiple-valued results.

The notation used for integrals and for integration is the following:

$\int f(x)dx=F(x)+C$; this is read: "The integral of $f(x)dx$ is $F(x)$ plus some constant." The constant C is spoken of as an *arbitrary constant*, because it may be any constant whatever; it must be written if the relation above is to be used as an equation.

The relation $\int f(x)dx=F(x)$, written without the arbitrary constant, is used for formulas, etc., and when so written means: "*An* integral of $f(x)dx$ is $F(x)$."

In the expression $\int f(x)dx$, $f(x)dx$ is called the *integrand*.

There is no systematic theory of integration as there is of differentiation; finding an integral is more a matter of search and discovery than of computation, but there are principles and rules to aid in the search. Moreover, as any formula of differentiation can be read as a formula of integration, we can begin by making the following table of fundamental integrals, in which u is any variable, and may be a function of the independent variable:

(1) $\int c\,du = cu.$

(2) $\int u^n du = \frac{u^{n+1}}{n+1}.$

(3) $\int e^u du = e^u.$

(4) $\int a^u du = \frac{a^u}{\log a}.$

(5) $\int \frac{du}{u} = \log u.$

(6) $\int \cos u\,du = \sin u.$

(7) $\int \sin u\,du = -\cos u.$

(8) $\int \sec^2 u\,du = \tan u.$

(9) $\int \csc^2 u\,du = -\cot u.$

(10) $\int \sec u \tan u\,du = \sec u.$

(11) $\int \csc u \cot u\,du = -\csc u.$

(12) $\int \cot u\,du = \log \sin u = -\log \csc u.$

(13) $\int \tan u\,du = \log \sec u = -\log \cos u.$

(14) $\int \csc u\,du = \log \tan \frac{u}{2} = -\log \cot \frac{u}{2}.$

(15) $\frac{du}{\sqrt{1-u^2}} = \sin^{-1} u \text{ or} = -\cos^{-1} u.$

(16) $\int \frac{du}{1+u^2} = \tan^{-1} u \text{ or } = -\cot^{-1} u.$

(17) $\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u \text{ or } = -\csc^{-1} u.$

(18) $\int \frac{du}{\sqrt{2u-u^2}} = \text{versin}^{-1} u.$

(19) $\int (f_1(u) + f_2(u) + f_3(u) + \ldots\ldots)du$
$= \int f_1(u)du + \int f_2(u)du + \int f_3(u)du + \ldots\ldots$

(20) $\int u\,dv = uv - \int v\,du.$

Any formula of integration (for instance, any of the identities above) can be verified by differentiation, with the aid of the definition of an integral:

$$d\int f(x)\cdot dx = f(x)\,dx.$$

Exercise.—Verify each of the twenty fundamental formulas.

115. Direct Integration.—It requires experience and careful observation to recognize these fundamental forms in all cases.

$$\int (a^2-x^2)^{\frac{1}{2}} d(a^2-x^2)$$

is an obvious instance of

$$\int u^n du;$$

but the precisely equivalent expression

$$-2\int x\sqrt{a^2-x^2}\cdot dx,$$

or the expression

$$\int x\sqrt{a^2-x^2}\cdot dx,$$

which is $-\frac{1}{2}$ as much, is not so obvious.

The real difficulty is that it is not enough to be able to say that $d(a^2-x^2)$ is $-2xdx$; the mere presence together of (a^2-x^2) and x must suggest this fact.

As another instance of the same formula, consider

$$\int \sin\theta \cos\theta \, d\theta.$$

A mere reminder that $d\sin\theta = \cos\theta\, d\theta$ is sufficient to make it evident that

$$\int \sin\theta\cos\theta\, d\theta = \int \sin\theta\, d(\sin\theta) = \frac{(\sin\theta)^2}{2}.$$

These will serve to point out the fact that, to practice integration successfully, the student must be able to remind himself of the differential formula that will be useful.

In order to keep track of constant factors, it sometimes is worth while to abbreviate the integrand by introducing a new variable.

For instance, required $\int \frac{x^2}{x^6+4}\,dx.$

We notice that x^2dx is a constant multiple of $d(x^3)$, and that $x^6=(x^3)^2$. Then let $x^3=y$, so that $3x^2dx=dy$, and we have

$$\int \frac{x^2dx}{x^6+4}=\int \frac{\frac{1}{3}\,dy}{y^2+4}=\tfrac{1}{6}\int \frac{d(y/2)}{1+(y/2)^2}=\tfrac{1}{6}\tan^{-1}\frac{y}{2}$$
$$=\tfrac{1}{6}\tan^{-1}\frac{x^3}{2}.$$

Substituting $y=\frac{x^3}{2}$ would have made the work still more mechanical.

116. Trigonometric Functions.—The fundamental formulas of trigonometry must also be so well known that useful relations will readily come to mind. For instance, the three important integrals

$$\int \sin^2\theta\,d\theta,\quad \int \cos^2\theta\,d\theta,\quad \text{and}\quad \int \sin\theta\cos\theta\,d\theta$$

can be evaluated directly by means of the relations

$$\sin^2\theta=\tfrac{1}{2}(1-\cos 2\theta),$$
$$\cos^2\theta=\tfrac{1}{2}(1+\cos 2\theta),$$
$$\sin\theta\cos\theta=\tfrac{1}{2}\sin 2\theta,$$

and

$$\int \cos 2\theta\,d\theta=\tfrac{1}{2}\int \cos 2\theta\,d(2\theta)=\tfrac{1}{2}\sin 2\theta.$$

Again

$$\int \sin^3\theta\,d\theta=-\int (1-\cos^2\theta)\,d(\cos\theta),$$
$$\int \tan\theta\sec^2\theta\,d\theta=\int \tan\theta\,d(\tan\theta)=\int \sec\theta\,d(\sec\theta).$$

117. *Examples.*

Evaluate the following:

1. $\int \sqrt{x}dx.$ Ans. $\frac{2}{3}\sqrt{x^3}.$
2. $\int gtdt.$ Ans. $\frac{1}{2}gt^2.$

3. $\int \frac{dx}{x^2}$. Ans. $-\frac{1}{x}$.

4. $\int bx\sqrt{\frac{x}{a}}$. Ans. $\frac{2b}{5}\sqrt{\frac{x^5}{a}}$.

5. $\int (x^2-3x+4)\,dx$. Ans. $\frac{1}{6}(2x^3-9x^2+24x)$.

6. $\int \frac{dx}{(x+1)^{\frac{3}{2}}}$. Ans. $\frac{-2}{(x+1)^{\frac{1}{2}}}$.

7. $\int \frac{dx}{\sqrt{3x+2}}$. Ans. $\frac{2}{3}\sqrt{3x+2}$.

8. $\int \frac{x\,dx}{(a^2+x^2)^{\frac{5}{2}}}$. Ans. $\frac{-1}{3(a^2+x^2)^{\frac{3}{2}}}$.

9. $\int x\sqrt{a^2-x^2}\,dx$. Ans. $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$.

10. $\int x^2(a^3+x^3)^{\frac{1}{3}}dx$. Ans. $\frac{1}{4}(a^3+x^3)^{\frac{4}{3}}$.

11. $\int \frac{dx}{3+x}$. Ans. $\log(3+x)$.

12. $\int \frac{x\,dx}{a^2-x^2}$. Ans. $\log\frac{1}{\sqrt{a^2-x^2}}$.

13. $\int a^{nx}dx$. Ans. $\frac{a^{nx}}{n\log a}$.

14. $\int xe^{x^2}dx$. Ans. $\frac{1}{2}e^{x^2}$.

15. $\int \frac{1-\cos\theta}{\sin\theta}\,d\theta = \int \tan\frac{\theta}{2}\,d\theta$. Ans. $2\log\sec\frac{\theta}{2}$.

16. $\int \frac{\sin\theta}{\cos\theta}\,d\theta$. Ans. $-\log\cos\theta = \log\sec\theta$.

17. $\int \sin^2\theta\,d\theta$. Ans. $\frac{\theta}{2}-\frac{1}{4}\sin 2\theta = \frac{1}{2}(\theta-\sin\theta\cos\theta)$.

18. $\int \cos^2\theta\,d\theta$. Ans. $\frac{\theta}{2}+\frac{1}{4}\sin 2\theta = \frac{1}{2}(\theta+\sin\theta\cos\theta)$.

19. $\int \sin\theta\cos\theta\,d\theta$. Ans. $\frac{1}{2}\sin^2\theta$, $-\frac{1}{2}\cos^2\theta$, or $-\frac{1}{4}\cos 2\theta$.

20. $\int \sec^2\theta\tan\theta\,d\theta$. Ans. $\frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$.

21. $\int \sin^3\theta\,d\theta$. Ans. $\frac{\cos^3\theta}{3}-\cos\theta$.

22. $\int \cos^3\theta\,d\theta$. Ans. $\sin\theta-\frac{\sin^3\theta}{3}$.

23. $\int \tan^3\theta\,d\theta$. Ans. $\frac{\sec^2\theta}{2}-\log\sec\theta$.

24. $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-(x-2)^2}}$ Ans. $\sin^{-1}\frac{x-2}{3}$.

25. $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{4+(x+1)^2}$ Ans. $\frac{1}{2}\tan^{-1}\frac{x+1}{2}$.

26. $\int \frac{dx}{(3-x)\sqrt{x^2-6x+8}}$. Ans. $-\sec^{-1}(3-x)$.

27. $\int \frac{dx}{\sqrt{1-6x-x^2}}$. Ans. $\sin^{-1}\frac{x+3}{\sqrt{10}}$.

28. $\int \frac{dx}{x^2-x+1}$. Ans. $\frac{2}{3}\sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

29. $\int \frac{dx}{(1+x)\sqrt{x^2+2x}}$. Ans. $\sec^{-1}(1+x)$.

30. $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$. Ans. $-\csc\theta$.

31. $\int \frac{d\theta}{\sin\theta\cos\theta}$. Ans. $\log\tan\theta$.

32. $\int\sqrt{1-\cos\theta}\,d\theta$. Ans. $-2\sqrt{2}\cos\frac{\theta}{2} = -2\sqrt{1+\cos\theta}$.

33. $\int\sqrt{1+\cos\theta}\,d\theta$. Ans. $2\sqrt{1-\cos\theta}$.

34. $\int\sec\theta\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\cdot d\theta$. Ans. $\log(\sec\theta+\tan\theta)$.

35. $\int(\sec\theta+\tan\theta)^n\sec\theta\,d\theta$. Ans. $\frac{1}{n}(\sec\theta+\tan\theta)^n$.

118. Integration by Substitution.—An integral that bears no evident resemblance to any one of the fundamental forms can often be made recognizable by the introduction of a new variable. For instance, if in

$$\int\sqrt{a^2-x^2}\,dx,$$

we put $x=a\sin\theta$, we have

$$\sqrt{a^2-x^2}=a\cos\theta, \quad dx=a\cos\theta\,d\theta,$$

and the integral becomes

$$a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}[\theta+\sin\theta\cos\theta].$$

(Ex. 18, Art. 117.)

Hence

$$\int\sqrt{a^2-x^2}dx=\frac{a^2}{2}\left[\sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]$$

or

$$\int\sqrt{a^2-x^2}dx=\tfrac{1}{2}[a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}].$$

The integral of any expression irrational merely through the presence of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ can be rationalized by an appropriate trigonometric substitution: $x=a\sin\theta$, $x=a\tan\theta$, or $x=a\sec\theta$. The resulting trigonometric integral is often recognizable.

The integral of an expression irrational merely through the presence of $\sqrt{ax+b}$ is generally recognizable, but is made simpler by putting $\sqrt{ax+b}=y$, $dx=\frac{2y}{a}dy$.

119. Rational Fractions.—The integral of any rational fraction is made simpler by separating the fraction into partial fractions. (See Algebra, Art. 136.)

For instance,

$$\int\frac{dx}{x^2(x^2+1)}=\int\frac{dx}{x^2}-\int\frac{dx}{1+x^2}=-\frac{1}{x}-\tan^{-1}x.$$

120. *Examples.*

1. $\int\frac{adx}{x\sqrt{a^2-x^2}}$. Ans. $\log\frac{a-\sqrt{a^2-x^2}}{x}$.

2. $\int\frac{dx}{\sqrt{a^2+x^2}}$. Ans. $\log(x+\sqrt{a^2+x^2})$.

3. $\int\frac{dx}{\sqrt{x^2-a^2}}$. Ans. $\log(x+\sqrt{x^2-a^2})$.

4. $\int\frac{dx}{\sqrt{2ax+x^2}}$. Ans. $\log(x+a+\sqrt{2ax+x^2})$.

5. $\int x^2\sqrt{a^2-x^2}\,dx.$

Ans. $\frac{a^4}{8}\sin^{-1}\frac{x}{a}-\frac{x}{8}\sqrt{a^2-x^2}(a^2-2x^2).$

6. $\int x\sqrt{a+x}\,dx.$ Ans. $\frac{2}{5}(a+x)^{\frac{5}{2}}-\frac{2a}{3}(a+x)^{\frac{3}{2}}.$

7. $\int\frac{x}{\sqrt{a+x}}\,dx.$ Ans. $\frac{2}{3}(a+x)^{\frac{3}{2}}-2a(a+x)^{\frac{1}{2}}.$

8. $\int\frac{a+x}{a-x}\,dx.$ Ans. $-x-2a\log(a-x).$

9. $\int\frac{dx}{a^2-x^2}.$ Ans. $\frac{1}{2a}\log\frac{a+x}{a-x}.$

10. $\int\frac{x^2dx}{(x^2-1)^2}.$ Ans. $\log\sqrt[4]{\frac{x-1}{x+1}}-\frac{x}{2(x^2-1)}.$

121. Areas Found by Integration.—Let it be required to find the area bounded by the parabola $y^2=4x$, the axis of x, and the line $x=4$; the area OAB of Fig. 45.

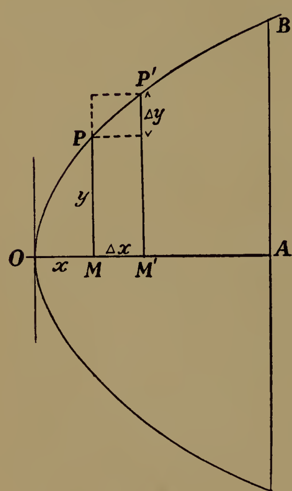


FIG. 45.

We shall proceed as follows: Drawing an ordinate, PM, through any point, $P(x, y)$, of the parabola, we cut off an area, OPM, the extent of which is determined by the value of $OM=x$, and varies when we vary x. This area, OPM, is therefore a function of x, as yet unknown. Call it $F(x)$.

$$\text{Area } OPM=F(x).$$

We shall determine $F(x)$; then the value of $F(x)$ for any value of x will give the area cut off by the ordinate corresponding to that value of x. In particular, when $x=0$, the area shrinks to nothing; and when $x=4$, the area becomes the required area OAB.

$$F(0)=0 \quad \text{and} \quad F(4)=\text{required area } OAB.$$

To determine $F(x)$, we first find its derivative according to the general definition of a derivative.

Extend OM to M', thus giving to x the increment $MM'=\Delta x$, and draw the corresponding ordinate $P'M'=y+\Delta y$.

Then

$$F(x)=\text{area } OPM, \quad F(x+\Delta x)=\text{area } OP'M'.$$
$$\Delta F(x)=\text{area } MPP'M'.$$

Complete the rectangles PM' and $P'M$.

Evidently

$$PM'<\Delta F(x)<P'M.$$
$$\frac{PM'}{\Delta x}<\frac{\Delta F(x)}{\Delta x}<\frac{P'M}{\Delta x}.$$

or

$$y<\frac{\Delta F(x)}{\Delta x}<(y+\Delta y).$$

Now the desired derivative

$$\frac{dF(x)}{dx}=\left[\frac{\Delta F(x)}{\Delta x}\right]_{\Delta x=0}=y,$$

for as Δx approaches zero, $(y+\Delta y)$ approaches y, and the value of $\dfrac{\Delta F(x)}{\Delta x}$ is always between $(y+\Delta y)$ and y.

Finally

$$dF(x)=ydx;$$

and as $y^2=4x$,

$$dF(x)=2\sqrt{x}dx,$$

and

$$F(x)=\int 2\sqrt{x}dx=\tfrac{4}{3}x^{\frac{3}{2}}+C.$$

We have not yet entirely determined $F(x)$, nor could we expect to do so by merely determining its derivative; for the process of finding the derivative would have been precisely the same if $F(x)$ had represented the part of OAB between any fixed ordinate and AB. The ordinate at which $F(x)$ becomes zero will, however, complete the determination of $F(x)$ by determining the arbitrary constant C. For since $F(0)=0$

$$O=F(0)=\tfrac{4}{3}\cdot O+C;$$

and $C=0$, so that

$$F(x)=\tfrac{4}{3}x^{\frac{3}{2}}.$$

Finally:

The required area $OAB=F(4)=\frac{4}{3}(4)^{\frac{3}{2}}=\frac{32}{3}$.

122. If it had been required to find the area bounded by $y^2=4x$, $y=0$, $x=1$, and $x=4$, the discussion would have been just the same, except that the variable area $F(x)$ would have been zero when $x=1$; i. e., we should have had $F(1)=0$. Consequently, although we should have had

$$F(x)=\tfrac{4}{3}x^{\frac{3}{2}}+C$$

as before, the value of C would have been different, since

$$O=F(1)=\tfrac{4}{3}+C$$

gives

$$C=-\tfrac{4}{3}.$$

Then

$$F(x)=\tfrac{4}{3}(x^{\frac{3}{2}}-1)$$

would have been the general expression for the variable area, and

$$F(4)=\tfrac{4}{3}(8-1)=\tfrac{28}{3}$$

would have been the required area.

123. Consider now the general process exemplified in the preceding article. Let it be required to find the area $AKLB$, Fig. 46, bounded by any given curve, $y=f(x)$, the axis of x, and the ordinates corresponding to the abscissas a and b. Consider, first, the area $AKPM$ bounded similarly, with the ordinate corresponding to the variable abscissa x in place of the one corresponding to the abscissa b. Denote this area, which is a function of x, by $F(x)$. Then

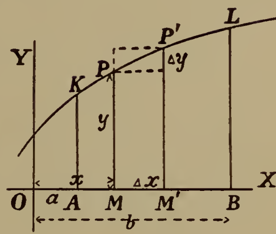


FIG. 46.

$$F(a)=0, \quad F(b)=\text{area } AKLB.$$

Increase x by $MM'=\Delta x$; y is correspondingly increased by

$$\Delta y=P'M'-PM,$$

and $F(x)$ by

$$\Delta F(x)=\text{area } MPP'M'.$$

Complete the rectangles PM' and $P'M$. Evidently

$$PM'<\Delta F(x)<P'M.$$

Divide by Δx:

$$y<\frac{\Delta F(x)}{\Delta x}<y+\Delta y.$$

Therefore,

$$\left[\frac{\Delta F(x)}{\Delta x}\right]_{\Delta x=0}=y=f(x),$$

$$\frac{dF(x)}{dx}=f(x), \quad d(F(x))=f(x)\,dx,$$

[$dF(x)$ is called the *element of integration.*]

$$F(x)=\int f(x)\,dx+C;$$

i. e., $F(x)$ is any one of the integrals of $f(x)dx$ plus some constant.

Since $F(a)=0$,

$$O=F(a)=[\int f(x)\,dx]_{x=a}+C,$$
$$C=-[\int f(x)\,dx]_{x=a},$$
$$F(x)=[\int f(x)\,dx]-[\int f(x)\,dx]_{x=a};$$

and the expression for the required area $AKLB$ is

$$\text{Area}=F(b)=[\int f(x)\,dx]_{x=b}-[\int f(x)\,dx]_{x=a}.$$

In this expression for the area $\int f(x)dx$ may be any one of the integrals of $f(x)dx$, but must of course be the same one in both brackets. Evidently, changing this integral by a constant will make compensatory changes in the two brackets.

For convenience, the expression for the area is more briefly written

$$F(b)=[\int f(x)\,dx]_{x=a}^{x=b},$$

or, most conveniently,

$$\text{Area}=F(b)=\int_a^b f(x)\,dx.$$

The last form is read: "The *definite integral* from a to b of $f(x)dx$." In distinction, $\int f(x)dx$ is called an *indefinite integral* of $f(x)dx$.

124. Definite Integrals.—Definite integrals have many other uses besides the determination of areas; a general definition will therefore be useful for future reference.

Definition.—If $dF(x)=f(x)dx$ (i. e., if $F(x)$ is an indefinite integral of $f(x)dx$), and if $F(a)=0$, then any other value of $F(x)$ is

$$F(b)=[\int f(x)\,dx]_{x=b}-[\int f(x)\,dx]_{x=a}=\int_a^b f(x)\,dx,$$

the definite integral from a to b of $f(x)dx$.

An indefinite integral, $\int f(x)\,dx$, is a function of x; the definite integral, $\int_a^b f(x)\,dx$, if a and b are given values, is the difference between two particular values of this function, and so is a constant. If a and b are supposed to vary, $\int_a^b f(x)\,dx$ will vary correspondingly; for instance, in the problem of the preceding article, any change in the abscissas a and b will cause the area $\int_a^b f(x)\,dx$ to vary. In other words:

A definite integral is a function of its limits.

Besides the values of the limits, nothing affects the value of a definite integral, $\int_a^b f(x)\,dx$, except the *form* of $f(x)$. For it clearly makes no difference whether x is written throughout the integrand $f(x)\,dx$, or some other letter, since this letter, whatever it is, will be replaced by the limits when the integral is evaluated.

For instance, each of the definite integrals,

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} \quad \text{and} \quad \int_0^a \frac{dz}{\sqrt{a^2-z^2}}$$

is $(\sin^{-1} 1 - \sin^{-1} 0)$ or $\frac{\pi}{2}$.

It is implied in the definition that the limits b and a written at top and bottom of the integral sign in a definite integral are values of the variable whose differential occurs in the integrand. This is important when a change of variable is made in evaluating the integral. For instance, suppose we are to find the area of the circle $x^2+y^2=a^2$ or $y=\sqrt{a^2-x^2}$ in the first quadrant of the coördinate axes. From the preceding article, this is

$$A=\int_0^a \sqrt{a^2-x^2}\,dx.$$

In order to integrate, let $x=a\sin\theta$; then

$$dx=a\cos\theta\,d\theta; \quad \sqrt{a^2-x^2}=a\cos\theta;$$

when $x=0$,

$$\theta=\sin^{-1} 0=0;$$

when $x=a$,

$$\theta=\sin^{-1}\frac{a}{a}=\frac{\pi}{2};$$

then

$$A=a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta=\frac{a^2}{2}\Big[\theta+\sin\theta\cos\theta\Big]_0^{\frac{\pi}{2}}=\frac{a^2}{2}\Big[\ \Big]_0^{\frac{\pi}{2}}=\frac{\pi a^2}{4}.$$

(It is convenient to write the values at the upper and lower limits in the positions of those limits, and then subtract.)

It is more expeditious to change the limits in this way when the variable is changed than to express A as a function of x and then substitute the original limits. The necessity of remembering the algebraic integral is also avoided.

The notation of the definite integral can be used to express any particular one of the indefinite integrals of a $f(x)\cdot dx$; thus

$$\int_a^x f(x)\,dx$$

represents the indefinite integral of $f(x)\,dx$ that becomes zero when $x=a$.

Thus we have seen that $\int\sin\theta\cos\theta\, d\theta$ may be written

$$\int\sin\theta\, d(\sin\theta), \quad -\int\cos\theta\, d(\cos\theta), \quad \text{or} \quad \tfrac{1}{4}\int\sin 2\theta\, d(2\theta),$$

and is therefore

$$\frac{\sin^2\theta}{2}, \quad -\frac{\cos^2\theta}{2}, \quad \text{or} \quad -\frac{\cos 2\theta}{4}.$$

These three values differ by constants, and of course there are any number of others. The particular one that becomes zero when $\theta=0$ is

$$\int_0^\theta \sin\theta\cos\theta\, d\theta=\frac{\sin^2\theta}{2}\Big]_0^\theta=\frac{\sin^2\theta}{2},$$

or

$$-\frac{\cos^2\theta}{2}\Big]_0^\theta = -\frac{\cos^2\theta}{2} + \tfrac{1}{2} = \frac{\sin^2\theta}{2},$$

or

$$-\frac{\cos 2\theta}{4}\Big]_0^\theta = -\frac{\cos 2\theta}{4} + \tfrac{1}{4} = \frac{\sin^2\theta}{2}.$$

125. *Examples.*

Evaluate the following definite integrals:

1. $\int_0^{\frac{\pi}{2}} \cos^3 x \sin x dx.$ Ans. $\frac{1}{4}$.

2. $\int_0^{\frac{\pi}{2}} \cos^3 x dx.$ Ans. $\frac{2}{3}$.

3. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{(a^2-x^2)}}.$ Ans. $\frac{\pi}{6}$.

4. $\int_0^{a} \frac{x dx}{\sqrt{(a^2-x^2)}}.$ Ans. a.

5. $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta} d\theta.$ Ans. $\frac{1}{2}$.

6. $\int_0^{1} \frac{1-3x}{\sqrt{1-x}} dx.$ Ans. -2.

7. Trace the curve $a^2y = ax^2 - x^3$ and find the area of the segment cut off by the x-axis. Ans. $\frac{a^2}{12}$.

8. Find the area enclosed by the curve, the x-axis and one of its asymptotes, given $y^2(a^2-x^2) = a^2x^2$. Ans. a^2.

9. Find the area enclosed by the first arch of the curve $y = \sin x$ and the x-axis. Ans. 2.

10. Find the area enclosed by the curve $y = e^x$, the y-axis and the left half of the x-axis, and also the area bounded by the curve, the axes, and $x = 1$. Ans. 1 and $e - 1$.

11. $\int_0^a \sqrt{2ax-x^2}dx.$ Ans. $\frac{\pi a^2}{4}.$

12. $\int_0^{2a} \sqrt{(2ax-x^2)}\,xdx.$ Ans. $\frac{\pi a^3}{2}.$

13. $\int_1^5 \frac{dx}{\sqrt{(6x-x^2-5)}}$ Ans. $\pi.$

14. $\int_0^a \frac{dx}{\sqrt{(x^2+a^2)}}$ Ans. $\log(\sqrt{2}+1).$

15. $\int_0^{\frac{a}{2}} \sqrt{(a^2-x^2)}\,dx.$ Ans. $\frac{a^2}{24}(2\pi+3\sqrt{3}).$

126. Areas.—The result of Art. 123 may be stated:

The area generated by an ordinate of the curve $y=f(x)$ in moving from the line $x=a$ to the line $x=b$ is

$$\int_a ydx=\int_a^b f(x)\,dx.$$

It follows from a proof precisely similar to that of Art. 123 that:

The area generated by an abscissa of the curve $x=f(y)$ in moving from the line $y=a$ to the line $y=b$ is

$$\int_a^b xdy=\int_a^b f(y)\,dy.$$

Thus the area bounded by $y^2=4x$, $y=4$, and $y=0$, is

$$\int_0^4 xdy=\int_0^4 \frac{y^2}{4}dy=\frac{y^3}{12}\Big]_0^4=\frac{16}{3}.$$

Since the axes and the lines $x=4$ and $y=4$ bound an area of 16 units, the area computed in Art. 121 is again seen to be

$$\left(16-\frac{16}{3}\right)=\frac{32}{3}.$$

In practice, the required area can generally be found by taking the element of integration parallel to either axis; the choice is determined by the relative difficulty of evaluation of $\int y dx$ and $\int x dy$.

When a curve has convenient parametric equations, it is always best to use them, as a change of variable is thereby avoided. For instance, the area of the circle $x^2+y^2=a^2$ in the first quadrant of the coördinate axes might have been found from the parametric equations

$$x=a\cos\phi, \quad y=a\sin\phi,$$

where $dx=-a\sin\phi\, d\phi$.

The generating ordinate moves across this quadrant from left to right as ϕ varies from $\frac{\pi}{2}$ to 0; hence

$$\int_0^a y dx=\int_{\frac{\pi}{2}}^0 a\sin\phi(-a\sin\phi\, d\phi)=-a^2\int_{\frac{\pi}{2}}^0 \sin^2\phi\, d\phi$$

$$=-\frac{a^2}{2}\Big[\phi-\sin\phi\cos\phi\Big]_{\frac{\pi}{2}}^0=-\frac{a^2}{2}\begin{bmatrix}0-0\\ \frac{\pi}{2}-0\end{bmatrix}=\frac{\pi a^2}{4}.$$

127. Sign of the Definite Integral.—If $\int f(x)dx=F(x)$,

$$\int_a^b f(x)dx=F(b)-F(a) \quad \text{and} \quad \int_b^a f(x)dx=F(a)-F(b).$$

Hence:

Reversing the order of the limits in a definite integral changes the sign of the result.

For instance, $-a^2\int_{\frac{\pi}{2}}^0 \sin^2\phi\, d\phi$ in the preceding article is the same as $a^2\int_0^{\frac{\pi}{2}} \sin^2\phi\, d\phi$.

In deducing the expression $\int_a^b f(x)dx=\int_a^b y dx$ for the area, it was tacitly assumed that $f(x)$ or y was always positive, and that x increased from a to b. If $f(x)$ is negative, the sign of the integral is reversed; and if x decreases from a to b, dx is negative, and again the sign of the integral is reversed.

In other words, if the generating ordinate is negative or moves in the negative direction, $\int_a^b f(x)\,dx = \int_a^b y\,dx$ has a negative value and $-\int_a^b y\,dx$ or $\int_b^a y\,dx$ gives the positive area. Of course, if both of these happen at once, $\int_a^b y\,dx$ remains positive.

To avoid trouble in practice, it is advisable in any problem in which such questions may arise to divide the area into separate parts so that y shall have one sign throughout each part, and then take the sum of the positive values of the separate areas.

Of course, all these remarks apply to $\int x\,dy$ as well.

128. *Examples.*

1. Find the total area of the ellipse in four ways, using the single equation and the two parametric equations, and in each case taking the element parallel to the x-axis and also parallel to the y-axis. Ans. πab.

2. Trace the curve $x(a^2+y^2)=a^3$, and find the whole area between the curve and its asymptote. Ans. πa^2.

3. Trace the curve $x^2y^2(x^2-a^2)=a^6$, and find the area between the curve and the asymptote, $x=a$. Ans. πa^2.

4. Find the whole area between the curve $y(a^2+x^2)=a^3$ and its asymptote. Ans. πa^2.

5. What is the whole area between the curve $x(a^2+y^2)^2=a^5$ and its asymptote?

Ans. $\dfrac{\pi a^2}{2}$.

6. Find the area between the witch $xy^2-2ay^2+4a^2x=0$ and its asymptote. Ans. $4\pi a^2$.

7. Find the area between one branch of the cycloid, $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, and the x-axis. Ans. $3\pi a^2$.

8. Find the area included between the cissoid $x^3-2ay^2+xy^2=0$ and its asymptote. Ans. $3\pi a^2$.

129. Volumes of Revolution.—The area bounded by a curve $y=f(x)$, two ordinates $x=a$ and $x=b$, and the axis of x is revolved about the axis of x, generating a surface. It is required to find the volume enclosed.

Let $P(x, y)$ be any point of the curve; then the volume generated by the area bounded by the curve, the axis, the ordinate $x=a$ and the ordinate of P is a function of x, the abscissa of P. Call this volume $F(x)$. To obtain its derivative, increase x by Δx; $F(x)$ is increased by the element of integration $\Delta F(x)$, the

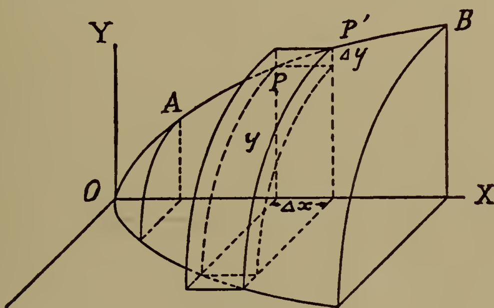


FIG. 47.

volume generated by the area bounded by the curve, the axis, and the ordinates of P and $P'(x+\Delta x, y+\Delta y)$. P and P' generate circles whose radii are y and $(y+\Delta y)$; the cylinder having the smaller circle as base and Δx as altitude is smaller than the volume $\Delta F(x)$, and the cylinder having the larger circle as base and the same altitude Δx is larger than $\Delta F(x)$; hence

$$\pi y^2 \Delta x < \Delta F(x) < \pi(y+\Delta y)^2 \Delta x,$$

$$\pi y^2 < \frac{\Delta F(x)}{\Delta x} < \pi(y+\Delta y)^2;$$

so that

$$\frac{dF(x)}{dx} = \left.\frac{\Delta F(x)}{\Delta x}\right]_{\Delta x=0} = \pi y^2,$$

or

$$dF(x) = \pi y^2 dx.$$

Since $F(a) = 0$, the required volume is therefore

$$V = F(b) = \int_a^b \pi y^2 dx.$$

The revolution about the axis of x of the area bounded by a curve $y = f(x)$, two ordinates $x = a$ and $x = b$, and the axis of x generates a solid of revolution having the volume

$$V = \pi \int_a^b y^2 dx.$$

In the same way, it can be proved that:

The revolution about the axis of y of the area bounded by a curve $x = f(y)$, two abscissas $y = a$ and $y = b$, and the axis of y generates a solid of revolution having the volume

$$V = \pi \int_a^b x^2 dy.$$

130. *Examples.*

1. Find the volume of the cone formed by revolving the line $y = \frac{b}{h} x$ about the x-axis, the altitude of the cone being h.

Ans. $\frac{\pi b^2 h}{3}$.

2. Find the volume of the sphere formed by revolving the circle $x^2 + y^2 = a^2$ about the x-axis, and also by revolving about the y-axis. Ans. $\frac{4}{3} \pi a^3$.

3. Find the volume required in example 2, using the parametric equations of the circle.

4. Find the volume of the ellipsoid formed by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a) about the x-axis, (b) about the y-axis.

Ans. $\frac{4}{3} \pi b^2 a$; $\frac{4}{3} \pi a^2 b$.

5. Find the volume formed by revolving about the x-axis the parabolic segment having the double ordinate $2b$ for its base and altitude a. Ans. $\frac{\pi b^2 a}{2}$.

6. Find the volume of the hour-glass-shaped figure formed by revolving about the y-axis the area enclosed between the parabola of example 5 and the lines $y=b$ and $y=-b$.

Ans. $\frac{2\pi a^2 b}{5}$.

7. Find the volume formed by revolving the witch

$$y^2x+a^2x-a^3=0$$

about its asymptote. Ans. $\frac{\pi^2 a^3}{2}$.

8. Find the volume produced by revolving about the x-axis the segment of the cissoid $y^2(2a-x)=x^3$ cut off by $x=a$.

Ans. $8a^3\pi(\log 2-\frac{2}{3})=0.6655a^3$.

9. Find the volume formed by revolving about the x-axis the part of the curve $y=e^x$ lying to the left of the origin.

Ans. $\frac{\pi}{2}$.

10. Find the volume of a capstan $2b$ in height, the curved surface of which is formed by the revolution about the y-axis of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Ans. $\frac{8}{3}\pi a^2 b$.

131. Further Methods of Integration: Integration by Parts.— The formula $\int udv=uv-\int vdu$ (20, Art. 114) is of great service in the integration of transcendental functions.

For instance, required $\int x^2 \log x dx$:

Let $\log x=u$, and $x^2dx=dv$; then

$$du=\frac{dx}{x}, \quad v=\int x^2dx=\frac{x^3}{3};$$

therefore,

$$\int [\log x]\,[x^2dx]=\int udv=uv-\int vdu$$

$$=\tfrac{1}{3}x^3\log x-\int \tfrac{1}{3}x^3\frac{dx}{x}=\frac{x^3}{3}(\log x-\tfrac{1}{3}).$$

Again, required $\int \sin^{-1} x \cdot dx$:

Let $\sin^{-1} x = u$, and $dx = dv$; then

$$du = \frac{dx}{\sqrt{1-x^2}}, \quad \text{and} \quad v = x;$$

therefore

$$\begin{aligned}\int \sin^{-1} x \cdot dx &= x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \tfrac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(1-x^2) \\ &= x \sin^{-1} x + \sqrt{1-x^2}.\end{aligned}$$

132. *Examples.*

Deduce by means of this formula the values of:

1. $\int \sin^2 \theta \, d\theta$, $\int \cos^2 \theta \, d\theta$, $\int \sin \theta \cos \theta \, d\theta$.
2. $\int x \sin x dx = \sin x - x \cos x$.
3. $\int \tan^{-1} x dx = x \tan^{-1} x - \log \sqrt{1+x^2}$.
4. $\int x \sin^{-1} x dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4}(\sin^{-1} x - x\sqrt{1-x^2})$.
5. $\int x^n \log x dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right)$.
6. $\int \cot \theta \cos \theta \, d\theta = \cos \theta + \log \tan \frac{\theta}{2}$.

133. Trigonometric Functions. $\int \sin m\theta \cos n\theta \, d\theta$, etc.— These integrals are important in problems of Mathematical Physics. $\int \sin m\theta \cos n\theta \, d\theta$ is readily evaluated if we note that $\sin (m+n)\theta + \sin (m-n)\theta = 2 \sin m\theta \cos n\theta$. For instance, to find $I = \int \sin 3\theta \cos 2\theta \, d\theta$. Since

$$\sin (3\theta + 2\theta) = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta$$

and

$$\sin (3\theta - 2\theta) = \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta,$$

the sum gives

$$\sin 5\theta + \sin \theta = 2 \sin 3\theta \cos 2\theta.$$

Thus

$$I = \tfrac{1}{2}\int \sin 5\theta \, d\theta + \tfrac{1}{2}\int \sin \theta \, d\theta = -\tfrac{1}{10} \cos 5\theta - \tfrac{1}{2} \cos \theta.$$

$\int \sin m\theta \sin n\theta \, d\theta$ and $\int \cos m\theta \cos n\theta \, d\theta$ are treated in the same way.

134. Integrals Containing Powers of the Trigonometric Functions.—We have in Art. 114 the integrals of the first powers of all the trigonometric functions, for

$$\int \csc \theta \, d\theta = \log \tan \tfrac{\theta}{2} = \log(\csc \theta - \cot \theta),$$

and, since $\csc(\frac{\pi}{2}+\theta) = \sec \theta$ and $d(\frac{\pi}{2}+\theta) = d\theta$,

$$\int \sec \theta \, d\theta = \log \tan(\tfrac{\pi}{4} + \tfrac{\theta}{2}) = \log(\sec \theta + \tan \theta).$$

We can integrate any positive odd power of sine or cosine, any positive integral power of tangent or cotangent, and any positive even power of secant or cosecant by using the formulas connecting the squares of the trigonometric functions:

$$\sin^2 + \cos^2 = 1, \quad \sec^2 - \tan^2 = 1, \quad \csc^2 - \cot^2 = 1.$$

In the following discussion, n represents a positive whole number, so that $2n+1$ is an odd number, $2n$ an even number.

$$\int \sin^{2n+1} \theta \cdot d\theta = \int \sin^{2n} \theta \sin \theta \, d\theta = \int (1 - \cos^2 \theta)^n \sin \theta \, d\theta,$$

which can now be expanded and integrated.

In the same way,

$$\int \cos^{2n+1} \theta \, d\theta = \int (1 - \sin^2 \theta)^n \cos \theta \, d\theta.$$

$$\begin{aligned}\int \tan^n \theta \, d\theta &= \int \tan^{n-2} \theta (\sec^2 \theta - 1) \, d\theta \\ &= \int \tan^{n-2} \theta \, d(\tan \theta) - \int \tan^{n-2} \theta \, d\theta \\ &= \frac{\tan^{n-1} \theta}{n-1} - \int \tan^{n-2} \theta \, d\theta.\end{aligned}$$

Reducing $\int \tan^{n-2} \theta \, d\theta$ in the same way, and so keeping on, until we reach either

$$\int \tan^0 \theta \, d\theta = \int d\theta = \theta \quad \text{or} \quad \int \tan \theta \, d\theta = \log \sec \theta,$$

the integral is completed. In the same way,

$$\begin{aligned}\int \cot^n \theta \, d\theta &= -\int \cot^{n-2} \theta \, d(\cot \theta) - \int \cot^{n-2} \theta \, d\theta \\ &= -\frac{\cot^{n-1} \theta}{n-1} - \int \cot^{n-2} \theta \, d\theta.\end{aligned}$$

$$\int \sec^{2n}\theta\, d\theta = \int (1+\tan^2\theta)^{n-1}\sec^2\theta\, d\theta.$$

$(1+\tan^2\theta)^{n-1}$ may be expanded and the powers of $\tan\theta$ integrated by the method just described.

135. We can integrate any even power of sine or cosine by using the formulas

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta); \quad \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta).$$

$$\int \cos^{2n}\theta\, d\theta = \frac{1}{2^n}\int (1+\cos 2\theta)^n d\theta = \frac{1}{2^{n+1}}\int (1+\cos\phi)^n d\phi,$$

where $\phi = 2\theta$. The even powers of $\cos\phi$, got by expanding $(1+\cos\phi)^n$, may be treated again in the same way, and the odd powers may be treated by the method above.

$$\int \sin^{2n}\theta\, d\theta = \frac{1}{2^{n+1}}\int (1-\cos\phi)^n d\phi,$$

where $\phi = 2\theta$, is similar.

136. *Examples.*

1. $\int_0^{\frac{\pi}{4}} \tan^3\theta\, d\theta = \frac{1}{2} - \frac{1}{2}\log 2.$

2. $\int_0^{\frac{\pi}{4}} \sec^4\theta\, d\theta = \frac{4}{3}.$

3. $\int_0^{\frac{\pi}{3}} \sec^4 x \tan x dx = \frac{15}{4}.$

4. $\int_0^{\frac{\pi}{3}} \tan^5 x dx = \frac{3}{4} + \log 2.$

5. $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^3\theta\, d\theta = \frac{2}{15}.$

6. $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^3\theta\, d\theta = \frac{2}{35}.$

7. $\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x dx = \frac{8}{15}.$

8. $\int_0^{\frac{\pi}{4}} \tan^4\theta\, d\theta = \frac{\pi}{4} - \frac{2}{3}.$

9. $\int \sin 2\theta \sin\theta\, d\theta = \frac{1}{2}\sin\theta - \frac{1}{6}\sin 3\theta.$

10. $\int \cos 2\theta \cos 4\theta = \frac{1}{4}\sin 2\theta + \frac{1}{12}\sin 6\theta.$

11. $\int_0^{\frac{\pi}{4}} \cos^4\theta\, d\theta = \dfrac{3\pi+8}{32}.$

12. Find the area of the segment of the curve $x^2(a^2-y^2) = a^4$ cut off by $x = a\sqrt{2}$. Ans. $2a^2(1-\frac{\pi}{4})$.

13. Find the area enclosed by $y(a^2-x^2)=a^3$, $y=0$, $x=-\frac{a}{2}$, and $x=\frac{a}{2}$. Ans. $a^2 \log 3 = 1.1a^2$, about.

137. The only powers of trigonometric functions left are odd powers of secant and cosecant. These can be integrated by using the relations $\sec^2-\tan^2=1$ (or $\csc^2-\cot^2=1$) and integration by parts. The integration of $\sec^3\theta$ follows:

$$I=\int\sec^3\theta=\int\sec\theta\cdot\sec^2\theta\,d\theta,$$

and in $\int udv=uv-\int vdu$, if

$$u=\sec\theta,\quad dv=\sec^2\theta\,d\theta,$$

we have

$$du=\sec\theta\tan\theta\,d\theta,\quad v=\tan\theta,$$

and

$$I=\sec\theta\tan\theta-\int\sec\theta\tan^2\theta\,d\theta;$$

and since $\tan^2\theta=\sec^2\theta-1$,

$$I=\sec\theta\tan\theta-\int\sec^3\theta\,d\theta+\int\sec\theta\,d\theta$$
$$=\sec\theta\tan\theta-I+\int\sec\theta\,d\theta;$$

so

$$2I=\sec\theta\tan\theta+\int\sec\theta\,d\theta=\sec\theta\tan\theta+\log(\sec\theta+\tan\theta),$$
$$I=\int\sec^3\theta\,d\theta=\tfrac{1}{2}[\sec\theta\tan\theta+\log(\sec\theta+\tan\theta)].$$

138. *Examples.*

Prove in this way:

1. $\int\csc^3\theta\,d\theta=\frac{1}{2}[-\csc\theta\cot\theta+\log(\csc\theta-\cot\theta)]$.
2. $\int\sec\theta\tan^2\theta\,d\theta=\frac{1}{2}[\sec\theta\tan\theta-\log(\sec\theta+\tan\theta)]$.
3. $\int\csc\theta\cot^2\theta\,d\theta=\frac{1}{2}[-\csc\theta\cot\theta-\log(\csc\theta-\cot\theta)]$.
4. Prove each of the last four formulas by beginning with the relation of the squared functions and following with integration by parts.
5. Show that $\int_0^a\sqrt{a^2+x^2}dx=\frac{a^2}{2}[\sqrt{2}+\log(1+\sqrt{2})]$.

6. Show that $\int \sec^{2n+1}\theta\, d\theta = \frac{1}{2n}\sec^{2n-1}\theta\cdot\tan\theta$

$$+\frac{2n-1}{2n}\int\sec^{2n-1}\theta\cdot d\theta.$$

7. Prove $\int\sec^5\theta\cdot d\theta = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}[\sec\theta\tan\theta$

$$+\log(\sec\theta+\tan\theta)].$$

8. Show that the segment cut from $y^2=2ax+x^2$ by $x=2a$ has for its area $a^2(6\sqrt{2}-\log(3+2\sqrt{2}))$.

139. The method of Art. 137 furnishes a convenient formula for certain definite integrals, for instance:

$$\int_0^{\frac{\pi}{2}}\sin^n\theta\, d\theta, \text{ and } \int_0^{\frac{\pi}{2}}\cos^n\theta\, d\theta,$$

where n is any positive integer greater than unity.

$$I=\int_0^{\frac{\pi}{2}}\sin^n\theta\cdot d\theta = \int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\cdot\sin\theta\cdot d\theta.$$

Call $u=\sin^{n-1}\theta$, $dv=\sin\theta\, d\theta$; then

$$du=(n-1)\sin^{n-2}\theta\cos\theta\, d\theta;\quad v=-\cos\theta.$$

$$I=\Big[-\sin^{n-1}\theta\cos\theta\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}\theta\cos^2\theta\, d\theta.$$

The bracket is zero, and $\cos^2\theta=1-\sin^2\theta$.

$$I=0+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\cdot d\theta-(n-1)\int_0^{\frac{\pi}{2}}\sin^n\theta\, d\theta;$$

$$I=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\cdot d\theta-(n-1)I;$$

$$n\cdot I=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\cdot d\theta;$$

$$I=\int_0^{\frac{\pi}{2}}\sin^n\theta\, d\theta=\frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta\, d\theta.$$

By this formula,

$$\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta = \frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4}\theta\, d\theta;$$

so

$$\int_0^{\frac{\pi}{2}} \sin^{n}\theta\, d\theta = \frac{(n-1)(n-3)}{n(n-2)}\int_0^{\frac{\pi}{2}} \sin^{n-4}\theta\, d\theta.$$

If this process is kept up, the last step will finally be:
If n is even,

$$\int_0^{\frac{\pi}{2}} \sin^{0}\theta\, d\theta = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.$$

If n is odd,

$$\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = 1.$$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^{n}\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots.1}{n(n-2)(n-4)\ldots.2} \times \frac{\pi}{2}, \text{ if } n \text{ is even};$$

$$\int_0^{\frac{\pi}{2}} \sin^{n}\theta\, d\theta = \frac{(n-1)(n-3)(n-5)\ldots.2}{n(n-2)(n-4)\ldots.3}, \text{ if } n \text{ is odd}.$$

$$\int_0^{\frac{\pi}{2}} \cos^{n}\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{n}\left(\tfrac{\pi}{2}-\theta\right) d\theta = -\int_{\frac{\pi}{2}}^{0} \sin^{n}\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sin^{n}\phi\, d\phi,$$

if $\frac{\pi}{2} - \theta = \phi$ and $d\theta = -d\phi$.

Hence

$$\int_0^{\frac{\pi}{2}} \cos^{n}\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{n}\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sin^{n}\theta\, d\theta.$$

These results should be memorized; they are very often useful.

140. *Examples.*

1. Evaluate the following definite integrals by the formulas of Art. 139.

(a) $\int_0^{\frac{\pi}{2}} \sin^4\theta = \frac{3\pi}{16}$, (b) $\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta = \frac{8}{15}$,

(c) $\int_0^{\pi} \sin^5\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin^5\theta\, d\theta$,

(d) $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$, (e) $\int_0^{\pi} \cos^5\theta\, d\theta$, (f) $\int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta$.

2. Find the volume generated by revolving the cycloid

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi)$$

about its base. [Hint, in y and dx, change $(1-\cos\phi)$ to $2\sin^2\frac{\phi}{2}$, and $\frac{\phi}{2}$ to θ.] Ans. $5\pi^2a^3$.

3. Find the volume formed by the revolution of the curve $xy^2 + 4a^2x = 8a^3$ about its asymptote. Ans. $4\pi^2a^3$.

141. Similar formulas may be developed for the integral of the product of a power of sine by a power of cosine.

$$I = \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^m\theta \cos^{n-1}\theta \cos\theta\, d\theta.$$

Call $\sin^m\theta \cos\theta\, d\theta = dv$, $\cos^{n-1}\theta = u$; then

$$v = \frac{\sin^{m+1}\theta}{m+1}, \quad du = -(n-1)\cos^{n-2}\theta \cdot \sin\theta\, d\theta.$$

$$I = \left\{ \left[\frac{\sin^{m+1}\theta \cos^{n-1}\theta}{m+1}\right]_0^{\frac{\pi}{2}} = 0 \right\} + \int_0^{\frac{\pi}{2}} \frac{\sin^{m+2}\theta}{m+1}(n-1)\cos^{n-2}\theta \cdot d\theta$$

$$= \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m+2}\theta \cos^{n-2}\theta\, d\theta.$$

By the same formula,

$$\int_0^{\frac{\pi}{2}} \sin^{m+2}\theta \cos^{n-2}\theta\, d\theta = \frac{n-3}{m+3}\int_0^{\frac{\pi}{2}} \sin^{m+4}\theta \cos^{n-4}\theta\, d\theta.$$

Continuing this process, we finally have:

If n is odd,

$$I=\frac{(n-1)(n-3)\ldots .2}{(m+1)(m+3)\ldots .(m+n-2)}\int_0^{\frac{\pi}{2}}\sin^{m+n-1}\theta\cos\theta\,d\theta;$$

where

$$\int_0^{\frac{\pi}{2}}\sin^{m+n-1}\theta\cos\theta\,d\theta=\left[\frac{\sin^{m+n}\theta}{m+n}\right]_0^{\frac{\pi}{2}}=\frac{1}{m+n};$$

and thus,

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta=\frac{(n-1)(n-3)\ldots .2}{(m+1)(m+3)\ldots .(m+n)}.$$

If n is even,

$$I=\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta$$

$$=\frac{(n-1)(n-3)(n-5)\ldots .1}{(m+1)(m+3)(m+5)\ldots .(m+n-1)}\int_0^{\frac{\pi}{2}}\sin^{m+n}\theta\cos^0\theta\,d\theta,$$

and two different cases arise, according as m is odd or even.

If m is odd, $(m+n)$ is also odd, and

$$I=\frac{(n-1)(n-3)\ldots .1}{(m+1)(m+3)\ldots .(m+n-1)}\times\frac{(m+n-1)(m+n-3)\ldots .2}{(m+n)(m+n-2)\ldots .3}.$$

If m is *even*, $(m+n)$ is also even, and

$$I=\frac{(n-1)(n-3)(n-5)\ldots .1}{(m+1)(m+3)(m+5)\ldots .(m+n-1)}\times\frac{(m+n-1)(m+n-3)\ldots .1}{(m+n)(m+n-2)\ldots .2}\times\frac{\pi}{2}.$$

In the last two formulas, $(m+1)(m+3)\ldots .(m+n-1)$ is the product of the highest of the factors in

$$(m+n-1)(m+n-3)\ldots .(1\text{ or }2);$$

hence the formulas are:

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta =$$

$$\frac{(n-1)(n-3)(n-5)\ldots.1\cdot(m-1)(m-3)(m-5)\ldots.1}{(m+n)(m+n-2)(m+n-4)\ldots.2} \times \frac{\pi}{2},$$

when m and n are both even;

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta =$$

$$\frac{(n-1)(n-3)(n-5)\ldots.(2 \text{ or } 1)\cdot(m-1)(m-3)(m-5)\ldots.(2 \text{ or } 1)}{(m+n)(m+n-2)(m+n-4)\ldots.(2 \text{ or } 1)},$$

when m and n are not both even.

These formulas should also be memorized; they contain the formulas for $\int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta$ and $\int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$ as special cases in which one of the two exponents in $\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta$ is zero.

142. *Examples.*

1. $\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta \, d\theta = \frac{2\cdot 3\cdot 1}{7\cdot 5\cdot 3} = \frac{2}{35} = \int_0^{\frac{\pi}{2}} \cos^3 \theta \sin^4 \theta \, d\theta.$

2. $\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^5 \theta \, d\theta = \frac{8}{105}.$

3. $\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^6 \theta \, d\theta = \frac{3\pi}{512}.$

4. $\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^5 \theta \, d\theta = \frac{1}{24}.$

5. $\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta \, d\theta = \frac{\pi}{16}.$

6. Find the area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$, using parametric equations. (See Art. 87.) Ans. $\frac{3}{8}\pi ab$.

7. Find by means of parametric equations the areas of the curves $a^8y^2=x^4(a^2-x^2)^3$ and $a^8y^2=x^8(a^2-x^2)$.

Ans. $\frac{\pi a^2}{8}$ for each.

8. Find the area between $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$ and the coördinate axes. Ans. $\frac{ab}{6}$.

143. The indefinite integral of the product of two powers of sine and cosine can always be got directly, unless both powers are even. For

$$\int \sin^{2m+1}\theta \cos^n\theta \, d\theta = -\int (1-\cos^2\theta)^m \cos^n\theta \, d\cos\theta$$

or

$$\int \cos^{2m+1}\theta \sin^n\theta \, d\theta = \int (1-\sin^2\theta)^m \sin^n\theta \, d\sin\theta.$$

When both powers are even the use of the double angle will always simplify the integral. For if $m<n$,

$$I=\int \sin^{2m}\theta \cos^{2n}\theta \, d\theta = \int (\sin^{2m}\theta \cos^{2m}\theta)\cos^{2(n-m)}\theta \, d\theta;$$

since

$$\sin\theta\cos\theta=\frac{\sin 2\theta}{2}, \quad \cos^2\theta=\frac{1+\cos 2\theta}{2},$$

$$I=\frac{1}{2^{2m+(n-m)+1}}\int \sin^{2m}(2\theta)\,(1+\cos 2\theta)^{n-m} d(2\theta)$$

$$=\frac{1}{2^{m+n+1}}\int \sin^{2m}\phi\,(1+\cos\phi)^{n-m}\,d\phi,$$

where $\phi=2\theta$.

A similar reduction gives, if $m>n$,

$$I=\int \sin^{2m}\theta\cos^{2n}\theta\,d\theta=\frac{1}{2^{m+n+1}}\int \sin^{2n}\phi\,(1-\cos\phi)^{m-n}\,d\phi,$$

where $\phi=2\theta$.

Whenever the powers are equal, it is well to note that

$$I = \int \sin^n \theta \cos^n \theta \, d\theta = \frac{1}{2^{n+1}} \int \sin^n \phi \, d\phi,$$

where $\phi = 2\theta$.

For instance,

$$I_1 = \int \sin^2 \theta \cos^3 \theta \, d\theta = \int \sin^2 \theta (1 - \sin^2 \theta) \, d(\sin \theta)$$

$$= \frac{\sin^3 \theta}{3} - \frac{\sin^5 \theta}{5}.$$

$$I_2 = \int \sin^2 \theta \cos^4 \theta \, d\theta = \int (\sin^2 \theta \cos^2 \theta)(\cos^2 \theta) \, d\theta$$

$$= \frac{1}{2^4} \int \sin^2 \phi (1 + \cos \phi) \, d\phi,$$

or

$$I_2 = \frac{1}{2^5}(2\theta - \sin 2\theta \cos 2\theta) + \frac{1}{2^4.3} \sin^3(2\theta).$$

$$I_3 = \int \sin^2 \theta \cos^2 \theta \, d\theta = \frac{1}{2^3} \int \sin^2 \phi \, d\phi$$

$$= \frac{1}{2^4}(2\theta - \sin 2\theta \cos 2\theta).$$

144. *Examples.*

1. Show $\int \sin^3 \theta \cos^3 \theta \, d\theta = \frac{1}{2^4}(\frac{1}{3} \cos^3 2\theta - \cos 2\theta)$.

2. Show $\int \sin^4 \theta \cos^4 \theta \, d\theta = \frac{1}{2^8}(6\theta - 2 \sin 4\theta + \frac{1}{4} \sin 8\theta)$.

3. Find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 \theta \sin^3 \theta \, d\theta$. Ans. $\frac{17}{480}$.

4. Find $\int_{\frac{\pi}{6}}^{\frac{5\pi}{3}} \sin^4 \theta \cos^4 \theta \, d\theta$. Ans. $\frac{9\pi}{256}$.

145. Quotients of powers of sine and cosine may be somewhat similarly handled, but are often best expressed in terms of secants, tangents, cosecants, cotangents.

For instance, taking advantage of the fact that

$$d \tan \theta = \sec^2 \theta \, d\theta, \quad d \cot \theta = -\csc^2 \theta \, d\theta,$$

$$\int \frac{\sin^3 \theta}{\cos^5 \theta} \, d\theta = \int \tan^3 \theta \sec^2 \theta \, d\theta = \int \tan^3 \theta \, d \tan \theta = \frac{\tan^4 \theta}{4}.$$

This method serves when the degree of the denominator is any even number greater than that of the numerator.

Taking advantage of the fact that

$$d \sec \theta = \sec \theta \tan \theta \, d\theta, \quad d \csc \theta = -\csc \theta \cot \theta \, d\theta,$$

we can handle any quotient of this sort having a numerator of odd degree, and a denominator of higher degree. For instance,

$$\int \frac{\sin^3 \theta}{\cos^4 \theta} \, d\theta = \int \tan^2 \theta \sec \theta \tan \theta \, d\theta$$

$$= \int (\sec^2 \theta - 1) \, d \sec \theta = \frac{\sec^3 \theta}{3} - \sec \theta.$$

$$\int \frac{\sin^3 \theta}{\cos^5 \theta} \, d\theta = \int \tan^2 \theta \sec \theta \tan \theta \sec \theta \, d\theta$$

$$= \int (\sec^3 \theta - \sec \theta) \, d \sec \theta = \frac{\sec^4 \theta}{4} - \frac{\sec^2 \theta}{2}.$$

These methods cover so far all cases where the denominator is the term of higher degree, except the one where the numerator is of even degree and the denominator of odd degree. These may be handled as follows:

$$\int \frac{\sin^2 \theta}{\cos^5 \theta} \, d\theta = \int \frac{\sin^2 \theta}{\cos^6 \theta} \cos \theta \, d\theta = \int \frac{x^2}{(1-x^2)^3} \, dx,$$

where $x = \sin \theta$. This fraction may be broken into partial fractions by the ordinary algebraic methods and the parts integrated.

Another way is to write the integral as

$$\int \tan^2 \theta \sec^3 \theta \, d\theta = \int (\sec^5 \theta - \sec^3 \theta) \, d\theta.$$

As any power of secant can be integrated, this method is also invariably feasible.

146. There remain the cases in which the numerator is of higher degree. The following examples show methods which can always be made to work:

$$\int \frac{\sin^5\theta}{\cos^2\theta}\,d\theta = \int \frac{\sin^4\theta}{\cos^2\theta}\sin\theta\,d\theta = -\int \frac{(1-\cos^2\theta)^2}{\cos^2\theta}\,d\cos\theta$$

$$= -\int \frac{(1-x^2)^2}{x^2}\,dx = -\int\left(\frac{1}{x^2}-2+x^2\right)dx,$$

where $x=\cos\theta$.

$$\int \frac{\sin^4\theta}{\cos^2\theta}\,d\theta = \int \frac{(1-\cos^2\theta)^2}{\cos^2\theta}\,d\theta = \int(\sec^2\theta - 2 + \cos^2\theta)\,d\theta.$$

$$\int \frac{\sin^4\theta}{\cos^3\theta}\,d\theta = \int \tan^3\theta\sin\theta\,d\theta = -\cos\theta\tan^3\theta + \int 3\tan^2\sec\theta\,d\theta$$

(integrating by parts).

147. *Examples.*

1. $\int \frac{\sin^2\theta}{\cos^3\theta}\,d\theta = \int(\sec^3\theta - \sec\theta)\,d\theta$

   $= \frac{1}{2}[\sec\theta\tan\theta - \log(\sec\theta + \tan\theta)].$

2. $\int \frac{\cos^3 x}{\sin^6 x}\,dx = \frac{\csc^3 x}{3} - \frac{\csc^5 x}{5}.$

3. $\int \frac{dx}{\sin x} = \int \frac{\sin x\,dx}{1-\cos^2 x} = \log(\csc x - \cot x).$

4. $\int \frac{d\theta}{\cos^3\theta} = \frac{1}{2}[\sec\theta\tan\theta + \log(\sec\theta + \tan\theta)].$

5. $\int \frac{d\theta}{\sin^4\theta} = -\cot\theta - \frac{\cot^3\theta}{3}.$

6. $\int \frac{\cos^5\theta}{\sin^3\theta}\,d\theta = \int \frac{\cos\theta(1-\sin^2\theta)^2}{\sin^3\theta}\,d\theta = \int(\cot\theta\,\text{cosec}^2\theta - 2\cot\theta +$

   $\cos\theta\sin\theta)\,d\theta.$

148. Integrals of the form $\int \frac{d\theta}{a+b\cos\theta}$ are readily handled by means of the functions of the half-angle; an example will show the method.

$$\int \frac{d\theta}{2+3\cos\theta}: \qquad \begin{aligned} 2 &= 2\cos^2\tfrac{\theta}{2} + 2\sin^2\tfrac{\theta}{2} \\ 3\cos\theta &= 3\cos^2\tfrac{\theta}{2} - 3\sin^2\tfrac{\theta}{2} \\ \hline 2+3\cos\theta &= 5\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} \end{aligned}$$

Let $\frac{\theta}{2} = \phi$; then

$$I = \int \frac{d\theta}{2+3\cos\theta} = \int \frac{2\,d\phi}{5\cos^2\phi - \sin^2\phi}$$
$$= \int \frac{2\sec^2\phi\, d\phi}{5-\tan^2\phi} = 2\int \frac{dx}{5-x^2},$$

where $x = \tan\phi = \tan\frac{\theta}{2}$.

$$I = \frac{2}{2\sqrt{5}} \log \frac{\sqrt{5}+x}{\sqrt{5}-x} = \frac{1}{\sqrt{5}} \log \frac{\sqrt{5}+\tan\frac{\theta}{2}}{\sqrt{5}-\tan\frac{\theta}{2}}$$

$$\int \frac{d\theta}{a+b\sin\theta} = \int \frac{d\phi}{a-b\cos\phi},$$

where $\phi = \frac{\pi}{2} + \theta$.

The only very important integrals of this type are

$$\int \frac{d\theta}{1+\cos\theta}, \quad \int \frac{d\theta}{1-\cos\theta}, \quad \int \frac{d\theta}{1+\sin\theta}, \quad \int \frac{d\theta}{1-\sin\theta},$$

all of which are directly integrable.

149. *Examples.*

Integrate the following:

1. $\int \frac{d\theta}{5+3\cos\theta} = \frac{1}{2}\tan^{-1}(\frac{1}{2}\tan\frac{\theta}{2})$.

2. $\int \frac{d\theta}{3+5\cos\theta} = \frac{1}{4}\log \frac{2+\tan\frac{\theta}{2}}{2-\tan\frac{\theta}{2}}$.

3. $\int \frac{d\theta}{1+\cos\theta} = \tan\frac{\theta}{2} = \csc\theta - \cot\theta$.

4. $\int \frac{d\theta}{1-\cos\theta} = -\cot\frac{\theta}{2} = -\csc\theta - \cot\theta$.

5. $\int \frac{d\theta}{1+\sin\theta} = -\cot(\frac{\pi}{4}+\frac{\theta}{2}) = \tan\theta - \sec\theta$.

6. $\int \frac{d\theta}{1-\sin\theta} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \tan\theta + \sec\theta.$

150. Representation of an Integral by an Area.—We have seen that the area bounded by a curve, $y=f(x)$, the two ordinates $x=a$ and $x=b$, and the x-axis is given by the definite integral $A=\int_a^b f(x)\cdot dx$. (See Arts. 124 and 127.)

Conversely, this area may be used as a graphic representation of the definite integral. The integral whose value is thus represented by an area may be itself an area, a volume, or any other of the many sorts of quantities that are computed by integration.

151. The Limit of a Certain Sort of Sum.—Quantities such as we have determined by integration can be found in another way. For instance,

To Find the Area Bounded by $y=\frac{x^2}{a}$, $y=0$, and $x=a$.—

Divide the area into n strips by equidistant ordinates $\Delta x = \frac{a}{n}$ apart. Call the distances from the origin to the points of division in OX:

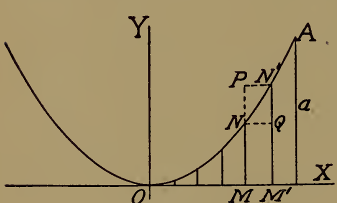


FIG. 48.

$$x_0=0,\ x_1=\Delta x,\ x_2=2\Delta x,\ \ldots\ldots,\ x_n=n\Delta x=a,$$

and the corresponding ordinates

$$y_0=0=\frac{x_0^{\,2}}{a},\ y_1=\frac{x_1^{\,2}}{a},\ y_2=\frac{x_2^{\,2}}{a},\ \ldots\ldots,\ y_n=\frac{x_n^{\,2}}{a}=a.$$

Inscribe a rectangle in each strip and circumscribe one about it; the sum of the inner rectangles is less than the required area; the sum of the outer ones is greater; and the two sums,

$$y_0\Delta x + y_1\Delta x + y_2\Delta x + \ldots\ldots + y_{n-1}\Delta x$$

and

$$y_1\Delta x + y_2\Delta x + \ \ldots\ldots\ + y_n\Delta x$$

differ by

$$(y_n - y_0)\Delta x.$$

As Δx approaches zero as a limit, this difference approaches the limit zero, and each of the sums therefore approaches the required area. The first sum is

$$0\cdot\Delta x + \frac{(\Delta x)^2}{a}\Delta x + \frac{(2\Delta x)^2}{a}\Delta x + \ \ldots\ldots\ + \frac{[(n-1)\Delta x]^2}{a}\Delta x$$
$$= \frac{(\Delta x)^3}{a}[0 + 1^2 + 2^2 + 3^2 + \ \ldots\ldots\ + (n-1)^2]$$
$$= \frac{\left(\frac{a}{n}\right)^3}{a}\left(\frac{n-1}{6}\right)(2n-1)(n) = a^2\frac{\left(1-\frac{1}{n}\right)}{6}\left(2-\frac{1}{n}\right)(1).$$

The second sum is

$$\frac{(\Delta x)^2}{a}\Delta x + \frac{(2\Delta x)^2}{a}\Delta x + \frac{(3\Delta x)^2}{a}\Delta x + \ \ldots\ldots\ + \frac{(n\Delta x)^2}{a}\Delta x$$
$$= \frac{(\Delta x)^3}{a}(1^2 + 2^2 + 3^2 + \ \ldots\ldots\ + n^2)$$
$$= \frac{\left(\frac{a}{n}\right)^3}{a}\left(\frac{n}{6}\right)(2n+1)(n+1) = a^2(\tfrac{1}{6})\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right).$$

The common limit approached by the sums as Δx approaches the limit zero and n consequently increases indefinitely is

$$a^2(\tfrac{1}{6})(2)(1) = \frac{a^2}{3},$$

the required area.

The same method can be used for volumes; indeed, except for the notation, this is the method already made familiar in the study of Solid Geometry, where it was used to find the volume of the pyramid, cone, and hemisphere. Any quantity that can be determined by a definite integral can be regarded as the limit of a sum of this sort. The advantage of the new conception of such problems lies in the saving of much preliminary discussion. The determination of the limit by algebraic means would be laborious and in many cases impracticable; but it is also unnecessary, for, as we shall see in the next article, the actual computation can be made by evaluating a definite integral precisely as in the earlier method.

152. The Definite Integral as the Limit of a Sum.—Any of the problems of which we have been speaking may be described as follows: A variable x is divided at the values x_0, x_1, x_2, x_3, ..., x_n, so that the quantity $(x_n - x_0)$ is divided into n parts, namely,

$$\Delta x_0 = x_1 - x_0, \quad \Delta x_1 = x_2 - x_1, \quad \Delta x_2 = x_3 - x_2, \; \ldots\ldots, \quad \Delta x_{n-1} = x_n - x_{n-1},$$

and either one of two sums is formed:

$$f(x_0)\Delta x_0 + f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \;\ldots\ldots\; + f(x_{n-1})\Delta x_{n-1}, \quad (1)$$

in which each term is the value of $f(x)$ at one of the points of division multiplied by the following Δx, or

$$f(x_1)\Delta x_0 + f(x_2)\Delta x_1 + f(x_3)\Delta x_2 + \;\ldots\ldots\; + f(x_n)\Delta x_{n-1}, \quad (2)$$

in which each term is the value of $f(x)$ at one of the points of division multiplied by the preceding Δx.

The number, n, of parts is then supposed to increase indefinitely, the sum of all the parts remaining equal to $(x_n - x_0)$ and the value of each part approaching zero as a limit. In any such case, each of the sums (1) and (2) will approach as its limit the value

$$\int_{x_0}^{x_n} f(x) \cdot dx.$$

To prove this, consider the graph of $f(x)$, and the area A bounded by the curve $y = f(x)$, the x-axis, and the two ordinates

$x=x_0$ and $x=x_n$. (Fig. 49.) Divide the part of the x-axis from x_0 to x_n into n parts corresponding to the values Δx_0, Δx_1, ..., Δx_{n-1}, and erect at the points of division the $(n+1)$ ordinates, of which the lengths are $f(x_0)$, $f(x_1)$, ..., $f(x_n)$. The area A is thus divided into n strips.

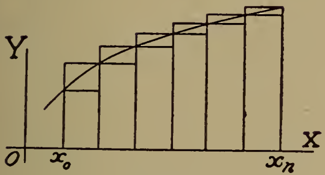


FIG. 49.

Each term of the sum (1) is the area of a rectangle inscribed in one of these strips, and the corresponding term of the sum (2) is the area of the rectangle circumscribed about the same strip. (Fig. 50.) The first sum is therefore the area of the polygon inscribed in A, and so is less than A; and the second sum is the area of the polygon circumscribed about A, and so is greater than A.

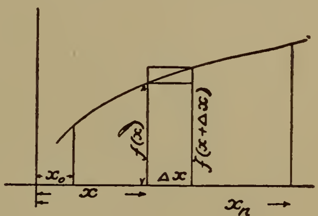


FIG. 50.

The difference between the two sums is the area of the small rectangles. If the divisions along the x-axis are all equal, this difference is

$$\Delta x_0[f(x_n)-f(x_0)].$$

If the divisions are unequal, let the largest of them be Δx_i; then this difference is less than

$$\Delta x_i[f(x_n)-f(x_0)].$$

In either case, as n is indefinitely increased, Δx_0 or Δx_i approaches zero as a limit, and the difference between the two sums therefore approaches zero as a limit. Consequently, as one of the sums is always greater than A, and the other always less, each of them approaches A as a limit. But the value of A is already known to be

$$A = \int_{x_0}^{x_n} f(x)\,dx.$$

Hence the limit of either the sum (1) or the sum (2), as n is indefinitely increased, is

$$\int_{x_0}^{x_n} f(x)\cdot dx.$$

153. As an illustration of the use of the definite integral regarded as the limit of a sum, consider the following problem It is required to find the volume produced by revolving about the axis of x the part of the parabola $\frac{y^2}{b^2} = \frac{x}{a}$ between $x = \frac{a}{4}$

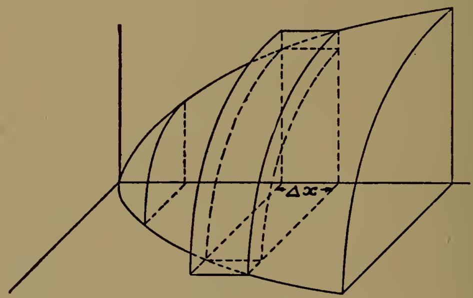


FIG. 51.

and $x=a$. Divide the volume into n slices by planes perpendicular to the axis, at the following distances from the origin:

$$\frac{a}{4} = x_0,\ x_1,\ x_2,\ \ldots\ldots,\ x_{n-1},\ x_n = a,$$

and let

$$x_1 - x_0 = \Delta x_0,\quad x_2 - x_1 = \Delta x_1,\quad \ldots\ldots,\quad x_n - x_{n-1} = \Delta x_{n-1}.$$

The corresponding ordinates of the parabola are

$$y_0 = \sqrt{\left(\frac{b^2}{a}x_0\right)},\quad y_1 = \sqrt{\left(\frac{b^2}{a}x_1\right)},\ \ldots\ldots,\quad y_n = \sqrt{\left(\frac{b^2}{a}x_n\right)}.$$

Inscribe in each slice a cylindrical disc; the sum of all these discs is

$$\pi y_0{}^2 \Delta x_0 + \pi y_1{}^2 \Delta x_1 + \ \ldots \ + \pi y^2{}_{n-1} \Delta x_{n-1}$$

or

$$\frac{\pi b^2}{a} x_0 \Delta x_0 + \frac{\pi b^2}{a} x_1 \Delta x_1 + \ \ldots \ + \frac{\pi b^2}{a} x_{n-1} \Delta x_{n-1}.$$

The limit of this sum, as n increases without limit and each division of the x-axis decreases without limit, is

$$\int_{x_0}^{x_n} \frac{\pi b^2}{a} x dx = \frac{\pi b^2}{a} \int_{a/4}^{a} x dx = \frac{\pi b^2}{a} \left. \frac{x^2}{2} \right]_{a/4}^{a} = \frac{15 \pi a b^2}{32}.$$

It is obvious that the limit of the sum of all the inscribed discs is the volume of revolution required; it is, however, logically conceivable that it may be something less than the required volume. But if we consider in the same way the discs circumscribed about the n slices, we have for the sum of their volumes:

$$\pi \frac{b^2}{a} x_1 \Delta x_0 + \frac{\pi b^2}{a} x_2 \Delta x_1 + \ \ldots \ + \frac{\pi b^2}{a} x_n \Delta x_n.$$

The limit of this sum is also

$$\int_{x_0}^{x_n} \frac{\pi b^2}{a} x dx = \frac{\pi b^2}{a} \int_{a/4}^{a} x dx = \frac{15 \pi a b^2}{32},$$

and is either the required volume or something greater; hence $\frac{15 \pi a b^2}{32}$ must be the required volume.

We shall not take the trouble in later applications to remove this logical doubt, as it never interferes with the clearness of the discussion, and can always be treated in the same way. Furthermore, we shall abbreviate the discussion by speaking of only a typical term of the sum whose limit we take. The preceding proof, thus abbreviated, is as follows:

Divide the volume into slices by planes perpendicular to the x-axis, Δx apart, and in each slice inscribe a cylindrical disc, the volume of which is

$$\pi y^2 \Delta x = \frac{\pi b^2}{a} x \Delta x.$$

The sum of all these discs is an approximation to the volume required, and its limit,

$$\int_{a/4}^{a} \frac{\pi b^2}{a} x dx,$$

is exactly the volume. Hence the volume is

$$V = \int_{a/4}^{a} \frac{\pi b^2}{a} x dx = \frac{15 \pi a b^2}{32}.$$

154. The typical term of the sum whose limit is the definite integral is precisely the *element of integration* of which we spoke in using the earlier method of finding areas and volumes. In this later method, the parts into which the approximate area or volume is divided, the values of which are the terms of the sum, are all called *elements of integration.*

dx, dx_0, dx_1, dx_2, etc., are frequently written in place of Δx, Δx_0, Δx_1, etc., for the infinitesimal factor of the element of integration. When all these are equal, and each is represented by dx, the typical term of the smaller sum is $f(x) \cdot dx$ and the corresponding term of the larger sum is $f(x+dx) \cdot dx$. It will be seen by what has just been done in the preceding example that in order to be sure that the proof of any such problem can be completed rigorously, it is merely necessary that the true value of any one of the parts into which we have divided the area or volume that we are computing shall be intermediate in value between the corresponding elements, $f(x) dx$ and $f(x+dx) dx$.

155. The conception of the definite integral as the limit of a sum simplifies the consideration of certain general principles of integration, notably the connection between the integrals:

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta, \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta, \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta,$$

and the integrals of the same functions when the limits are whole multiples of $\frac{\pi}{2}$.

An example or two will show the method. Consider

$$\int_{-\frac{\pi}{2}}^{\pi} \sin^3\theta \cos^4\theta \, d\theta.$$

It represents the limit of a sum. The function

$$\sin^3\theta\cos^4\theta$$

passes through a certain set of values as θ varies, the sign of each term of the sum depending on $\sin^3\theta$, since $\cos^4\theta$ is always positive. Each term, therefore, from $-\frac{\pi}{2}$ to 0, is negative, and the terms are repeated in reverse order with positive sign from 0 to $\frac{\pi}{2}$, the corresponding terms of the two quadrants thus cancelling each other. The quadrant $\frac{\pi}{2}$ to π remains, each term of which is positive, and equal to the corresponding term in the quadrant from 0 to $\frac{\pi}{2}$; hence

$$\int_{-\frac{\pi}{2}}^{\pi} \sin^3\theta\cos^4\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta\,d\theta = +\frac{2\cdot 3\cdot 1}{7\cdot 5\cdot 3} = +\frac{2}{35}.$$

By the same sort of reasoning

$$\int_0^{\pi} \sin^3\theta\cos^3\theta\,d\theta = 0,$$

as in the second quadrant $\cos\theta$ is negative; but

$$\int_0^{\frac{3\pi}{2}} \sin^4\theta\cos^4\theta\,d\theta = 3\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^4\theta\,d\theta$$
$$= 3\,\frac{3\cdot 1\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{9\pi}{256},$$

since the even powers are positive everywhere. It is necessary in such cases merely to observe the sign of $\sin^m\theta\cos^n\theta$ in each quadrant. Where this sign is negative, the integration through the quadrant gives

$$-\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta;$$

and where it is positive,

$$+\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta;$$

156. *Examples.*

1. Draw, very roughly, the graphs of

$$\sin^3 x \cos^4 x = y, \text{ from } -\tfrac{\pi}{2} \text{ to } \pi;$$
$$\sin^3 x \cos^3 x = y, \text{ from } 0 \text{ to } \pi;$$
$$\sin^2 x \cos^6 x = y, \text{ from } 0 \text{ to } \frac{3\pi}{2}.$$

Note how the relations just discussed are exhibited by the graphical representation of the integrals. (See Art. 150.)

Find the following integrals:

2. $\int_{-\pi}^{\frac{\pi}{2}} \sin^2 \theta \cos^3 \theta \, d\theta = \frac{2}{15}$. $\int_0^{\pi} \sin^2 \theta \cos^3 \theta \, d\theta = 0$.

3. $\int_0^{\pi} \sin^2 \theta \cos^4 \theta \, d\theta = \frac{\pi}{16}$.

4. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta \, d\theta = 0$; $\int_0^{\frac{3\pi}{2}} \sin^3 \theta \cos^2 \theta \, d\theta = \frac{2}{15}$.

5. $\int_{\pi}^{\frac{3\pi}{2}} \sin^3 \theta \cos^3 \theta \, d\theta = \frac{1}{12}$; $\int_{\frac{n\pi}{2}}^{(n+1)\frac{\pi}{2}} \sin^3 \theta \cos^3 \theta \, d\theta = \pm\frac{1}{12}$.

157. Areas with Curvilinear Boundaries.—The following problems are merely an extension of the ordinary problem of finding areas.

I. To Find the Area of the Ellipse $4y^2 - 4xy + 17x^2 + 12y - 86x + 73 = 0$.—Solving the equation for y, we get

$$y = \tfrac{1}{2}(x-3) \pm 2\sqrt{-x^2+5x-4}.$$

The curve is sketched in Fig. 52. Divide the area into vertical strips Δx apart; the height of the strip corresponding to any value of x is the sum of the two equal distances of the two corresponding points of the curve from the diameter $y=\frac{1}{2}(x-3)$, or is

$$2\times 2\sqrt{-x^2+5x-4}.$$

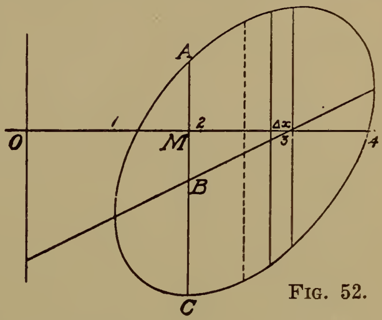


FIG. 52.

We might get at this as follows: The area is bounded above by the graph of the single-valued function

$$y_1=\tfrac{1}{2}(x-3)+2\sqrt{-x^2+5x-4},$$

and below by the graph of

$$y_2=\tfrac{1}{2}(x-3)-2\sqrt{-x^2+5x-4}.$$

The height of a strip is the algebraic difference of corresponding ordinates of the two graphs, or

$$y_1-y_2=4\sqrt{-x^2+5x-4}.$$

The limits of integration with respect to x are the least and greatest values of x corresponding to points within the ellipse; as $\sqrt{-x^2+5x-4}=\sqrt{(x-1)(4-x)}$, these values are 1 and 4.

The area of the strip corresponding to any value of x is $4\sqrt{-x^2+5x-4}\cdot\Delta x$, approximately, and the required area is exactly

$$A=\int_1^4 4\sqrt{-x^2+5x-4}\cdot dx.$$

Since

$$\sqrt{-x^2+5x-4}=\sqrt{(\tfrac{3}{2})^2-(x-\tfrac{5}{2})^2},$$

let

$$x-\tfrac{5}{2}=\tfrac{3}{2}\sin\theta.$$

Then

$$dx=\tfrac{3}{2}\cos\theta\,d\theta,\quad \sqrt{(\tfrac{3}{2})^2-(x-\tfrac{5}{2})^2}=\tfrac{3}{2}\cos\theta;$$

when

$$x=4,\quad \theta=\tfrac{\pi}{2},$$

when

$$x=1,\quad \theta=-\tfrac{\pi}{2}.$$

Therefore

$$A=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{3}{2}\cos\theta\cdot\tfrac{3}{2}\cos\theta\,d\theta=9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{9\pi}{2}.$$

II. To Find the Areas into which $x^2+y^2=a^2$ is Divided by $x^4-y^2(a^2-x^2)=0$, and the Area Between the Latter Curve and one of its Asymptotes.—The curves are shown in Fig. 53; they meet where $\sqrt{a^2-x^2}=\dfrac{x^2}{\sqrt{a^2-x^2}}$ or where $x=\pm\dfrac{a}{\sqrt{2}}$, $y=\pm\dfrac{a}{\sqrt{2}}$.

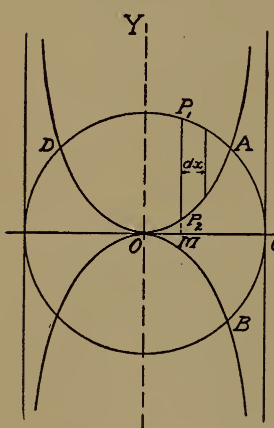


FIG. 53.

Of the parts into which the circle is divided, the easiest to get is AOD, as it is bounded by the graphs of only two single-valued functions,

$$y_1=\sqrt{a^2-x^2}\text{ above,}$$

and

$$y_2=\frac{x^2}{\sqrt{a^2-x^2}}\text{ below.}$$

Divide the area AOD in the usual way into strips dx wide; the height of the strip corresponding to any value of x is

$$P_1M-P_2M=y_1-y_2=\sqrt{a^2-x^2}-\frac{x^2}{\sqrt{a^2-x^2}}=\frac{a^2-2x^2}{\sqrt{a^2-x^2}},$$

and its area is approximately $\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\,dx$; the area required is exactly

$$AOD=\int_{-\frac{a}{\sqrt{2}}}^{\frac{a}{\sqrt{2}}}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}\,dx.$$

$$\left.\begin{array}{l}\text{Let } x=a\sin\theta\\ \quad dx=a\cos\theta\,d\theta\end{array}\right\}\begin{array}{l}\sqrt{a^2-x^2}=a\cos\theta\\ a^2-2x^2=a^2(1-2\sin^2\theta)\end{array}\Bigg\}$$

When $x=-\frac{a}{\sqrt{2}},\ \theta=-\frac{\pi}{4}$.

When $x=\frac{a}{\sqrt{2}},\ \theta=\frac{\pi}{4}$.

$$AOD=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{a^2(1-2\sin^2\theta)}{a\cos\theta}\,a\cos\theta\,d\theta$$

$$=a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta\,d\theta=a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\phi\,\tfrac{1}{2}d\phi=\left[\frac{a^2}{2}\sin\phi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=a^2.$$

It is evident from symmetry that the area $ACBO$ is $\frac{\pi a^2}{2}-a^2$ $=0.5708a^2$, about. The area between the curve $x^4-y^2(a^2-x^2)=0$ and the asymptote $x=a$ is obtained by summing strips $\frac{2x^2}{\sqrt{a^2-x^2}}$ high, and is

$$A_2=\int_0^a\frac{2x^2}{\sqrt{a^2-x^2}}\,dx=\int_0^{\frac{\pi}{2}}2a^2\sin^2\theta\,d\theta=\frac{\pi a^2}{2}.$$

158. *Examples.*

1. Trace the ellipse $y=\frac{2}{3}x-3\pm\frac{1}{4}\sqrt{15+2x-x^2}$, and find its area. Ans. $A=4\pi$.

2. Trace the curve $y=\frac{1}{2}x-4\pm\frac{2}{3}\sqrt{-x^2+20x-91}$ and find the area enclosed by the curve. Ans. $A=6\pi$.

3. Find the area between $ay^2 = x^3$ and $y = x$. Ans. $\frac{a^2}{10}$.

4. Find the area between $y^2 = 3ax$ and $x^2 = 3ay$. Ans. $3a^2$.

5. Find the area between $x^2(a^2 - y^2) = a^4$ and $x^2 = 4y^2$.
Ans. $a^2(\pi - 2)$.

6. Find the area between $x^2 + y^2 = a^2$ and $2y^2 = 3ax$.
Ans. $\frac{a^2}{12}(4\pi + \sqrt{3})$.

CHAPTER VI.

Space Coördinates.

159. Space Coördinates.—Although no analytic treatment of the geometry of three dimensions is to be attempted in this book, some of the notation and ideas of the subject will be useful in our later work.

160. Rectangular Coördinates.—A point, P, may be located in space by three coördinates, measured as follows: Three

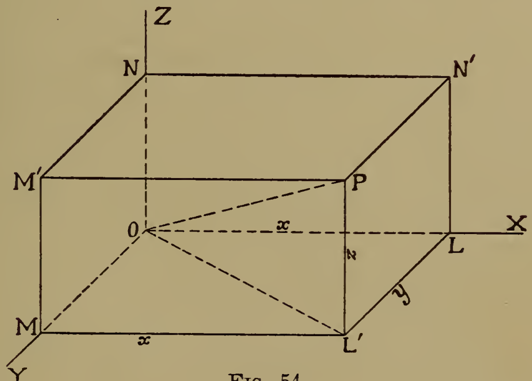


Fig. 54.

straight lines, OX, OY, and OZ, are given, each perpendicular to the other two at their common intersection, O. These are called the *coördinate axes*, and the three planes determined by them, each of which is perpendicular to the others, are called the coördinate planes. The distances $M'P = x$, $N'P = y$, $L'P = z$ of P from the three coördinate planes, are the *coördinates* which determine the position of P. (See Fig. 54.)

The coördinates are measured from the planes to the point, and are positive when directed to the *right, forward,* or *upward,* and negative in the opposite directions.

161. Direction Cosines.—The position of any point, P, in space may be fixed by giving its distance ρ from the origin and the angles α, β, γ made with the axes of x, y, z by the line OP. (See Fig. 54.) It is evident from the figure that the relations between the rectangular coördinates (x, y, z) of P and the coördinates $(\rho, \alpha, \beta, \gamma)$ are

$$x = OP \cos POL = \rho \cos \alpha,$$
$$y = OP \cos POM = \rho \cos \beta,$$
$$z = OP \cos PON = \rho \cos \gamma.$$

Since

$$\rho^2 = x^2 + y^2 + z^2 = \rho^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma),$$

it is evident that the relation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

always holds among the coördinates α, β, γ of any point, so that $(\rho, \alpha, \beta, \gamma)$ amount to but three coördinates.

162. Cylindrical Coördinates.—The location of the point P may be described by saying that it is at the distance z from the x-y plane, and directly over the point whose rectangular coördinates in that plane are (x, y). The point P may also be located by giving its distance z from the plane XOY and the polar coördinates (r, θ) of L', the foot of the perpendicular PL' from P to XOY. (See Fig. 55.)

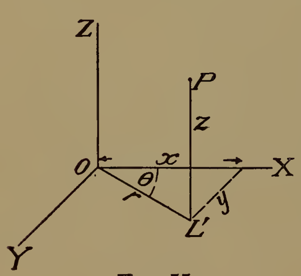


FIG. 55.

163. Spherical Coördinates.—Any point, P, may be located, as in Fig. 56, by giving its distance ρ from the origin O, the angle ϕ made by $OP=\rho$ with the axis OZ, and the diedral angle θ made by the plane ZOP with the plane ZOX. This is equivalent to giving the radius ρ of the sphere centered at the origin and passing through P, and then locating P on the sphere by giving two surface-coördinates ϕ and θ, analogous to the colatitude and longitude by which a location is fixed on the surface of the earth.

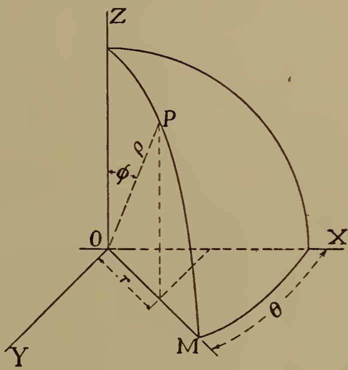


FIG. 56.

The relations between rectangular and spherical coördinates are evidently

$$x=r\cos\theta=\rho\sin\phi\cos\theta,$$
$$y=r\sin\theta=\rho\sin\phi\sin\theta,$$
$$z=\rho\cos\phi.$$

164. Equations in Three Dimensions.—If no restriction is put upon the values of its coördinates, the point (x, y, z) may, of course, occupy any position in space whatever; if it is given that $x=a$, the point is clearly constrained to move in a plane parallel to YOZ, at the distance a from it; if in addition it is given that $y=b$, the point is further restricted to a line of this plane—the line parallel to OZ, at a distance $\sqrt{a^2+b^2}$ from it; if, finally, it is given that $z=c$, the position of the point is definitely fixed.

In any system of space-coördinates, a single relation among the coördinates restricts the point to some surface, two relations to a curve, the intersection of two surfaces; and three relations fix it at the common intersection of three surfaces. The locus of a single equation in space coördinates is thus a surface, and the locus of a pair of equations is a curve.

The following instances of surfaces are evident from the definitions and fundamental theorems of elementary geometry.

$x=y$ is the plane bisecting the diedral $X-OZ-Y$.

$x^2+y^2+z^2=a^2$ or $\rho=a$ is the sphere with its center at O and the radius a.

$y=mx$ is a plane through OZ, making the angle $\tan^{-1} m$ with the plane XOZ, etc.

165. Cylindrical Surfaces.—An equation in rectangular coördinates, which lacks one of the coördinates, represents a right

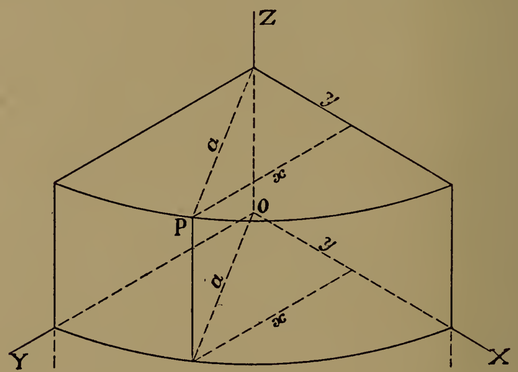


FIG. 57.

cylindrical surface of which the elements are parallel to the axis corresponding to the missing coördinate. For suppose the equation is $f(x, y)=0$; it is satisfied by the points of the curve $f(x, y)=0$ in the $x-y$ plane, and if this curve, keeping parallel to its first position, moves in the direction of OZ, its x and y coördinates will not change, so the equation will still be satisfied. $f(x, y)=0$ is thus the equation of the cylindrical surface so generated. As an example, consider the cylinder $x^2+y^2=a^2$ in Fig. 57.

In the same way an equation $f(r, \theta)=0$ represents a cylindrical surface having elements parallel to OZ.

166. Analysis of Equations by Plane Sections.—Suppose we wish to learn the form of the surface represented by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

By making $z=0$, we see that its trace on the $(x\text{-}y)$ plane is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; similarly, its traces on the other co-

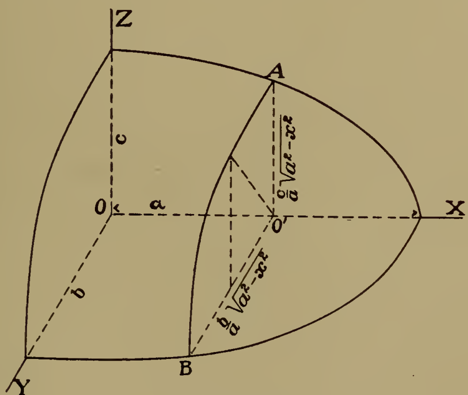


FIG. 58.

ördinate planes are ellipses. If we cut it by a plane $z=k$, the equation of the section, referred to axes parallel to OX and OY, is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{c^2-k^2}{c^2},$$

or

$$\frac{x^2}{\frac{a^2}{c^2}(c^2-k^2)}+\frac{y^2}{\frac{b^2}{c^2}(c^2-k^2)}=1.$$

The section is thus an ellipse with semi-axes

$$\frac{a}{c}\sqrt{c^2-k^2} \quad \text{and} \quad \frac{b}{c}\sqrt{c^2-k^2}.$$

As k is given a larger and larger value, beginning with zero, this ellipse clearly diminishes, vanishing when $k=c$, and thereafter being imaginary. The section is not affected by changing the sign of k.

Sections parallel to the other coördinate planes give similar results. The form of the solid, which is called an ellipsoid, is now evident. (See Fig. 58, which illustrates any section parallel to ZOY.)

Any equation can be analyzed by these same principles.

167. Surfaces of Revolution.—An equation in the form $f(r, z)=0$, where $r=\sqrt{x^2+y^2}$, represents a surface formed by revolving about the axis of z the trace of the surface on the x-z plane or the trace on the y-z plane. This is because the intersection of $f(r, z)=0$ and any plane $z=k$ perpendicular to the z-axis is given by $f(r, k)=0$, which when solved will give $r=$ a constant, the equation of a circle. Thus $x^2+y^2=az$ represents the surface formed by revolving about OZ either of the equal parabolas, $y^2=az$ in the y-z plane, or $x^2=az$ in the x-z plane.

In the same way, any equation in the form $f(r', x)=0$, where $r'=\sqrt{y^2+z^2}$ represents a surface of revolution about the x-axis, and $f(r'', y)=0$, where $r''=\sqrt{z^2+x^2}$, represents a surface of revolution about the y-axis.

168. Projections of Space Curves.—The orthogonal projection of a space curve upon a plane is the intersection with the plane of a cylindrical surface containing the curve and having its elements perpendicular to the plane. The equation of the projection upon one of the coördinate planes is obtained by eliminating from the two equations of the curve the corresponding coördinate.

For instance, the two cylinders

$$x^2+y^2=a^2 \tag{1}$$

and

$$y^2+z^2=b^2 \tag{2}$$

(the first of which has OZ, for its axis, and the second OX) determine by their intersection a space curve. The projections of this curve upon the planes of x-y and y-z are clearly the traces of the two given cylinders. If we subtract so as to eliminate y^2, the resulting equation

$$x^2-z^2=a^2-b^2 \tag{3}$$

represents a third cylinder, with OY for its axis. This cylinder contains the curve in question, because the coördinates of any point of the curve satisfy the original pair of equations (1) and (2) from which (3) is derived and so satisfy (3). Hence the trace of (3) on the x-z plane is the projection on this plane of the curve. This projection is a pair of straight lines if $a=b$, otherwise a rectangular hyperbola.

169. *Examples.*

Discuss and sketch the surfaces represented by the following equations:

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ Elliptic cylinder,

2. $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1.$ Spheroid.

3. $\frac{x^2+y^2}{a^2}-\frac{z^2}{b^2}=1.$ Hyperboloid of revolution of one sheet.

4. $\frac{x^2+y^2}{a^2}-\frac{z^2}{b^2}=0.$ Asymptotic cone of (3).

5. $\dfrac{x^2}{a^2} - \dfrac{y^2+z^2}{b^2} = 1.$ Hyperboloid of revolution of two sheets.

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$ Hyperboloid of one sheet.

7. $\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = \dfrac{x}{a}.$ Paraboloid (elliptic).

8. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^4}{c^4} = 1.$ Bull-headed ellipsoid.

Find the projections on the coördinate planes of the following curves:

9. $\left\{\begin{array}{l} x^2+y^2+z^2=4a^2 \\ x^2+y^2=2ax \end{array}\right\}$ Ans. $z^2+2ax=4a^2$ on ZOX, $z^4-4a^2(z^2-y^2)=0$ on ZOY.

10. $\left\{\begin{array}{l} x^2+y^2=az \\ x^2+y^2=2ax \end{array}\right\}$ Ans. $z=2x$ on ZOX, $z^2+4y^2=4az$ on ZOY.

11. $r=a\cos\theta$. Circular cylinder.

12. $z=r\cot\alpha$. Cone of revolution.

13. $az=h(a-r)$. Cone of revolution.

CHAPTER VII.

AREAS, VOLUMES, ARCS, AND SURFACES.

170. Areas.—The computation of areas by means of (x, y) coördinates has already been explained; the methods are collected here on account of their relation to what follows.

If the equation of a curve in rectangular coördinates is $y=f(x)$, y being written as an explicit function of x, the area $APQB$ in Fig. 59a is given by

$$\int_a^b ydx,$$

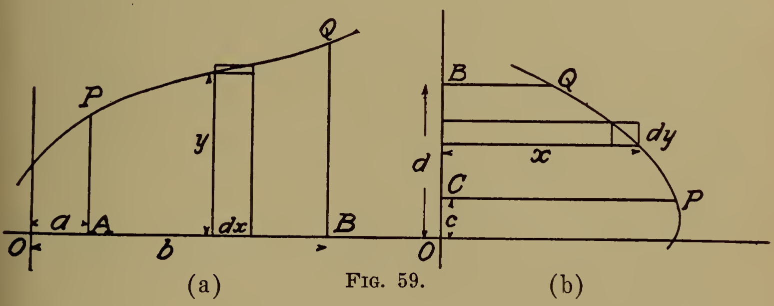


FIG. 59.

an abbreviation for the process of dividing the area into vertical strips dx wide and taking the limit of the sum of such strips from $x=a$ to $x=b$ as their number is indefinitely increased and dx approaches the limit zero.

If x is given as an explicit function of y, $x=\phi(y)$, the area $CPQB$ in Fig. 59b is given by

$$\int_c^d xdy,$$

an abbreviation for a similar process where horizontal strips are used as elements of integration.

An area bounded by $y_1=f_1(x)$, $y_2=f_2(x)$, $x=a$ and $x=b$ is given by

$$\int_a^b (y_1-y_2)\,dx,$$

the limit of the sum of vertical elementary divisions of the area. An area bounded by $x_1=\phi_1(y)$, $x_2=\phi_2(y)$, $y=c$ and $y=d$ is similarly

$$\int_c^d (x_1-x_2)\,dy.$$

When the bounding curve is given by parametric equations, these formulas still hold.

In any definite integral, $\int_p^q f(x)\cdot dx$, dx is positive if x increases from p to q, negative if x decreases from p to q; where $f(x)$ and dx have the same sign, the integration gives a positive result; where they have opposite signs, it gives a negative result. In most computations, as for instance in the computation of areas, we desire the limit of a sum of the positive values of certain elements; for such a purpose the integration should if necessary be done in pieces in each of which neither $f(x)$ nor dx changes sign, and the order of integration should be so chosen in each piece that $f(x)$ and dx have the same sign. (See Art 127).

Whatever the variable x in $\int_p^q f(x)\cdot dx$ may stand for, it may be used for the abscissa of a graph representing $y=f(x)$, and then the value of $\int_p^q f(x)\cdot dx$ will be represented by some area or sum of areas like that in Fig. 59a. (See Art. 150.)

171. *Examples.*

1. Find the area bounded by $y=\sin x$, $y=\cos x$, $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$. Ans. $2\sqrt{2}$.

2. The *strophoid* $x(x^2+y^2)+a(x^2-y^2)=0$ has a loop and a vertical asymptote. Find the area of the loop and the area between the curve and the asymptote, getting the parts below and above the axis of x separately.

Ans. $A_1=2a^2\left(1-\frac{\pi}{4}\right)$; $A_2=2a^2\left(1+\frac{\pi}{4}\right)$.

3. Find the areas bounded by $y=\sin x$ and $y=\cos 2x$.

Ans. $\frac{3}{2}\sqrt{3}$ and $\frac{3}{4}\sqrt{3}$.

4. Find the area between $y^2=4a(x+a)$ and $27ay^2=4(x-a)^3$.

Ans. $\frac{22}{15}a^2(16\sqrt{2}.)$

5. Find the area common to $a^8y^2=x^4(a^2-x^2)^3$ and $a^8y^2=x^8(a^2-x^2)$.

Ans. $\frac{a^2}{2}\left(\frac{\pi}{4}-\frac{1}{3}\right)$.

6. Find the area cut off from a loop of $x=a\phi-b\sin\phi$, $y=a-b\cos\phi$ by the x-axis. $(b>a.)$

Ans. $(2a^2+b^2)\cos^{-1}\frac{a}{b}-3a\sqrt{b^2-a^2}$.

172. Sectorial Areas by the $y=mx$ Method.—Parametric equations in terms of $m=\frac{y}{x}$ (see Art. 110) are used chiefly for computing sectorial areas bounded by the curve and two lines of given slope through the origin. For this purpose special formulas are better than those already given.

Let it be required to find the area of the sector bounded by the curve $x=f(m)$, $y=mf(m)$ and the lines $y=m_1x$ and $y=m_2x$ (Fig. 60.)

Divide the difference (m_2-m_1) into any number of parts, each equal to dm, and divide the sector into elementary sectors by the lines

$$y=(m_1+dm)x,\ y=(m_1+2dm)x,$$
$$y=(m_1+3dm)x,\ \ldots\ldots,$$
$$y=(m_2-2dm)x,\ y=(m_2-dm)x.$$

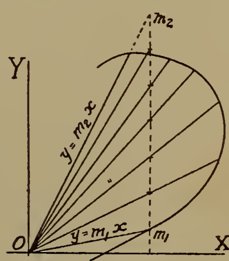


FIG. 60.

Consider any one of these radial lines, $y=mx$, reaching to the point $P(x, y)$, and the next one, $y=(m+dm)x$, reaching to the point $Q(x+\Delta x, y+\Delta y.)$ (Fig. 61.)

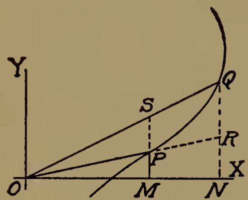


FIG. 61.

Draw MPS parallel to the y-axis.

The altitude of the triangle OPS is $OM=x$; the base is $PS=MS-MP$ $=(m+dm)x-mx=xdm$.

The area of OPS is therefore $\frac{1}{2}x^2dm$.

The sum of all the triangles inscribed in the sector, of which OPS is a type, is an approximation to the area of the sector, and the area is

$$A=\frac{1}{2}\int_{m_1}^{m_2} x^2dm.$$

(See Art. 154.)

In the application of this formula x^2 is written as a function of m. For the area of a loop having its double point at the origin, the bounding lines of the sector are the tangents at the origin between which the loop lies, and the sector is the whole loop.

The area bounded by two curves

$$x_1=f_1(m), \quad y_1=mf_1(m)$$

and

$$x_2=f_2(m), \quad y_2=mf_2(m)$$

and the lines $y=m_1x$ and $y=m_2x$ is similarly seen to be

$$A=\frac{1}{2}\int_{m_1}^{m_2} (x_1{}^2-x_2{}^2)\,dm.$$

The element of integration in this case is a trapezoid.

173. *Examples.*

1. Find the area of the circle $y^2=2ax-x^2$ by the method of Art. 172.

2. Find the area of the curve $y^2=x^2(1-x^2)$: (a) by the integral of ydx, (b) by the integral of $\frac{1}{2}x^2dm$, (c) by substitut-

ing $x=\cos\theta$ in (a) and changing limits, (d) by substituting $x=\cos\theta$ in the equation of the curve and thence finding parametric equations of the curve, and then integrating ydx expressed in terms of θ. Ans. $\frac{4}{3}$.

Find the following areas by the method of Art. 172:

3. The loop of $y^2=x^2+x^3$. Ans. $\frac{8}{15}$.

4. The curve $x^4+x^2y^2+y^2-3x^2=0$. Ans. $\frac{8\pi}{3}-2\sqrt{3}$.

5. The loop of $x^3+ay^2-axy=0$. Ans. $\frac{a^2}{60}$.

6. Between $x^3=a(x^2+y^2)$ and $x=2a$. Ans. $\frac{32}{15}a^2$.

7. The loop of the strophoid $x(x^2+y^2)+a(x^2-y^2)=0$.

Ans. $\frac{a^2}{2}(4-\pi)$.

8. Between the strophoid and its asymptote.

Ans. $\frac{a^2}{2}(4+\pi)$.

9. Sector of ellipse $x=a\cos\phi$, $y=b\sin\phi$ between two conjugate diameters. (See example 2, Art. 91.) Ans. $\frac{\pi ab}{4}$.

174. Areas by Polar Coördinates.—*Problem:* To find the area of the sector bounded by a curve $r=f(\theta)$, and the lines $\theta=\alpha$ and $\theta=\beta$. Divide the difference $(\beta-\alpha)$ into any number of parts, each $=d\theta$, and draw the radii vectores

$$\theta=\alpha+d\theta,\ \theta=\alpha+2d\theta,\ \ldots,\ \theta=\beta-d\theta$$

to meet the curve. (Fig. 62.)

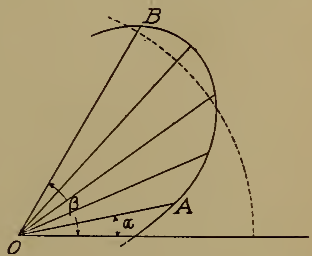


FIG. 62.

Let $P(r,\theta)$ be one of the points of division, $Q(r+dr,\theta+d\theta)$ the next (Fig. 63). Draw a circular arc with the origin as center cutting across the angle $d\theta$ from P.

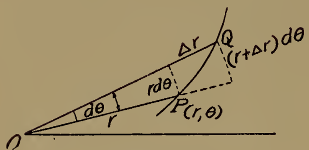


FIG. 63.

This arc is $rd\theta$ in length, and cuts off a circular sector $\frac{1}{2}r^2d\theta$ in area. The sum of all the circular sectors of which this one is a type is an approximation to the area required, and the required area is

$$A=\frac{1}{2}\int_\alpha^\beta r^2d\theta. \quad \text{(Art. 154.)}$$

The area bounded by two curves,

$$r_1=f_1(\theta) \quad \text{and} \quad r_2=f_2(\theta),$$

and the lines $\theta=\alpha$ and $\theta=\beta$ is in the same way

$$A=\frac{1}{2}\int_\alpha^\beta (r_1^2-r_2^2)\,d\theta,$$

the element of integration being in this case the difference between two circular sectors.

175. *Examples.*

1. Find the area of the lemniscate $r^2=a^2\cos 2\theta$. Ans. a^2.
2. Find the area of the cardioid $r=a(1-\cos\theta)$, or

$$r=2a\sin^2\tfrac{1}{2}\theta.$$

Ans. $\frac{3}{2}\pi a^2$.

3. Find the area of one loop of $r=a\sin 3\theta$. What is the total area? Ans. $\dfrac{\pi a^2}{12}, \dfrac{\pi a^2}{4}$.

4. Find the area between the curve and its asymptote, given $r=2a(\sec\theta-\cos\theta)$. Ans. $3\pi a^2$.

5. Find the total area of $r=a\cos\theta$, $r=a\cos 2\theta$, $r=a\cos n\theta$.

Ans. $\dfrac{\pi a^2}{4}$, if n is odd; $\dfrac{\pi a^2}{2}$, if n is even.

6. Find the area of each of the loops of the curve

$$r = a \cos\theta \cos 2\theta.$$

Ans. $0.3630\,a^2$ and $0.0148\,a^2$.

7. Find the areas of example 2, Art. 171, and examples 7 and 8, Art. 173, from the polar equation of the strophoid

$$r \cos\theta = a \cos 2\theta.$$

176. Volumes of Revolution.—If a curve is given by an equation that can be solved in the form $y=f(x)$ or in the form

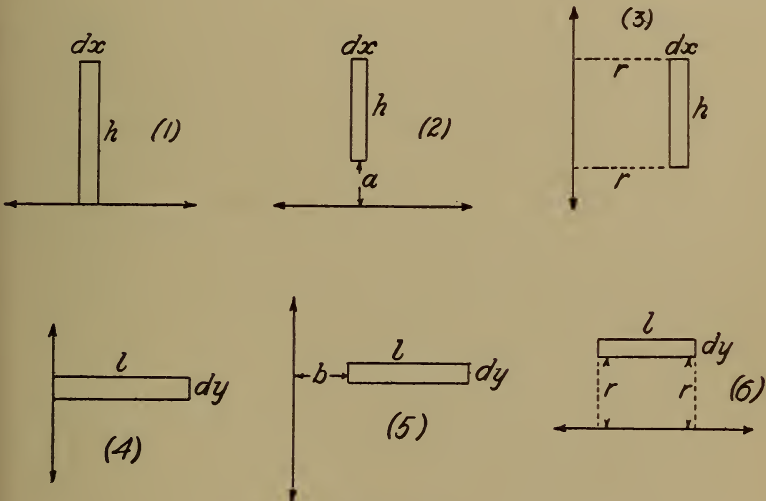


FIG. 64.

$x=f(y)$, the volume produced by revolving the curve about either axis of coördinates, or about any line parallel to either axis, may be got by an integration that sums up the volumes produced by the revolution about the axis in question of elementary strips of the area of the type, $y\,dx$, $x\,dy$, $(y_1-y_2)\,dx$, or $(x_1-x_2)\,dy$, the limits of integration being the extreme values for the solid of the variable of integration.

In the figures, h, the height of a vertical element, is an *ordinate* or a *difference of ordinates; l*, the length of a horizontal element, is an *abscissa* or a *difference of abscissas.*

The axes of revolution are marked with arrow-heads.

The lengths a and b may be constant or variable; the lengths r are always variable.

The solids generated in (1) and (4) are thin discs; those in (2) and (5) are thin discal rings; those in (3) and (6) are thin cylindrical shells.

The elements of volume are:

(1)	(2)	(3)
$\pi h^2 dx$	$\pi[(a+h)^2-a^2]dx$	$2\pi rhdx$
(4)	(5)	(6)
$\pi l^2 dy$	$\pi[(b+l)^2-b^2]dy$	$2\pi rldy$

The forms (1-6) should *not* be memorized; the elementary volume produced should in any problem be got by actual computation; but it should be noted that the volume given for either of the thin shells (3) or (6) is the area of its inner curved surface multiplied by its thickness dx or dy. The volume of (3) is actually $2\pi\left(r+\frac{dx}{2}\right)hdx$, a value intermediate between $2\pi rhdx$ and $2\pi(r+dx)hdx$; so the integral of the simpler form gives the correct total volume. (See Art. 154.) The other elements are volumes of cylinders or the difference between two such volumes.

177. *Examples.*

1. Find the volume formed by revolving about the x-axis the area included between the parabolas $y^2=ax$ and $x^2=ay$.

Ans. $\frac{3}{10}\pi a^3$.

2. Find the volume formed by revolving the cissoid $y^2(2a-x) = x^3$ about its asymptote, using as the element of volume a thin cylindrical shell of radius $(2a-x)$. Ans. $2\pi^2a^3$.

Find the volume formed by revolving the area of each of the following about the axis indicated:

3. $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}}$ about the x-axis. Ans. $\frac{32}{105}\pi ab^2$.

4. Area between axes and $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ about y-axis.

Ans. $\frac{\pi a^2 b}{15}$.

5. Area between $ay^2 = x^3$ and $x = y$ about y-axis.

Ans. $\frac{2}{21}\pi a^3$.

6. Area between $\frac{y^2}{b^2} = \frac{x}{a}$ and $x = a$ about $x = a$.

Ans. $\frac{16}{15}\pi a^2 b$.

7. Cycloid $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$ about $y = a$.

Ans. Large spindle, $\frac{\pi a^3}{6}(3\pi + 8)$; small spindle, $\frac{\pi a^3}{6}(3\pi - 8)$.

8. Area between $y^2 = 4a(x + a)$ and $27ay^2 = 4(x - a)^3$ about x-axis. Ans. $80\pi a^3$.

9. The distance from a point (x, y) to the line $x = y$ is $\pm\frac{1}{2}\sqrt{2}(x - y)$. Find the volume of the spindle produced by revolving the area common to $x^2 = ay$ and $y^2 = ax$ about their common chord. (Divide the solid by planes perpendicular to the axis of revolution.) Ans. $\frac{\pi a^3}{30\sqrt{2}}$.

178. Successive Integrations.—In some problems of integration, the value of the element of integration is not immediately evident, and is itself determined by an integration. This may be done even in the simpler problems that we have already discussed, though to no advantage except for a convenient method of writing general formulas which is of value in subsequent work.

To explain the process more simply, we will first show how it can be applied to some of the familiar problems.

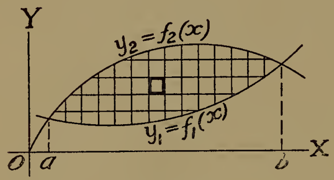


FIG. 65.

Let it be required to find the area between two curves,

$$y_1=f_1(x)$$

and

$$y_2=f_2(x).$$

Divide the area into rectangles by lines dx apart, parallel to OY, and lines dy apart, parallel to OX. (Fig. 65.) Any one of the vertical strips is thus divided into small rectangles, each $dxdy$ in area, so that the area of the whole strip is $dx\int dy$. In this integration dx is the same throughout the strip (is constant), and the limits are the least and greatest values of y for the strip; i. e., y_1 or $f_1(x)$ and y_2 or $f_2(x)$. As these limits are functions of x, the area of the strip,

$$dx\int_{f_1(x)}^{f_2(x)} dy,$$

is itself dx times a function of x. Now if a is the least and b the greatest value of x for the area, the complete area is

$$A=\int_a^b\left[dx\int_{f_1(x)}^{f_2(x)} dy\right],$$

or, as it is commonly more briefly written:

$$A=\int_a^b dx\int_{f_1(x)}^{f_2(x)} dy.$$

There is nothing new in this result; $\int_{f_1(x)}^{f_2(x)} dy$ is merely $f_2(x)-f_1(x)$ or y_2-y_1, so that this is the familiar formula

$$A=\int_a^b (y_2-y_1)\,dx.$$

In the same way, the area bounded by $x_1=F_1(y)$ and $x_2=F_2(y)$ (Fig. 66) is

$$A=\int_c^d dy\int_{F_1(y)}^{F_2(y)} dx.$$

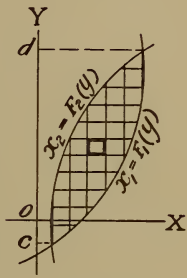


FIG. 66.

Either of these processes may be regarded as piling up rectangles to make a strip, then adding together such strips (each stretching across the area) to fill the area approximately, and, finally, by indefinitely increasing the number of rectangles, obtaining the area. The process is briefly referred to as *integrating dxdy over the area,* and is indicated by $\int\int dxdy$ *over the area;* where $\int\int dxdy$ is called the *double integral of dxdy.*

Suppose again that the area of Fig. 65 or of Fig. 66 is to be revolved about the axis of x. Then each element $dxdy$ will generate a ring of rectangular cross-section and of inner perimeter $2\pi y$. Its volume is thus approximately $2\pi ydxdy$, and the total volume generated is

$$\int\int 2\pi ydxdy \text{ over the generating area.}$$

In the same way, the volume generated by an area revolving about the axis of y is

$$\int\int 2\pi xdxdy \text{ over the generating area.}$$

The actual volume of one of the generating rings is in the first case

$$\pi dx(y+dy)^2-\pi dxy^2=2\pi dx\left(y+\frac{dy}{2}\right)dy,$$

but in order to establish rigorously the correctness of the integration with respect to y, it is merely necessary to observe that this volume is intermediate in value between $2\pi dx(ydy)$ and $2\pi dx(y+dy)dy$. (See Art. 154.)

As another example, let it be required to find the area between $r_1=f_1(\theta)$ and $r_2=f_2(\theta)$. Divide the area by radii vectores $d\theta$ apart and concentric circles dr apart. (Fig. 67.)

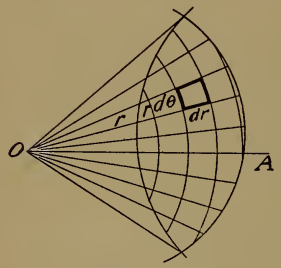


FIG. 67.

Any of the small sectors is thus divided into figures of a nearly rectangular shape, a typical one of which is bounded by two straight sides dr in length and two circular arcs $rd\theta$ and $(r+dr)d\theta$ in length. Its area is very nearly $rd\theta dr$; the area of the whole sector is

$$d\theta\int_{f_1(\theta)}^{f_2(\theta)} rdr;$$

and the whole area is

$$A=\int_\alpha^\beta d\theta\int_{f_1(\theta)}^{f_2(\theta)} rdr$$

if α and β are the extreme values of θ. More briefly, the area is

$$\int\int rd\theta dr \text{ over the area.}$$

The actual area of the small rectangular division is

$$\tfrac{1}{2}(r+dr)^2d\theta-\tfrac{1}{2}r^2d\theta=\left(r+\frac{dr}{2}\right)d\theta dr;$$

but as this is intermediate between $rd\theta dr$ and $(r+dr)d\theta dr$, the integral is rigorously correct. (See Art. 154.)

Completing the first integration of course gives the familiar formula

$$A=\tfrac{1}{2}\int_\alpha^\beta (r_1^2-r_2^2)\,d\theta.$$

180. Volumes of Revolution by Polar Coördinates.—In finding the volume generated by the revolution of a curve given by its polar equation, successive integrations are actually needed. Let it be required to find the volume generated by revolving about the initial line the area between $r_1=f_1(\theta)$ and $r_2=f_2(\theta)$. Divide the area as in the preceding article and consider any elementary division having $P(r, \theta)$ for one corner. (See Fig. 68.) The area of this element is to be taken $rd\theta dr$ as before. As was the case in rectangular coördinates, this area multiplied by the distance $(2\pi r\sin\theta)$ traversed by P gives a sufficiently close approximation to the volume of the ring generated by the revolution of the elementary division, so that the volume generated by the sector is

$$\int_{f_1(\theta)}^{f_2(\theta)} 2\pi r\sin\theta\cdot rd\theta dr=2\pi\sin\theta d\theta\int_{f_1(\theta)}^{f_2(\theta)} r^2dr;$$

and the total volume generated by the area is

$$V=2\pi\int_\alpha^\beta \sin\theta\, d\theta\int_{f_1(\theta)}^{f_2(\theta)} r^2dr,$$

if α and β are the extreme values of θ.

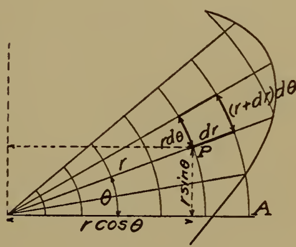


FIG. 68.

In the same way, since P, in revolving about the perpendicular to the initial line, describes a path $2\pi r\cos\theta$ in length, the volume of the elementary ring is $2\pi r\cos\theta\, rd\theta dr$, and the total volume is

$$2\pi\int_\alpha^\beta \cos\theta\, d\theta\int_{f_1(\theta)}^{f_2(\theta)} r^2dr.$$

More briefly, the volumes formed by revolving an area are:

$2\pi\int\int r^2\sin\theta\, d\theta dr$ (revolution about initial line),

$2\pi\int\int r^2\cos\theta\, d\theta dr$ (revolution about $\perp$ to initial line),

the integral in each case being taken over the generating area.

181. *Examples.*

1. Find the volume of the solid formed by revolving the circle $r = 2a\cos\theta$ (a) about the initial line, (b) about the perpendicular to the initial line. Ans. (b) $2\pi^2 a^3$.

2. Find the volume generated by the revolution of the cardioid $r = 2a\sin^2\frac{1}{2}\theta$ about the initial line. Ans. $\frac{8}{3}\pi a^3$.

3. Find the volume generated by the revolution of the lemniscate $r^2 = a^2\cos 2\theta$ about the perpendicular to the initial line. Ans. $\frac{1}{8}\pi^2 a^3\sqrt{2}$.

4. The arc of a cardioid $r = 2a\sin^2\frac{\theta}{2}$ revolves about the perpendicular to the initial line; find the volume enclosed by the outer surface so formed, and the volume of the double spindle inside. Ans. $\frac{\pi a^3}{4}(16 + 5\pi)$ and $\frac{\pi a^3}{4}(16 - 5\pi)$.

5. The area to the right of the perpendicular to the init line between $r = a(1 - \cos\theta)$ and $r = a(1 + \cos\theta)$ revolves abo this perpendicular, and again, about the initial line. Show th the volumes so generated are $\frac{5}{2}\pi^2 a^3$ and $\frac{7}{3}\pi a^3$.

6. The lemniscate $r^2 = a^2\cos 2\theta$ revolves about the initial lin Show that it generates the volume

$$\tfrac{1}{12}\pi a^3\sqrt{2}[3\log(1 + \sqrt{2}) - \sqrt{2}].$$

182. Volumes by Parallel Sections.—The methods that w have used for finding the volume of a solid of revolution in rec tangular coördinates amount to dividing the solid by planes per pendicular to the axis of revolution, computing the volume of cylinder inscribed between two of the planes, and finally inte grating to find the limit of the sum of all such cylinders; i. e. the total volume. The process is made easy in these cases by the simplicity with which the volume of the typical element can be computed (its base being a circle), and is equally easy in any case where the area of a section parallel to one of the coördinate planes can be simply expressed in terms of its distance from the plane.

If the area, A, of a section of a given solid by a plane parallel to the coördinate plane YOZ at a distance x from YOZ is a given function of $x[A=f(x)]$, the volume of the elementary cylinder between two planes at distances x and $x+dx$ is

$$Adx \quad \text{or} \quad f(x)\cdot dx,$$

and the total volume is the limit of the sum of all such cylinders, or

$$V=\int_{x_1}^{x_2} f(x)\,dx,$$

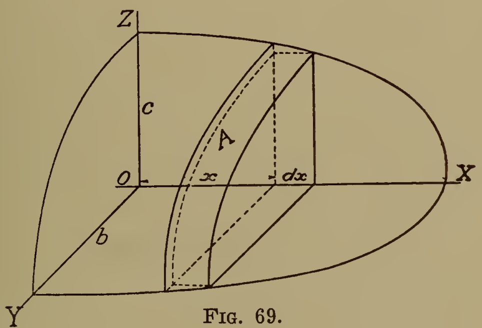


FIG. 69.

x_1 and x_2 being the minimum and maximum values of x for the solid.

If the area cut by a plane parallel to XOY or to ZOX can be expressed in terms of z or y respectively, the volume may be found by a similar integration with respect to z or y.

The section of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

made by a plane parallel to YOZ at a distance x from YOZ is an ellipse, the semi-axes of which are the value of z when $y=0$ and the value of y when $z=0$ obtained from the equation of the ellipsoid, or

$$\frac{c}{a}\sqrt{a^2-x^2} \quad \text{and} \quad \frac{b}{a}\sqrt{a^2-x^2}.$$

The area of the section is thus

$$\frac{\pi bc}{a^2}(a^2-x^2).$$

In the same way, a section of this ellipsoid by a plane parallel to XOY at a distance z from XOY is

$$\frac{\pi ab}{c^2}(c^2-z^2),$$

and of a section parallel to ZOX at a distance y from ZOX is

$$\frac{\pi ca}{b^2}(b^2-y^2).$$

Thus the volume of the ellipsoid is

$$2\int_0^a \frac{\pi bc}{a^2}(a^2-x^2)\,dx$$

or

$$2\int_0^b \frac{\pi ca}{b^2}(b^2-y^2)\,dy$$

or

$$2\int_0^c \frac{\pi ab}{c^2}(c^2-z^2)\,dz;$$

i. e.,

$$V=\tfrac{4}{3}\pi abc.$$

183. Volumes by Successive Integrations.—The method of parallel sections can be extended to cases in which the area of a typical section is not immediately evident, for this area can in any case be found by a preliminary integration. If the preliminary process is made a double integration, the complete solution of the problem will necessitate three successive integrations, and may be expressed briefly by

$$\iiint dxdydz \textit{ throughout the volume,}$$

where the symbol $\iiint$ is read *triple integral.*

For the ellipsoid in the preceding article, the process might be

$$V=8\int_0^a dx\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz,$$

which would be described as building up, out of the small elements, each $(dxdydz)$ in volume, a column to reach from XOY to the surface of the ellipsoid, adding such columns to make a slice parallel to YOZ, reaching from XOZ, where $y=0$, to the trace of the ellipsoid on XOY, where $y=b\sqrt{1-\frac{x^2}{a^2}}$ (z being zero), and finally adding all such slices to build up the solid from YOZ, where $x=0$, to the end, where $x=a$.

As x and y have constant values for the whole column, and x the same value for the whole slice, these quantities are treated as constants in the corresponding integrations. The integral is thus reduced as follows:

$$V=8\int_0^a dx\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy$$

$$\left[\frac{y}{b}=\sqrt{1-\frac{x^2}{a^2}}\sin\theta\right],$$

$$V=8\int_0^a dx\cdot c\int_0^{\frac{\pi}{2}} b\left(1-\frac{x^2}{a^2}\right)\cos^2\theta\,d\theta,$$

$$V=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=\tfrac{4}{3}\pi abc.$$

Advantage should always be taken of knowledge already gained, so as to reduce the number of integrations as much as possible. The more elaborate process is shown here because the notation will be convenient later in many connections, and the process in a few.

184. *Examples.*

1. Find the volume between the vertex and the plane $x=a$, of the elliptical paraboloid

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x}{a},$$

by taking an element between two planes parallel to YOZ.

Ans. $\dfrac{\pi abc}{2}$.

2. Find the volume cut from a right circular cylinder of radius a, by a plane passing through a tangent to the base and making an angle α with its plane. Ans. $\pi a^3 \tan\alpha$.

3. What is the volume in example 2, if the plane passes through the center of the base of the cylinder?

Ans. $\frac{2}{3}a^3 \tan\alpha$.

4. An isosceles triangle of constant altitude c has for its base a double ordinate of the circle $x^2+y^2=a^2$, and its plane is perpendicular to the plane of the circle. Find the volume of the *conoid* generated as it moves across the circle. Ans. $\dfrac{\pi a^2 c}{2}$.

5. Find the volume of a conoid generated as in example 4, except that the triangle moves parallel to the y-axis of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and also when it moves parallel to the x-axis.

Ans. $\dfrac{\pi abc}{2}$ for each.

6. Find the volume common to two equal right cylinders of radius a which intersect at right angles. What is the common volume when the two axes are inclined at the angle α?

Ans. $\frac{16}{3}a^3 \csc\alpha$.

7. A cylindrical hole, diameter $2b$, is bored out of a sphere, diameter $2a$; the axis of the cylinder is a diameter of the sphere. Find the volume left. Ans. $\frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$.

185. Cylindrical Volumes.—The volume enclosed by a cylindrical surface and secant surfaces can be found by the method of Art. 183, which can moreover be readily extended so that the equation of the cylinder may be used in cylindrical coördinates. (See Fig. 70.)

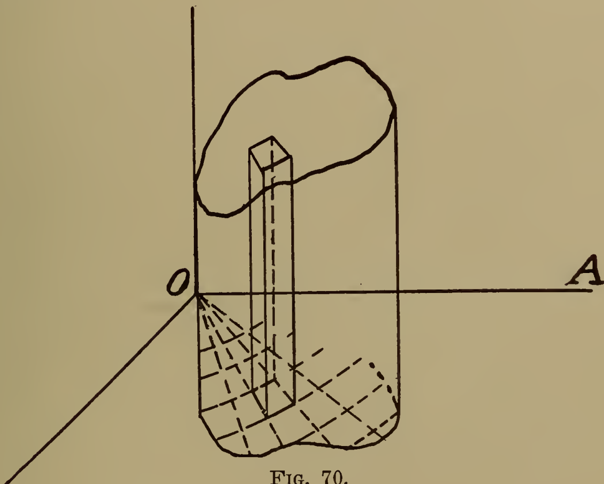


FIG. 70.

For instance, the volume cut out from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $y^2=ax-x^2$, two opposite elements of which are a diameter and a tangent of the sphere, may be obtained by the integration,

$$4\int_0^a dx\int_0^{\sqrt{ax-x^2}}\sqrt{a^2-x^2-y^2}\,dy.$$

Here we build up a column $z=\sqrt{a^2-x^2-y^2}$ in height on the element $dxdy$ of the base of the cylinder to reach from XOY to the sphere, integrate for y to form a slice across the cylinder, and finally sum these slices to form the complete volume.

We can as well take $r=a\cos\theta$ as the equation of the cylinder, and $r^2+z^2=a^2$ as the equation of the sphere. Then, building up a column of height $z=\sqrt{a^2-r^2}$ on the element $rd\theta dr$ of the base of the cylinder, integrating for r to determine the wedge from the z-axis across the cylinder, and for θ to sum up all these wedges, we have for the volume:

$$V=4\int_0^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} \sqrt{a^2-r^2}\cdot rdr=\tfrac{4}{3}a^3(\tfrac{\pi}{2}-\tfrac{2}{3}).$$

The same result can be obtained from the form in rectangular coördinates, but not so readily.

186. *Examples.*

1. Find the volume of the sphere $z^2+r^2=a^2$, using cylindrical coördinates.

2. Find the volume cut from the sphere of example 1 by the cylinder $r^2=a^2\cos 2\phi$. Ans. $\frac{2}{9}a^3(20-16\sqrt{2}+3\pi)$.

3. Find the volume common to a right cone the altitude of which is h and the radius of whose base is a, and a right cylinder having the radius of the cone for its diameter.

Ans. $ha^2(\frac{\pi}{4}-\frac{4}{9})$.

4. Find the volume of the cylinder included between the plane $mx+ny+c=z$, and the plane of xy, the equation of the cylinder being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Ans. πabc.

5. Find the volume cut from the cylinder $y^2=2ax-x^2$ by the paraboloid $x^2+y^2=az$ and the x-y plane. Ans. $\frac{3}{2}\pi a^3$.

187. In the following figure are collected the most important elements of area and volume with all the dimensions infinitesimal.

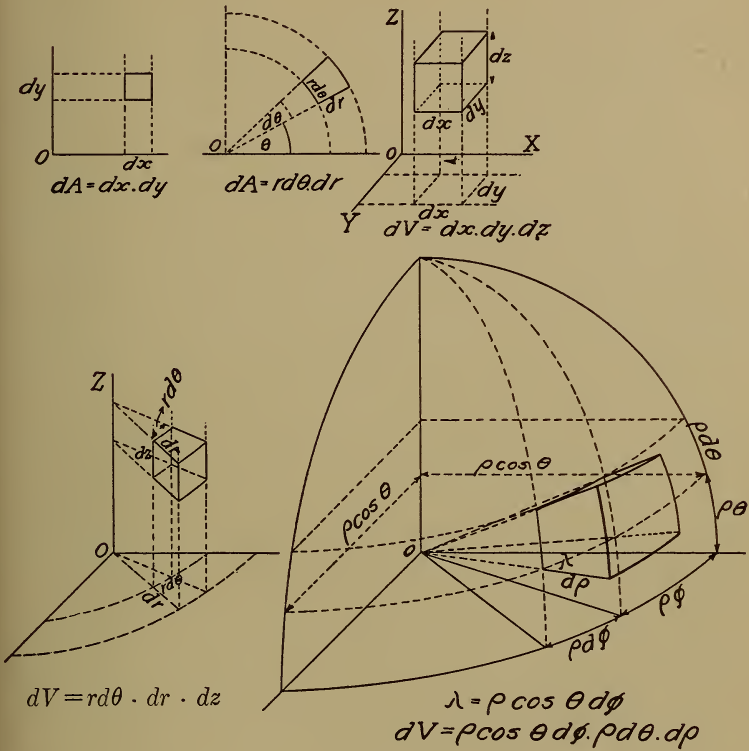


FIG. 71.

188. Length of Arc.—If s is the length of the arc of a curve from any fixed point to any variable point, ds may be expressed in some such form as $F(x)\,dx$, $F(y)\,dy$, $F(\phi)\,d\phi$, $F(\theta)\,d\theta$, etc., so that we have directly (see Art. 124):

The length of the arc of a curve from the point A to the point B is $s = \int_A^B ds$, *where the limits A and B are the values corresponding to the points A and B of the variable in the integrand.*

Thus the circle of radius a, having its center at the origin may be represented by $x^2+y^2=a^2$, by $x=a\cos\phi$ and $y=a\sin\phi$, or by $r=a$; and we have as corresponding expressions for ds:

$$ds=\pm\frac{adx}{\sqrt{a^2-x^2}},\quad \pm\frac{ady}{\sqrt{a^2-y^2}},\quad \pm ad\phi \text{ or } \pm ad\theta.$$

The length of the quadrant between the points represented in rectangular coördinates by $(a, 0)$ and $(0, a)$ is

$$\int_a^0\frac{-adx}{\sqrt{a^2-x^2}}=\int_0^a\frac{ady}{\sqrt{a^2-y^2}}=\int_0^{\frac{\pi}{2}}ad\phi=\int_0^{\frac{\pi}{2}}ad\theta=\frac{\pi a}{2}.$$

[Each of the integrands is positive, since from $(a, 0)$ to $(0, a)$, x decreases and all the other coördinates increase.]

189. Surfaces of Revolution.—The surface S, generated by the arc between two points, A and B, of a curve when the curve revolves about either coördinate axis, is readily found. This surface is the limit of the surface generated by a broken line inscribed in the arc as the chords of which it is composed increase indefinitely in number and decrease indefinitely in size.

The area generated by the chord from (x, y) to $(x+\Delta x, y+\Delta y)$ as it revolves about the x-axis is by elementary geometry

$$\Delta S=2\pi\left(y+\frac{\Delta y}{2}\right)\sqrt{(\Delta x^2)+(\Delta y)^2};$$

hence the differential of the surface is

$$dS=2\pi y\cdot\sqrt{(dx)^2+(dy)^2}=2\pi yds.$$

In the same way, if the curve revolves about the axis of y,

$$dS=2\pi x\sqrt{(dx)^2+(dy)^2}=2\pi xds.$$

Hence the surfaces formed by revolving the arc from A to B of a given

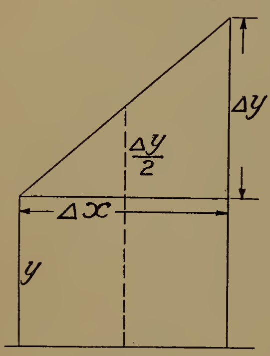


FIG. 72.

curve about the coördinate axes are

$$S = 2\pi \int_A^B y\,ds \text{ (revolution about } OX\text{, or the initial line),}$$

$$S = 2\pi \int_A^B x\,ds \text{ (revolution about } OY\text{, or the } \perp \text{ to the initial line).}$$

If the equation of the curve is more convenient in polar coördinates, x may be replaced by $r \cos \theta$, y by $r \sin \theta$.

190. Cylindrical Surfaces.—To find the area of a cylindrical surface included between two secant surfaces, let the axis of z be parallel to the elements of the cylindrical surface, so that the equation of this surface is in the form

$$f(x, y) = 0.$$

The problem will of course be solved if we find the area of the part of the cylindrical surface between either secant surface and the x-y plane. Let the equation of this secant surface be $z = F(x, y)$.

Call the section of the cylindrical surface by the x-y plane the *base;* then the equation of the base is $f(x, y) = 0$. (See Art. 165.)

Divide the perimeter of the base into elements of arc each equal to ds, and through the points of division draw elements of the cylinder, thus dividing the required area into strips. (See Fig. 73.)

Suppose the cylindrical surface to be cut along an element and developed on a plane: the base thus becomes a straight line which may be regarded as an axis of abscissas divided into parts ds in length, and the space curve in which the cylinder and the secant surface intersect becomes a plane curve of which the ordinates are the z-coördinates of the space curve. The required area becomes an area between a

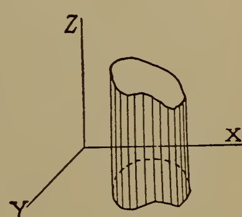


FIG. 73.

curve, the axis of abscissas, and two ordinates, divided into strips in the usual way, so that its value is

$$S = \int z ds.$$

In this integration, z is the z-coördinate of the *curve*

$$\begin{Bmatrix} z = F(x, y) \\ f(x, y) = 0 \end{Bmatrix},$$

and so may be expressed as a function of x by eliminating y from the two equations of the curve; s is the length of the arc of the base, so that ds may be expressed in terms of x through the equation of the base, $f(x, y) = 0$. The integral may therefore be put in the form $\int \phi(x) \cdot dx$. In the same way, it may be put in the form $\int \theta(y) \cdot dy$. The limits are the same that would be used in finding the perimeter of the base.

Again, if the surfaces have convenient equations in cylindrical coördinates, $f(r, \theta) = 0$ for the cylinder, $z = F(r, \theta)$ for the secant surface, the same form

$$S = \int z ds.$$

can be similarly reduced to the integral of $d\theta$ times a function of θ.

191. *Examples.*

1. Find the length of the arc of $y^2 = 8x$ from $(2, -4)$ to $(8, 8)$. Ans. $4\sqrt{5} + 2\sqrt{2} + \log \frac{9 + 4\sqrt{5}}{3 - 2\sqrt{2}}$.

2. Find the length of a quadrant of the circle $x^2 + y^2 = a^2$ by the form $s = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$, by direct integration.

3. Find the length of one branch of the cycloid

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi).$$

Ans. $8a$.

4. Find the length of the catenary $y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right)$ from $x = -a$ to $x = a$. Ans. $c(e^{a/c} - e^{-a/c})$.

5. Find the length of $ay^2=x^3$ between $x=0$ and $x=5a$.

Ans. $\frac{670}{27}a$.

6. Find the total area of the sphere formed by revolving $x^2+y^2=a^2$ about the x-axis.

7. Find the surface of the spindle formed by revolving the area between a semicircle, the tangents at the extremities of its diameter, and a perpendicular tangent, about the latter as an axis. Ans. $2\pi a^2(\pi-2)$.

8. Find the surface formed by revolving a branch of the cycloid about its base. Ans. $\frac{64}{3}\pi a^2$.

9. Find the surface formed by revolving one branch of the curve $x=a(\phi-\sin\phi)$, $y=a(1+\cos\phi)$ about the x-axis.

Ans. $\frac{32}{3}\pi a^2$.

10. The cycloid of example 9 revolves about the line $y=a$, forming a succession of spindles alternately smooth and ridged. Find the surface of a spindle of each type.

Ans. $\frac{16}{3}\pi a^2\sqrt{2}$ and $\frac{16}{3}\pi a^2(\sqrt{2}-1)$.

11. Find the whole length of the cardioid $r=a(1-\cos\theta)$, and the length from the cusp to the highest point.

Ans. $8a$ and $2a$.

12. Find the surface of the solid formed by revolving $r=2a\sin^2\frac{\theta}{2}$ about the initial line. Ans. $\frac{32}{5}\pi a^2$.

13. Find the surface of the solid formed by revolving $r=2a\sin^2\frac{\theta}{2}$ about the perpendicular to the initial line: (a) the inner surface, (b) the outer surface.

Ans. $\dfrac{16\pi a^2}{5}(3\sqrt{2}-4)$ and $\frac{48}{5}\pi a^2\sqrt{2}$.

14. Find the total surface formed by the revolution of the circle $r=2a\cos\theta$ about the perpendicular to the initial line.

Ans. $4\pi^2a^2$.

15. Find the lateral surfaces of the sections cut from the cylinder in examples 2 and 3 of Art. 184.

Ans. $2\pi a^2\tan\alpha$ and $2a^2\tan\alpha$.

16. Find the lateral surface of the portion of the right cylinder, having the radius a of a sphere for the diameter of its base, which is included within the sphere. Ans. $4a^2$.

17. Find the lateral surface of the cylinder $r=a\sin^2\frac{\theta}{2}$ included within the sphere $z^2+r^2=a^2$. Ans. $\frac{8}{3}a^2(2\sqrt{2}-1)$.

18. Find the lateral surface of $r=2a\cos\theta$ included between $z=0$ and $r^2=az$. Ans. $4\pi a^2$.

192. Other Curved Surfaces.—Let it be required to find the area cut out from a surface $F_1(x, y, z)=0$ by a surface $F_2(x, y, z)=0$. This area lies on $F_1=0$ and is bounded by a space curve of which the equations are $F_1=0$ and $F_2=0$. If we eliminate z from the two equations, getting $f(x, y)=0$, the new equation will represent a surface (cylindrical) also cutting $F_1=0$ in the space curve just mentioned, so we can simplify the problem by replacing $F_2=0$ by $f=0$ in the original statement. Moreover, $f(x, y)=0$ is the equation of the projection on the x-y plane of the space curve bounding the required area. (See Art. 168.)

Divide the plane area bounded by this projection into elements infinitesimal in both dimensions (e. g., $dxdy$ or $rd\theta\, dr$), and on each element erect a prism by drawing ordinates parallel to the z-axis. Let P be any point of XOY within the projection $f(x, y)=0$ and Q the corresponding point directly above it on $F_1(x, y, z)=0$. Let the prism corresponding to P and Q cut out the area dS from the plane tangent to $F_1=0$ at Q. The desired area will be the limit of the sum of elements of the type of dS. Representing the element of the projection by dA, we have by elementary geometry:

$$dS=\sec\gamma\cdot dA,$$

where γ is the inclination to the x-y plane of the plane tangent to $F_1(x, y, z)=0$ at Q.

The required area is then

$$S=\int\sec\gamma\, dA,$$

taken over the area bounded by $f(x, y)=0$, the projection of the area S on the x-y plane.

The projection of S on either of the other coördinate planes can of course be used in the same way. If cylindrical coördinates are used, $F(r, \theta, z)=0$ and $f(r, \theta)=0$ replace $F_1(x, y, z)=0$ and $f(x, y)=0$ in the preceding discussion.

As an example, consider the area cut out from the sphere $x^2+y^2+z^2=a^2$ by the parabolic cylinder $z^2=-a(x-a)$. In this case the secant surface is already a cylinder and can be used directly; but if we eliminate z, we get as a new secant surface, cutting out the same area, $x^2+y^2=ax$, a circular cylinder, which is still simpler. In cylindrical coördinates, the sphere and the circular cylinder are $r^2+z^2=a^2$ and $r=a\cos\theta$.

It is geometrically evident that for this sphere,

$$\sec\gamma=\frac{a}{z}=\frac{a}{\sqrt{a^2-r^2}};$$

for the circle $r=a\cos\theta$, $dA=rd\theta\,dr$, and the required area is

$$S=4a\int_0^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\frac{rdr}{\sqrt{a^2-r^2}}=2a^2(\pi-2).$$

193. *Examples.*

1. The cone $z=mr$ or $z=\cot\alpha\cdot r$ is cut by the sphere $r^2+z^2=2ar\cos\theta$. Find the area of the surface cut from one nappe of the cone. Ans. $\pi a^2\sin^3\alpha$.

2. Show that the area cut from the sphere in example 1 is $4a^2(\alpha-\sin\alpha\cos\alpha)$.

3. Show that the surface of the solid bounded by the cylinders $y^2+z^2=a^2$ and $x^2+z^2=a^2$ is $16a^2$.

4. Show that the area of each piece cut out from $y^2+z^2=a^2$ by $x^2+y^2=b^2$ is $4a\int_0^b\sqrt{\frac{b^2-y^2}{a^2-y^2}}\,dy$ or $4a\int_0^b\sin^{-1}\frac{\sqrt{b^2-x^2}}{a}\,dx$.

194. The Loxodrome or Rhumb-Line.—Suppose the earth to be a sphere a miles in radius, and let L and λ be the latitude and longitude of a point, M, on its surface. Let λ be measured westward from the Meridian of Greenwich, and let north latitude be positive, south latitude negative, so that $\phi=\frac{\pi}{2}-L$ is the colatitude of M. ϕ and λ are thus coördinates of M.

Let $M(\phi, \lambda)$ and $M'(\phi+\Delta\phi, \lambda+\Delta\lambda)$ be adjacent points on a course which passes through M, making the angle α with the meridian. (See Fig. 74.) Let the parallel of latitude through M' (radius $a \sin\phi$) meet the meridian through M (radius a) at m; then

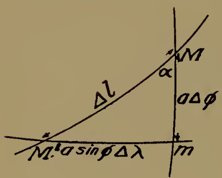


FIG. 74.

$$Mm = a\Delta\phi, \quad mM' = a \sin\phi \Delta\lambda.$$

Let l be the distance sailed from any fixed point of the course to M, and $\Delta l = MM'$.

In the limiting form of the triangle MmM',

$$\cot\alpha = \frac{d\phi}{\sin\phi \, d\lambda}, \quad dl = a \sec\alpha \, d\phi.$$

For a *rhumb-line* or *loxodrome*, α is constant, so that the distance from $M_1(\phi_1, \lambda_1)$ to $M_2(\phi_2, \lambda_2)$ is

$$l = \int_{\phi_1}^{\phi_2} (dl = a \sec\alpha \, d\phi) = a \sec\alpha(\phi_2 - \phi_1). \tag{1}$$

Integrating the two members of

$$\frac{d\phi}{\sin\phi} = \cot\alpha d\lambda$$

between corresponding limits gives

$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sin\phi} = \cot\alpha \int_{\lambda_1}^{\lambda_2} d\lambda,$$

or

$$\log \frac{\tan\frac{\phi_2}{2}}{\tan\frac{\phi_1}{2}} = \cot\alpha(\lambda_2 - \lambda_1). \tag{2}$$

The direction of the rhumb-line between any two ports can be determined from (2), and the distance from (1); or if the distance run on a given course from a given position is known, (1) will determine the colatitude and (2) the longitude of the position reached.

195. Mercator's Projection of the Sphere.—Suppose the earth to be mapped on a terrestrial globe, and a cylinder of revolution constructed tangent to this globe along the equator; and suppose each parallel of latitude to be projected upon the cylinder by a conical surface having its vertex on the axis of the globe. The points of the axis chosen as vertices of the cones which project the various parallels will influence the form of the projection, but in any case if the cylindrical surface is developed into a rectangle, the meridians and parallels will form two sets of parallel straight lines, mutually perpendicular.

Mercator's projection is one of this sort, so designed as to show any loxodrome as a straight line, having its angle with the meridian unchanged by projection.

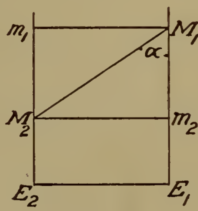


FIG. 75.

Fig. 75 represents a Mercator's projection, showing a loxodrome, M_1M_2, two meridians, M_1E_1 and M_2E_2, two parallels of latitude, M_1m_1 and M_2m_2, and the equator E_1E_2.

If the colatitudes and longitudes of M_1 and M_2 are (ϕ_1, λ_1) and (ϕ_2, λ_2), then

$$E_1E_2 = a(\lambda_1 - \lambda_2),$$

since lengths along the equator are unchanged. Therefore

$$m_2M_2 = E_1E_2 = a(\lambda_1 - \lambda_2).$$

As the angle α is unaltered, and the loxodrome M_1M_2 is a straight line,

$$\cot\alpha = \frac{M_1m_2}{m_2M_2} = \frac{M_1m_2}{a(\lambda_1 - \lambda_2)}.$$

But

$$\cot\alpha = \frac{1}{\lambda_2 - \lambda_1} \log \frac{\tan\frac{\phi_2}{2}}{\tan\frac{\phi_1}{2}}$$

(Art. 194); hence

$$M_1m_2 = a \log \frac{\tan\frac{\phi_2}{2}}{\tan\frac{\phi_1}{2}}.$$

Consequently, in a Mercator's projection of the terrestrial sphere the distance between the equator and the parallel of which the latitude is L is the value of M_1m_2 when $\phi_1 = \frac{\pi}{2} - L$, $\phi_2 = \frac{\pi}{2}$, or is

$$y = a \log \frac{\tan\frac{\pi}{4}}{\tan\left(\frac{\pi}{4} - \frac{L}{2}\right)} = a \log \cot\left(\frac{\pi}{4} - \frac{L}{2}\right) = a \log \tan\left(\frac{\pi}{4} + \frac{L}{2}\right).$$

If lines are drawn on a Mercator's projection representing parallels separated by equal intervals of latitude, the distance between them will increase rapidly with the latitude. (The rate of increase is $\frac{dy}{dL} = a \sec L$.) The formula $y = a \log \tan\left(\frac{\pi}{4} + \frac{L}{2}\right)$ is therefore called "the law of increased latitudes for the terrestrial sphere." It is also called "the law of meridional parts for the sphere."

196. The Terrestrial Spheroid.—The earth is much more nearly a spheroid (ellipsoid of revolution) than a sphere.

Fig. 76 represents a meridian with its eccentricity much exaggerated. $OA = a$ is the equatorial radius, the normal at M making with a the angle L, the latitude of M. The colatitude is

$$\phi = \frac{\pi}{2} - L = \frac{\pi}{2} - \left(\tau - \frac{\pi}{2}\right) = \pi - \tau.$$

x is the radius of the parallel of latitude through M. If e is the eccentricity,

$$y = \sqrt{1 - e^2}\sqrt{a^2 - x^2};$$

$$-\tan\phi = \tan\tau = \frac{dy}{dx} = \frac{-x\sqrt{1-e^2}}{\sqrt{a^2-x^2}};$$

$$\sec^2\phi = \frac{a^2-e^2x^2}{a^2-x^2};$$

$$x = \frac{a\sin\phi}{\sqrt{1-e^2\cos^2\phi}}. \tag{1}$$

Hence

$$dx = \frac{a(1-e^2)\cos\phi\, d\phi}{(1-e^2\cos^2\phi)^{\frac{3}{2}}}.$$

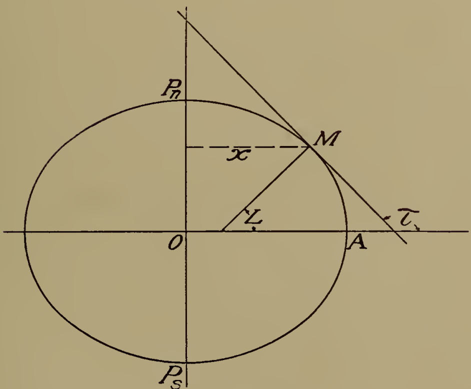


FIG. 76.

If the arc is measured in the direction $P_N A$,

$$\frac{ds}{dx} = -\sec\tau = \sec\phi, \quad ds = dx\sec\phi,$$

$$ds = \frac{a(1-e^2)\,d\phi}{(1-e^2\cos^2\phi)^{\frac{3}{2}}}. \tag{2}$$

197. The loxodrome and the Mercator's projection for the terrestrial spheroid can now be treated as for the sphere, x taking the place of $a\sin\phi$ as the radius of the parallel, and ds taking the place of $ad\phi$ as the element of arc of the meridian. We have:

$$\cot\alpha = \frac{ds}{xd\lambda};$$

which, from (1) and (2), is:

$$\cot\alpha = \frac{(1-e^2)\,d\phi}{(1-e^2\cos^2\phi)\sin\phi\,d\lambda}, \tag{3}$$

and

$$dl = \sec\alpha\,ds;$$

or

$$dl = \frac{a(1-e^2)\sec\alpha\,d\phi}{(1-e^2\cos^2\phi)^{\frac{3}{2}}}. \tag{4}$$

(3) is integrated as follows:

$$\frac{(1-e^2)\,d\phi}{(1-e^2\cos^2\phi)\sin\phi} = \cot\alpha\,d\lambda;$$

$$\lambda\cot\alpha = \int\frac{1-e^2\sin^2\phi - e^2\cos^2\phi}{(1-e^2\cos^2\phi)\sin\phi}d\phi$$

$$= \int\frac{d\phi}{\sin\phi} - \int\frac{e^2\sin\phi\,d\phi}{1-e^2\cos^2\phi};$$

$$\lambda\cot\alpha = \log\tan\frac{\phi}{2} + \frac{e}{2}\log\frac{1+e\cos\phi}{1-e\cos\phi}.$$

Hence

$$(\lambda_2-\lambda_1)\cot\alpha = \log\frac{\tan\frac{\phi_2}{2}}{\tan\frac{\phi_1}{2}} + \frac{e}{2}\log\frac{(1+e\cos\phi_2)(1-e\cos\phi_1)}{(1-e\cos\phi_2)(1+e\cos\phi_1)}.$$

Therefore, in the Mercator's projection,

$$M_1m_2 = a(\lambda_2-\lambda_1)\cot\alpha$$

becomes, when $\phi_1 = \frac{\pi}{2} - L$, $\phi_2 = \frac{\pi}{2}$:

$$y = a\left[\log\tan\left(\frac{\pi}{4} + \frac{L}{2}\right) + \frac{e}{2}\log\frac{1-e\sin L}{1+e\sin L}\right].$$

This is the law of increased latitudes for the spheroid; if we introduce an auxiliary angle, θ, such that $e \sin L = \cos \theta$, we shall have

$$\tfrac{1}{2} \log \frac{1 - e \sin L}{1 + e \sin L} = \log \tan \tfrac{\theta}{2},$$

and

$$y = a\left[\log \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) + e \log \tan \tfrac{\theta}{2}\right].$$

The use of infinite series will enable us to integrate (4), and will give a more practical form for

$$\int \frac{e^2 \sin \phi \, d\phi}{1 - e^2 \cos^2 \phi} = -\int \frac{e^2 \cos L dL}{1 - e^2 \sin^2 L} = \frac{e}{2} \log \frac{1 - e \sin L}{1 + e \sin L}.$$

(See Art. 206.)

198. *Examples.*

1. Show that if we use denary logarithms, and if λ^0 represents the difference in longitude in degrees, $(\phi_2 - \phi_1)'$ the difference of colatitude in minutes, and D the distance in knots, the rhumb-line course and distance on the terrestrial sphere is given by

$$\cot \alpha = \frac{A}{\lambda^0}\left[\log_{10} \tan \frac{\phi_2}{2} - \log_{10} \tan \frac{\phi_1}{2}\right],$$

$$D = \sec \alpha (\phi_2 - \phi_1)'.$$

where $\log_{10} A = 2.12034$.

2. Find the course and distance by rhumb-line and by great circle (assuming the earth a sphere) from San Francisco, $8^h\, 9^m\, 43^s$ W., 37° 47′ 28″ N., to Manila, $8^h\, 3^m\, 50^s$ E., 14° 35′ 25″ N. Ans.:

	Course.	Distance.
Rhumb-Line	W. 12° 37.6′ S.	6368.2
Great Circle	W. 28° 14.4′ N.	6051.0

3. Find the intervals between the parallels of 0° and 5°, 30° and 35°, 60° and 65° on a Mercator's projection of a sphere of radius a, and on a Mercator's projection of a spheroid of equatorial radius a, eccentricity $e = 0.081697$.

CHAPTER VIII.

Series.

199. In the study of functions it is often the case that the development of a function in power-series gives an expression more readily handled than the original form. This is particularly true when we attempt to integrate functions whose integrals are very complicated or cannot be expressed at all in terms of familiar functions.

200. Development in Series.—Suppose that a function of x can be developed into a power-series

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots;$$

then, assuming that the method of finding the derivative of a finite power-series applies also to an infinite power-series:

$$f'(x) \equiv a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots + na_n x^{n-1} + \dots,$$
$$f'(0) = a_1;$$
$$f''(x) \equiv 2a_2 + 2\cdot 3a_3 x + 3\cdot 4a_4 x^2 + \dots + (n-1)na_n x^{n-2} + \dots,$$
$$f''(0) = 2a_2;$$
$$f'''(x) \equiv 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4 x + \dots + (n-2)(n-1)na_n x^{n-3} + \dots,$$
$$f'''(0) = 2\cdot 3a_3;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots;$$
$$f^{(n)}(x) \equiv 2\cdot 3\cdot 4\dots(n-2)(n-1)na_n + \text{multiples of powers of } x,$$
$$f^{(n)}(0) = a_n \lfloor n.$$

Since $f(0) = a_0$, we have *Maclaurin's Series:*

$$f(x) \equiv f(0) + x\cdot f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \dots + \frac{x^n}{\lfloor n} f^{(n)}(0) + \dots.$$

Exercises.

Use Maclaurin's Series to obtain the series for $(a+x)^m$, $\sin x$, $\cos x$, e^x and a^x already developed in the Trigonometry and in the Algebra.

201. Practical Methods.—We can develop $\sec x$ directly. Since $\csc(0)=\infty$, $\csc x$ cannot be thus developed; by noticing that $\csc x=\sec(x-\frac{\pi}{2})$, we can develop $\csc x$ in powers of $(x-\frac{\pi}{2})$. It is, however, much simpler to replace $\cos x$ and $\sin x$ in the formulas $\sec x=\dfrac{1}{\cos x}$, $\csc x=\dfrac{1}{\sin x}$ by the first few terms of their developments, and carry out the divisions to determine an equal number of terms of $\sec x$ and $\csc x$.

Maclaurin's Series is used in practice to develop such functions as it readily applies to; then the development of any simple function of these can be obtained by elementary processes.

202. *Examples.*

1. Write four terms of the development of $\cos 2x$, and thence obtain four terms of the development of $\sin^2 x$ and of $\cos^2 x$.

$$\text{Ans.}\quad \begin{cases} \sin^2 x = x^2 - \dfrac{x^4}{3} + \dfrac{2x^6}{45} - \dfrac{x^8}{315} + \cdots. \\[2ex] \cos^2 x = 1 - x^2 + \dfrac{x^4}{3} - \dfrac{2x^6}{45} + \cdots. \end{cases}$$

2. Find three terms of the development of $\csc x$ from three terms of the development of $\sin x$. Ans. $\dfrac{1}{x} + \dfrac{x}{6} + \dfrac{7x^3}{360}$.

3. Given c. m. $30° = 0.5236$, compute to four decimals (by logarithms) the first three terms of $\csc x$, and compare their sums with $\csc 30°$.

4. Find three terms of the development of $\tan x$ (from $\sin x$ and $\cos x$). Ans. $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots.$

5. Find the developments of $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ from the development for e^x.

$$\text{Ans.}\quad \begin{cases} 1 + \dfrac{x^2}{\underline{|2}} + \dfrac{x^4}{\underline{|4}} + \dfrac{x^6}{\underline{|6}} + \cdots, \\[2ex] x + \dfrac{x^3}{\underline{|3}} + \dfrac{x^5}{\underline{|5}} + \dfrac{x^7}{\underline{|7}} + \cdots. \end{cases}$$

6. Find the development of $\sqrt{1-\cos x}$, considering $\sqrt{1-\cos x}$ as a function of $\frac{x}{2}$. Ans. $\sqrt{2}\left[\frac{x}{2}-\frac{x^3}{2^3\underline{|3}}+\frac{x^5}{2^5\underline{|5}}-\ldots\right]$.

7. Find the development of $\cos^3 x$ from the identity $\cos^3 x \equiv \frac{3}{4}\cos x+\frac{1}{4}\cos 3x$. Ans. $1-\frac{3}{2}x^2+\frac{7}{8}x^4-\frac{61}{240}x^6+\ldots$.

203. If the derivative of a function is easier to develop than the function itself, it should be developed, and the resulting series integrated on the assumption that an infinite and a finite power-series obey the same laws of calculus. The arbitrary constant that appears can always be determined from the value of the function for some one value of its argument. Often the development of the derivative follows from some familiar theorem. For instance, to develop $\tan^{-1} x$:

By the Binomial Theorem,

$$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\ldots,$$

$$\tan^{-1} x=\int[1-x^2+x^4-x^6+x^8-\ldots]dx$$

$$=C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\ldots,$$

$$\tan^{-1} 0=C=0,$$

$$\tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots.$$

204. *Examples.*

1. Develop $x\tan^{-1} x-\log\sqrt{1+x^2}$ by first developing its derivative. Ans. $\frac{x^2}{2}-\frac{x^4}{3.4}+\frac{x^6}{5.6}-\frac{x^8}{7.8}+\ldots$.

2. Given $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$, find the value of π to eight figures from the series for $\tan^{-1} x$.

Ans. $\pi=3.1415926$.

3. Develop $\sin^{-1} x$ by first using the Binomial Theorem to develop its derivative.

$$\text{Ans.}\quad x+\frac{1}{2.3}x^3+\frac{1.3}{2^2\underline{|2}}\frac{x^5}{5}+\frac{1.3.5}{2^3\underline{|3}}\frac{x^7}{7}+\ldots.$$

4. Compute $\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$ to four figures from the series of example 3.

205. Approximate Integration.—The method of the preceding article enables us to find an expression in series for an integral if we can develop the integrand, and so is of great advantage in the evaluation of an integral which is not a known function.

For instance, the length of a quadrant of the ellipse $x=a\cos\phi$, $y=a\sqrt{1-e^2}\sin\phi$ is

$$Q=a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\cos\phi}\cdot d\phi$$

$$=a\int_0^{\frac{\pi}{2}}\left[1-\frac{e^2}{2}\cos^2\phi-\frac{e^4}{2^2.2}\cos^4\phi-\frac{1.3}{2^3\underline{|3}}e^6\cos^6\phi\right.$$
$$\left.-\frac{1\cdot3\cdot5}{2^4\underline{|4}}e^8\cos^8\phi-\ldots.\right]d\phi;$$

$$Q=\frac{\pi a}{2}\left[1-\frac{e^2}{4}-\frac{3e^4}{4^3}-\frac{5e^6}{4^4}-\frac{5^2\cdot7e^8}{4^7}-\frac{7^2\cdot9\cdot e^{10}}{4^8}-\ldots.\right].$$

206. *Examples.*

1. Show that if $a=10$, $e=0.1$, $Q=15.669$ is the length of an elliptic quadrant.

2. Show $\int x^n e^{-x^2}dx=x^{n+1}\left[\frac{1}{n+1}-\frac{x^2}{n+3}+\frac{x^4}{(n+5)\underline{|2}}\right.$
$$\left.-\frac{x^6}{(n+7)\underline{|3}}+\frac{x^8}{(n+9)\underline{|4}}-\ldots.\right].$$

3. Show that the area in example 4, Art. 193, is

$$S=4a\int_0^b\sqrt{\frac{b^2-y^2}{a^2-y^2}}\,dy=4b^2\int_0^{\frac{\pi}{2}}\cos^2\phi\left(1-\frac{b^2}{a^2}\sin^2\phi\right)^{-\frac{1}{2}}d\phi:$$

$\left(\text{if } m=\frac{b}{a}\right)$

$$=\pi b^2\left(1+\frac{m^2}{2^2\,\underline{|2}}+\frac{(1\cdot 3)^2}{2^4\,\underline{|2}\,\underline{|3}}m^4+\frac{(1\cdot 3\cdot 5)^2}{2^6\,\underline{|3}\,\underline{|4}}m^6+\ldots\right);$$

and if $a=2b$,

$$S=1.035\pi b^2.$$

4. Show that the second term in the law of increased latitudes for the terrestrial spheroid (Art. 197)

$$\frac{e}{2}\log\frac{1-e\sin L}{1+e\sin L}=-\int\frac{e^2\cos L dL}{1-e^2\sin^2 L}$$

$$=-e^2\sin L(1+\tfrac{1}{3}e^2\sin^2 L+\tfrac{1}{5}e^4\sin^4 L+\ .\ .\ .\ .\ ).$$

5. Given that $a=3437.7$ minutes of equatorial arc,

$$\log_e N=2.3026\log_{10} N,\quad e=0.081697,$$

show that in Art. 197

$$y=7915.7\log_{10}\tan\left(45°+\frac{L}{2}\right)-23\sin L,\ \text{approx.},$$

in minutes of arc.

6. Show that the length of the loxodrome on the terrestrial spheroid (Art. 197) is

$$l=a(1-e^2)\sec\alpha\int_{L_1}^{L_2}[1+\tfrac{3}{2}e^2\sin^2 L+\tfrac{15}{8}e^4\sin^4 L+\ .\ .\ .\ .\ ]dL.$$

207. Differential Equations.—An equation involving derivatives of a function is called a *differential equation.*

Sometimes such an equation is formed for the sake of developing a function; as an example, suppose we wish to find the law of coefficients for the development of $\tan x$. We let $y=\tan x$, so that

$$\frac{dy}{dx}=\sec^2 x=1+y^2,$$

and our differential equation is formed:

$$\frac{dy}{dx}=1+y^2.$$

Since $\tan(-x) = -\tan(x)$, no even powers can occur; assume, therefore,

$$y \equiv a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \dots.$$

$$\begin{aligned} y^2 \equiv a_1^2 x^2 &+ a_3^2 x^6 + a_5^2 x^{10} + a_7^2 x^{14} + \dots. \\ &+ 2a_1a_3x^4 + 2a_1a_5x^6 + 2a_1a_7x^8 + 2a_1a_9x^{10} + 2a_1a_{11}x^{12} + 2a_1a_{13}x^{14} + \dots. \\ &+ 2a_3a_5x^8 + 2a_3a_7x^{10} + 2a_3a_9x^{12} + 2a_3a_{11}x^{14} + \dots. \\ &+ 2a_5a_7x^{12} + 2a_5a_9x^{14} + \dots. \\ &+ \dots\dots\dots \end{aligned}$$

$$\frac{dy}{dx} \equiv a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + 9a_9x^8 + 11a_{11}x^{10} + 13a_{13}x^{12} + \dots.$$

Comparing coefficients of like powers of x, we have

$$a_1 = 1, \quad 3a_3 = a_1^2, \quad 5a_5 = 2a_1a_3, \quad 7a_7 = 2a_1a_5 + a_3^2,$$
$$9a_9 = 2a_1a_7 + 2a_3a_5, \quad 11a_{11} = 2a_1a_9 + 2a_3a_7 + a_5^2.$$

The computation for several terms follows:

$$a_1 = 1.$$

$$3a_3 = a_1^2 = 1. \qquad a_3 = \tfrac{1}{3}.$$

$$5a_5 = 2a_1a_3 = \tfrac{2}{3}. \qquad a_5 = \tfrac{2}{15}.$$

$$7a_7 = 2a_1a_5 + a_3^2 = \frac{4}{15} + \frac{1}{9} = \frac{12+5}{45} = \frac{17}{45}. \qquad a_7 = \frac{17}{5 \cdot 7 \cdot 9}.$$

$$9a_9 = \frac{34}{5 \cdot 7 \cdot 9} + \frac{4}{9 \cdot 5} = \frac{34+28}{5 \cdot 7 \cdot 9} = \frac{62}{5 \cdot 7 \cdot 9}. \qquad a_9 = \frac{62}{5 \cdot 7 \cdot 9^2}.$$

$$\tan x = x + \tfrac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{5 \cdot 7 \cdot 9}x^7 + \frac{62}{5 \cdot 7 \cdot 9^2}x^9 + \dots.$$

Exercises.

1. Since $\sec(0) = 1$ and $\sec(-x) = \sec x$, assume

$$y \equiv \sec x \equiv 1 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \dots,$$

and show that $\dfrac{d^2y}{dx^2} = 2y^3 - y$.

Substitute, and determine a_2, a_4, a_6:

$$\sec x = 1 + \tfrac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots.$$

2. Find five terms of the expansion of log cos x.

$$\text{Ans.}\quad -\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\frac{17x^8}{2520}-\frac{31x^{10}}{14175}\cdots$$

208. Elementary Series.—The following developments of elementary functions are collected here for reference:

$$(1+x)^n=1+nx+n(n-1)\frac{x^2}{2!}+n(n-1)(n-2)\frac{x^3}{3!}+\cdots$$

$$a^x=1+x\log a+\frac{1}{2!}(x\log a)^2+\frac{1}{3!}(x\log a)^3+\cdots$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots$$

$$\csc x=\frac{1}{x}+\frac{x}{6}+\frac{7x^3}{360}+\frac{31x^5}{15120}+\cdots$$

$$\sec x=1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots$$

$$\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\cdots$$

$$\log\sin x=\log x-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\cdots=-\log\csc x.$$

$$\log\sec x=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots=-\log\cos x.$$

$$\log\tan x=\log x+\frac{x^2}{3}+\frac{7x^4}{90}+\frac{62x^6}{2835}+\cdots=-\log\cot x.$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \dots \quad = \tfrac{\pi}{2} - \cos^{-1} x.$$

$$\csc^{-1} x = \frac{1}{x} + \frac{1}{6x^3} + \frac{3}{40x^5} + \frac{5}{112x^7} + \dots = \tfrac{\pi}{2} - \sec^{-1} x.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad = \tfrac{\pi}{2} - \cot^{-1} x.$$

$$\cot^{-1} x = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \dots \quad = \tfrac{\pi}{2} - \tan^{-1} x.$$

209. Development when $f^{(n)}(0) = \infty$.—If $f(x)$, $f'(x)$ or any higher derivative is infinite when $x=0$, $f(x)$ cannot be developed by Maclaurin's Series. For instance, an attempt to apply the method directly to $\log x$ will fail; $\log(1+x)$, however, is readily developed. If

$$\begin{aligned} f(x) &= \log(1+x), & f(0) &= 0; \\ f'(x) &= (1+x)^{-1}, & f'(0) &= 1; \\ f''(x) &= -(1+x)^{-2}, & f''(0) &= -1; \\ f'''(x) &= 2(1+x)^{-3}, & f'''(0) &= 2; \\ &\dots\dots\dots, & &\dots\dots\dots; \end{aligned}$$

$$f^{(n)}(x) = (-1)^{n-1} \lfloor n-1 \, (1+x)^{-n},$$

$$f^{(n)}(0) = (-1)^{n-1} \lfloor n-1 :$$

$$\log(1+x) \equiv x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + \dots.$$

From this development the series actually used for computing logarithms can be derived, as in the Algebra, Art. 162.

$$\log(n+h) = \log n + 2\left[\frac{h}{2n+h} + \frac{1}{3}\left(\frac{h}{2n+h}\right)^3 + \frac{1}{5}\left(\frac{h}{2n+h}\right)^5 + \dots\right]$$

210. Taylor's Series.—The procedure that was necessary in the case of $\log(1+x)$ is useful in many connections. Suppose we wish to compare a particular value of $f(x)$, say $f(x_0)$, with adjacent values. All values of $f(x)$ may be represented by $f(x_0+z)$, where z is variable, and these values will be nearer to the value of $f(x_0)$ as z diminishes. To develop $f(x_0+z)$, which is a function of z (x_0 being a constant), we assume

$$f(x_0+z) \equiv a_0+a_1z+a_2z^2+a_3z^3+\ldots\ldots+a_nz^n+\ldots\ldots$$

Now, since x_0 is a constant, $d(x_0+z)=dz$; hence

$$\frac{d}{dz}f(x_0+z) \equiv \frac{d}{d(x_0+z)}f(x_0+z) \equiv f'(x_0+z),$$

and similarly for the higher derivatives. Consequently, if we take successive z-derivatives of both members of the assumed identity, and subsequently make $z=0$ in each of the resulting identities, we have:

$$f'(x_0+z) \equiv a_1+2a_2z+3a_3z^2+\ldots\ldots+na_nz^{n-1}+\ldots\ldots,$$
$$f''(x_0+z) \equiv 2a_2+3\cdot 2a_3z+\ldots\ldots+n(n-1)a_nz^{n-2}+\ldots\ldots,$$
$$f'''(x_0+z) \equiv \underline{|3}\cdot a_3+\ldots\ldots+n(n-1)(n-2)a_nz^{n-3}+\ldots\ldots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$f^{(n)}(x_0+z) \equiv \underline{|n}\cdot a_n+\ldots\ldots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$f(x_0)=a_0,\quad f'(x_0)=a_1,\quad f''(x_0)=2a_2,$$
$$f'''(x_0)=\underline{|3}\cdot a_3,\quad \ldots\ldots,\quad f^{(n)}(x_0)=\underline{|n}a_n,\quad \ldots\ldots,$$

so that

$$f(x_0+z) \equiv f(x_0)+zf(x_0)+\frac{z^2}{\underline{|2}}f''(x_0)$$
$$+\frac{z^3}{\underline{|3}}f'''(x_0)+\ldots\ldots+\frac{z^n}{\underline{|n}}f^{(n)}(x_0)+\ldots\ldots$$

This development is called *Taylor's Series;* it is often spoken of as the development of the function $f(x)$ in the neighborhood of the value x_0 of its argument x. Maclaurin's Series is a special case of Taylor's, the development in the neighborhood of the value zero.

Taylor's Series is especially adapted to the study of the increase of a function of x when x is increased from a particular value x_0 by the difference or increment dx.

Call $f(x)=y$, $f(x_0)=y_0$, $f(x_0+dx)=y_0+\Delta y$; then

$$y_0+\Delta y=f(x_0+dx)=f(x_0)+dxf'(x_0)+\frac{(dx)^2}{\underline{|2}}f''(x_0)$$
$$+\frac{(dx)^3}{\underline{|3}}f'''(x_0)+\ldots.+\frac{(dx)^n}{\underline{|n}}f^{(n)}(x_0)\ldots.$$

211. Finite Differences.—The development

$$\Delta y=dx\cdot f'(x_0)+\frac{(dx)^2}{\underline{|2}}f''(x_0)+\frac{(dx)^3}{\underline{|3}}f'''(x_0)+\ldots.$$
$$+\frac{(dx)^n}{\underline{|n}}f^{(n)}(x_0)+\ldots.$$

gives, to any required degree of approximation, the increase in the function y caused by the increase dx in its argument. The number of terms needed will depend on the nature of the function, the size of dx, and the accuracy required. In many practical cases, one term is sufficient; the resulting formula,

$$\Delta y=dx\cdot f'(x_0) \quad \text{or} \quad \Delta y=dy,$$

is merely the assumption that the increment of y and the differential of y do not differ in the decimal places that it is desired to have correct. Graphically this means that the curve $y=f(x)$ and the tangent to it at (x_0, y_0) have ordinates differing by a negligible amount when $x=x_0+dx$. The second term,

$$\frac{(dx)^2}{2}f''(x_0),$$

will generally show whether this difference is really negligible.

In any case, enough terms of the Taylor's Series are computed so that the last one does not affect the decimal places retained in the computation, and generally one or two more places are retained in the computation than are desired in the result.

For instance, if $y=\log_{10} x$,

$$\Delta y = dx\frac{\mu}{x_0} - \frac{(dx)^2}{2}\frac{\mu}{x_0^{\,2}} + \frac{(dx)^3}{3}\frac{\mu}{x_0^{\,3}} - \dots + (-1)^{n-1}\frac{(dx)^n}{n}\frac{\mu}{x_0^{\,n}} + \dots,$$

or

$$\Delta y = \mu\left[\frac{dx}{x_0} - \frac{1}{2}\left(\frac{dx}{x_0}\right)^2 + \frac{1}{3}\left(\frac{dx}{x_0}\right)^3 - \dots + \frac{(-1)^{n-1}}{n}\left(\frac{dx}{x_0}\right)^n + \dots\right].$$

($\mu = \log_{10} e = 0.43429.$)

If dx is positive, this is an alternating series, so that the error committed by taking $\Delta y = dy = \frac{\mu dx}{x_0}$ is less than $\frac{\mu}{2}\left(\frac{dx}{x_0}\right)^2$, and the value of Δy so taken is too large.

212. *Examples.*

1. If it is assumed that $\log_{10}(N+n) = \log_{10} N + \frac{\mu n}{N}$, how large a fractional part of N may n be if the error in the assumption is not to affect the fifth place in the mantissa? [Error <0.000005.] How many figures of μ will be useful in the approximate computation?

Ans. $\left(\frac{n}{N}\right)_{\!\lrcorner} < .00480$; three figures; use $\mu = 0.434$.

$$\log_{10}(N+n) = \log_{10} N + \frac{.434n}{N}.$$

2. Find by Taylor's Series the value to 5 decimals of sin (30° 30′) from the functions of 30°.

3. Find the increase in $\log_{10} \sin x$, when $x=30°$, $dx=1'$, to 7 decimals, and check by tabular difference between $\log_{10} \sin 30°$ and $\log_{10} \sin 30°\ 1'$.

4. Find the increase in $\log_{10} \tan x$, $x=45°$, $dx=1'$.

5. Develop $y = \log_{10} \sec x$ in the neighborhood of the value $x_0 = \tan^{-1} \frac{3}{4}$ and thence find to seven decimals the values of Δy when $dx = 1'$, $dx = 10'$ and $dx = 1° \ 40'$.

Ans. 0.0000948, 0.0009504, 0.0097663.

6. Develop $y = \sin^2 x$ in the neighborhood of $x_0 = 45°$, and thence find to five decimals the values of $\sin^2 46°$, $\sin^2 50°$, $\sin^2 60°$, and $\sin^2 75°$.

Ans. 0.51745, 0.58683, 0.75000, 0.93301.

213. Small Changes in the Astronomical Triangle.—The formula $\Delta y = dy$ gives valuable approximations to the changes produced by small variations in the parts of the astronomical triangle. Of the five parts, L, d, h, t, and Z, three must be given to determine the triangle, and then the other two can be found. Each of the other two is thus a function of the three given. For instance, any one of the four parts, L, d, h, t, can be expressed as a function of the other three by means of the equation

$$\sin h = \sin d \sin L + \cos d \cos L \cos t. \qquad (1)$$

The results obtained from (1) can be simplified by means of the equations

$$\cos h \sin Z = \cos d \sin t, \qquad (2)$$

$$\cos h \cos Z = \cos L \sin d - \sin L \cos d \cos t. \qquad (3)$$

For instance, suppose L, d, h are given, and it is desired to find the effect on t of small errors in the data. We will suppose each of the given parts in turn to vary, the other two being constant, and thus find three errors produced in t, the sum of which will be the total approximate variation of t.

First, let L and d be constant, h and t variable; then, differentiating (1) we find:

$$\cos h dh = 0 + \cos d \cos L(-\sin t dt), \qquad (1')$$

or, by (2):

$$\cos h dh = -\cos L \sin Z \cos h dt,$$

$$dt = -\csc Z \sec L dh.$$

Next, let d and h be constant, L and t variable,

$$0=\sin d\cos L dL-\cos d\cos t\sin L dL-\cos d\cos L\sin t dt.$$

Then by (3):

$$\cos d\cos L\sin t dt=(\sin d\cos L-\cos d\sin L\cos t)dL$$
$$=\cos h\cos Z dL,$$

or, by (2):

$$\cos d\cos L\sin t dt=\cos Z\frac{\sin t\cos d}{\sin Z}dL,$$
$$dt=\cot Z\sec L dL.$$

In like manner, differentiating (1) on the assumption that t and d are the only variables, and reducing the result by means of

$$\cos h\cos M=\cos d\sin L-\sin d\cos L\cos t \qquad (4)$$

and

$$\cos L\sin t=\cos h\sin M, \qquad (5)$$

we find

$$dt=\cot M\sec d\cdot dd.$$

Finally, if L, d, h are given, subject to errors ΔL, Δd, Δh, the error in t is approximately

$$\Delta t=\cot Z\sec L\Delta L+\cot M\sec d\Delta d-\csc Z\sec L\Delta h.$$

In the same way, if d, t, h are given, subject to errors Δd, Δt, Δh, the error in L is approximately

$$\Delta L=-\cos M\sec Z\Delta d+\tan Z\cos L\Delta t+\sec Z\Delta h.$$

In the navigation problems of the time-sight and the latitude-sight, the error in declination is negligible, and these formulas become, for the time sight,

$$\Delta t=\cot Z\sec L\Delta L-\csc Z\sec L\Delta h$$

and for the latitude-sight,

$$\Delta L=\tan Z\cos L\Delta t+\sec Z\Delta h.$$

In either of these, the error Δh arises from inaccuracy in observation. In the time-sight, ΔL is the error in the assumed (dead-reckoning) latitude. If the time-sight is taken when the sun is near the prime-vertical, Z is nearly $90°$, $\cot Z$ and $\csc Z$ are small, and the error in longitude, $\Delta t = \sec L(\cot Z\Delta L - \csc Z\Delta h)$, is small, so that the longitude found from the observation is more accurate than the dead-reckoning longitude. The latitude-sight, on the other hand, should be taken when the sun is near the meridian, for then Z is nearly 0, and $\tan Z$ and $\sec Z$ are small. The effect of the error in the dead-reckoning longitude and the error in h are thus both made small, so that the latitude found is more accurate than the dead-reckoning latitude.

Formulas involving Z in place of t can be derived in the same way from

$$\sin d = \sin h \sin L + \cos h \cos L \cos Z. \qquad (6)$$

214. *Examples.*

1. The time and amplitude of sunrise or sunset, computed on the assumption that $h = 0°$, are given by

$$\sin A = \sin d \sec L, \quad \cos t = -\tan d \tan L.$$

Supposing the assumed values of d and L to be accurate, show that the change $\Delta h = -50'$, due to the mean semi-diameter of the sun (16′) and the mean refraction (34′), will cause a change, $\Delta t = \sec A \sec L \cdot 50'$.

2. Show that, at the Naval Academy (Lat. 38° 58′ 53″ N.) the apparent solar times of sunrise and sunset are about as follows:

	Winter Solstice, Dec. 21. $d = 23° 27' 07''$ S.		Equinoxes, Mar. 21, Sept. 21. $d = 0$.		Summer Solstice, June 21. $d = 23° 27' 07''$ N.	
	hrs.	min.	hrs.	min.	hrs.	min.
Sun rises . . .	7	17.2 a.m.	5	55.7 a.m.	4	32.8 a.m.
Sun sets	4	42.8 p.m.	6	04.3 p.m.	7	27.2 p.m.

3. Show that the error made in computing h from assumed values of t, d, L is

$$\Delta h = \cos Z\Delta L - \sin Z \cos L\Delta t \text{ approx.},$$

if $\Delta d=0$. It follows from this that a line of position can always be laid off by St. Hilaire's method on a large scale chart, regardless of the azimuth of the sun.

4. Show that, if $\Delta d=0$,

$$\Delta Z=\tan h \sin Z\Delta L-\cos M \sin Z \csc t\Delta t.$$

Use $\sin t \cot Z=\cos L \tan d-\sin L \cos t$, and simplify the results by using $\cos t \cos Z-\sin L \sin t \sin Z=-\cos M$, and the formulas (1) and (2) already given.

215. Simpson's Rules.—When an area or a volume is to be found and the equations of the bounding curves or surfaces are not known, it is possible to determine the desired result or a fair approximation to it from measurements of a number of ordinates. More generally, when a definite integral is to be found, it is possible to express it, sometimes exactly, more often approximately, in terms of a few values of the integrand, even if the form of the integrand is not known.

The problem in this method is: To find $\int_a^b f(x)\cdot dx$, given the values $f(x_0), f(x_1), f(x_2), \ldots\ldots, f(x_n)$ of $f(x)$ for a number of values of x uniformly distributed in the range from a to b.

216. Simpson's First Rule.—Suppose three values of the unknown function to be given; shift the origin so that the middle one corresponds to $x=0$, and call $b-a=H$. In this case we wish to find $\int_{-\frac{H}{2}}^{\frac{H}{2}} f(x)\cdot dx$, given $f(-\frac{H}{2}), f(0), f(\frac{H}{2})$. For the sake of brevity, call these given values y_1, y_2, y_3 respectively.

Assume the function to be of the form

$$f(x)\equiv A+Bx+Cx^2+Dx^3;$$

then

$$y_1=f\left(-\frac{H}{2}\right)=A-\frac{BH}{2}+\frac{CH^2}{4}-\frac{DH^3}{8}; \qquad (1)$$

$$y_2=f(0)=A; \qquad (2)$$

$$y_3 = f\left(\frac{H}{2}\right) = A + \frac{BH}{2} + \frac{CH^2}{4} + \frac{DH^3}{8}. \qquad (3)$$

If, in Fig. 77, the heavy line $PQRS$ is the graph of $f(x)$, the unknown function, and the dotted line $pQRS$ is the graph of $y = A + Bx + Cx^2 + Dx^3$, the true integral and the approximate integral we are about to find are represented by the areas $AQRSBA$, the first bounded by the full line, the second by the dotted line. The function $A + Bx + Cx^2 + Dx^3$ and its graph are of sufficient flexibility, even with the three values at Q, R, S fixed, to conform pretty closely to any given function and its graph.

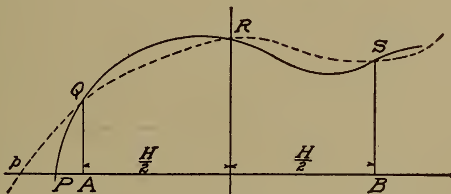


FIG. 77.

We have for the approximate integral:

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} (A + Bx + Cx^2 + Dx^3)\,dx = \left[Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} + \frac{Dx^4}{4}\right]_{-\frac{H}{2}}^{\frac{H}{2}}$$

$$= AH + \frac{CH^3}{12} = H\left(A + \frac{CH^2}{12}\right).$$

Adding relations (1) and (3) above,

$$2A + \frac{CH^2}{2} = y_1 + y_3; \quad A = y_2;$$

so

$$\frac{CH^2}{2} = y_1 + y_3 - 2A, \quad \frac{CH^2}{12} = \frac{y_1 + y_3 - 2y_2}{6};$$

hence

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} (A+Bx+Cx^2+Dx^3)\,dx = H\left(A+\frac{CH^2}{12}\right) = \tfrac{H}{6}(y_1+4y_2+y_3).$$

Therefore, approximately,

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} f(x)\cdot dx = \tfrac{H}{6}[y_1+4y_2+y_3] = \tfrac{H}{6}[f(-\tfrac{H}{2})+4f(0)+f(\tfrac{H}{2})].$$

The position of the origin relatively to a and b has no effect on the value of the integral; it has merely been necessary for a definite discussion to fix it somewhere. As H represents $(b-a)$, and the value of x midway between a and b is $\frac{a+b}{2}$,

$$\int_a^b f(x)\cdot dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

approximately, or

$$\int_a^b f(x)\cdot dx = \frac{b-a}{6}[y_1+4y_2+y_3],$$

where y_1, y_2, y_3 are the values of $f(x)$ at the beginning, the middle, and the end of the range from a to b. This formula is known as *Simpson's First Rule.*

The assumption which makes the result approximate is that $f(x)$ is a polynomial of the third or lower degree; if this is really so, the integration is exact.

217. *Examples.*

1. If two perpendiculars, y_1 and y_3 in length, a distance h apart, are dropped from points of a curve to a straight line, and a third perpendicular, y_2 in length, is drawn midway between them from a point of the curve to the straight line, show (a) that the area bounded by the curve, the straight line, and the perpendiculars y_1 and y_3 is approximately

$$A = \frac{h}{6}(y_1+4y_2+y_3),$$

and (b) that the volume produced by revolving this area about the straight line is approximately

$$V=\frac{\pi h}{6}(y_1{}^2+4y_2{}^2+y_3{}^2).$$

2. Find the following integrals by Simpson's First Rule:

$$\int_1^2 (2+x+2x^2+3x^3)\,dx=\tfrac{233}{12}$$

and

$$\int_{-1}^1 (7x^3+3x^2+2x+5)\,dx=12.$$

Find the following volumes by Simpson's First Rule:

3. The volume of a sphere.

4. The volume formed by revolving about $y=0$ the segment from $x=0$ to $x=a$ of the parabola $ay^2=b^2x$. Ans. $\frac{1}{2}\pi ab^2$.

5. The volume of a barrel formed by the revolution about the major axis of the segment of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ between the ordinates through its foci. Ans. 56.64π.

6. The volume of a barrel formed by the revolution about the major axis of the segment between the ordinates through its foci, if the length of the barrel is $2h$ and the diameter at the bung is $2b$. Ans. $\frac{2}{3}\pi b^2h\,\frac{3b^2+2h^2}{b^2+h^2}$.

7. The volume of a spherical segment, altitude h, radii of bases, b and c. Ans. $\frac{1}{6}\pi h(3b^2+3c^2+h^2)$.

8. The volume of a capstan in the form of a hyperboloid of revolution, each base of radius b, circle of gorge of radius g, altitude h. Ans. $\frac{\pi h}{3}(b^2+2g^2)$.

9. The volume of a conoid of height h having a circle of radius a as base. Ans. $\frac{1}{2}\pi a^2h$.

218. Closer Approximations.—When the approximation of $A+Bx+Cx^2+Dx^3$ to $f(x)$ is not sufficiently exact, there are two ways of coming closer to the true value of $\int_a^b f(x)\cdot dx$; one is to break up the range into equal parts, and to apply the method just described to each part; the other is to assume $f(x)$ equal to a polynomial of higher degree and to determine correspondingly more values of $f(x)$ from which to compute the integral. Either method involves an increased number of measurements of $f(x)$, or of computations of $f(x)$, for cases in which $f(x)$, though not integrated, is known.

If we follow the first method, dividing the range $(b-a)$ into two equal parts, and applying the method to each of them, $y_1, y_2, y_3, y_4, y_5,$

1 2 3 4 5

a b

being the values of $f(x)$ at the beginning, the points of quadrisection, and the end, we have

$$\int_a^{x_3} f(x)\,dx = \frac{\frac{1}{2}(b-a)}{6}(y_1+4y_2+y_3),$$

$$\int_{x_3}^{b} f(x)\,dx = \frac{\frac{1}{2}(b-a)}{6}(y_3+4y_4+y_5),$$

$$\int_a^{b} f(x)\,dx = \frac{b-a}{12}(y_1+4y_2+2y_3+4y_4+y_5).$$

In the same way, if the range $(b-a)$ is divided into any even number n of parts and $y_1, y_2, y_3, \ldots, y_n$ are the values of the function at the beginning, the points of division, and the end,

$$\int_a^{b} f(x)\,dx = \frac{b-a}{3n}(y_1+4y_2+2y_3+4y_4+2y_5+\ldots+y_{n+1}).$$

In the parenthesis, the first coefficient is 1, the last is 1, the others are alternately 4 and 2, beginning with 4. There are $\frac{n}{2}$ 4's, $\frac{n-2}{2}$ 2's, and 2 1's:

$$4\left(\frac{n}{2}\right)+2\left(\frac{n-2}{2}\right)+2(1)=3n.$$

The sum of the coefficients is three times the number of divisions of the range. The rule is often abbreviated:

$$\int_a^b f(x)\,dx = \frac{(b-a)}{1+4+2+\ldots\ldots+1}\,(1,\ 4,\ 2,\ \ldots\ldots,\ 1).$$

The simpler rule is similarly written:

$$\int_a^b f(x)\,dx = \frac{b-a}{1+4+1}\,(1,\ 4,\ 1).$$

219. *Examples.*

1. The segment from $x=0$ to $x=a$ of $ay^2=b^2x$ revolves about $x=a$. Apply Simpson's First Rule, dividing the solid by planes perpendicular to the axis of revolution, first, $\frac{b}{2}$ apart, second, $\frac{b}{4}$ apart; and show that the errors are $\frac{1}{64}$ and $\frac{1}{1024}$ of the correct volume, $\frac{16}{15}\pi a^2 b$.

2. Treat as in example 1 the volume formed by revolving the same area about the tangent at the vertex, and show that the errors are $\frac{1}{96}$ and $\frac{1}{1536}$ of the correct volume, $\frac{8}{5}\pi a^2 b$.

3. Find $\int_1^3 \frac{dx}{x}$ and $\int_1^4 \frac{dx}{x}$ by Simpson's First Rule, using values of the integrand for $x=1, \frac{5}{4}, \frac{3}{2}$, etc., and compare the results with $1.09861 = \log 3$ and $1.38629 = \log 4$.

220. Simpson's Second Rule.—Using the same assumption for $f(x)$, as in the First Rule, but dividing the range into three equal parts, taking the origin half-way between the ends of the range, and having given the values of $f(-\frac{H}{2})$, $f(-\frac{H}{6})$, $f(\frac{H}{6})$, and $f(\frac{H}{2})$, where $H=b-a$, we can prove by a precisely similar discussion that

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} f(x)\,dx = \frac{H}{8}\left[f(-\tfrac{H}{2}) + 3f(-\tfrac{H}{6}) + 3f(\tfrac{H}{6}) + f(\tfrac{H}{2})\right],$$

or that

$$\int_a^b f(x)\,dx = \frac{b-a}{8}\,(y_1 + 3y_2 + 3y_3 + y_4),$$

the y's being the values of $f(x)$ at the beginning, the points of *trisection,* and the end of the interval. This method may also be applied if we take $3n$ equal divisions of the range, giving

$$\int_a^b f(x)\,dx = \frac{b-a}{8n}(y_1+3y_2+3y_3+2y_4 \\ +3y_5+3y_6+2y_7+\ldots\ldots+y_{3n+1}),$$

where the y's are the values of $f(x)$ corresponding to the points of division.

These are briefly written:

$$\int_a^b f(x)\,dx = \frac{b-a}{1+3+3+1}[1,\ 3,\ 3,\ 1],$$

or

$$= \frac{b-a}{1+3+3+2\ldots\ldots+1}[1,\ 3,\ 3,\ 2,\ \ldots\ldots,\ 1].$$

They are known as *Simpson's Second Rule.*

221. Employment of the assumption that $f(x)$ is expressed by a polynomial of degree n gives, for various values of n, other approximations; e. g.,

$$n=4:\ \int_a^b f(x)\,dx = \frac{b-a}{90}[7,\ 32,\ 12,\ 32,\ 7].$$

$$n=6:\ \int_a^b f(x)\,dx = \frac{b-a}{840}[41,\ 216,\ 27,\ 272,\ 27,\ 216,\ 41];$$

a slight change in the ratios of these coefficients gives

$$\frac{b-a}{840}[42,\ 210,\ 42,\ 252,\ 42,\ 210,\ 42],$$

or

$$\frac{b-a}{20}[1,\ 5,\ 1,\ 6,\ 1,\ 5,\ 1],$$

a very valuable rule, known as *Weddles' Rule.*

222. *Examples.*

1. Prove that if $c-b=b-a$, so that a, b, and c are equally spaced values of x, and $f(a)=y_1$, $f(b)=y_2$, $f(c)=y_3$, then

$$\int_a^b f(x)\cdot dx=\frac{b-a}{12}(5y_1+8y_2-y_3).$$

This is known as "the *five-eight rule.*"

2. Prove in detail Simpson's Second Rule.

3. Find the volume of the larger segment cut by the plane $x=\frac{a}{2}$ from the sphere formed by revolving $x^2+y^2=a^2$ about the x-axis. (Use Simpson's Second Rule.) *Ans.* $\frac{9}{8}\pi a^3$.

4. Find the value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's First Rule, making ten divisions of the range, and compare your result with the value of $\pi=3.14159265$.

5. To find $4b^2\int_0^{\frac{\pi}{2}}\frac{\cos^2\phi\, d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}$ (example 4, Art. 193, and example 3, Art. 206), compute the function at intervals of $7\frac{1}{2}°$ and apply both the first and the second of Simpson's Rules. In computing the values of the integrand, use an auxiliary angle θ such that $\sin^2\phi=2\sin^2\theta$; then the values of the function can be computed from the form:

ϕ	log sin		2 log cos
		9.84949 − 10	
θ	log sin		log sec
$f(\phi)$			log

223. **Evaluation of $\frac{0}{0}$**.—*Problem:* To determine the value to assign to a fraction $\frac{u(x)}{v(x)}$ when $x=a$ if $u(a)=0$ and $v(a)=0$, in order to complete the definition of an otherwise continuous function. (See Algebra, Art. 45.)

We have by Taylor's Series:

$$u(x) = u[a + (x-a)] = u(a) + (x-a)u'(a) + \frac{(x-a)^2}{\underline{|2}} u''(a) + \ldots,$$

$$v(x) = v[a + (x-a)] = v(a) + (x-a)v'(a) + \frac{(x-a)^2}{\underline{|2}} v''(a) + \ldots.$$

Since $u(a) = 0$ and $v(a) = 0$,

$$\left[\frac{u(x)}{v(x)}\right]_{x=a} = \left[\frac{(x-a)u'(a) + \frac{(x-a)^2}{\underline{|2}} u''(a) + \ldots}{(x-a)v'(a) + \frac{(x-a)^2}{\underline{|2}} v''(a) + \ldots}\right]_{x=a}$$

$$= \left[\frac{u'(a) + \frac{x-a}{\underline{|2}} u''(a) + \ldots}{v'(a) + \frac{x-a}{\underline{|2}} v''(a) + \ldots}\right]_{x=a} = \frac{u'(a)}{v'(a)}.$$

Thus we have, if $u(a) = 0$ and $v(a) = 0$,

$$\left[\frac{u(x)}{v(x)}\right]_{x=a} = \left[\frac{u'(x)}{v'(x)}\right]_{x=a}.$$

If the fraction $\frac{u'(x)}{v'(x)}$, resulting from differentiation of the terms of $\frac{u(x)}{v(x)}$, is indeterminate when $x=a$ (that is, if $u'(a)=0$ and $v'(a)=0$), the same method gives $\left[\frac{u''(x)}{v''(x)}\right]_{x=a}$ as its value and so on. If differentiation of this sort is kept up after the fraction becomes determinate, the results will be correct only by accident.

It is often more convenient, especially when $a=0$, to write the actual developments of $u(x)$ and $v(x)$; for if these developments are known, all the differentiation is avoided. Only the first non-vanishing term of the development need be written.

For example:

(a) $$\left[\frac{\sin x}{\log(1+x)}\right]_{x=0} = \frac{0}{0} = \left[\frac{\cos x}{\frac{1}{1+x}}\right]_{x=0} = \frac{1}{1} = 1;$$

or

$$\frac{\sin x}{\log(1+x)} = \frac{x+}{x+} = 1.$$

(b) $$\left[\frac{\theta - \frac{\pi}{2}\sin\theta}{1+\cos\theta+\cos 2\theta}\right]_{\theta=\frac{\pi}{2}} = \frac{0}{0} = \left[\frac{1-\frac{\pi}{2}\cos\theta}{-\sin\theta - 2\sin 2\theta}\right]_{\theta=\frac{\pi}{2}}$$
$$= \frac{1}{-1} = -1.$$

Or, putting $\theta = \frac{\pi}{2} - \phi$, the fraction is

$$\left[\frac{\frac{\pi}{2} - \phi - \frac{\pi}{2}\cos\phi}{1+\sin\phi - \cos 2\phi}\right]_{\phi=0} = \left[\frac{\frac{\pi}{2} - \phi - \frac{\pi}{2}\left(1 - \frac{\phi^2}{2} + \right)}{1+\phi - 1 + \frac{(2\phi)^2}{2} + }\right]_{\phi=0} = -1.$$

224. Evaluation of $\frac{\infty}{\infty}$.—When $\frac{u(x)}{v(x)}$ defines a continuous function except for $x=a$, and $u(a)=\infty$ and $v(a)=\infty$, the proper value to be given to the fraction in order to complete the definition may be found by writing

$$\frac{u(x)}{v(x)} = \frac{\frac{1}{v(x)}}{\frac{1}{u(x)}},$$

which has the form $\frac{0}{0}$ when $x=a$, and so may be treated by the methods just given.

For example,

$$\left[\frac{\sec\theta}{\sec 5\theta}\right]_{\theta=\frac{\pi}{2}} = \frac{\infty}{\infty} = \left[\frac{\cos 5\theta}{\cos\theta}\right]_{\theta=\frac{\pi}{2}} = \frac{0}{0} = \left[\frac{5\sin 5\theta}{\sin\theta}\right]_{\theta=\frac{\pi}{2}} = 5.$$

225. The following theorem facilitates the evaluation in many cases:

If $u(a) = \infty$ and $v(a) = \infty$,

$$\left[\frac{u(x)}{v(x)}\right]_{x=a} = \frac{\infty}{\infty} = \left[\frac{u'(x)}{v'(x)}\right]_{x=a}.$$

In the following proof, the *x's* are omitted for brevity:

$$\left[\frac{u}{v}\right]_a = \left[\frac{\frac{1}{v}}{\frac{1}{u}}\right]_a = \left[\frac{\frac{-dv}{v^2}}{\frac{-du}{u^2}}\right]_a = \left[\frac{\frac{1}{v}}{\frac{1}{u}} \cdot \frac{\frac{1}{v}}{\frac{1}{u}} \cdot \frac{dv}{du}\right]$$

$$= \left[\frac{u}{v}\right]_a \cdot \left[\frac{u}{v}\right]_a \cdot \left[\frac{dv}{du}\right]_a.$$

Hence

$$\left[\frac{du}{dv}\right]_a = \left[\frac{u}{v}\right]_a,$$

or

$$\left[\frac{u(x)}{v(x)}\right]_{x=a} = \left[\frac{du(x)}{dv(x)}\right]_{x=a} = \left[\frac{u'(x)\,dx}{v'(x)\,dx}\right]_{x=a} = \left[\frac{u'(x)}{v'(x)}\right]_{x=a}.$$

This theorem should be made use of only when it gives forms simpler to invert than the original ones; reduction to $\frac{0}{0}$ is necessary at some time in handling $\frac{\infty}{\infty}$ fractions, for without it, every derived expression will remain $\frac{\infty}{\infty}$.

For example,

$$\left[\frac{\log(x-1)}{\log(x^2-1)}\right]_{x=1} = \frac{\infty}{\infty} = \left[\frac{\frac{1}{x-1}}{\frac{2x}{x^2-1}}\right]_{x=1} = \left[\frac{x+1}{2x}\right]_{x=1} = 1.$$

226. Any factor of an indeterminate form may be evaluated separately, unless the factor is zero or infinite.

For example:

$$\left[\frac{\cos 2\theta}{1-\tan\theta}\right]_{\theta=\frac{\pi}{4}}=\frac{0}{0}=\left[\frac{(\cos^2\theta-\sin^2\theta)\cos\theta}{\cos\theta-\sin\theta}\right]_{\theta=\frac{\pi}{4}}$$

$$=(\cos\theta+\sin\theta)\left[\frac{(\cos\theta-\sin\theta)\cos\theta}{(\cos\theta-\sin\theta)}\right]_{\theta=\frac{\pi}{4}}=1.$$

$$\left[\frac{1-\cos x}{x\log(1+x)}\right]_{x=0}=\left[\frac{1-\cos x}{x^2}\,\frac{x}{\log(1+x)}\right]_{x=0}$$

$$=\tfrac{1}{2}\left[\frac{x}{\log(1+x)}\right]_{x=0}=\tfrac{1}{2}\left[\frac{1}{\frac{1}{1+x}}\right]_{x=0}=\tfrac{1}{2}.$$

227. The Forms $0\times\infty$, 1^∞, 0^0, and ∞^0.—Other types of indeterminate forms may be treated by first reducing them to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as follows:

$0\times\infty$: To evaluate $[\phi(x)\cdot\theta(x)]_{x=a}$ if $\phi(a)=0$, $\theta(a)=\infty$; write

$$[\phi(x)\cdot\theta(x)]_{x=a}=\left[\frac{\phi(x)}{[\theta(x)]^{-1}}\right]_{x=a}=\frac{0}{0}$$

or

$$=\left[\frac{\theta(x)}{[\phi(x)]^{-1}}\right]_{x=a}=\frac{\infty}{\infty}.$$

1^∞: To evaluate $[[\phi(x)]^{\theta(x)}]_{x=a}$ if $\phi(x)=1$, $\theta(x)=\infty$, evaluate

$$[\log[\phi(x)]^{\theta(x)}]_{x=a}=[\theta(x)\cdot\log\phi(x)]_{x=a}=\infty\cdot 0.$$

If the result is b,

$$[[\phi(x)]^{\theta(x)}]_{x=a}=e^b.$$

0^0, ∞^0: To evaluate $[[\phi(x)]^{\theta(x)}]_{x=a}$ if $\phi(a)=0$ or ∞, $\theta(a)=0$, evaluate

$$[\log[\phi(x)]^{\theta(x)}]_{x=a}=[\theta(x)\cdot\log\phi(x)]_{x=a}=0(\mp\infty).$$

If the result is b,

$$[[\phi(x)]^{\theta(x)}]_{x=a}=e^b.$$

228. *Examples.*

Determine the following:

1. $\left[\dfrac{\tan x - x}{x - \sin x}\right]_{x=0} = 2.$

2. $\left[\dfrac{1-x}{\log x}\right]_{x=1} = -1.$

3. $\left[\dfrac{x^2}{1-\cos mx}\right]_{x=0} = \dfrac{2}{m^2}.$

4. $\left[\dfrac{a^x - b^x}{x}\right]_{x=0} = \log \dfrac{a}{b}.$

5. $\left[\dfrac{e^x - e^{-x} - 2x}{x - \tan x}\right]_0 = -1$

6. $[(1-x)^{1/x}]_0 = \dfrac{1}{e}.$

7. $\left[\log \dfrac{\tan x}{x}\right]_0 = 0.$

8. $\left[\left(\dfrac{\tan x}{x}\right)^{1/x^2}\right]_0 = e^{\frac{1}{3}}.$

9. $\left[\dfrac{m \sin x - \sin mx}{x(\cos x - \cos mx)}\right]_{x=0} = \dfrac{m}{3}.$ (Use series.)

10. $\left[\dfrac{\tan nx - n \tan x}{n \sin x - \sin nx}\right]_{x=0} = 2.$ (Use series.)

11. $[x \tan x - \frac{\pi}{2} \sec x]_{x=\frac{\pi}{2}} = -1.$

12. $[\sec \theta - \tan \theta]_{\theta=\frac{\pi}{2}} = 0.$

13. $[\sin x^{\tan x}]_{x=0} = [\tan x^{\sin x}]_{x=0} = 1.$

14. $[(\sin x)^{\sec^2 x}]_{x=\frac{\pi}{2}} = \sqrt{\dfrac{1}{e}}.$

15. $\left[(\cos mx)^{n/x^2}\right]_{x=0} = \dfrac{1}{\sqrt{e^{nm^2}}}.$

16. Trace the curve $y = x^{2 \log x}$.

17. Trace the curve $y = x^{-3 \log x}$.

CHAPTER IX.

MEAN VALUES.

229. The sum of a set of quantities divided by their number is the *average* or *mean* of the quantities. That is, if $a_1, a_2, a_3, \ldots,$ a_n are n quantities, their mean is

$$\frac{1}{n}(a_1+a_2+a_3+\ldots+a_n).$$

Consider the following problem: Two straight roads, AB and AC, make an angle of 30°; from A to B is 1 mile, and along AB telegraph poles are set 110 feet apart; to find the average distance of these poles from AC.

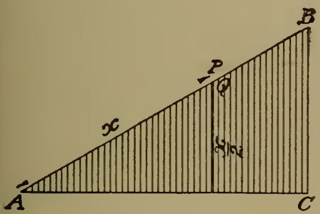


FIG. 78.

Let any pole be P, its distance from AC be PP', and let $AP=x$; then $PP'=\frac{x}{2}$. There are 48 distances, 55 feet, 110 feet, 165 feet. 220 feet, etc.; and their mean is

$$\tfrac{1}{48}\times 55\times(1+2+3+4+\ldots+48)=1347\tfrac{1}{2}\text{ ft.}$$

Now suppose that instead of 48 posts 110 feet apart there were n posts $\frac{5280}{n}$ feet apart; their mean distance from AC would be

$$\frac{1}{n}\left[\frac{5280}{2n}(1+2+3+\ldots+n)\right]=1320\left(1+\frac{1}{n}\right)\text{ft.}$$

If the mile of road AB is bordered by a fence having pickets 2 inches apart, the mean distance of all the pickets from AC is 1320 ft. $\frac{1}{2}$ in.

It is natural to call the mean distance of the roadside from AC,

$$\left[1320+\frac{1}{n}\right]_{n=\infty}=1320\text{ ft.}$$

230. This last result might have been got as follows: Using x as before, suppose points uniformly distributed along AB, k to the foot; and suppose AB divided into parts each dx feet long. If PQ is any one of these parts, it contains kdx points, each of which is approximately $\frac{x}{2}$ feet from AC; the sum of the distances from AC of all the points in PQ is approximately $\frac{x}{2}kdx$ feet, and the number of these distances is k. The sum of the distances from AC of all the points of AB is approximately

$$\sum_{x=0}^{x=5280}\frac{x}{2}\,kdx\text{ feet,}$$

and the number of these distances is approximately

$$\sum_{x=0}^{x=5280}kdx.$$

The mean distance in feet is approximately

$$\frac{\sum\frac{x}{2}\,kdx}{\Sigma kdx}=\frac{\sum\frac{x}{2}\,dx}{\Sigma dx},$$

and is exactly

$$\frac{\int_0^{5280}\frac{x}{2}\,dx}{\int_0^{5280}dx}=\frac{\frac{(5280)^2}{4}}{5280}=1320.$$

We might, with the same result, have supposed each of the distances for points in the element to be $\frac{x}{2}$ feet, and have called the number of points in the element equal rather than proportional to the length of the element; when the idea at the basis of the process is clear, it is well to abbreviate in this way.

231. Illustrative Examples.—Required the mean distance from the base of points on a semicircumference of radius a: (1) regarded as the limit of such a mean distance for points distributed uniformly along the arc, (2) regarded as the limit of such a mean distance for points whose projections are uniformly distributed along the base.

The arc in (1) or the base in (2) is called the *region of uniform distribution.*

In (1), radii to the points bound equal angular divisions; call one of the divisions $d\theta$; the equal divisions of the arc, the region of uniform distribution, are each $a d\theta$ long. Let points be uniformly distributed along the arc so that there are $k a d\theta$ in each element; the distance for each is approximately $a \sin\theta$, the sum of the distances for all the points in the element is $a \sin\theta \cdot k a d\theta$ approximately; the exact value of the mean,

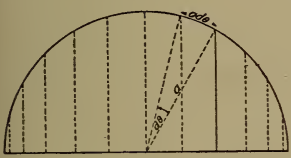


FIG. 79.

$$M = \frac{\int_0^\pi a \sin\theta \cdot k \cdot a d\theta}{\int_0^\pi ka \cdot d\theta} = a \frac{\int_0^\pi \sin\theta\, d\theta}{\int_0^\pi d\theta} = a\frac{2}{\pi} = \frac{2a}{\pi}.$$

In (2), divide the region of uniform distribution, the base, into parts each $= dx$, each containing $k dx$ points, for each of which the distance is approximately $\sqrt{a^2 - x^2}$:

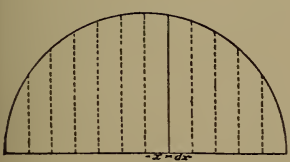


FIG. 80.

Sum of distances for an element $= k\sqrt{a^2 - x^2}\,dx$.

Sum of distances for all the element $= \sum_{-a}^{a} k\sqrt{a^2 - x^2} \cdot dx$.

Number of distances taken in all the elements $= \overset{a}{\underset{-a}{\Sigma}} k\,dx.$

Mean value $= \dfrac{\Sigma\sqrt{a^2-x^2}\cdot dx}{\Sigma dx}$ approximately.

Limit, or mean distance required

$$= \frac{\int_{-a}^{a}\sqrt{a^2-x^2}\cdot dx}{\int_{-a}^{a} dx} = \frac{a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta}{2a} = \frac{\pi}{4}a.$$

The mean distance in (1) is about $0.6366a$; in (2), is about $.7854a$. It is clear that the distances of points near the base, where the arc is nearly vertical, have counted much less heavily in (2) than in (1).

Note that *a mean value is not defined unless the region of uniform distribution is given.*

232. In order to find the mean value of a given function for a given region of uniform distribution, divide this region into elements throughout each of which the function may be assumed to have the same value; find the integral of the product of the function by the element throughout the region, and the integral of the element itself with the same limits. The mean value will be the quotient of the first integral divided by the second.

The region of uniform distribution may be, as in the cases cited, a length, or it may be an area or a volume. Other functions than functions of position are treated in the same way; for instance, the average distance fallen by a body in a given time may be computed as the limit of the average of the distances fallen as it reaches points distributed at uniform distances from top to bottom of its path, or as the limit of the average of the distances fallen at instants of time distributed at uniform intervals from start to finish of its fall.

Lines passing through a fixed point and uniformly distributed about it (so as to divide the space around about into equal plane or solid angles), if they lie in one plane, mark points uniformly distributed around any circumference centered at the fixed point; and if not so confined, mark points uniformly distributed over the surface of any sphere centered at the fixed point.

233. *Examples.*

1. Find the average distance of points of a circle from the center, and the average distance of points of a sphere from the center, the regions of uniform distribution being the area and volume respectively. Ans. $\frac{2}{3}a$ and $\frac{3}{4}a$.

2. Show that the mean distance of points on the circumference of a circle from a fixed point on the circumference is $\frac{4a}{\pi}$, region of uniform distribution the circumference.

3. Show that the mean distance of points of a circle from a fixed point of the circumference is $\frac{32a}{9\pi}$, points uniformly dense over the surface.

4. Find the average distance of points of the surface of a hemisphere from the base, the region of uniform distribution being (a) the hemispherical surface, (b) the base.

Ans. $\frac{a}{2}$, $\frac{2a}{3}$.

5. Find the mean distance of points uniformly distributed over the surface of a sphere from a fixed point on the surface.

Ans. $\frac{4a}{3}$.

6. Find the mean length of chords drawn from a fixed point on the circumference of a circle and uniformly distributed about the point. Ans. $\frac{4a}{\pi}$.

7. Find the mean distance of points uniformly distributed throughout a sphere from a fixed point on the surface.

Ans. $\frac{6a}{5}$.

8. Find the mean length of chords drawn from a fixed point on the surface of a sphere and uniformly distributed about the point. Ans. a.

9. Examples 2 and 6 have the same result; examples 5 and 8, different results. Why?

10. Show that the average distance from the base of points uniformly distributed through the volume of a hemisphere is $\frac{3}{8}$ of the radius.

11. Show that the average latitude of points uniformly distributed over the surface of the northern hemisphere of the earth is about 32° 42′.

12. Show that the mean distance of points uniformly distributed along the perimeter of a square from one corner is 0.8239 of the side.

13. Show that the mean distance from one corner of a square of points uniformly distributed over the area is 0.7652 of the side.

14. A line is divided at random into two parts. Find the mean of the product of the segments so formed, the points of division being uniformly distributed. Ans. $\frac{a^2}{6}$.

15. Show that for the ellipse $x=a\cos\phi$, $y=b\sin\phi$, if the length of a quadrant is Q, the mean radius of curvature for points uniformly distributed along the arc is

$$\frac{\pi}{16abQ}(3a^4+2a^2b^2+3b^4).$$

MECHANICAL APPLICATIONS.

CHAPTER X.

KINEMATICS.

234. Displacement.—Kinematics treats of the relation between change of position and the time in which change of position takes place. A change of position from a point A to a point B is measured by what is called the *displacement AB*, which is determined by the *distance AB* and *the direction of B from A*. Just as the distance from A to B is defined as the length of the straight line joining A and B, and is, therefore, independent of whatever path may actually be traced by a point moving from A to B, so the displacement AB has nothing to do with the path of motion, but is wholly determined by the relative positions of A and B. The direction of a displacement is as important as its numerical magnitude or distance.

235. Two displacements are equal if they produce the same change of position; that is, if they have the same distance and direction. The sum of two displacements is the single displacement which causes the same change of position as the two displacements combined. It is found as follows: Let the given displacements be p and q, Fig. 80a; let the point A be given the displacement p, which carries it to B, determined by drawing AB of the length and direction of p; let the displacement q be given to the point B, so that it is carried to C (BC = and $\|$ to q and in the same direction). Then A is carried by two successive displacements which result in the new position C. This change of position is measured by the single displacement AC, or r, which is thus the sum of the displacements p and q.

$$p+q=r.$$

If the displacement q is made first, and followed by the displacement p, A will be carried to the same final position C, as is evident from the parallelogram $ABCD$ of Fig. 80a. The sum of two displacements is, therefore, independent of the order in which they are combined.

Indeed, the sum of two displacements is the same even if the two take place wholly or partly in the same interval of time. For instance, suppose a man to be rowing a boat in a river, and suppose q in Fig. 80a to be the displacement that would result from his rowing if there were no current, and p to be the displacement that would be caused by the current if he did no rowing; then r is the actual displacement due to the combined causes.

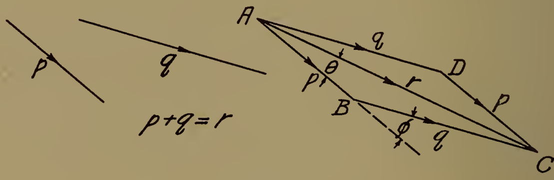


FIG. 80a.

Then, in order to construct graphically the sum of two displacements, choose some convenient scale of distances, and draw a triangle ABC, giving AB and BC the directions of the given displacements, and making their lengths correspond to the distances of the displacements; the direction of the required sum will be from A to C, and its distance, or numerical magnitude, will be represented, to the chosen scale, by the length of AC.

The sum r, of two displacements p and q, is ordinarily called the *resultant* of the displacements, and the displacements p and q are called *components* of r. The addition of displacements is called *composition;* thus, the displacements p and q are said to be *compounded* into the resultant r.

The angle from one displacement to another is the angle through which the first displacement must be turned in order to make it point in the same direction as the second; thus in Fig. 80a the angle from p to q is $\phi = 180° - B$.

The distance and direction of the resultant (or sum) of two component displacements can be computed by the Law of Cosines and the Law of Sines of Trigonometry from the formulas

$$r^2 = p^2 + q^2 + 2pq \cos \phi = p^2 + q^2 - 2pq \cos B,$$

$$\sin \theta = q \frac{\sin \phi}{r}.$$

236. If two displacements involve the same distance, and are opposite in direction ($\phi = 180°$), their combined effect is to cause

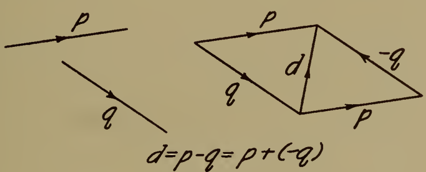


FIG. 81.

no change of position; that is, their sum or resultant is zero. In this case, each displacement is the negative of the other.

If we are given the resultant of two displacements and one of the component displacements and are required to find the other component—in other words, if we are required to subtract a displacement, we can do so by adding its negative.

For instance, if we are to determine the difference $p - q$ of the displacements p and q, in Fig. 81, we have, from the right-hand triangle, $p + (-q) = d$, and evidently d is the required difference, since, from the left-hand triangle, $q + d = p$.

237. If we are given the resultant of two displacements and the two directions of the components, we can determine the components as follows: Let r (Fig. 82) be the given displacement, and let the required directions of its components be those of the lines a and b. Draw to convenient scale and in the proper direc-

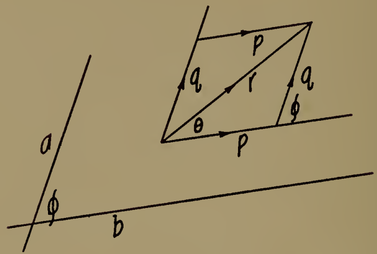


FIG. 82.

tion a line to represent r, and from its ends draw parallels to a and b. In either of the triangles thus formed, we have two displacements, p and q, whose sum is r, as required. The lengths of p and q can be computed by the Law of Sines, since

$$p = r\frac{\sin(\phi-\theta)}{\sin\phi} \text{ and } q = r\frac{\sin\theta}{\sin\phi}.$$

In this case, r is said to be *resolved* along the given directions a and b into the two components p and q. When the given directions are perpendicular, ($\phi = 90°$), the resulting components, $p = r\cos\theta$ and $q = r\sin\theta$, are called *resolved parts*.

238. The most usual way of treating displacements, especially in problems that are at all complicated, is to resolve each displacement along the directions of a pair of rectangular coördinate axes, as in Fig. 83; the resolved parts are then evidently: $x = r\cos\theta$, $y = r\sin\theta$; and the resultant is determined from its components by: $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{r} = \cos^{-1}\frac{x}{r}$. Two such resolved parts, referred to a given

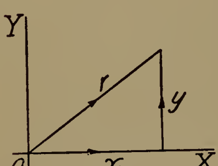


FIG. 83.

pair of axes, determine the resultant displacement completely. In fact, the distance and direction which define the displacement are merely the polar coördinates of the point to which the origin is carried by the displacement; the resolved parts, on the other hand, are the rectangular coördinates of the same point.

The resultant of any number of displacements is found by adding (or compounding) some two, adding the sum of these to a third, and so on, until each of the given displacements has been used.

It is possible to compute the distance and direction of the resultant of a number of displacements from the trigonometric relations already given, but this is by no means the easiest way. It is much simpler to choose a convenient pair of rectangular axes, resolve each of the given displacements along the two axes, combine separately the x-components and the y-components of all the displacements, and finally determine the resultant of the two perpendicular displacements thus found.

For instance, let it be required to find the sum or resultant of the following displacements: p_1 2 mls. N. 20° E., p_2 $2\frac{1}{2}$ mls. N. 35° W., p_3 3 mls. N. 15° E., p_4 $1\frac{1}{2}$ mls. N. 20° W., p_5 45 mls. W. 40° S. Taking the positive direction of the x-axis due east, we have

r.	θ.	$x = r\cos\theta$		$y = r\sin\theta$.	
$r_1=2$	$\theta_1=70°$	$x_1=0.684$		$y_1=1.879$	
$r_2=2.5$	$\theta_2=125°$	$x_2=$	-1.434	$y_2=2.048$	
$r_3=3$	$\theta_3=75°$	$x_3=0.776$		$y_3=2.897$	
$r_4=1.5$	$\theta_4=110°$	$x_4=$	-0.513	$y_4=1.410$	
$r_5=45$	$\theta_5=-140°$	$x_5=$ ——	-34.47	$y_5=$ ——	-28.93
		1.460	-36.42	8.234	$+8.23$
			$+1.46$		
		$X=\Sigma x=$	-34.96	$Y=\Sigma y=$	-20.70

$$\Theta = \tan^{-1}\frac{Y}{X} = \tan^{-1}\frac{20.70}{34.96} = 210° \ 38'.$$

$$R = \sqrt{X^2 + Y^2} = X \sec \Theta = Y \csc \Theta = 40.63.$$

The combined effect of the five displacements is thus the same as that of a single displacement of 40.63 mls. W. 30° 28′ S.

239. *Examples.*

1. A point undergoes the following displacements: 40 ft. N. 60° E.; 50 ft. S.; and 60 ft. N. 60° W.; find the resultant displacement. Ans. $10\sqrt{3}$ ft. W.

2. Show that two component displacements represented by two chords of a circle drawn from any point P and at right angles, are equivalent to a single displacement represented by a diameter of the circle.

3. A ship makes 40 miles S. 30° E., 60 miles S. 60° W., and 50 miles N. 30° W.; find the resultant displacement.
Ans. 60.83 miles N. 159° 28′ W.

4. A particle suffers five successive displacements of magnitudes a, $2a$, $3a$, $4a$ and $5a$, parallel to the sides of a regular hexagon taken in succession; what is the resultant displacement?
Ans. $6a$, making 180° with the first.

5. A steamer is carried by her propeller 12 miles N., and by the wind 3 miles S. 15° E., finds that her displacement is 15 miles NE.; find the displacement due to a current, which is unknown.
Ans. 9.94 miles N. 81° 18′ E.

6. A carriage wheel, 16 inches radius, rolls along a horizontal road; find the displacement of a point originally in contact with the road after the wheel has made a quarter revolution.
Ans. (approx.) 18.4 in., at angle of 60° 17′ with the horizontal.

SPEED, VELOCITY AND ACCELERATION.

240. Speed.—The *speed* of a point moving in a straight line was defined in Arts. 7-10 as follows:

If a point moves Δs feet in Δt seconds, $v = \frac{\Delta s}{\Delta t}$ is its *mean* *speed.* If this mean speed is constant throughout the motion, the point moves with the *uniform speed,* $v = \frac{\Delta s}{\Delta t}$.

If the mean speed is variable, the actual speed of the point at any instant is the value at that instant of the time-derivative of the distance traversed, *i. e.*, $v = \frac{ds}{dt}$.

These are also the definitions of uniform speed, mean speed and speed for motion of any kind.

It should be observed that speed is in any case a *distance;* in the case of uniform speed it is the distance actually traversed in each second; the mean speed is a uniform speed, and a variable speed is the limit of a mean speed.

The speed of a moving point is, therefore, represented graphically by a length corresponding to the distance traversed in one second.

For instance, if a point moves in 5 seconds over the quadrant of a circle of 10 feet radius, its mean speed is $\frac{15.708}{5} = 3.1416$ f/s, and would be represented by a length corresponding (in accordance with a chosen scale) to 3.1416 feet.

241. Velocity.—The definition of *velocity* is similar to the definition of *speed,* but with the important difference that the *displacement* given to the moving point takes the place of the *distance* traversed by the point.

If the velocity is constant, the same displacement must be given to the moving point in any two equal intervals of time; that is, the point must move the same distance and *in the same direction;* in other words, it must move with uniform speed in a straight line. Motion with constant velocity is, therefore, uniform rectilinear motion.

If a point moves in a straight line with variable speed, its mean velocity during any interval after a given instant, and its actual velocity at the given instant, are evidently determined in magnitude by the corresponding speeds and in direction by the direction of the motion.

The velocity of a point moving in any way is represented graphically by a displacement corresponding to the *displacement* given to the point in 1 second. In rectilinear motion, the mean velocity and the velocity are represented by displacements whose distances are the lengths representing the corresponding speeds, and whose directions are the direction of the motion.

242. The velocity of curvilinear motion is always variable, for whether the speed changes or not, the direction must; consequently, although the magnitude of the actual velocity at any instant is the speed at that instant, the magnitude of a mean velocity is not the corresponding mean speed.

Consider, for instance, the example of Art. 240, and suppose, for convenience, that the point moves over the quadrant of the circle of 10 feet radius in 5 seconds at *uniform speed.*

Then, in moving over the quadrant AB (Fig. 84), the point covers 15.708 feet in 5 seconds at the uniform speed (which is also its mean speed) of 3.1416 f/s. This motion, however, gives it a displacement AB of 14.1428 feet in the direction from A to B. Its mean velocity is, therefore, 2.8286 f/s in a direction making an angle of 135° with OA. The mean speed is represented by a length corresponding to 3.1416 feet drawn regardless of direction; the mean velocity by a distance corresponding to 2.8286 feet at the angle 135° with oA, as AM.

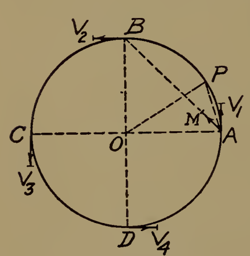


FIG. 84.

To find the actual velocity at A, consider the motion from A to P, calling the angle $AOP = \Delta\theta$. Then the arc $AP = \Delta s = 10\Delta\theta$, and as $\Delta s = 3.1416\Delta t$, $\Delta t = \dfrac{10\,\Delta\theta}{3.1416}$ is the number of seconds required to reach P. The displacement AP is $20 \sin \frac{\Delta\theta}{2}$ in magnitude, and makes the angle $\left(90° - \frac{\Delta\theta}{2}\right)$ with OA The corresponding mean velocity is a displacement, $20 \sin \frac{\Delta\theta}{2} \div \frac{10\,\Delta\theta}{3.1416} =$

$$3.1416 \left(\frac{\sin \frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right)$$

in magnitude and makes the angle $\left(90° - \frac{\Delta\theta}{2}\right)$ with OA. Since $\left[\frac{\sin x}{x}\right]_{x=0} = 1$, the limit approached by this mean velocity as Δt and hence $\Delta\theta$ becomes zero, is the displacement 3.1416 at 90° with OA, represented by AV_1 in the figure. The velocity at A thus has for its magnitude the speed at A, and for its direction the direction of the motion at A. In the same way, the velocity at any point of the path will be seen to have the speed of the motion for its magnitude and the direction of the motion for its direction. If the motion continues around the circle, the velocities at B, C, D are represented by the displacements BV_2 CV_3, DV_4, from which it appears that the change in velocity between opposite points of the circle causes a reversal of direction, and amounts, therefore, to 6.2832 f/s.

243. There is no material difference in the most general case of curvilinear motion. Let a point move in any way along a curve AB; to determine its velocity when, after t seconds, it reaches the point P. Consider its motion from P to P'; call the arc $PP' = \Delta s$, the chord $PP' = \Delta c$ and the time taken to move from P to $P' = \Delta t$. Then the mean velocity is $\frac{\Delta c}{\Delta t}$ in the direction of the chord PP'; and the mean speed is $\frac{\Delta s}{\Delta t}$; the limit of the magnitude of the mean velocity when $\Delta t = 0$ is

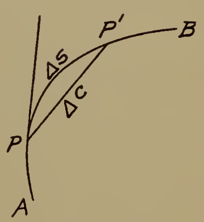


FIG. 85.

$\left.\frac{\Delta c}{\Delta t}\right]_{\Delta t=0} = \frac{ds}{dt}$, the speed at P, and the limiting direction of the velocity is that of the tangent at P or of the path of motion at P.

In any case of curvilinear motion, then, the velocity at any point of the path has for its magnitude the speed at that point, and for its direction the direction of motion.

244. Since a velocity is a displacement (the displacement that would take place if the moving point preserved its motion unchanged for 1 second), the addition and subtraction of two velocities, or the composition and resolution of any number, has already been discussed in the treatment of displacements.

For instance, in the discussion just preceding, we might have referred the motion of the point P to rectangular coördinates. Then the displacement PP' might have been determined by its two components, found by resolving it along the axes. If P has the coördinates (x, y) and P' the coördinates $(x+\Delta x, y+\Delta y)$, the components of the displacement PP' are Δx and Δy in the directions of the axes; the mean component velocities are $\frac{\Delta x}{\Delta t}$ and $\frac{\Delta y}{\Delta t}$ in these directions and the velocity at P has for its components in the directions of the axes the limiting values $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The resultant velocity is, therefore,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

in the direction making the angle

$$\tan^{-1}\left(\frac{dy}{dt} \div \frac{dx}{dt}\right)$$

with the x-axis, or is $\frac{ds}{dt}$ in magnitude, and makes the angle $\tan^{-1}\frac{dy}{dx}$ with the x-axis, and

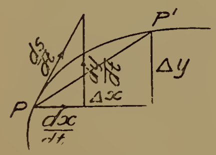


FIG. 86.

so is again seen to have the magnitude of the speed and the direction of the motion.

245. *Examples.*

1. A ship sailing north at a speed of 8 m/h is carried east by a current of 4 m/h; find the resultant velocity.

Ans. 8.94 m/h, N. 26° 34′ E.

2. A point moves N. 30° E. 60 feet in 10 seconds, then west 30 feet in 20 seconds; find the mean speed and the mean velocity.

Ans. Mean speed 3 f/s, mean velocity $\sqrt{3}$ f/s N.

3. What will be the apparent velocity of rain drops falling vertically 20 f/s to a person in a train having a speed of 30 m/h?

Ans. 43.33 f/s, at 24° 27′ with the horizontal.

4. A ship steaming 8 m/h due east has an apparent north wind of 6 m/h; what is the velocity of the wind?

Ans. 10 m/h N. 53° 8′ W. (nearly).

5. If the mean speed of the earth in its path around the sun is 18.6 m/s, and the speed of light is 186,000 m/s, what is the apparent angular displacement of the sun, due to the combined motion of the two?

Ans. 20″.6 in a direction opposite to the earth's motion.

6. A ship heading east at 10 knots an hour, makes 10 knots an hour NE. due to a current; find its velocity.

Ans. 7.65 knots an hour N. 22½° W.

7. Find the velocity of a point on the rim of a wheel, radius a, rolling along a straight line.

Ans. $v=4a\sin\theta\dfrac{d\theta}{dt}$, in a direction $90°-\theta$ to the given line.

($\theta=\frac{1}{2}$ the angle of revolution measured from the lowest point.)

8. Find the velocity of the point of problem 7, when the center of the wheel moves uniformly at a speed of 60 m/h; $a=4$ feet and $\phi=2\theta=\dfrac{\pi}{2}$ and π.

Ans. $v=88\sqrt{2}$ f/s for $\phi=\dfrac{\pi}{2}$, $v=176$ f/s for $\phi=\pi$.

9. A sailing ship heads NE. and makes a speed of 15 m/h, at the same time the wind carries her 2 m/h SE., and a current 4 m/h N. 30° W.; what is her velocity over the ground?

Ans. 16.14 m/h N. 38° 23′ E. (approx.).

246. Acceleration.—Acceleration was defined in Art. 75 for rectilinear motion as the time-rate of speed: $v=\frac{ds}{dt}$, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$. This special case of acceleration is the distance a second that the speed increases in each second, or, if the acceleration is variable, is the limit of such an increase.

Acceleration is defined in general as the time-rate of increase of velocity, and so is the displacement a second by which the velocity increases in each second, or the limit of such an increase. As an acceleration is thus a displacement, the composition, resolution, etc., of accelerations is merely another special case of the performance of these operations on displacements.

247. Acceleration in Rectilinear Motion.—In the case of rectilinear motion with uniform speed, the acceleration is zero, since there is no change of velocity. For rectilinear motion with variable speed, the change in velocity in any interval of time is a velocity in or opposite to the direction of motion—for if the increase in velocity were in any different direction, its addition would change the direction of motion.

Hence, for rectilinear motion, the mean change in velocity during the interval, or the mean acceleration, is in the direction of motion for any interval of time, and its limit, the acceleration, is also in the direction of motion.

It is, however, *only* in the case of rectilinear motion, that is, motion in which the *direction of the velocity does not change,* that the acceleration is in the direction of motion.

248. Acceleration in Uniform Circular Motion.—Consider, for instance, the case next in simplicity to rectilinear motion: uniform circular motion, in which a point moves along the circumference of a circle at a constant speed. Let the radius of the circle be a feet, and the constant speed z f/s. The velocity at any point in the path has for its magnitude the speed, z f/s, and for its direction the direction of the motion, and so may be represented by a tangent to the path at the point in question, marked in the direction of the motion, of length to represent z feet. Let PV represent in this way the velocity v at a point P, and $P'V'$ the velocity $v+\Delta v$ at a point P', the arc PP' being Δs feet and subtending the central angle $\Delta\theta$. Construct the displacement $P'R$ to represent $\Delta v = v + \Delta v + (-v)$. Then $P'R = \Delta v$ is the change that occurs in the velocity of the moving point as it goes from P to P'. Dividing Δv by the number of seconds the motion takes, we shall have the mean acceleration in the corresponding interval. The distance $PP' = \Delta s = a\Delta\theta$ is covered uniformly at z f/s, taking $\dfrac{a\Delta\theta}{z}$ seconds. The mean acceleration is, therefore, $\dfrac{z\Delta v}{a\Delta\theta}$. In the triangle $P'V'R$ each of the sides $P'V'$ and RV' represents z feet, and the angle $V' = \Delta\theta$. Hence the side $P'R$ represents $2z \sin\dfrac{\Delta\theta}{2}$ feet. The angle $RP'V' = 90° - \dfrac{\Delta\theta}{2}$. Thus the magnitude of the mean acceleration is

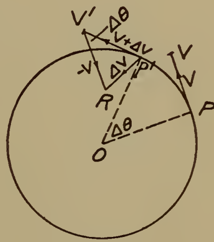


FIG. 87.

$$\frac{z\Delta v}{a\Delta\theta} = \frac{z \times 2z \sin\dfrac{\Delta\theta}{2}}{a\Delta\theta} = \frac{z^2}{a}\,\frac{\sin\dfrac{\Delta\theta}{2}}{\dfrac{\Delta\theta}{2}},$$

and its direction is at the angle $90° - \frac{\Delta\theta}{2}$ with the direction of motion or is at the angle $\frac{\Delta\theta}{2}$ with the radius $P'O$ (directed from P' to the center O). In the limiting case of the acceleration at P, ($\Delta\theta = 0$) we therefore have: The acceleration at any point of the path in uniform circular motion is $\frac{z^2}{a}$ in magnitude, and is directed toward the center of the circle; z being the speed of the motion, a the radius of the circle.

249. It should be observed that in all the preceding discussions we have used foot and second as units of length and time; the statements would have been more general if we had written "unit of length" and "unit of time" in all cases, but the definite units are simpler. Our results, of course, apply in the case of any units, but it is necessary that the units used in any particular discussion should be the same throughout.

250. The treatment of uniform circular motion by rectangular components is instructive. Let the point move as before in a circle of radius a feet at a uniform speed of z f/s. Refer the motion to a pair of perpendicular axes, OX and OY, drawn through the center of the circle, and, P being any position of the moving point, let (x, y) be its coördinates and the angle $XOP = \theta$. Let us find the accelerations of M and N, the projections of P on OX and OY. $x = a\cos\theta$, $y = a\sin\theta$, arc $AP = a\theta$; and since $\frac{d(a\theta)}{dt} = z$, $\frac{d\theta}{dt} = \frac{z}{a}$, a constant.

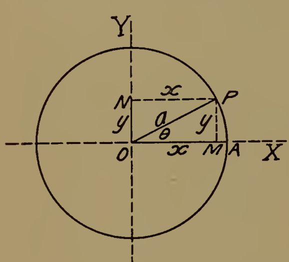


FIG. 88.

Call $\frac{z}{a}=\frac{d\theta}{dt}=\omega$.

$$\frac{dx}{dt}=-a\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt}=a\cos\theta\ \frac{d\theta}{dt},$$

$$\frac{dx}{dt}=-a\omega\sin\theta, \quad \frac{dy}{dt}=a\omega\cos\theta,$$

$$\frac{d^2x}{dt^2}=-a\omega\cos\theta\ \frac{d\theta}{dt}, \quad \frac{d^2y}{dt^2}=-a\omega\sin\theta\ \frac{d\theta}{dt},$$

$$\frac{d^2x}{dt^2}=-a\omega^2\cos\theta, \quad \frac{d^2y}{dt^2}=-a\omega^2\sin\theta.$$

Then the velocity of motion is given by the components $-a\omega\sin\theta$ in the direction of the x-axis and $a\omega\cos\theta$ in the direction of the y-axis, or is $a\omega\sqrt{\sin^2\theta+\cos^2\theta}=a\omega=z$ in the direction making $\tan^{-1}\frac{a\omega\cos\theta}{-a\omega\sin\theta}=\sin^{-1}\frac{a\omega\cos\theta}{a\omega}=\tan^{-1}(-\cot\theta)=\sin^{-1}(\cos\theta)=$ $90^\circ+\theta$ with the x-axis. This merely reproduces the given conditions.

The acceleration has for its components $-a\omega^2\cos\theta$ in the direction of the x-axis, $-a\omega^2\sin\theta$ in the direction of the y-axis, and so is $a\omega^2$ in magnitude and is directed at the angle $\tan^{-1}\frac{-a\omega^2\sin\theta}{-a\omega^2\cos\theta}=$ $\sin^{-1}\frac{-a\omega^2\sin\theta}{a\omega^2}=\tan^{-1}\tan\theta=\sin^{-1}(-\sin\theta)=180^\circ+\theta$ with the x-axis. Since $\omega=\frac{z}{a}$ and $a\omega^2=\frac{z^2}{a}$, the results of the earlier discussion are thus verified.

251. In the general case of motion in a plane curve, the acceleration is determined as follows: Let a point moving in a plane curve (Fig. 89) be at P and Δt seconds later be at P'. Let the arc $PP'=\Delta s$, and let the velocities at P and P' be v and $v+\Delta v$, represented by the displacements PV and $P'V'$. Construct the increment $\Delta v=P'R$; then the mean acceleration in the interval of Δt seconds has the direction of $P'R$ and the magnitude $\frac{P'R}{\Delta t}$.

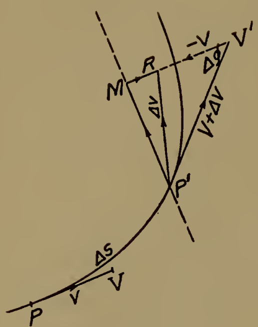


FIG. 89.

It is convenient to treat the components of the acceleration in the direction of motion at P and in the perpendicular direction. Resolving Δv in these directions, we have a mean component acceleration of magnitude $\frac{MR}{\Delta t}$ in the direction of motion at P, and a mean component acceleration of magnitude $\frac{P'M}{\Delta t}$ in the direction of the *interior* normal at P. Let $\Delta\phi$ be the angle between the tangents at P and P'; then the angle $P'V'M = \Delta\phi$, and

$$MR = P'V' \cos\Delta\phi - RV' = (v + \Delta v)\cos\Delta\phi - v$$
$$= \Delta v \cos\Delta\phi - 2v\sin^2\frac{\Delta\phi}{2}.$$

The magnitude of the component of the mean acceleration in the direction of the tangent at P is thus

$$\frac{MR}{\Delta t} = \frac{\Delta v}{\Delta t}\cos\Delta\phi - v\sin\frac{\Delta\phi}{2} \cdot \frac{\sin\frac{\Delta\phi}{2}}{\frac{1}{2}\Delta\phi} \cdot \frac{\Delta\phi}{\Delta t}.$$

The limit of this, when $\Delta t = 0$, is $\frac{dv}{dt} \times 1 - v \times 0 \times 1 \times \frac{d\phi}{dt} = \frac{dv}{dt}$, since, when $\Delta t = 0$, $\Delta\phi = 0$.

The component in the direction of motion at P of the acceleration at P is thus $\frac{dv}{dt}$, where v represents the magnitude of the velocity, or the *speed*, $\frac{ds}{dt}$, at P. This component, $\frac{dv}{dt} = \frac{d^2s}{dt^2}$, is commonly called the *tangential* acceleration or acceleration *in the path of motion*.

Further,

$$P'M = P'V' \sin \Delta\phi = (v + \Delta v) \sin \Delta\phi.$$

The magnitude of the component of the mean acceleration in the direction of the *interior* normal at P is thus

$$\frac{P'M}{\Delta t} = v \frac{\sin \Delta\phi}{\Delta t} + \frac{\Delta v}{\Delta t} \sin \Delta\phi = v \frac{\sin \Delta\phi}{\Delta\phi} \frac{\Delta\phi}{\Delta t} + \frac{\Delta v}{\Delta t} \sin \Delta\phi.$$

The limit of this, when $\Delta t = 0$, is

$$v \frac{d\phi}{dt} = v \frac{ds}{dt} \frac{d\phi}{ds} = v^2 \frac{d\phi}{ds} = \frac{v^2}{\rho},$$

since, by Art. 80, $\frac{ds}{d\phi}$ is the radius of curvature, ρ.

The component in the direction of the interior normal at P of the acceleration at P is thus $\frac{v^2}{\rho}$, where v represents the magnitude of the velocity, or the speed, $\frac{ds}{dt}$, at P, and ρ is the radius of curvature of the path of motion at P. This component is commonly called the *normal* acceleration at P. The normal acceleration is always directed along the *interior* normal, hence the positive value must always be used for ρ in the formula.

Finally, then, the tangential and normal components of the acceleration of a point moving in a plane curve are $a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ in the direction of motion, and $a_n = \frac{v^2}{\rho}$ along the interior normal, $v = \frac{ds}{dt}$ being the *speed.*

Consequently, the total acceleration is

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \frac{v^4}{\rho^2}}$$

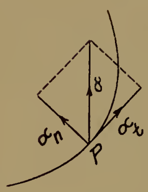


FIG. 90.

in magnitude, and is directed at the angle $\tan^{-1}\frac{a_n}{a_t}=\tan^{-1}\frac{v^2}{\rho\frac{d^2s}{dt^2}}$ with the direction of motion, on the same side of the tangent as the curve itself (Fig. 90).

In the special case of rectilinear motion, we have $\rho=\infty$, $a_n=0$, so that $a_t=\frac{d^2s}{dt^2}$ is the total acceleration.

In the special case of uniform curvilinear motion, we have v constant, $\frac{dv}{dt}=\frac{d^2s}{dt^2}=0$, so that $a_n=\frac{v^2}{\rho}$ is the total acceleration. As a particular example of this case, we have the uniform circular motion already discussed in Art. 248.

In the general case of curvilinear motion, the normal acceleration, $a_n=\frac{v^2}{\rho}$, arises from the changing *direction* of the velocity; the tangential acceleration, $a_t=\frac{d^2s}{dt^2}$, arises from the variation of the *speed*, $v=\frac{ds}{dt}$, or magnitude of the velocity. To emphasize this relation, a_n is sometimes called the *shunt*, a_t the *spurt* of the acceleration.

252. *Examples.*

1. A point moves in a circle of 4 feet radius with a uniform speed of 4 f/s; show that the magnitude of the acceleration is 4 f/s^2.

2. Derive from the component velocities and accelerations the total acceleration in example 1, and show that it is directed towards the center (assume the path of motion $x=4\cos\theta$, $y=4\sin\theta$).

3. A point moving in a given direction with a speed of 600 f/s, one minute later is moving with the same speed in a direction of

60° with the first; what is the total change in velocity and the mean acceleration?

Ans. 600 f/s and 10 f/s^2 in a direction of 120° with the first.

4. The earth's equatorial radius is 3962.8 miles; find the velocity and the acceleration of a point on the equator due to the earth's rotation on its axis.

Ans. 1522 f/s and 0.111 f/s^2.

5. If the value of g on the equator is 32.1 f/s^2, find what horizontal velocity a point must have in the plane of the equator to go around the earth.

Ans. 27,438 f/s E. or 24,394 f/s W.

6. The mean distance of the moon from the earth is 238,800 miles, and its mean period of revolution about the earth $27\frac{1}{3}$ days (approximately); show that its mean speed is 3354.6 f/s and its acceleration 0.0089 f/s^2 = 0.107 inch per second.

7. Show that tangential and normal accelerations of a point on the rim of a wheel of radius a rolling along a straight line are $a_t = 4a \sin\theta \frac{d^2\theta}{dt^2} + 4a\cos\theta\left(\frac{d\theta}{dt}\right)^2$ and $a_n = 4a\sin\theta\left(\frac{d\theta}{dt}\right)^2$, where θ is one-half the angle through which the wheel has turned.

8. If the center of the wheel in example 7 moves at the uniform speed of 60 m/h, and $a = 4$ feet, show that the total acceleration has the constant magnitude 1936 f/s^2 and is directed toward the center of the wheel.

CHAPTER XI.

FORCES.

253. Words are often used as scientific terms in senses rather different from their ordinary meanings; for instance, the distinction between *velocity* and *speed* that we have been obliged to make for the sake of accuracy is foreign to every-day use. The term *force,* however, has in mechanics the meaning with which everybody is familiar. The forces most commonly in evidence are probably the pushes, pulls and twists exerted by one's own muscles and the pull or attraction of the earth, the force of gravity, which is exerted on all material bodies. The simplest force with which to experiment is the pull of a stretched coiled spring or rubber band, for if such a spring is kept stretched to a fixed length, it exerts a constant pull. The force of gravity is not so simple, because it acts more strongly on some bodies than on others.

If a material body of any sort is placed on a very smooth horizontal table and pulled with a constant force (by a spring, for instance), the pull will be practically the only force acting. Under these circumstances, the body will move with increasing velocity, but constant acceleration. If this body and another just like it are acted upon together by the same pull, the acceleration will be half as great, and a pull strong enough to move both bodies with the original acceleration will move one of them alone with twice the acceleration. Again, suppose we have a number of bodies all alike, made of steel, say, and a second set all alike among themselves, made of wood. Suppose the wooden bodies are large enough so that when ten of them together are acted upon by a certain pull they acquire the same acceleration that is given by this pull to three of the steel bodies. Then it will be found that the force required to give a certain acceleration to one wooden body will give $\frac{3}{10}$ this acceleration to one steel body.

As a result of experiments of this sort, though of course most elaborately and carefully made, certain facts and definitions have been established, on the basis of which the science of mechanics has been developed.

254. First, it is found that any force, acting by itself on any material body, gives to the body an acceleration in the *direction* of the force. Also, if the magnitudes of different forces are defined to be proportional to the accelerations they give to the same body, it is found that the same relative magnitudes will thus be assigned to all forces, no matter what body is used in testing them.

As a corollary of this law, it appears that if no force acts on a body, the acceleration of the body will be zero; that is, the body will be at rest or else moving with uniform speed in a straight line.

It is found that the size and material of a body affect the magnitude of the acceleration it receives from the action of a force, and that if the *masses* of different bodies are defined as magnitudes inversely proportional to the accelerations given to the bodies by the same force, the relative magnitudes thus assigned will be independent of the force used in the tests.

255. Law of Motion.—All the preceding is summed up in the *Law of Motion:*

If a force f, acting on a body of mass m, gives to it an acceleration α, the magnitude of f is proportional to the product of the magnitudes of m and α, and the direction of f is the direction of α.

This relation of the magnitudes of f, m and α is expressed in the *Equation of Motion:*

$$f = km\alpha.$$

256. Units.—The value that must be given to the constant k will of course depend upon the units chosen for acceleration, mass and force. It is desirable to make the equation of motion as simple as possible; consequently the relation between the units is always made such that the *unit force will give unit acceleration to a body of unit mass;* then $1 = k \times 1 \times 1$, or $k = 1$, and the equation of motion becomes

$$f = ma.$$

There are four systems of units in use; in the first place, the English Systems take 1 f/s (one foot a second each second) as the unit of acceleration, and the Metric Systems take 1 cm/s². Further, in the Gravitational or Engineer's Systems, a familiar force is chosen as the unit of force, so that the mass to which this force will give unit acceleration must be the unit of mass, while in the Absolute Systems, the mass of a well-known body is taken as the unit of mass, so that the force which will give it unit acceleration is the unit of force. The basis of all these systems is the fact, experimentally established, that the earth's attraction (the force of gravity), if unimpeded, will give the same acceleration to any two bodies in the same situation. The value of this acceleration is indicated by g; it varies for different situations, and is not even constant over the surface of the earth; its surface-value, however, is never far from 32.2 f/s² or 981 cm/s².

The force with which gravity acts on a body (under certain standard conditions) is called the *weight* of the body; the weight of a piece of platinum kept in the Standards Office in London is called *one pound,* and the weight of a piece of platinum in the Palais des Archives in Paris is called *one kilogram.*

I. In the English Gravitational System, the unit of force is the *pound,* the unit of acceleration is 1 f/s², and the unit of mass is the mass to which a force of 1 pound would give an acceleration of 1 f/s². Now if a body weighing 1 pound contains m units of mass, the equation of motion, $f = ma$, gives, for the force with which gravity acts on the body, $1 = m \cdot g$, whence $m = \dfrac{1}{g}$; that is, a body weighing 1 pound contains $\dfrac{1}{g}$ (about $\frac{1}{32}$) units of mass, or the unit of mass is the mass of a body weighing g (about 32) pounds. Consequently, there are $\frac{W}{g}$ units of mass in a body weighing W pounds.

II. In the Metric Gravitational System the unit of force is 1 *gram*, the unit of acceleration is 1 cm/s², and the unit of mass, to which the unit force gives unit acceleration, weighs 981 grams.

III. In the English Absolute System the unit of mass is the mass of the body (the piece of platinum spoken of earlier) which weighs 1 pound, the unit of acceleration is 1 f/s², and the unit of force is the force which gives unit acceleration to the unit mass. Then, if the force with which gravity acts on a body weighing 1 pound (the force of 1 pound) contains f of the units of force of this system, the equation of motion, $f=ma$, gives for the force with which gravity acts on this body:

$$f=1\cdot g, \text{ or } f=g;$$

that is, the force of 1 pound contains g (about 32) of the units of force of this system. The unit of force in the English Absolute System is called a *poundal;* 1 poundal $=\frac{1}{g}$ pound, about $\frac{1}{32}$ pound, or half an ounce. A force of x pounds is thus a force of gx poundals—about $32x$ poundals; a body weighing W pounds or Wg poundals contains W absolute units of mass and $\frac{W}{g}$ gravitational units of mass.

IV. In the Metric Absolute System the unit of mass is the mass of a body weighing 1 gram, the unit of acceleration is 1 cm/s², and the unit of force is the force which gives unit acceleration to a unit mass. The absolute metric unit of force is called a *dyne;* it is $\frac{1}{981}$ gram, or 1 gram $=981$ dynes.

There is no name for any one of the four different units of mass; this is because in practice the mass of a body is always expressed in terms of the weight of the body. Thus the equation of motion for a body weighing W pounds, acted upon by a force of f pounds, and acquiring an acceleration of a f/s², is $f = \frac{W}{g} a$; since the body contains $\frac{W}{g}$ gravitational units of mass.

Again, a body weighing W' poundals, acted upon by a force of f' poundals, and acquiring an acceleration of a' f/s², has for its equation of motion $f' = \frac{W'}{g} a'$; since the body weighs $\frac{W'}{g}$ pounds, and so contains $\frac{W'}{g}$ absolute units of mass.

Consequently, the equation of motion for a body in terms of the *weight* of the body, the force acting, and the acceleration acquired, can always be written $f = \frac{W}{g} a$, provided that f and W are both expressed in terms of the same unit, either the pound or the poundal, and that a and g are expressed in terms of the same unit, the foot a second each second.

The same is of course true of the metric units, with centimeter, gram and dyne in place of foot, pound and poundal. Indeed, this equation, $\frac{f}{W} = \frac{a}{g}$, is merely the assertion that the magnitudes of the accelerations given to the same body by two different forces are proportional to the magnitudes, f and W, of the forces themselves.

257. Composition and Resolution of Forces.—It is found by experiment that when a number of forces act at once on the same body, the effect of each is independent of the effect of the others; that is, that the acceleration actually given by all the forces is the sum or resultant of the several accelerations that would be given by the different forces, each acting by itself. If each of the forces acting on a body is represented by a line having the direction of the force and having a length proportional to the magnitude of the force, all these lines will be in the directions of the accelerations given by the forces, and their lengths will be proportional to the magnitudes of the accelerations. Hence these directed lengths will be displacements that represent (to different scales) either the forces or the accelerations given by them, and the composition and resolution of the forces can be effected by the methods already described for displacements, velocities and accelerations.

That is, the triangle construction, or the corresponding computations, can be used to combine two or more of the forces acting on a body, or to separate any one of the forces into components acting in given directions; then the effect on the motion of a body of such a resultant force is precisely the same as the effect of its components.

258. It should be observed that this treatment of forces is concerned merely with directions and magnitudes. A line drawn in the direction of a force from the point at which the force is applied is called the *line of action* of the force. Shifting the point of application from one point to another along the line of action is found to have no effect if the two points are rigidly connected; but any other change in the point of application does alter the effect of the force. Consequently, in compounding and resolving forces a resultant and its components are treated as having the same point of application.

259. Resolved Parts of a Force.—Perpendicular components of a force are called *resolved parts* of the force, as is the case with

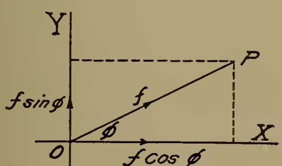


FIG. 91.

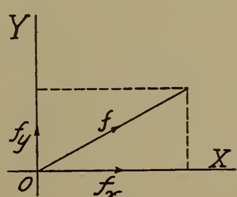


FIG. 92.

accelerations. For instance, a force f, which causes an acceleration a in a given direction OP, may be resolved in two perpendicular directions, OX and OY, giving, if the angle XOP is called ϕ, the resolved part $f \cos \phi$ in the direction OX and the resolved part $f \sin \phi$ in the direction OY (Fig. 91).

Note that the acceleration a in the direction OP amounts to an acceleration $a \cos \phi$ in the direction OX and an acceleration $a \sin \phi$ in the direction OY, and that these are the accelerations that would be due to the forces $f \cos \phi$ and $f \sin \phi$ acting individually.

Again, in order to determine a force f from its resolved parts, f_x in the direction OX and f_y in the direction OY, we have $\sqrt{f_x^2+f_y^2}=f$ for the magnitude of f, and $\phi=\tan^{-1}\frac{f_y}{f_x}=\sin^{-1}\frac{f_y}{f}$ for the direction of f.

260. Equation of Motion for a Given Direction.—The equation of motion, $f=ma$ or $f=\frac{W}{g}a$, expresses the relations between the *magnitudes* of f, m and a for any moving body; the fact that f and a have the same *direction* is of equal importance, but can be no more than implied in the equation. Since, however, a resolved part of the acceleration is due to the resolved part in the same direction of the force, we can always write the equation of motion for the resolved parts in two different directions, thus obtaining two algebraic relations which involve both the equality of magnitudes of f and ma and the identity of directions of f and a. The Law of Motion is consequently utilized in practice in the following form:

The (algebraic) sum of the resolved parts in any given direction of all the forces acting on a body is equal to the mass of the body multiplied by the resolved part of the acceleration in that direction.

This is formulated: $f_1 \cos \phi_1 + f_2 \cos \phi_2 + f_3 \cos \phi_3 + \ldots =$

ma', or $\Sigma f \cos \phi = ma'$; $\phi_1, \phi_2, \phi_3, \ldots$ etc., representing the angles between the direction of a' and the directions of $f_1, f_2, f_3, \ldots$ etc.

The Equation of Motion may be written for any two directions that prove convenient; there is no reason why the two should be perpendicular. It is often desirable to write an equation that shall fail to contain one of the forces; it is evident that this can be done by choosing the direction perpendicular to that of the force in question.

261. *Examples.*

Use $g=32$ unless otherwise indicated.

1. If 1 pound $=453.59$ grams, and 1 meter $=3.281$ feet, convert a pound into dynes. $g=32.19$. Ans. 445,000.

2. If a force of 10 pounds produces in a body an acceleration of 20 f/s², find the weight and the mass of the body.
Ans. 16 pounds; $\frac{1}{2}$.

3. A weight of 10 pounds lies in the scale pan of a spring balance, hanging from the top of an elevator; if the elevator starts up with an acceleration of 5 f/s², what would the balance indicate?
Ans. 11 pounds 9 ounces.

4. A weight of 40 pounds is hanging vertically by means of a long string; what force applied horizontally would give the weight an acceleration of 4 f/s²? Ans. 5 pounds.

5. A weight of 10 pounds hanging vertically by a string is pushed by a horizontal force so that the string makes an angle of 30° with the vertical; find the force and the tension of the string.
Ans. $f=5.77$ pounds and tension $=11.55$ (approx.).

6. A weight of 100 pounds is suspended from two pegs, placed in a horizontal line 5 feet apart, by two cords 3 and 4 feet long, respectively; find the tension in each cord, by construction and also by the equations of the forces resolved horizontally and vertically.
Ans. Shorter cord 80 pounds, the other 60 pounds.

7. A weight of 2000 pounds is suspended by two ropes making angles of 30° and 45° with the vertical, respectively; find t_1 and t_2, the corresponding tensions in the ropes.
Ans. $t_1=1464.1$, $t_2=1035.3$ (approx.).

8. Two forces of 3 and 5 pounds acting at a point have a resultant of 7 pounds; find the angle between the forces and also between each force and the resultant.

Ans. 60°, 21° 47′ and 38° 13′ (approx.).

9. If a man can lift 180 pounds when standing on the ground, how much can be lift in an elevator ascending with an acceleration of 8 f/s^2? When it is descending with the same acceleration?

Ans. 144 and 240 pounds.

10. An anchor weighing 5000 pounds hangs vertically from the end of a boom which makes an angle of 45° with the vertical mast where it is hinged; the outer end is supported by a lift making an angle of 60° with the mast; find by resolution of the forces acting at the end of the boom, the tension on the lift and the thrust on the boom.

Ans. $T = 3660$ pounds, $P = 4482$ pounds.

CHAPTER XII.

Motion of a Heavy Particle.

262. Definitions.—The equation of motion can be used to determine an unknown force when the acceleration is given, or to determine the acceleration when the forces are given. In one large class of problems it is required that there shall be no motion, *i. e.,* that the acceleration, and, therefore, any resolved part of the acceleration, shall be zero. The study of such problems is called *Statics.* The study of the motion caused by given forces is called *Dynamics.* In a problem in dynamics, when the acceleration has been determined from the equation of motion, there remains a problem in integration, the determination of the velocity and position of the moving body at any time. This part of the problem is treated in accordance with the principles of Kinematics. We have discussed kinematics only so far as concerns the acceleration, velocity and displacement of a moving *point,* and so are prepared to treat only those problems in which all the forces may be considered to act at a single point, and in which the motion is the same as if all the mass of the moving body were concentrated at that point. This is not so narrow a restriction as would appear at first sight, for as we shall see later, a very large class of problems can be treated on this assumption. For instance, we shall prove that the force of gravity acts on any body precisely as it would if all the mass of the body were concentrated at a definite point within the body, called its center of gravity.

When all the forces acting on a body are considered to act at one point, at which the whole mass of the body is concentrated, the body is said to be considered as a *heavy particle.* We shall confine ourselves for the present to the *Dynamics of a Heavy Particle.*

263. Rectilinear Motion under Gravity.—A force having a constant direction produces an acceleration in the same constant direction; a force constant in magnitude as well as direction produces a constant acceleration in a fixed direction. The simplest case of accelerated motion is motion in a straight line with constant acceleration. As an example of uniformly accelerated rectilinear motion consider the motion of a body falling under the action of gravity alone. This is either the motion of a body falling in a vacuum, or approximately the motion of a body falling through the air.

Suppose the body to be H feet above the ground, moving vertically upward with a speed of V f/s, and let the weight of the body be W pounds. Then the force is W pounds, acting vertically downward, and the mass is $\frac{W}{g}$, so that the equation of motion gives, if α is the vertical downward acceleration, $W = \frac{W}{g}\alpha$, whence $\alpha = g$.

Then if, at the end of t seconds, the body is s feet above the ground and moving v f/s vertically upward,

$$-\alpha = +\frac{dv}{dt} = -g, \qquad (1)$$

as the upward velocity is decreased by the downward acceleration.

Integrating, we have $v = -gt + C$. Here C is some constant; since, by the given conditions, $v = V$ when $t = 0$,

$$V = -g \cdot 0 + C, \text{ or } C = V;$$

hence

$$v = V - gt. \qquad (2)$$

As s, the distance above the ground, is increased by the upward velocity v, $v = +\frac{ds}{dt}$. Hence, $\frac{ds}{dt} = V - gt$. Integrating,

$$s = Vt - \tfrac{1}{2}gt^2 + K.$$

K is an undetermined constant; but by the conditions, $s = H$ when $t = 0$; hence,

$$H = V \cdot 0 - \tfrac{1}{2}g \cdot 0 + K, \text{ or } K = H.$$

Finally,

$$s = H + Vt - \tfrac{1}{2}gt^2. \qquad (3)$$

Equations (2) and (3) give the velocity and the position of the body after it has been in motion any given number of seconds; they apply to the case of a body having an initial velocity downward, if V is taken negative.

If we eliminate t from (2) and (3), we derive an expression for v in terms of s; the same result can also be obtained by a different integration of the equation of motion, which is important because it is the only feasible method in a large class of problems involving variable forces.

Since $a = \frac{dv}{dt}$ and $v = \frac{ds}{dt}$, $a = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$; so that we have $v\frac{dv}{ds} = -g$, or $vdv = -gds$.

Integrating, $\frac{1}{2}v^2 = -gs + C$, and since $v = V$ when $s = H$,

$$\tfrac{1}{2}V^2 = -gH + C; \quad C = \tfrac{1}{2}(V^2 + 2gH);$$

hence

$$v^2 = V^2 + 2g(H - s). \qquad (4)$$

Thus the increase in the square of the speed is $2g$ times the distance the body has fallen, and, in particular, the speed is the same at a given height whether the body is rising or falling.

264. We have made no use, in the foregoing discussion, of the equation of motion for the horizontal direction. The horizontal component of gravity is zero, hence the horizontal acceleration is zero. As the horizontal velocity is constant, and is zero at the start, there is no horizontal motion. In this case, then, the simple initial conditions cause the direction of motion to be the same as the direction of the acceleration, or of the force. In the example that follows, the force (and, therefore, the acceleration) is vertical, but the path of motion is a curve.

265. The Parabolic Trajectory.—Suppose that a body acted upon by gravity alone starts from a point H feet above the ground, moving with a speed of V f/s at an angle ϕ above the horizontal. Consider the vertical and horizontal motion separately, with reference to axes drawn through the initial position of the body. Then if (x, y) are the coördinates of the body at any time, it will have moved x feet horizontally and y feet vertically, and will be at the height $(y+H)$ feet. The resolved parts of the acceleration and the velocity are $\frac{d^2y}{dt^2}$ and $\frac{dy}{dt}$ vertically, $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ horizontally.

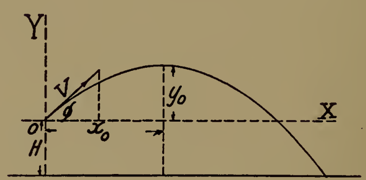


FIG. 93.

Just as in the simpler case, the equation of motion gives for the vertical direction:

$$\frac{d^2y}{dt^2} = -g,$$

and for the horizontal direction,

$$\frac{d^2x}{dt^2} = 0.$$

The initial values, however, are $V \cos \phi$ for $\frac{dx}{dt}$, $V \sin \phi$ for $\frac{dy}{dt}$, and zero for x and for y. The integration of the equations, therefore, gives

$$v_x = \frac{dx}{dt} = V \cos \phi, \quad v_y = \frac{dy}{dt} = V \sin \phi - gt; \qquad (1)$$

$$x = t \cdot V \cos \phi, \quad y = t \cdot V \sin \phi - \tfrac{1}{2} g t^2. \qquad (2)$$

The velocity of the body after t seconds is, therefore, $v = \sqrt{v_x^2 + v_y^2}$ in magnitude, in a direction making the angle $\tan^{-1} \frac{v_y}{v_x} = \cos^{-1} \frac{v_x}{v}$ with the horizontal

The curve traced by the body, called its *trajectory*, is given by the pair of parametric equations (2), which furnish the readiest means of solving most problems involving the trajectory. Eliminating t, we find as the single equation of the path of motion, since $t = \frac{x}{V \cos \phi}$,

$$y = \frac{-gx^2}{2V^2 \cos^2 \phi} + x \tan \phi.$$

The trajectory for a body acted upon by gravity alone is thus a parabola. The highest point (x_0, y_0) of the parabola is given by

$$\frac{dy}{dx} = \frac{V \sin \phi - g t_0}{V \cos \phi} = 0; \quad t_0 = \frac{V \sin \phi}{g};$$

$$x_0 = t_0 \cdot V \cos \phi = \frac{V^2 \sin \phi \cos \phi}{g} = \frac{V^2 \sin 2\phi}{2g}$$

$$y_0 = t_0 \cdot V \sin \phi - \tfrac{1}{2} g t_0^2 = \frac{V^2 \sin^2 \phi}{g} - \frac{V^2 \sin^2 \phi}{2g} = \frac{V^2 \sin^2 \phi}{2g}$$

$$= \frac{V^2 (1 - \cos 2\phi)}{4g}.$$

Transforming to (x_0, y_0) as a new origin, putting $x = x_0 + x'$, $y = y_0 + y'$, $t = t_0 + t'$, and dropping primes, we have

$$x = t \cdot V \cos \phi, \quad y = -\tfrac{1}{2} g t^2, \quad \text{or} \quad y = -\frac{g}{2V^2 \cos^2 \phi} x^2.$$

The highest point is, therefore, the vertex; the axis is vertical, and the parameter is $\frac{V^2 \cos^2 \phi}{2g}$.

In the case of a projectile fired from a gun, the greatest height reached, $\frac{V^2}{4g}(1-\cos 2\phi)$, will be a maximum when the elevation of the gun is 90°, as this value makes $\cos 2\phi$ a minimum.

The horizontal distance traversed when the projectile regains its original level is called the *horizontal range;* it is evidently $R=2x_0=\frac{V^2 \sin 2\phi}{g}$, and is a maximum when $\sin 2\phi$ is a maximum, or $\phi=45°$.

266. Rectilinear Motion under any Constant Force.—The methods of integration used in determining the motion of a freely falling body can be applied directly to any case of rectilinear motion under a constant force. For if the constant force is F pounds, the equation of motion is:

$F=\frac{W}{g}a$, whence $a=\frac{F}{W}g$, so that the constant $\frac{F}{W}g$ takes the place of the constant g throughout the discussion. For instance, suppose a body weighing 20 pounds is placed on a board so inclined that if the body is started down the board it will move with constant velocity, showing that there is no force acting on it (*i. e.,* that the sum of any forces acting is zero), and suppose that it is drawn down the plane (starting from rest) by a spring balance kept at a constant tension of 2 pounds. Then the equation of motion for the direction down the plane is: ($g=32$ f/s²),

$2=\frac{20}{32}a$, whence $a=3.2$ f/s².

Integrating, and determining the constant of integration in accordance with the initial conditions, we have for the speed and the distance after t seconds:

$$v=3.2t, \quad s=1.6t^2.$$

Again, suppose the board on which the weight rests is elevated at an angle of 30° to the horizontal. To simplify the problem, we will suppose that gravity is the only force acting in a downward direction along the board, neglecting the effect of any roughness of the board. The resolved part of the force of gravity down the board is $W \sin 30° = 20 \times \frac{1}{2} = 10$ pounds, so that for the acceleration down the board we have

$$10 = \tfrac{20}{32} a, \quad a = 16 \text{ f/s}^2.$$

Hence, t seconds after motion starts, the velocity and distance down the board are

$$v = 16t \text{ and } s = 8t^2.$$

267. *Examples.*

1\. A weight is dropped from a balloon ascending with uniform speed of 20 f/s, and is observed to strike the ground 4 seconds later; find the height of the balloon and the velocity with which the weight strikes. Ans. $h = 176$ feet, $v = 108$ f/s.

2\. A steamer approaching a dock with engines reversed, producing uniform retardation, is observed to go 300 feet in 20 seconds after reversing engines and 100 feet in the next 10 seconds; find a and v_0 and the distance and time before coming to rest.

Ans. $a = -\frac{1}{3}$ f/s², $v_0 = \frac{55}{3}$ f/s, $t = 55$ seconds, $s = 504\frac{1}{6}$ feet.

3\. A body moves 12 feet while being uniformly retarded from 24 f/s to 6 f/s; find the time and acceleration.

Ans. $t = \frac{4}{5}$ seconds, $a = -22.5$ f/s².

4\. A projectile fired from the top of a tower at an elevation of 45° strikes the ground 60 feet from the foot of the tower at the end of 4 seconds; find the height of the tower, also the time before striking if the projectile had been fired horizontally.

Ans. $h = 196$, $t_2 = 3\frac{1}{2}$ seconds.

5\. A body is thrown at an elevation of 60° with a velocity 150 f/s; find the coördinates of its position at the end of 5 seconds, and its velocity at that instant.

Ans. $y = 249.5$ feet, $x = 375$ feet, $v = 80.82$ f/s, $\tau = -21° 52'$.

6\. Show that the time of descent of a body down any chord drawn from the highest point of a circle is the same as the time of falling down the vertical diameter.

Ans. $t^2 = \dfrac{4a}{g}$ for all chords.

7. A car starts from rest with an acceleration of 4 f/s^2; at what angle must a man lean forward to keep himself in equilibrium? Ans. $\tan^{-1}\frac{1}{8} = 7°\ 8'$ (nearly).

8. A train running at 30 m/h against a constant resistance of 12 pounds per ton of 2000 pounds shuts off steam just as it strikes an up-grade of 1 ft. in 200 ft.; find the time and distance before the train comes to rest. Ans. 125 seconds, $s = 2750$ feet.

9. Find the distance required in example 8 by direct integration of the equation of motion (Art. 266).

10. Show that the speed acquired in sliding down a smooth plane from rest, under the force of gravity, is the same as for falling freely through the height of the plane.

11. A body weighing 20 pounds is pulled along a smooth horizontal plane by a horizontal force of 5 pounds. Find its motion. How is this motion affected if the pull is inclined at the angle ϕ to the horizontal?

Ans. The body moves with constant acceleration; $\alpha = 8$ f/s^2, $\alpha' = 8 \cos\phi$ f/s^2.

12. A body weighing 20 pounds, on a smooth plane inclined at the angle $\tan^{-1}\frac{3}{4}$ to the horizontal, is pulled by a force of 14 pounds acting up the plane and parallel to the plane. Find its motion. How is this motion affected if the pull is inclined at the angle ϕ to the plane?

Ans. The body moves with constant acceleration up the plane: $\alpha = 3.2$ f/s^2, $\alpha' = 3.2(7\cos\phi - 6)$ f/s^2.

13. A body weighing 64 pounds, on a smooth plane inclined at the angle $\sin^{-1}\frac{1}{8}$ to the horizontal, is pulled for 3 seconds by a force of 12 pounds acting up the plane and parallel to the plane. How far up the plane will it go, and with what speed will it pass its initial position in its descent? Ans. 9 feet, $6\sqrt{2}$ f/s.

14. A canal-boat weighing $7\frac{1}{2}$ tons is brought from rest to a speed of 3 m/h in one minute by a constant pull of 200 pounds, making the angle $\cos^{-1}\frac{24}{25}$ with the direction of motion. Find the resistance, assumed constant.

Ans. 163.4 pounds, making the angle 200° 3′ with the direction of motion.

268. Rectilinear Motion under Two or More Forces and under Variable Forces.—When a force is applied to a body, it is generally impossible to prevent other forces from acting, so that the motion is caused, not by the applied force alone, but by this force combined with one or more resistances. For instance, the motion of a body under the action of gravity is affected by the resistance of the air through which it falls, by the resistance of a rough surface down which it slides, or by the resistance of a coiled spring or mass of sand upon which it falls. Either an applied force or a resistance may vary as it acts. Any force has laws of action, more or less accurately determined by experiment, from which its effect can be deduced. We shall next consider a few of the most familiar forces.

269. Hooke's law for the force exerted by an elastic rod, cord, coiled spring, etc., holds for all practical purposes provided the stretching or compression is not great enough to destroy the elasticity. It is formulated as follows:

The natural length being l, and the length when stretched $l+\Delta l$, the force exerted is proportional to Δl.

The actual value of the force depends upon both the material and the form of the elastic object.

For example, suppose a spring 10 inches long, which exerts a pull of 2 pounds when stretched to a length of 11 inches, to be hung up with its axis vertical, and a body weighing 5 pounds to be attached to its lower end. Then when the spring is stretched s feet it exerts a pull of $24s$ pounds. The equation of motion for the forces acting on the body in the downward direction (if we neglect the weight of the spring and the resistance of the air) is:

$$5-24s=\tfrac{5}{32}\, v\, \frac{dv}{ds},$$

since $a=v\,\dfrac{dv}{ds}$ (see Art. 263).

Separating the variables, we have:

$$vdv=32\left(1-\tfrac{24}{5}s\right)ds.$$

Integrating, and determining the arbitrary constant (s and v are zero together):

$$v^2 = 64(s - \tfrac{12}{5}s^2) = \left(\frac{ds}{dt}\right)^2.$$

Again separating the variables,

$$dt = \frac{ds}{8\sqrt{s - \frac{12}{5}s^2}}, \text{ or } 16\sqrt{\tfrac{3}{5}}dt = \frac{ds}{\sqrt{\frac{5}{12}s - s^2}}.$$

Integrating, and determining the arbitrary constant,

$$t = \frac{\sqrt{15}}{48}\cos^{-1}\frac{5-24s}{5}, \qquad (1)$$

whence

$$s = \tfrac{5}{24}\left(1 - \cos\frac{16t\sqrt{15}}{5}\right) = \tfrac{5}{12}\sin^2\frac{8t\sqrt{15}}{5}. \qquad (2)$$

The motion is, therefore, an oscillation $\frac{5}{12}$ feet or 5 inches down and back, repeated indefinitely. During a complete oscillation, $\cos\frac{16t\sqrt{15}}{5}$ goes through a complete cycle of its values, from 1 through 0, -1, 0 back to 1, and $\cos^{-1}\frac{5-24s}{5}$ increases from $2k\pi$ to $2(k+1)\pi$; hence the time of a complete oscillation is

$$T = \frac{\sqrt{15}}{48}2\pi = \frac{\pi\sqrt{15}}{24} = .51 \text{ second (nearly)}. \qquad (3)$$

The neglected resistances will actually shorten the oscillations until the body comes to rest $2\frac{1}{2}$ inches below its initial position, the spring then being stretched by a steady pull of 5 pounds.

270. The Force of Gravity.—According to the law of gravitation (which is universal) any two particles of matter attract each other with a force proportional to their masses and inversely proportional to the square of the distance between them. It follows from this that a sphere of which the density is the same at all points equidistant from its center attracts a particle at any distance s from its center as if all the mass of the sphere were concentrated at its center, with a force inversely proportional to s^2 if the particle is outside the sphere, directly proportional to s if the particle is inside the sphere.

This is a very close approximation to the law of the attraction of the earth, or of the force of gravity. If we denote the force of gravity on a particle of mass m at a distance s feet from the center of the earth by f pounds, the value of f at the surface of the earth by mg, and the radius of the earth by a,

$$f=\frac{bm}{s^2} \text{ if } s>a, \quad f=cms \text{ if } s<a, \quad f=mg \text{ if } s=a.$$

Hence, taking $g=32.2$ f/s², $a=3960$ mls., $b=ga^2=1.41\times10^{16}$, $c=\dfrac{g}{a}=1.54\times10^{-6}$.

For a body falling directly toward the center of the earth from a great distance, we have, neglecting resistances

$$f=\frac{bm}{s^2}=-ma, \text{ or } a=\frac{-b}{s^2}.$$

For a body falling vertically inside the earth (down a mine shaft, for instance) we have, again neglecting resistances,

$$f=cms=-ma, \text{ or } a=-cs.$$

These equations are integrated in essentially the same way as the equation of motion under Hooke's Law.

271. *Examples.*

1, 2. An elastic cord, of natural length l feet, is fastened at a point l feet vertically below a hole in a smooth horizontal table. A particle P, weighing W pounds, is attached to the free end of the cord and placed on the table. Let the particle be s feet from the hole at the end of t seconds.

1. Given $W=32$ pounds, find the motion if the particle is drawn back 3 feet from the hole and let go, the tension of the cord being 12 pounds when the particle is started.

Ans. $s=3\cos 2t$; the particle oscillates back and forth, completing an oscillation in 3.14 seconds.

2. Given $W=4$ pounds, find the motion if the particle is started from the hole with an initial (horizontal) speed of 10 f/s, a steady pull of 2 pounds being needed to hold the particle 1 foot from the hole.

Ans. $s=\frac{5}{2}\sin 4t$; the particle makes a complete oscillation in 1.57 seconds.

3. A coiled spring is set up on a firm support with its axis vertical; the natural length of the spring is 5 feet; under a steady pressure of 10 pounds, the spring is compressed to a length of $4\frac{1}{2}$ feet. A weight of 10 pounds is placed upon the spring. Find the motion of the weight.

Ans. If the top of the spring is at C under a steady pressure of 10 pounds, the weight will be s feet below C, t seconds after it has passed C, where $s=\frac{1}{2}\sin 8t$; and will oscillate from 6 inches below C to 6 inches above and back, making a complete oscillation in $T=0.79$ seconds (nearly).

4. If the weight in example 3 is dropped upon the spring from a point 2 feet above, find the motion.

Ans. Using s and t as before, $s=\frac{3}{2}\sin 8t$, $T=.79$ second.

5. Given that the earth's attraction for a body outside the earth is inversely proportional to the square of the distance of the body from the center of the earth, show that when a body weighing W pounds is s feet from the center, it is attracted with a force of $\frac{Wa^2}{s^2}$ pounds, the radius of the earth being a feet, and that if the body is at rest when $s=H$, its speed is v in general, and v_1 when it reaches the earth, where

$$v^2=\frac{2a^2g}{Hs}(H-s), \quad v_1{}^2=2ag\left(1-\frac{a}{H}\right),$$

and the greatest value possible for v_1 is less than 7 m/s.

6. Given that the earth's attraction for a body inside the earth is directly proportional to the distance of the body from the center of the earth, show that when a body weighing W pounds is s feet from the center, it is attracted with a force of $\frac{Ws}{a}$ pounds, the radius of the earth being a feet, and that a body starting from

rest at one end of a hole bored diametrically through the earth would reach the other end in $\pi\sqrt{\frac{a}{g}}$ seconds = about $42\frac{1}{2}$ minutes.

272. Resistance of a Rough Surface.—The resistance offered to the motion of a body A by a body B with which it is in contact is subject to well-established laws. This resistance, the force with which B presses against A, is the same as the force with which A is pressed against B. The magnitude and direction of the resistance, determined from observation of their effect on the motion of the resisted body, are found to be subject to limitations due to the physical nature of the bodies in contact.

Thus, if a body is at rest on a firmly supported flat board, the resistance of the board is a force equal to the weight of the body, directed vertically upward, and as the body may remain at rest when the board is tilted, the inclination of the resistance to the resisting surface is capable of variation. This variation is limited by the roughness of the surface.

Friction.—The laws governing the resistance of a rough surface (commonly called the *Laws of Friction*) are expressed in terms of the resolved parts of the total resistance along and normal to the surface; the normal component R is called the *reaction* of the surface (sometimes the *normal reaction*); the component along the surface F is called the *frictional resistance* or the *friction* (sometimes the *tangential reaction*).

If a body is at rest on a surface inclined at an angle θ to the horizon, then, since the resultant of F and R, or the total resistance, is a vertical force, $\tan\theta = \frac{F}{R}$. The greatest angle θ that will allow the body to remain at rest is called the *angle of repose;* its tangent is the greatest value possible for the ratio $\frac{F}{R}$.

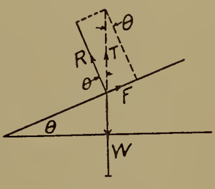


FIG. 94.

The Laws of Friction are:

I. For any two bodies in *stationary* contact, the ratio $\frac{F}{R}$ of the friction to the reaction is limited; it can never be greater than a certain proper fraction, called the *coefficient of statical friction,* and usually denoted by μ.

If the angle of repose is α, $\mu = \tan \alpha$.

II. The value of μ depends merely upon the roughness of the bodies in contact, being independent of the magnitude of the total resistance, and of the size and shape of the area over which the bodies touch.

III. For any two bodies in *moving* contact, the ratio $\frac{F}{R}$ is constant; its value may be represented by μ'. μ' is called the *coefficient of dynamical friction.*

IV. The value of μ', like that of μ, depends only upon the roughness of the surfaces in contact; it is independent of the speed. μ' is always less than μ, but never much less.

These laws are essentially exact except for very fine points or edges and very low or very high speeds. Values of μ and μ' are given in engineering hand books for various materials, according to their condition of polish and lubrication.

Two unknown quantities must be found in order to determine an unknown force; the laws of friction determine one of these in the case of the resistance of a rough surface if the resisted body is in motion along the surface, or at rest and just on the point of moving. In the case of a body at rest and not on the point of moving, the laws are of no assistance; they are also useless for determining the resistance of anything like a rough hinge, for which the direction normal to the surfaces in contact is indeterminate.

273. The following problems will illustrate the use of the laws of friction. A body is placed on an inclined plane; the statical coefficient of friction for the body and the plane is .45, the dynamical $\frac{2}{5}$; if the angle between the plane and the horizontal is θ, find what happens if $\theta=20°$, and if $\theta=30°$.

Since $\mu=\tan \alpha=.45$, the angle of repose, α, is between 24° and $24\frac{1}{2}°$; if $\theta=20°$, the body will, therefore, remain at rest. If $\theta=30°$, resolve the resistance into the normal and frictional components R and $\frac{2}{5}R$ and the weight W into the components $W\cos\theta=\frac{W}{2}\sqrt{3}$ normal to the plane and $W\sin\theta=\frac{W}{2}$ downward along the plane. If a is the acceleration with which the body

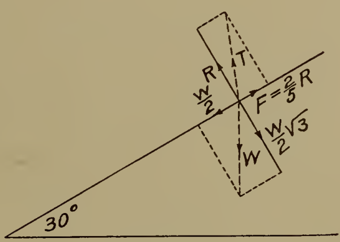


FIG. 95.

slides down the plane, the equation of motion gives, for the direction down the plane:

$$\frac{W}{2}-\tfrac{2}{5}R=\frac{W}{g}a,$$

and for the direction normal to the plane:

$$\frac{W}{2}\sqrt{3}-R=0.$$

From these, $a=g\left(\frac{1}{2}-\frac{\sqrt{3}}{5}\right)=.1536g.$

The body, therefore, moves down the plane with a constant acceleration of about 4.92 f/s², which is independent of the value of W.

274. Suppose the body in the preceding example is held from moving by a pull directed at an angle ϕ with the plane, and suppose first that the pull P_1 is just enough to keep the body from sliding down, and next that the pull P_2 is just too little to pull the body up the plane.

In either case, the pull P will have a component along the plane, tending to drag the body up, and a component normal to the plane which will increase the pressure against the plane if the pull is below the plane, and decrease it if the pull is above. Consequently, P_1 must be below the plane, and P_2 above it. It remains to determine the angle ϕ so as to balance to best advantage the changes in the dragging force and in the friction caused by changing ϕ.

In the first case, the resistance points up the plane as much as possible, so that the equations for motion along and normal to the plane are:

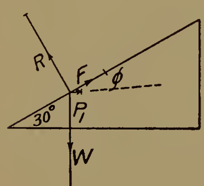


FIG. 96.

$$\left.\begin{aligned}\frac{W}{2}-.45R-P_1\cos\phi=0\\ \frac{W}{2}\sqrt{3}-R+P_1\sin\phi=0\end{aligned}\right\}$$

whence,

$$\frac{P_1}{W}=\frac{.1103}{\cos\phi+.45\sin\phi}.$$

In the second case, since the resistance points down the plane as much as possible, we have:

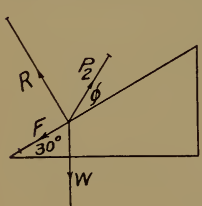


FIG. 97.

$$\left.\begin{aligned}\frac{W}{2}+.45R-P_2\cos\phi=0\\ \frac{W}{2}\sqrt{3}-R-P_2\sin\phi=0\end{aligned}\right\}$$

whence,

$$\frac{P_2}{W}=\frac{.8897}{\cos\phi+.45\sin\phi}.$$

In either case, the least value of P occurs when $\tan\phi = .45$, that is, when the angle made with the plane by the pull is equal to the angle of repose. The least values are, therefore, $P_1 = 0.1006W$, $P_2 = 0.8115W$. These are the limiting values desired, for in each case the required pull was the least pull that would call into play the maximum frictional component of the resistance.

275. No comprehensive statement exists for the law of atmospheric resistance. The statement that has generally been considered good enough for text-books is that this resistance is proportional to the projection, on a plane normal to the motion, of the surface bounding the moving body (proportional to the "opposed surface") and to the speed of motion for low speeds, the square of the speed for medium speeds, and the cube for higher speeds. In practice, it has been customary to determine the pounds resistance per square foot for a particular type of body at varied speeds, tabulating the results. Recent studies of the aeroplane have demanded some general law to be used as a basis of design, and some progress has been made—enough to show that the usual text-book law is by no means general. Sir Hiram Maxim has tested aeroplane struts of different cross-sections, and found that at a speed of 40 m/h, the resistances of the atmosphere varied from 9.12 pounds to 0.38 pound per square foot of opposed surface. We shall need some basis for our problems, and so, as it would be inconvenient to give the data in each case, we will agree to take the atmospheric resistance proportional to the speed, its square or its cube, according as the speed is less than 20 f/s, between 20 f/s and 500 f/s, or greater than 500 f/s, and to call the resistance 7.2 pounds to the square foot of opposed surface at a speed of 60 f/s. This is to be understood as merely a rule to do problems by; nothing but experiment with the body actually in question will at present give a useful formula.

The resistance offered by water to the motion of a ship has been more thoroughly studied, but with results that only serve to discourage any attempt to invent a simple law, further than the obvious one that the resistance increases with the speed of the ship until it is equal to the driving force. This is evident because the speed of the ship, at first accelerated, soon reaches a maximum uniform value.

276. To illustrate the effect of atmospheric resistance, consider the case of a sphere weighing 25 pounds, with a total surface of 3 square feet, let fall from a height in still air. The opposed surface is $\frac{3}{4}$ square foot, and when the sphere is moving v f/s, the atmospheric resistance in pounds per square foot of opposed surface is $.04v$ if v is less than 20 f/s, $.002v^2$ if v is between 20 f/s and 500 f/s, $.000004v^3$ if v is greater than 500 f/s. Let s be the number of feet that the sphere has fallen at the end of t seconds; then s, t and v are zero together.

While v is less than 20 f/s, the equation of motion is

$$25-(.04\times\tfrac{3}{4})v=\tfrac{25}{32}a=\tfrac{25}{32}\frac{dv}{dt}. \qquad (1)$$

Separating the variables, we have $1.28dt=\dfrac{dv}{25-.03v}$; integrating and determining the arbitrary constant,

$$-0.0384t=\log(1-0.0012v); \qquad (2)$$

whence,

$$v=\frac{ds}{dt}=833.33(1-e^{-0.0384t}). \qquad (3)$$

Integrating again, and determining the constant,

$$s=833.33t-21{,}701.5(1-e^{-0.0384t}). \qquad (4)$$

277. The law of the resistance changes when $v=20$ f/s,

$$t=0.6326 \text{ seconds and } s=6.33 \text{ feet.}$$

In the next stage of the problem, we have

$$25-(0.002\times\tfrac{3}{4})v^2=\tfrac{25}{32}\frac{dv}{dt}, \qquad (5)$$

and if s feet is the distance fallen in addition to the fall of 6.33 feet, the initial values are $t=0$, $v=20$, $s=0$.

Treating the equation as before, we have:

$$1.28dt = \frac{dv}{25-0.0015v^2}.$$

$$0.49574t = \log\left[0.73172\,\frac{129.1+v}{129.1-v}\right]. \qquad (6)$$

$$v = \frac{ds}{dt} = 129.1 \times \frac{1.3667e^{0.49574t}-1}{1.3667e^{0.49574t}+1}. \qquad (7)$$

$$s = 520.84 \log[0.5775e^{0.24787t} + 0.4225e^{-0.24787t}]. \qquad (8)$$

If we require the time when the body will have attained a given speed, we see that if v is greater than 129.1, formula (6) gives an imaginary value to t. But when $v=129.1$, the force acting on the sphere is from (5), $25-.0015v^2=0$, the resistance at this speed being exactly equal to the force of gravity. As a result, the sphere would fall with constant speed from this point on; but, according to the formulas, when $v=129.1$, t and s are infinite. The body falls with diminishing acceleration, its speed approaching the limiting value of 129.1 f/s. At the end of 15.22 seconds, $v=129$ f/s, $s=1679$ feet (the sphere has fallen from rest to 1685 feet in 15.85 seconds) and the acceleration is less than $\frac{1}{20}$ f/s².

278. Mechanical Connections.—Of somewhat the same nature as the topics just treated, is the effect of connecting bodies by means of strings or rods that are so light and inelastic that the effect of their mass and elasticity may be neglected without serious error. Such a rod will transmit a push or a pull without altering the magnitude of the transmitted force, merely changing its direction and point of application; a string will serve in the same way, but of course only to transmit pulls. The direction of a taut string is changed by passing it over a pulley, a peg or something of the sort, with some loss of force, which, however, may also be neglected in the case of a light pulley with smooth bearings or of a smooth peg. Under these conditions, the two tensions in a taut string are directed along the string in both directions at any point, are equal and constant throughout the length of the string, and so exert the same pull at each end. The tension or compression in a stiff light rod is in accordance with similar laws.

279. *Example.*

Two bodies, A and B, weighing 10 pounds and 25 pounds respectively, are connected by a light inextensible string. A is placed on an inclined plane ($\mu' = .2$) inclined at an angle of 60° to the horizontal; the string is passed over a pulley C at the top of the plane, so that the part AC is parallel to the plane, and the part BC is vertical. Find the motion. Let T be the tension of the string, F and R the frictional and normal components of the resistance of the plane, $W_1 = 10$ pounds and $W_2 = 25$ pounds, the weights of A and B.

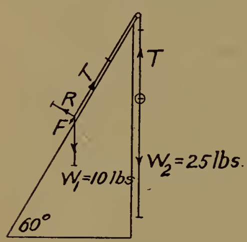


FIG. 98.

Then, as indicated in Fig. 98, if a is the acceleration of A up the plane, or of B vertically downward, the equations of motion are:

For A:

$$\left.\begin{aligned} T - F - W_1 \sin 60° &= \frac{W_1}{g} a, \\ R - W_1 \cos 60° &= 0, \\ F &= \frac{R}{5}. \end{aligned}\right\}$$

And for B:

$$W_2 - T = \frac{W_2}{g} a.$$

Thence,

$$T = .2 (W_1 \cos 60°) + W_1 \sin 60° + \frac{W_1}{g} a = W_2 - \frac{W_2}{g} a.$$

or

$$T=9.6603+\tfrac{10}{32}a=25-\tfrac{25}{32}a;$$

whence,

$$a=14.025 \text{ f/s}^2.$$

The pull of the string is $T=14.043$ pounds on each body.

280. *Examples.*

1. Two weights, W_1 and W_2, are connected by a thin inextensible cord which passes over a smooth pulley having a horizontal axle. Show that the weights move with the acceleration

$$a=\frac{W_1-W_2}{W_1+W_2}g.$$

2-3. A weight of W_1 pounds hangs from a thin inextensible cord l feet in length which passes over a smooth pulley at the edge of a smooth horizontal table and is attached at the other end to a weight of W_2 pounds lying on the table. The table top is h feet from the floor, and W_1 starts from rest at the edge of the table. Find the motions of the weights.

2. $W_1=4$, $W_2=21$, $l=h=4$.

Ans. W_1 reaches the floor in $1\frac{1}{4}$ seconds; W_2 reaches the edge of the table in $1\frac{1}{4}$ seconds and $\frac{1}{2}$ second later strikes the floor $3\frac{1}{5}$ feet from W_1.

3. $W_1=9$, $W_2=135$, $l=12$, $h=9$.

Ans. W_1 reaches the floor in 3 seconds, W_2 in $4\frac{1}{4}$ seconds, falling $4\frac{1}{2}$ feet from W_1.

4. Given the same conditions as in example 2, except that the table slopes at an angle of θ to the horizontal, the edge at which the pulley is situated being at the top of the slope and horizontal, find the value of θ if there is no motion, and the acceleration of the weights if $\sin\theta=\frac{1}{7}$, and if $\sin\theta=\frac{2}{7}$.

Ans. $\theta=\sin^{-1}\frac{4}{21}$; $a_1=1.28$ f/s², W_1 descending; $a_2=2.56$ f/s², W_2 descending.

5-6. Given the same conditions as in examples 2-3, except that the table is rough and slopes, as in example 4, θ being $\tan^{-1}\frac{3}{4}$; find for what value of μ there would be no motion, and find the acceleration of the weights if $\mu'=\frac{5}{12}$.

5. $W_1=32$, $W_2=30$.

Ans. μ must be $\frac{7}{12}$ or greater; $a=\frac{64}{31}$ f/s², W_1 descending.

6. $W_1=4$, $W_2=30$.

Ans. μ must be $\frac{7}{12}$ or greater; $a=\frac{64}{17}$ f/s², W_2 descending.

7. Given the same conditions as in examples 5-6, find for what values of W_1 there will be no motion if $W_2=30$ and $\mu=\frac{1}{2}$.

Ans. W_1 between 6 and 30.

8-9. A body weighing W pounds rests on a rough horizontal plane and is pulled by a force of P pounds directed at an angle ϕ above the plane.

8. $W=15$, $\mu=\frac{2}{5}$. Find P if the body is just on the point of moving, and $\phi=0$; again, $\phi=\tan^{-1}\frac{5}{12}$, $\phi=\tan^{-1}\frac{3}{4}$. For what value of ϕ is P least, and what is the least value of P?

Ans. $P_1=6$, $P_2=5\frac{4}{7}$, $P_3=5\frac{10}{13}$, $\phi=\tan^{-1}\frac{2}{5}$, $P_5=5.57$.

9. $W=12$, $\mu'=\frac{1}{3}$, $P=15$. Find the acceleration, a f/s², with which the body moves according as $\phi=0$, $\tan^{-1}\frac{3}{4}$; the value of ϕ for which a is greatest, and the greatest value of a.

Ans. $a_1=29\frac{1}{3}$, $a_2=29\frac{1}{3}$, $\phi=\tan^{-1}\frac{1}{3}$, $a_3=31.49$.

10-11. A weight of W_1 pounds rests on a rough horizontal table-top, h feet above the floor; a light inextensible string l feet long is attached to the weight at one end, passes over a smooth pulley at the edge of the table, and is attached at the other end to a weight of W_2 pounds, which hangs without other support. Find the motions of the weights if at the start the string is taut and W_2 is at the edge of the table.

10. $l=6$, $h=6$, $W_1=10$, $W_2=5$, $\mu'=\frac{2}{5}$.

Ans. W_2 reaches the floor in 2.37 seconds; 0.62 second later W_1 strikes the floor 3.1 feet from W_2.

11. $l=10$, $h=6$, $W_1=15$, $W_2=9$, $\mu'=\frac{1}{3}$.

Ans. W_2 reaches the floor in $1\frac{1}{2}$ seconds; $\frac{3}{4}$ second later W_1 stops 1 foot from the edge of the table.

12-13. The ridge of a peak roof is h feet higher than the eaves, and the breadth is $2b$ feet between the eaves. Two weights, of W_1 pounds and W_2 pounds, one on each slope of the roof, are connected by a taut rope l feet long, which passes over a smooth pulley at the ridge in a plane perpendicular to the ridge.

12. $W_1=120$, $W_2=300$ and W_1 is just about to slide up. If $h=10$, $b=15$, what is μ?

Ans. $\mu=\frac{2}{7}$.

13. $W_1=53$, $W_2=331$, $\mu'=\frac{1}{4}$, $h=9$, $b=12$. Find the motion of the weights, given that $l=15$, that W_2 starts at the ridge, that the rope slips from both weights when W_1 reaches the ridge, and that the eaves are $30\frac{5}{8}$ feet above the ground.

Ans. W_1 reaches the ridge in 2 seconds, and $1\frac{7}{8}$ seconds later strikes the ground $10\frac{1}{2}$ feet from the building; W_2 reaches the eaves in 2 seconds, and $1\frac{1}{8}$ seconds later strikes the ground $13\frac{1}{2}$ feet from the building.

14. A sphere d feet in diameter, weighing w pounds to the cubic foot, falls from rest; show that until its speed becomes 20 f/s, its acceleration is $\frac{g}{c}\ (c-v)$, where $c=\frac{2wd}{3k}$, $k=0.04$.

Show that a rain-drop 0.12 inch in diameter cannot acquire a speed of $10\frac{5}{12}$ f/s in a vertical fall, given $w=62.5$

Ans. $t=\frac{c}{g}\log\frac{c}{c-v}$, $c=\frac{125}{12}$.

15. If the initial velocity of the sphere in example 14 is vertically downward, and between 20 f/s and 500 f/s, show that until its speed becomes 500 f/s, its acceleration is $\frac{g}{a^2}\ (a^2-v^2)$, where $a^2=\frac{2wd}{3k}$, $k=0.002$.

Show that bird-shot 0.028 inch in diameter fired vertically upward in still air will fall to the ground with a speed of less than $23\frac{1}{3}$ f/s, given $w=700$.

Ans. $s=\frac{a^2}{2g}\log\left(\frac{a^2-v_0^{\,2}}{a^2-v^2}\right)$, $a=\frac{70}{3}$.

281. The Simple Pendulum.—The simple pendulum consists of a heavy particle suspended from a fixed point by a weightless inextensible string or rod, and moving in a vertical circle.

Let O be the fixed point, let the particle start from rest at A, and reach the position P after t seconds. Call the lowest point of the circle B, and let angle $BOA=\theta_0$, angle $BOP=\theta$, arc $BP=s$. Let the weight of the particle be W pounds, and call the length of the pendulum $OP=l$ feet.

The motion of the particle is caused by a tension N directed along PO, and the weight W directed vertically downward. For the motion in the path at P we have $\frac{W}{g}a_t = W \sin\theta$, since N has no tangential component.

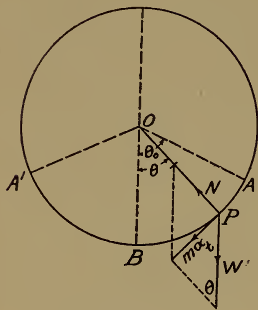


FIG. 99.

$$a_t = -\frac{d^2s}{dt^2}, \quad s = l\theta;$$

hence

$$a_t = -\frac{d^2s}{dt^2} = -l\frac{d^2\theta}{dt^2} = g\sin\theta.$$

If we call $\frac{d\theta}{dt} = \omega$, we have

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta};$$

and

$$\frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta} = -\frac{g}{l}\sin\theta.$$

Integrating, we have, since $\omega = 0$ when $\theta = \theta_0$,

$$\omega^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0).$$

Hence

$$\omega = \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)},$$

$$\sqrt{\frac{2g}{l}}\,dt = \frac{-d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

The function $\int\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$ is a new function somewhat like the $\int\sqrt{1 - e^2\cos^2\phi}\,d\phi$ that we found in Art. 205 for the length of an elliptical arc.

282. If θ_0 is small, we can get an approximation to the time required to swing through a given angle by putting

$$\cos\theta_0 = 1 - \frac{\theta_0^2}{2},\quad \cos\theta = 1 - \frac{\theta^2}{2},\quad \text{or}\quad \cos\theta - \cos\theta_0 = \tfrac{1}{2}(\theta_0^2 - \theta^2).$$

We then have, since $t=0$ when $\theta=\theta_0$,

$$\sqrt{\frac{g}{l}}\cdot t = \int\frac{-d\theta}{\sqrt{\theta_0^2-\theta^2}} = \cos^{-1}\frac{\theta}{\theta_0}.$$

An inverse cosine has any number of values, but since t starts from 0 and increases gradually throughout the motion, $\cos^{-1}\dfrac{\theta}{\theta_0} = \sqrt{\dfrac{l}{g}}\cdot t$ must do the same, starting from $\cos^{-1}\dfrac{\theta_0}{\theta_0}=0$ when the particle P is at A, becoming $\cos^{-1}\dfrac{0}{\theta_0}=\dfrac{\pi}{2}$ when P reaches B, $\cos^{-1}\dfrac{-\theta_0}{\theta_0}=\pi$ when P reaches A'. The motion from A to A' is called a *vibration;* at the end of the vibration,

$$\omega = \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}}\left(\cos(-\theta_0) - \cos\theta_0\right) = 0,$$

and the particle starts from A' under the same conditions as it left A. The pendulum therefore vibrates back and forth between A and A', and if T is the time in seconds of a vibration,

$$\sqrt{\frac{g}{l}}\,T = \pi,\quad T = \pi\sqrt{\frac{l}{g}}\ \text{seconds (approximately)}.$$

283. If θ_0 is not small enough for this approximation to be sufficiently accurate, we can obtain any required degree of approximation as follows: In $\sqrt{\dfrac{2g}{l}}\,T = \displaystyle\int_{\theta_0}^{-\theta_0}\frac{-d\theta}{\sqrt{\cos\theta-\cos\theta_0}}$, write $\cos\theta = 1-2\sin^2\dfrac{\theta}{2}$, $\cos\theta_0 = 1-2\sin^2\dfrac{\theta_0}{2}$; then, noting that the values of the integrand are the same for equal and opposite values of θ, we have:

$$\sqrt{\frac{2g}{l}}\,T=2\int_0^{\theta_0}\frac{d\theta}{\sqrt{2\left(\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}\right)}};$$

$$T=\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}.$$

To simplify the limits, put $\sin\frac{\theta}{2}=\sin\frac{\theta_0}{2}\sin\phi$; then

$$T=\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{2d\phi}{\cos\frac{\theta}{2}}=2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}.$$

Abbreviate $\sin\frac{\theta_0}{2}$ by k; then

$$T=2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Since

$$(1-k^2\sin^2\phi)^{-\frac{1}{2}}=1+\tfrac{1}{2}k^2\sin^2\phi+\frac{1\cdot 3}{2\cdot 4}k^4\sin^4\phi+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6\sin^6\phi+\dots$$

and

$$\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\,d\phi=\frac{1\cdot 3\cdot 5\ \dots\ (2n-1)}{2\cdot 4\cdot 6\ \dots\ (2n)}\times\frac{\pi}{2},$$

$$T=\pi\sqrt{\frac{l}{g}}\left[1+(\tfrac{1}{2})^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\dots\right]=\pi S\sqrt{\frac{l}{g}}.$$

284. Taking $g=32.16$ f/s², $\pi=3.1416$, the first approximation $T=\pi\sqrt{\frac{l}{g}}$ gives as the length of the second's pendulum,

$$(T=1)\quad l=\frac{g}{\pi^2}=39.10 \text{ inches}.$$

For the second approximation, this result must be divided by

$$\left(1+\frac{k^2}{4}\right)^2=\left(1+\tfrac{1}{4}\sin^2\frac{\theta_0}{2}\right)^2.$$

If $\theta_0=5°$ (it is never more for an accurate clock),

$$\frac{k^2}{4}=\tfrac{1}{4}\sin^2(2°\ 30')=0.0004757,$$

so that the more accurate value of l is

$$39.10\times(1-0.00095)=39.06 \text{ inches.}$$

The effect of the third term of the series is inappreciable. If a pendulum $\frac{g}{\pi^2 S^2}$ feet long vibrates in 1 second, a pendulum $\frac{g}{\pi^2}$ feet long vibrates in S seconds and in one day makes $86{,}400\div S=86{,}400(1+0.000476)^{-1}$ vibrations; hence a clock with a pendulum constructed according to the formula $\pi^2 l=g$ will lose about 41.1 seconds a day if it swings over a complete arc of 10°.

285. Variation of Gravity.—In this discussion of the simple pendulum, W, the weight of the moving particle, is the earth's attraction for the particle at mean sea-level in London (latitude 51°) and the acceleration given by this attraction at the same place is 32.19 f/s². These are the standard conditions mentioned in Art. 256 as the basis for comparing force, acceleration and mass. The earth's attraction for a body at mean sea-level varies with the latitude, and the acceleration it gives to the body varies in the same ratio. The values at sea-level for a few latitudes are:

$L=$	0°	40°	51°	90°
$g_0=$	32.09	32.16	32.19 $(=g)$	32.26
$\log g_0=$	1.50637	1.50729	1.50772	1.50860

The acceleration due to gravity also varies with the distance above or below sea-level, being $\left(\frac{a}{a+h}\right)^2 g_0$ at a place h feet above sea-level, and $\left(\frac{a-d}{a}\right) g_0$ at a place d feet below, a being the earth's radius in feet.

The earth's attraction for a body weighing W pounds is $mg' = \frac{W}{g} g'$, where g' is the acceleration due to gravity at the place where the body is situated, and $g = 32.19$. Then, in the problem of the simple pendulum, the equation of motion becomes

$$\frac{W}{g} a_t = \frac{W}{g} g' \sin\theta; \quad \text{or} \quad a_t = g' \sin\theta,$$

instead of $a_t = g \sin\theta$, so that g' takes the place of g throughout.

286. *Examples.*

1. What must be the value of θ_0 for a pendulum of length $l = \frac{6}{\pi^2}$ to lose 1 second in a day? Ans. $\theta_0 = 46' 47''$.

2. A pendulum that beats seconds when swinging through a very small arc is made to vibrate through an arc of $120°$ ($\theta_0 = 60°$); how many seconds will it lose in a day?
Ans. About 5890 seconds $= 1^h \, 38^m \, 10^s$.

3. A pendulum constructed to beat seconds in a locality where $g = 32.19$, is found to lose 72 seconds a day. What is the value of g where the pendulum is, and what change must be made in its length to adjust it?
Ans. $g' = 32.137$; length must be decreased 0.165 per cent.

4. If the pendulum of example 3 is on a mountain in latitude $40°$, how high is the mountain? (Take the earth's radius 4000 miles.)
Ans. About 1.4 miles.

5. A pendulum beats seconds at sea-level; how many seconds a day will it lose at the bottom of a mine 1056 feet deep?
Ans. 2.16 seconds.

6. A spring-balance is graduated in London at sea-level; what will it read when sustaining a 10-pound weight at sea-level at the equator? Ans. 9.969 pounds.

7. Show that the weight given by a lever-balance is independent of the variation of gravity.

287. Motion in a Vertical Circle.—If the pendulum of Art. 281 is started by giving the particle P an initial speed of v_0 f/s in the circle, or an initial angular rate, $\omega_0 = \frac{v_0}{l}$, the first integration of the equation of motion gives

$$\omega^2 = \omega_0^2 + \frac{2g}{l}(\cos\theta - \cos\theta_0).$$

The force N is determined from the equation of motion in the direction PO:

$$N - W\cos\theta = \frac{W}{g}a_n = \frac{W}{g}\frac{v^2}{l} = \frac{W}{g}l\omega^2;$$

whence

$$\frac{N}{W} = \cos\theta + \frac{l\omega^2}{g} = \frac{l\omega_0^2}{g} + 3\cos\theta - 2\cos\theta_0.$$

ω^2 becomes zero when $\theta = \theta_1$, if $\cos\theta_1 = \cos\theta_0 - \frac{l\omega_0^2}{2g}$.

N becomes zero when $\theta = \theta_2$, if $\cos\theta_2 = \frac{2}{3}\cos\theta_0 - \frac{l\omega_0^2}{3g} = \frac{2}{3}\cos\theta_1$.

θ_1 and θ_2 are in the same quadrant; if they are acute, θ_2 is greater than θ_1; if obtuse, θ_1 is greater than θ_2.

Hence if $\frac{l\omega_0^2}{2g} < \cos\theta_0$, the upward motion of P will cease before N can become zero, and the pendulum will vibrate.

If $\frac{l\omega_0^2}{2g} > \cos\theta_0$, N becomes zero before $\omega = 0$. If PO is a rod, it will keep moving until $\omega = 0$, in the meantime being under compression, and then will vibrate back. If PO is a string, however, P will start from the point (l, θ_2) with an initial velocity in a direction making the angle $\pi - \theta_2$ with the horizontal, and will move under the action of gravity alone (in a parabola) until stopped by the string.

If $\frac{l\omega_0^2}{2g} > 1 + \cos\theta_0$, it is impossible for ω to be zero, and if PO is a rod, the pendulum will make complete revolutions.

If $\frac{l\omega_0^2}{2g} > \frac{3}{2} + \cos\theta_0$, it is impossible for N to be zero, and the pendulum will make complete revolutions whether PO is a string or a rod.

288. In the problems so far discussed, we have been required to find the nature of the motion from data concerning the forces; in the next article we shall consider an example of the inverse problem of finding the relations of the forces when the motion is completely given.

289. Motion in a Horizontal Circle.—If a particle acted upon by gravity is constrained by any resistance of R pounds to move

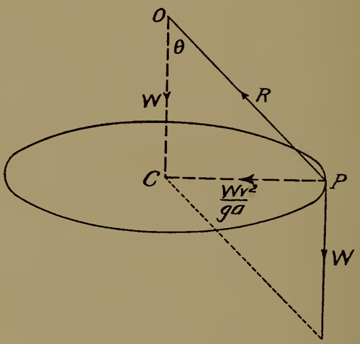


FIG. 100.

in a horizontal circle of radius a feet, with a uniform speed of v f/s, the resultant of R and the weight of the particle, W pounds, must be a horizontal force $\frac{W}{g}\frac{v^2}{a}$ pounds, directed toward the center of the circle, for the total acceleration of the particle is $\frac{v^2}{a}$ in this direction. Suppose the line of action of R to be PO, and the vertical through the center C of the circle to be OC; let the displacement PO represent the force R; then the displacements OC and PC must represent W and $\frac{Wv^2}{ga}$. If θ is the angle made by R with the vertical, $\tan\theta = \frac{v^2}{ga}$, and $R = W\sec\theta$ in magnitude and is always in the same vertical plane as PC.

In applications of this principle, the angle $\theta = \tan^{-1}\frac{v^2}{ga}$ is usually the only important value. For instance, along a railroad curve a feet in radius, over which trains pass at v f/s, the road-bed should have a lateral slope of $\tan^{-1}\frac{v^2}{ga}$ to the horizontal, so that the force R may be exerted wholly against the tires of the wheels rather than partly against the flanges.

Again, if the rods of a centrifugal governor are l feet in length, and if ω is the greatest angular speed that the balls should have, the governor is adjusted to shut off steam when the rods make the angle θ with the vertical, where

$$\tan\theta = \frac{v^2}{ga} = \frac{a\omega^2}{g} = \frac{l\sin\theta\omega^2}{g}, \quad \text{or} \quad \cos\theta = \frac{g}{l\omega^2}.$$

In the same way, it is found that if we wish to cause the particle in Art. 281 to move from A in a horizontal circle, we must give it the initial velocity $v_0 = l\omega_0 = \sqrt{gl\sec\theta_0}$ f/s in the horizontal direction perpendicular to AO. When the simple pendulum moves in this way it is called the *conical pendulum*.

290. *Examples.*

1. A lead is swung in a vertical circle of 30 inches radius with just enough force to make complete revolutions. What is its speed at the top of the circle, at the bottom, and when it is let go so as to start off at an angle of 45° to the horizontal? ($g = 32$.)
Ans. 8.944 f/s, 20 f/s, 18.792 f/s or 11.267 f/s.

2. A curve on a railway of 4′ 8″ gauge has a radius of 847 feet. How much must the outer rail be raised above the inner for trains going at the rates of 15 m/h, 30 m/h, 45 m/h and 60 m/h?
Ans. 1 inch, 4 inches, 9 inches, 16 inches.

3. The inner edge of a semi-circular curve on a running track has a radius of 105 feet, the outer edge a radius of 121 feet; the

track is flat and banked so that the outer edge is 2 feet higher than the inner. For what range of speeds is the track adapted?

Ans. From $20\frac{1}{4}$ f/s to 22 f/s.

4. What angle will be made with the vertical by a simple pendulum hung in a car which is going 60 m/h around a curve of 1000 feet radius?

Ans. $\tan^{-1} 0.242 = 13° \ 36'$.

5. If the gauge of the track in the preceding example is 4′ 8″ and the outer rail is 7″ higher than the inner, what must be the coefficient of friction between a box and the floor of the car if the box is not to slide? What if the two rails are on the same level?

Ans. .114, .242.

6. A particle is placed at the top of a fixed smooth sphere and slightly displaced. Show that it will leave the sphere when it has descended over an arc $= \cos^{-1} \frac{2}{3}$, and has acquired a speed of $\frac{1}{3}\sqrt{6ga}$ f/s, a being the radius of the sphere.

CHAPTER XIII.

Momentum, Work and Energy.

291. General results of great importance can be obtained from the equation of motion, $f = ma$. In the discussion to follow, f may be the resultant of all the forces acting, so that a is the total acceleration, or f may be the resultant of the resolved parts of all the forces in some given direction, and a the resolved part of the acceleration in the same direction.

292. Mean Speed under Constant Force.—From the relations

$$f = ma, \quad a = \frac{dv}{dt}, \quad v = \frac{ds}{dt},$$

it is readily seen that a *constant* force f will move the mass m a distance of s feet in t seconds, changing its speed from v_0 to v_1, where $v_1 = v_0 + at$, and $s = (v_0 + \frac{1}{2}at)t$.

The mean speed V, at which the same distance would be covered in the same time, is

$$V = \frac{s}{t} = v_0 + \tfrac{1}{2}at = \frac{v_0 + v_1}{2},$$

and is equal to the average of the initial and final speeds.

This relation also holds for curvilinear motion under a force having a constant tangential component.

293. Momentum.—The product, mv, of the mass of a body by its velocity, is called its momentum in the direction of the velocity. Momentum is thus a directed quantity, bearing the same relation to velocity that force bears to acceleration. Momenta can therefore be compounded and resolved by the methods given in Chapter X, and the relations between momentum and velocity treated in the same way as those connecting force and acceleration.

From the relations $f=ma=m\frac{dv}{dt}=\frac{d(mv)}{dt}$, it appears that the force acting to produce motion in a given direction is the time-rate of the momentum in that direction; that is, force is related to momentum as acceleration is related to velocity.

We may observe that $f=\frac{d(mv)}{dt}$ is the equation of motion for a body with variable mass.

294. Impulse.—Calling the momentum $mv=M$,

$$dM=d(mv)=fdt,$$

so that if the force f changes the momentum from M_0 to M_1 in t seconds, the total change in momentum is

$$M_1-M_0=\int_{M_0}^{M_1} dM=\int_0^t fdt. \tag{1}$$

This value is called the *impulse* given by the force f in the time t. In case the mass m is constant, the total change in momentum (or the impulse) is

$$m(v_1-v_0)=\int_0^t fdt; \tag{2}$$

and if the force f is also constant, $=F$, we have, as a check,

$$m(v_1-v_0)=Ft, \quad \text{or} \quad v_1-v_0=at; \tag{3}$$

where $a=\frac{F}{m}$ is constant.

The mean value of the force f, $\frac{\int_0^t fdt}{\int_0^t dt}$, is equal to $\frac{M_1-M_0}{t}$, or to the constant force which would produce the same change of momentum (the same impulse) as f in the same time.

Equation (1) or (2) can be used to compare the effects of one force on two different bodies, even when the force itself and the time during which it acts are both unknown. For instance, suppose a 1-ounce bullet is fired from a 15-pound rifle, leaving the muzzle with a speed of 2000 f/s, with what initial speed does the rifle recoil? Here the gaseous pressures against the bullet and against the end of the chamber are the same, though the intensity of the pressure is unknown; so that if we neglect the resistances to the motion of the bullet through the bore, we have for the bullet and for the rifle

$$\frac{1}{16\times 32}(2000-0)=\int_0^t fdt \text{ and } \frac{15}{32}\ (v-0)=\int_0^t fdt,$$

and f and t are the same in the two equations.

Consequently, the rifle recoils with a speed of $v=\frac{32}{15}\times\frac{2000}{16\times 32}$ $=8\frac{1}{3}$ f/s. If the recoil is stopped in $\frac{1}{10}$ second, the mean resistance is

$$F=\tfrac{15}{32}\times\tfrac{25}{3}\times\tfrac{10}{1}=39\tfrac{1}{16} \text{ pounds.}$$

Suppose the gun recoils 2 inches; then, if the retarding force were constant, the average speed would be $\frac{25}{6}$ f/s, and the recoil would take $\frac{2}{12}\times\frac{6}{25}=\frac{1}{25}$ second. The mean resistance is therefore

$$\tfrac{15}{32}\times\tfrac{25}{3}\times\tfrac{25}{1}=97.7 \text{ pounds.}$$

295. Impact.—As a further example of the use of momentum, consider what happens when two bodies collide. The pressures which the two bodies exert on each other change the velocities of the bodies as any forces do, but the time during which they act is often so short that it is impossible to make any observations for determining the rates of change of the velocities (the accelerations) or the rates of change of the momenta (the forces). Consequently the equation of motion, $f=ma$, is of no use.

The total changes in velocity and the total changes in momentum (the impulses) can be observed, however, and the laws that govern them can be used to determine the effect of a collision or impact.

The pressures exerted on each other by two bodies in collision are equal in magnitude and opposite in direction and act during the same interval of time; consequently, the impulses or changes in momentum are equal and opposite, that is, the sum of the momenta of the two bodies is unchanged by the collision. In case one of the bodies is so large that its change of motion is too small to be observed, this relation, though still true, is useless—as, for instance, when one of the bodies is "fixed"; *i. e.,* cannot move without moving the earth.

It is found by experiment that the same laws of friction hold for impulses as for forces; we can treat here, however, only cases in which friction is negligible. We shall consider the impact of a particle impinging on a smooth fixed surface, and the impact of two smooth particles moving in the same straight line. In these cases, a homogeneous sphere can be treated as a particle.

Experiment shows that the effect on two bodies of the impulse of their collision is to change their *relative* velocity by reversing its component in the direction of the impulse and multiplying the magnitude of this component by a number e, called the *coefficient of restitution,* the value of which depends on the elasticity of the bodies. The perpendicular component of the relative velocity is unchanged.

296. For instance, suppose a body weighing 12 pounds moves 10 f/s in a direction making an angle of 30° with a smooth fixed surface, and suppose the coefficient of restitution for the body and the surface to be $\frac{2}{3}$; find the motion after the body strikes the surface.

In this case, since the surface is fixed, the relative velocity of the body and the surface is the velocity of the body. Since the surface is smooth, the impulse is normal to it; the component of the velocity normal to the surface is 5 f/s toward the surface, and is changed by the impulse to $\frac{2}{3}\times 5=\frac{10}{3}$ f/s away from the surface; the component of the velocity along the surface is unchanged by the impulse. Hence the body leaves the surface in a direction making the angle $\tan^{-1}\frac{\frac{10}{3}}{5\sqrt{3}}=\tan^{-1}\frac{2\sqrt{3}}{9}=21°\ 3'$ with the surface, at $\sqrt{(5\sqrt{3})^2+(\frac{10}{3})^2}=9.28$ f/s.

Suppose that a ball weighing 12 pounds and moving 10 f/s to the right meets a ball weighing 3 pounds and moving in the same line 15 f/s to the left, and suppose the coefficient of restitution for the balls to be $\frac{4}{5}$.

To avoid confusion about direction and sign in a problem of this sort, it is well to fix upon one direction as positive, and to express

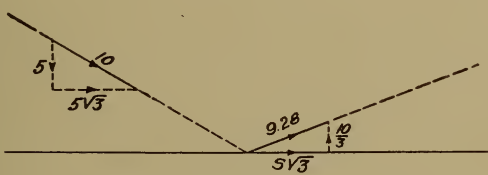


FIG. 101.

all velocities accordingly. The relative velocity must be found by subtracting the velocities of the two balls in the same order before and after impact. Taking the positive direction to the right, we have, if after the impact the 12-pound ball moves v_1 f/s and the 3-pound ball v_2 f/s (each to the right) :

$$\tfrac{12}{32}v_1+\tfrac{3}{32}v_2=\tfrac{12}{32}\times 10+\tfrac{3}{32}(\times -15)=\tfrac{75}{32},$$

since the sum of the momenta is unchanged, and

$$v_1-v_2=-\tfrac{4}{5}[10-(-15)]=-20,$$

since the relative velocity is reversed and multiplied by e. These equations,

$$\left.\begin{aligned}4v_1+v_2&=25\\ v_1-v_2&=-20\end{aligned}\right\}$$

give $v_1=1$ f/s, $v_2=21$ f/s, both balls moving to the right after the collision.

If these balls had been inelastic ($e=0$), we should have had $v_1-v_2=0$, $v_1=v_2=5$ f/s to the right; if e had been 1, we should have had $v_1=0$, $v_2=25$ f/s to the right.

297. *Examples.*

1. A particle impinges upon a smooth surface, moving along a line that makes the angle α with the surface, and rebounds along a line making the angle β with the surface. Prove that $\tan\beta = e \tan\alpha$.

2. Show that if a billiard ball is knocked around the table, its path across a corner is parallel to its path across the opposite corner.

3. A ball A, weighing 5 pounds, moving 7 f/s, is struck by a ball B, weighing 6 pounds, moving in the same direction; after impact the speed of A is doubled. $e=\frac{5}{6}$. Find the speed of B before and after impact. Ans. 14 f/s and $8\frac{1}{6}$ f/s.

4. Two balls, moving 25 f/s and 16 f/s in opposite directions, impinge directly upon each other. $e=\frac{2}{3}$. Find the distance between them $4\frac{1}{2}$ seconds after the collision. Ans. 123 feet.

298. Work.—If the point of application of a constant force F moves the distance s in the direction of the force, the product Fs is called the *work done by the force.* If the point of application of a variable force f moves through the distance s in the direction of the force, the work done by the force is $\int_0^s f ds$. In either case, if the motion is along a path of length s at an angle ϕ to the direction of the force, the work done is $\int_0^s f \cos\phi ds$.

It is evident from this definition that the work done by a resultant force is the sum of the work done by its components; if each of the forces acting on a particle is resolved along and perpendicular to the motion of the particle, each of the normal components will clearly do no work ($\cos\phi=0$), so that all the work is done by the components in the direction of motion. This is also evident directly from the definition, since if f represents any force acting on the body, $f \cos \phi$ is its component in the direction of motion.

The work done by a force is positive or negative according as the force has a positive or negative component in the direction in which its point of application moves. When a force does negative work on a body, the body is said to do positive work against the force.

The unit of work, in the English Gravitational System, is the foot-pound, the work done by a force of 1 pound as its point of application moves 1 foot in the direction of the force.

299. As an illustration of the definition of work, consider the work done by gravity when a body moves along the sides of a right triangle ABC, of which BC is vertical and AC is horizontal (Fig. 102). If the body weighs W pounds, $f = W$ pounds. If the motion is from B to C, the work done is directly Wa foot-pounds, or in $\int_0^a f \cos \phi ds$, $f = W$ pounds, $\phi = 0$.

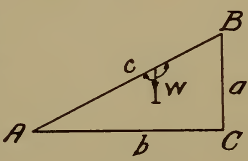


FIG. 102.

If the motion is from C to B, ϕ, the angle between the downward direction of gravity and the upward direction of motion, is 180°, so the work is $\int_0^a -Wds = -Wa$ foot-pounds. If the motion is from C to A or from A to C, $\phi = 90°$, and the work done by gravity is zero. In the motion from B to A, $\phi = \cos^{-1} \frac{a}{c}$, and the work is $\int_0^c W \frac{a}{c} ds = Wa$ foot-pounds; in the motion from A to B, $\phi = 180° - \cos^{-1} \frac{a}{c}$, and the work is $\int_0^c -\frac{Wa}{c} ds = -Wa$ foot-pounds. In any circuit of the triangle, gravity does as much negative work as positive, or a total of zero.

300. Work Done by a Constant Force.—If the point of application of a constant force f moves in a curve, let the axis of x be drawn in the direction of f; then, in the expression for work, $\int_0^s f\cos\phi ds$, ϕ is the angle made by the direction of motion with the axis of x, and $\cos\phi = \frac{dx}{ds}$. Therefore, if the abscissas of the initial and final points of the path are x_0 and x, the work is

$$\int_0^s f\cos\phi ds = \int_{x_0}^x f dx = (x - x_0)f.$$

The work is therefore independent of the path traversed, depending only upon the magnitude of f and the component in the direction of f of the displacement given to its point of application.

301. Work Done by a Central Force.—A central force is always directed toward a fixed point and varies in magnitude according to the distance of its point of application from the fixed point. Let O be the fixed point, and A the point of application of the central force f (Fig. 103). Then if A moves over any path, the angle ϕ between the direction of the force and the direction of motion is the angle between the radius vector and the path; hence $\cos\phi = \frac{dr}{ds}$. But by definition, the force is a function of r, say $F(r)$. Then the work done as A moves along the path is

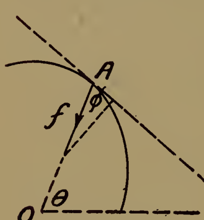


FIG. 103.

$$\int_0^s f\cos\phi ds = \int_{r_0}^r F(r)\,dr,$$

and depends merely upon the limits of integration, and the initial and final distances of A from O, being otherwise independent of the path of motion.

302. Work Done by Gravity.—The force of gravity is very nearly constant for differences of level less than a thousand feet, and is for all levels practically a central force directed toward the center of the earth. Thus, if a body weighing W pounds is lowered h feet, the work done by gravity is hW foot-pounds, no matter what path the body traverses to reach the lower level. Moreover, although at present the application of this law is apparently limited by the assumption that the body can be regarded as a particle, we shall see later (Art. 323) that in this connection any body can be treated as a heavy particle.

303. *Examples.*

1. A body, weight 6 pounds, starting from rest, is drawn by a cord up a rough inclined plane in 3 seconds, the cord being parallel to the plane and the motion uniformly accelerated. Inclination of plane 30°, length 10 feet, $\mu' = \frac{1}{5}$. Find the tension of the cord and the work done by it.

Ans. $T = 4.456$ pounds. Work $= 44.56$ foot-pounds.

2. An elastic cord is stretched by a gradually increasing pull until its natural length of 1 foot is increased to 18 inches, the pull then being 24 pounds. Find the work done by the pull, and the work that would be done by gravity in the first descent of a weight of 24 pounds hung on the end of the cord, the other end being supported. (See Art. 269.)

Ans. 6 foot-pounds and 24 foot-pounds.

304. Work and Energy.—If a particle of mass m moves in t seconds over a path of s feet, the speed changing from v_0 f/s to v f/s under the action of a force of F pounds whose component in the direction of motion is $F\cos\phi = f$ pounds, then since

$$f = ma = m\,\frac{dv}{dt},$$

we have

$$fds = m\,\frac{dv}{dt}\,ds = m\,\frac{ds}{dt}\,dv = mvdv,$$

and the

$$\text{Work} = \int_0^s f ds = \int_{v_0}^{v} mvdv = \tfrac{1}{2}m(v^2 - v_0^2).$$

Here f is positive if it accelerates the motion (ϕ acute), negative if it retards the motion (ϕ obtuse).

The product, $\frac{1}{2}mv^2$, of the mass of a body by one-half the square of its speed, is called the *kinetic energy* of the body. The change in the kinetic energy of a body is the same in magnitude and sign as the work done in changing the motion of the body. The kinetic energy of a moving body is thus the amount of negative work that must be done on the body to bring it to rest, or is the amount of work that the body, by virtue of its motion, can do against a resistance. The value mv^2, twice the kinetic energy, is often called *vis viva,* or active force.

305. For instance, if a constant pull of 12 pounds, exerted on a body weighing 9 pounds, lifts the body 3 feet, the pull does 36 foot-pounds of work on the body, gravity does -27 foot-pounds, and both together 9 foot-pounds. Or, the total work is done by the upward resultant of 3 pounds as its point of application moves 3 feet upward.

The forces therefore make a change of 9 foot-pounds in the kinetic energy of the body, so that

$$9 = \tfrac{1}{2}m(v^2 - v_0^2) = \tfrac{9}{64}(v^2 - v_0^2); \quad v^2 - v_0^2 = 64.$$

Starting from rest, the body would acquire an upward speed of 8 f/s, and if the pull were then released, it would rise by virtue of its 9 foot-pounds of kinetic energy, going up 1 foot against the action of gravity. In falling 4 feet back to its original level, the body would regain through the action of gravity the 36 foot-pounds of energy given to it by the upward pull.

Starting with an upward or downward speed of 6 f/s, the body would acquire an upward speed of 10 f/s, and if the pull were then released would rise $1\frac{9}{16}$ feet higher through its $\frac{225}{16}$ foot-pounds of kinetic energy, and in falling to its original level would regain $41\frac{1}{16}$ foot-pounds of kinetic energy, its initial $5\frac{1}{16}$ foot-pounds plus the energy given to it by the upward pull.

Again, suppose a body weighing 12 pounds is projected along a rough horizontal plane ($\mu'=\frac{1}{3}$) with an initial speed of 8 f/s. The body has $\frac{12}{64}\times 64$ foot-pounds of kinetic energy, and is acted upon by a retarding force of $\frac{12}{3}=4$ pounds, which will do -12 foot-pounds of work, reducing the energy to zero, while the body moves $\frac{12}{4}=3$ feet. The body therefore comes to rest after it has moved 3 feet and has done 12 foot-pounds of work against a resistance of 4 pounds.

306. Potential Energy.—There is an important physical difference in the nature of the forces in these two examples. In the first example no energy is lost, for the body has at any time, either in kinetic energy or through its position, the power of doing as much work as has been expended on it, and returning to its original level with its original kinetic energy left. The work it can do through its position is called *potential energy;* forces like those of this problem, which do no work that is not available in the form of either kinetic or potential energy are called *conservative forces.* The resistance of the rough plane in the second example is not a conservative force; the kinetic energy that it takes away is lost as far as mechanical effects are concerned.

If a body weighing W pounds rises h feet, its kinetic energy is decreased Wh foot-pounds by the action of gravity. At the same time, its potential energy is increased Wh foot-pounds, for in falling h feet the body would acquire a speed of $\sqrt{2gh}$ f/s and therefore $\frac{W}{2g}\,2gh=Wh$ foot-pounds of kinetic energy. Gravity thus causes no change in the total energy of a moving body, so that the change effected in the total energy by the combined action of gravity and any other forces is equal to the work done by the other forces.

307. *Examples.*

1. A 400-pound shot is fired with a muzzle velocity of 1800 f/s from a gun weighing 50 tons; find velocity of recoil and the ratio of the kinetic energy of the gun to that of the shot.

Ans. $6\frac{3}{7}$ f/s, $\frac{1}{280}$.

2. Prove that when a shot is fired from a gun, the kinetic energies of the shot and the gun are inversely proportional to their weights.

3. Determine the *mean* effective pressure in the bore of a 6-pound gun if the projectile travels 7 feet before leaving the muzzle at 2100 f/s. Ans. $59{,}062\frac{1}{2}$ pounds.

3. A 2-pound weight and a 3-pound weight on a rough horizontal table are connected by a thin inextensible cord and moved by a constant pull of 7 pounds in the direction of the cord. $\mu' = \frac{1}{3}$. From the work done by the pull and by friction, find the speed of the weights when they have moved 3 feet 9 inches from rest.

Ans. $v = 16$.

4. Show that when two bodies move with the same speed, their kinetic energies are proportional to their masses, and that when they are acted upon by the same force during the same time their kinetic energies are proportional to their speeds.

5. A weight of W_1 pounds rests on a rough horizontal table l feet from the edge, and is attached to one end of a light inextensible cord which passes over a smooth pulley at the edge of the table and supports a weight of W_2 pounds hanging h feet above the floor. From the work done by gravity and by friction, find the speed of the weights when the falling weight reaches the floor, and the speed with which the sliding weight reaches the edge of the table.

$$\text{Ans.} \quad v_1^2 = \frac{2gh(W_2 - \mu' W_1)}{W_2 + W_1}; \quad v_2^2 = v_1^2 - 2g\mu'(l - h).$$

6. Show that if, in example 5, $W_1 = 10$, $W_2 = 15$ and $\mu' = \frac{3}{5}$, in order that W_1 may just reach the edge of table l must be $\frac{8}{5}h$.

7. Show that the work done by gravity when a body weighing W pounds falls to the earth from an infinite distance is Wa foot-pounds, the radius of the earth being a feet, and that therefore the body will be going $\sqrt{2ga}$ f/s when it strikes the earth.

8. Two weights, of W_1 pounds and W_2 pounds, hang from the ends of a light inextensible string which passes over a smooth pulley. After t seconds, the heavier weight, W_2, has descended s feet and is going v f/s. Show that

$$(W_2 - W_1)s = \frac{1}{2g}(W_1 + W_2)v^2,$$

and by differentiating with respect to t, that the acceleration of the weights is

$$\frac{dv}{dt} = \frac{W_2 - W_1}{W_2 + W_1} g.$$

308. Power.—The *power* exerted by an engine is measured by its rate of doing work. The unit of power in the English Gravitational System is thus the foot-pound per second. 550 foot-pounds per second or 33,000 foot pounds per minute make one *horse-power* (H. P.).

Any unit of force combined with the corresponding unit of distance may be used as a unit of work, and with the unit of time will form a unit of power. We thus have foot-poundal and foot-poundal per second in the English Absolute System, gram-centimeter and gram-centimeter per second in the Metric Gravitational System. In the Metric Absolute System, the dyne-centimeter unit of work is called an *erg*, and 10,000,000 ergs a second constitute 1 watt.

If a force of F pounds moves a body weighing W pounds at a tangential acceleration of a f/s² against resistances having a tangential component of R pounds, we have, if the tangential component of F is $F\cos\phi = f$,

$$f - R = \frac{W}{g}a, \quad \text{or} \quad f = R + W\frac{a}{g}.$$

The work done by the force F (or by f) as its point of application moves over any arc s of the path is

$$\text{Work} = \int_0^s f\,ds = \int_0^s \left(R + W\frac{a}{g} \right) ds.$$

The rate of work is

$$\text{Power} = \frac{d(\text{work})}{dt} = \frac{d(\text{work})}{ds}\,\frac{ds}{dt} = v\left(R + W\frac{a}{g}\right).$$

If the body is being hauled up an inclined plane, making an angle θ with the horizontal, the part of the resistance due to the force of gravity is $W\sin\theta$ pounds, or the weight times the *grade;* frictional resistances are also proportional to the weight. If the total resistance of R pounds is proportional to the weight, and is given as k pounds for each pound weight, the source of power causing the motion develops

$$v\left(R + W\frac{a}{g}\right) = Wv\left(k + \frac{a}{g}\right) \text{ foot-pounds per second,}$$

or

$$\frac{Wv}{550}\left(k + \frac{a}{g}\right) \text{ H. P.}$$

309. For instance: At the foot of a 3 per cent grade, a 200-ton train is going 45 m/h, and at the top of the grade, which is 1 mile long, is going 30 m/h. If the pull of the locomotive is constant and the traction and atmospheric resistances amount to 20 pounds to the ton, what horse-power does the engine develop? The total resistance per pound is $\frac{20}{2240}$ pound plus the $\frac{3}{100}$ pound due to gravity. As the forces are all constant, a is constant, and the mile is covered (at a mean speed of $37\frac{1}{2}$ m/h) in $\frac{2\times 60\times 60}{75}$ seconds, with an increase in speed of -15 m/h $= -22$ f/s; therefore $a = \frac{-22\times 75}{2\times 60\times 60} = -\frac{11}{48}$ f/s², and $\frac{a}{g} = -\frac{11}{48\times 32}$. Hence

$$\text{H. P.} = \frac{200\times 2240}{550}\,v\left(\frac{3}{100} + \frac{20}{2240} - \frac{11}{48\times 32}\right),$$

$$\text{H. P.} = \frac{8539}{330}\,v.$$

At the foot of the grade, H. P. $= 1164.4$; at the top, H. P. $= 776.3$.

Again: The driving wheel of a locomotive is 8 feet in diameter, the pistons are 2 feet in diameter, the stroke is 5 feet and the mean steam pressure in the cylinders is 250 pounds per square inch. What H. P. is furnished by the steam pressure in two such cylinders when the locomotive is going 45 m/h?

The driving wheel makes $\frac{45 \times 5280}{8\pi \times 3600}$ revolutions a second. The mean steam pressure is $36{,}000\pi$ pounds, and in a complete revolution, the work done is $360{,}000\pi$ foot-pounds for each cylinder. Hence

$$\text{H. P.} = 2 \times \frac{45 \times 5280}{8\pi \times 3600} \times \frac{360{,}000\pi}{550} = 10{,}800.$$

If the engine of a locomotive works under a constant steam-pressure, the horse-power it develops is evidently proportional to the speed of the locomotive, so that $\frac{Wv}{550}\left(k + \frac{a}{g}\right) = cv$, where c is a constant. Consequently the acceleration a is constant, and the pull of the locomotive is constant.

If work is to be done continuously by a moving mass, either the speed of the moving body must be kept up by the continuous application of force, or else the mass must be renewed as fast as its kinetic energy is used up. Power is furnished in the first way by the fly-wheel of a stationary engine, in the second by a stream of water.

310. *Examples.*

1. At what uniform speed can an engine of 30 H. P. draw a train weighing 50 tons up a grade of 1′ in 280′ and against traction resistances of 7 pounds to the ton? Ans. 15 m/h.

2. An engine under constant steam pressure brings a 500-ton train from rest to a speed of 60 m/h in 2 miles against resistances of 11 pounds to the ton. What is its H. P. when it has gone 220 yards, and when it has reached its highest speed?

Ans. $733\frac{1}{3}$, $2933\frac{1}{3}$.

3. A pipe is delivering 11 cubic feet of water a second at the rate of 80 f/s; what horse-power is used if the speed is reduced one-half? Ans. 93.75 H. P.

311. Characteristics of Motion.—With the usual notation of m, v and f, with L for impulse, K for work, E for kinetic energy, P for power, the zero subscript to indicate initial values, and the subscript t to indicate tangential components, we have the following relations between the characteristics of motion:

$$M = mv, \qquad E = \tfrac{1}{2}mv^2 = \tfrac{1}{2}Mv,$$

$$L = M - M_0, \qquad K = E - E_0,$$

$$f = \frac{dM}{dt} = \frac{dL}{dt}, \qquad P = \frac{dE}{dt} = \frac{dK}{dt} = f_t \cdot v.$$

The relations on the left involve direction and line of action as well as magnitude; those on the right involve merely magnitude and sign. K is the work done by f upon the moving mass; the work done by the moving mass against f is $-K = E_0 - E$. The power of the moving mass is $-P$.

CHAPTER XIV.

RIGID BODIES.

312. Mass of a Body of Variable Density.—If any point P of a body is surrounded by a closed surface containing a volume ΔV of which the mass is Δm, the ratio $\frac{\Delta m}{\Delta V}$ is called the mean density of the body within the chosen surface. If the mean density is independent of the surface drawn and of the position of P, the body is said to be of uniform density, or to be homogeneous. The density of the body at any point P is defined as the limit of the mean density as the chosen surface, always enclosing P, contracts, and ΔV approaches zero.

$$\text{Density} = \rho = \frac{dm}{dV}.$$

Then if the density ρ, at every point of any non-homogeneous body is a given function of the position of the point, the mass of the body is $m = \int \rho dV$ taken throughout the body.

For instance, suppose the density of a sphere of radius a is 1 at the center and decreases by an amount proportional to the distance from the center, becoming $\frac{2}{3}$ at the surface. The density at a distance r from the center is $\rho = 1 - kr$, and as $\rho = \frac{2}{3}$ when $r = a$, $k = \frac{1}{3a}$, and $\rho = \frac{3a - r}{3a}$. A spherical shell of radius r and thickness dr therefore has for its mass $dm = \frac{4\pi}{3a} r^2 (3a - r) dr$, approximately, and the total mass is

$$m = \frac{4\pi}{3a} \int_0^a (3ar^2 - r^3) dr = \pi a^3.$$

313. Resultant of Like Parallel Forces.—We have so far considered only such forces as have concurrent lines of action. Two forces of which the lines of action are parallel are called *like* or *opposite parallel forces,* according as their directions are the same or opposite.

The resultant of two parallel forces can be found by combining the triangle construction with the principle that the point of application of a force acting on a rigid body can be shifted along the line of action without altering the effect of the force.

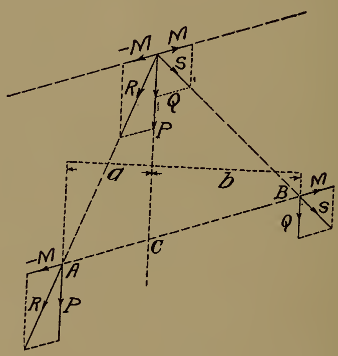


FIG. 104.

Let P and Q (Fig. 104) be two like parallel forces, A and B their respective points of application. Join AB, and let equal and opposite forces $-M$ and M be applied at A and B, having AB as their common line of action. These two forces neutralize each other, so that, R being the resultant of $-M$ and P, and S the resultant of M and Q, the pair of forces R and S is equivalent to the given pair, P and Q. In order to find the resultant of R and S, shift their points of application to the common point O of their lines of action, and resolve each of them in the direction of AB and that of the forces P and Q. Evidently the components of R are $-M$ and P, and of S are M and Q, so that the resultant of R and S, which is also the resultant of the forces P and Q in their original positions, is a force T, of magnitude $T=P+Q$, having the same direction as P and Q. Moreover, if T's line of action meets AB at C, we have from similar triangles,

$$\frac{AC}{OC}=\frac{M}{P}, \quad \frac{BC}{OC}=\frac{M}{Q},$$

or

$$\frac{AC}{BC}=\frac{Q}{P},$$

The magnitude, the direction and the line of action of the resultant of the given forces P and Q are thus completely determined.

If a common perpendicular to the lines of action of P, T and Q is divided by them into segments a and b,

$$\frac{a}{b}=\frac{AC}{BC}=\frac{Q}{P}, \quad \text{or} \quad Pa=Qb.$$

314. Moment of a Force.—The moment of a force about a straight line perpendicular to its line of action is defined as the product of the magnitude of the force by the length of the common perpendicular to the line of action and the given straight line if the rotation that the force tends to produce about the straight line as an axis is contra-clockwise, and as the negative of this product if the rotation is clockwise.

When the forces considered all lie in the same plane, so that the straight lines about which moments are taken (the *axes* of the moments) are all perpendicular to this plane, it is convenient to speak of the moment of a force about an axis as its moment about the point in which the axis cuts the plane.

315. Moments of Parallel Forces.—The moments of the forces P and Q of Art. 313 about a line perpendicular to their plane at any point of the line OC (about any point of OC) are Pa and

$-Qb$; the moment of T about the same axis is zero, or the sum of the moments of P and Q.

Consider the moments of P, Q and T about any point O' of their plane. For convenience, shift the forces along their lines of action until their points of application, A, B and C, fall on the perpendicular to their lines of action through O' (Fig. 105). Then $AC=a$, $CB=b$; call $O'A=x$.

The moments about O' are: Of P: $-Px$; of Q: $-Q(a+b+x)$;

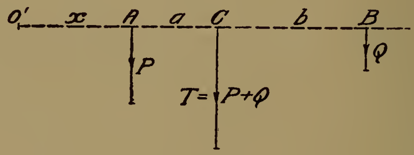


FIG. 105.

of $T=P+Q$: $-(P+Q)(a+x)$. The sum of the moments of P and Q, since $Qb=Pa$, is

$$-[Px+Qa+Qb+Qx]=-(P+Q)(a+x),$$

or the moment of T. The same result is obtained if O' is between A and B or on the other side of B.

Consequently, the resultant of two like parallel forces of magnitudes P and Q is a third like parallel force of magnitude $(P+Q)$, having for its moment about any axis perpendicular to the common plane the sum of the moments of the two given forces about the same axis.

It evidently follows that the resultant of any number of like parallel forces is a force having the same direction as its components, a magnitude equal to the sum of their magnitudes, and a line of action such that its moment about any axis perpendicular to the common direction of the forces is the sum of the moments of all the forces about the same axis.

316. Resultant of Opposite Parallel Forces.—The process of Art. 313, if applied to a pair of opposite parallel forces of different magnitudes, shows that their resultant has the same direction, and tends to produce rotation in the same direction, as the larger force; that its magnitude is the difference between the magnitudes of the two forces, and that its line of action is so situated in the plane of the two forces that the moment of the resultant about any point of the plane is the difference between the moments of the two given forces about the same point.

Consequently, the results stated in Art. 315 for like parallel forces will hold with one exception for any parallel forces if the magnitudes of forces in one of the two directions are represented by positive numbers, and those of forces in the opposite direction by negative numbers.

317. Couples.—The exception occurs when the forces reduce to two opposite parallel forces of the same magnitude, with different lines of action. Such a pair of forces, called a *couple,* can be replaced or balanced only by another couple. The sum of the moments of the forces constituting the couple, called the moment of the couple, is of course even in this case the same for any axis as the sum of the moments of the original forces; aside from this fact, we shall not for the present be concerned with the nature of couples.

318. Identical and Identically Opposed Forces.—Two forces, f_1 and f_2, having the same magnitude, the same line of action and the same direction, are identical; this is indicated by writing: $f_1 \equiv f_2$. Two forces f_1 and f_2, having the same magnitude, the same line of action, and opposite directions, are said to be equal and directly opposed, or balanced; this is indicated by writing: $f_1 \equiv -f_2$.

Note that $f_1 = f_2$ or $f_1 = -f_2$ imply magnitude and direction, but not line of action.

Two forces produce no effect on the motion of a body when and only when they are equal and directly opposed; in other words, if a body is at rest under the action of parallel forces, the algebraic sum of all the forces is zero, and the sum of the moments of all the forces about any axis is zero, for the resultant of the forces having one of the two directions must be equal and directly opposed to the resultant of the forces having the opposite direction.

319. Balanced Parallel Forces.—The lever furnishes the simplest instance of balanced parallel forces; the following problem is of the same nature.

A light circular table, 10 feet in diameter, is supported on three vertical legs, spaced at equal angular intervals, each 4 feet from the center. Let O be the center of the table-top, A, B, C the points directly over the legs. It is required to find the tensions or compressions in the legs if the table is stationary and of negligible weight and supports a weight of 60 pounds placed on the line OA: (1) on the edge of the table at the point nearest A, (2) at the point farthest from A, (3) between B and C, (4) at O, (5) at A.

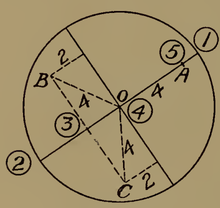


FIG. 106.

Consider the forces acting on the table-top; these are the weight of 60 pounds and the pressures or pulls of the legs, which are directly opposite to the compressions or tensions in the legs. Let A represent the pressure exerted by the leg at A (or the compression in A), and so for the other legs; then negative values will indicate pulls exerted by the legs, or tensions in the legs.

In any of the cases, we have

$$A+B+C=60,$$

and if we take moments about the line AO, we find $B=C$. Taking moments about the perpendicular to AO at O we find $M+4A-2(B+C)=0$, where M is the moment of the 60-pound weight, and is -300, 300, 120, 0 and -240 in the several cases. Solving the three equations in each case we find:

Case	1	2	3	4	5
$A=$	70	-30	0	20	60
$B=C=$	-5	45	30	20	0

320. *Examples.*

1. Show that in the case of any lever, if we neglect the weight of the lever, the work done by the power in any motion is numerically equal to that done by the resistance.

2. A triangular table, sides 6 feet, 8 feet and 10 feet, has a vertical leg under the middle point of each side. Neglecting the weight of the table, find the compressions in the legs when a weight of 300 pounds rests at a point 3 feet from each of the perpendicular sides.

Ans. 75 pounds, 0 and 225 pounds respectively.

3. Find the compressions if in example 2 the weight is placed 2 feet from each of the perpendicular sides.

Ans. 150 pounds, 100 pounds and 50 pounds respectively.

4. Solve example 3 with the legs at the corners.

Ans. 125 pounds at the right angle, 75 pounds at the smaller acute angle, 100 pounds at the larger.

321. Center of Gravity of a Body.—Suppose three mutually perpendicular coördinate planes to be fixed in a material body and the body to be placed so that the axis of z is vertical. Let the body be divided up into elements of volume, of which dV is typical, and let the density of the body at a point of this element be ρ, so that the mass of the element is $\rho \cdot dV = dm$.

Then the body may be conceived as made up of heavy particles of which the element of mass dm is a type, and the force of gravity acting on the body (the weight of the body) may be regarded as the resultant of a system of like parallel forces, the forces with which gravity acts on the constituent particles, of which gdm is a type. When these forces are summed by integration, the approximate assumptions become exact; consequently, the force with which gravity acts on the body has for its magnitude the integral of gdm taken throughout the body, or $g\int dm = gm$, m being the mass of the body, and $gm = W$, its weight. Suppose the line of action of the resultant force of gravity (which is vertical) to be at the intersection of the planes $x = x_0$, $y = y_0$, and consider the moments of the resultant about the axes of x and y, each of which is perpendicular to all the forces. The moments of an element gdm, situated at a point (x, y, z) of the body are $ygdm$ about the axis of x and $xgdm$ about the axis of y. The sum of all the elementary moments about either of the axes is the moment of the resultant, $W = mg$, about the same axis; hence

$$mgy_0 = \int ygdm; \quad mgx_0 = \int xgdm,$$

or

$$x_0 = \frac{\int xdm}{m} = \frac{\int xdm}{\int dm}, \quad y_0 = \frac{\int ydm}{m} = \frac{\int ydm}{\int dm},$$

the integrations in each case being taken throughout the body.

If we suppose the body to be set up with the x-axis or the y-axis vertical, we find y_0 or x_0 as before, and

$$z_0 = \frac{\int zdm}{m} = \frac{\int zdm}{\int dm}.$$

The point (x_0, y_0, z_0), through which the resultant weight of the body always acts, is called the *center of gravity* of the body. Each of its coördinates is the mean distance of points of the body from the corresponding coördinate plane, the distribution being proportional to the mass, or to both the volume and the density.

In the case of a homogeneous body, the factor ρ in $dm = \rho dV$ is constant, so that

$$x_0 = \frac{\rho\int xdV}{\rho\int dV} = \frac{\int xdV}{\int dV}, \quad y_0 = \frac{\int ydV}{\int dV}, \quad z_0 = \frac{\int zdV}{\int dV}.$$

The center of gravity in this case is called the center of gravity of the geometric solid occupied by the body.

322. Centers of Gravity of Areas and Arcs.—It is often important to find the center of gravity of a geometric surface or arc; in such problems the element of volume dV is replaced by the element of surface dS, or by the element of arc ds.

The coördinates of the center of gravity of a plane area, if dA is the element of area, are thus given by the integrals: $x_0 = \frac{\int x dA}{\int dA}$, $y_0 = \frac{\int y dA}{\int dA}$, taken over the area; and for an arc of a plane curve, the coördinates of the center of gravity are given by the integrals: $x_0 = \frac{\int x ds}{\int ds}$, $y_0 = \frac{\int y ds}{\int ds}$, taken along the arc.

It is only in the case of a plane surface that the center of gravity lies on the surface, and except in the case of a straight line, the center of gravity of an arc of a plane curve never lies on the curve.

If a body is in any way symmetrical, the location of its center of gravity is always partly evident; for instance, the center of gravity of any homogeneous solid of revolution is evidently on the axis of revolution, and the center of gravity of any homogeneous central figure is at the geometric center.

323. Work Done by Gravity on an Extended Body.—If an extended body weighing W pounds moves so that its center of gravity falls h feet the work done by gravity is $+Wh$ foot-pounds, for the resultant force of gravity is applied at the center of gravity, which moves in the direction of the force. In moving the body so that its center of gravity rises h feet, Wh foot-pounds of work are done against gravity. For instance, to up-end a 24-foot ladder weighing 50 pounds requires 600 foot-pounds of work, and to pump out a cistern 10 feet deep containing 120 cubic feet of water standing 4 feet deep, requires $62.5 \times 120 \times 8 = 60{,}000$ foot-pounds of work.

324. Computation of the Coördinates of Centers of Gravity.— As a coördinate of the center of gravity of a body is merely the mean distance of the points of a body from the corresponding plane, there is nothing new about the process of computing the coördinate by an integration. In the first illustrative example of Art. 231, and in examples 4a and 10 of Art. 233, we found the distance of the center of gravity from the base in the case of the semi-circumference, the hemispherical surface and the hemisphere of radius a to be $\frac{2a}{\pi}$, $\frac{a}{2}$ and $\frac{3a}{8}$ respectively. In the case of the semi-circle, the corresponding distance is $\frac{4a}{3\pi}$.

The center of gravity of a triangle is between each vertex and the mid-point of the opposite side, twice as far from the vertex as from the opposite side; the center of gravity of any pyramid or cone is between the vertex and the center of gravity of the base, three times as far from the vertex as from the center of gravity of the base.

Bearing in mind that the center of gravity is the point at which the resultant force of gravity acts, we can use these elementary results for the purpose of finding certain centers of gravity without integration.

For instance, let it be required to find the center of gravity of the trapezoidal area in Fig. 107. Divide the figure into a triangle and a rectangle, as shown. Let area = weight for convenience $\left(\rho = \frac{1}{g}\right)$, and consider the moments of the triangle and the rectangle about the 6′ and 4′ 6″ sides. The sums of these moments are the corresponding moments of the whole area. Then if x and y are the distances of the required center of gravity from the 6′ and 4′ 6″ sides, respectively,

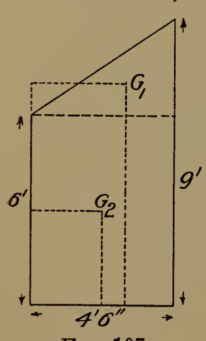


FIG. 107.

$$x[6\times\tfrac{9}{2}+\tfrac{1}{2}\times3\times\tfrac{9}{2}]=(\tfrac{1}{2}\times\tfrac{9}{2})(6\times\tfrac{9}{2})+(\tfrac{2}{3}\times\tfrac{9}{2})(\tfrac{1}{2}\times\tfrac{9}{2}\times3),$$
$$y[6\times\tfrac{9}{2}+\tfrac{1}{2}\times3\times\tfrac{9}{2}]=(\tfrac{1}{2}\times6)(6\times\tfrac{9}{2})+(6+\tfrac{1}{3}\times3)(\tfrac{1}{2}\times\tfrac{9}{2}\times3),$$

whence $x=2.4'$, $y=3.8'$

Again, to find the center of gravity of a solid consisting of a hemisphere and a cone having a common base, the vertical angle of the cone being 90°. Let the radius of the hemisphere be a; then the altitude of the cone is a. Take moments about a diameter of the base. [The figure shows the section of the solid by the plane through the geometric axis and perpendicular to the axis of moments.]

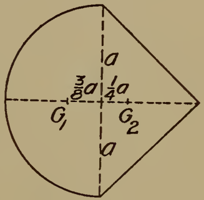


FIG. 108.

Let volume = weight, and let the distance of the required center of gravity from the common base be x. Then

$$x(\tfrac{2}{3}\pi a^3+\tfrac{1}{3}\pi a^3)=\frac{3a}{8}\times\tfrac{2}{3}\pi a^3-\frac{a}{4}\times\tfrac{1}{3}\pi a^3,$$
$$x=\frac{a}{6}.$$

325. *Examples.*

1-3. Find by integration the centers of gravity of the following homogeneous figures (see Art. 324):

1. Semi-circle. (Find the moment about the base of an element perpendicular to the base, considering its mass to be concentrated at its mid-point.)
2. Triangle.
3. Cone or pyramid.
4. Show that the solid mentioned in the second illustrative example of Art. 324 will stand with any point of the hemispherical surface in contact with a horizontal plane if the vertical angle of the cone is made 60°.
5. How much work is done by gravity in emptying a cone full of mercury through a hole in its base into a cylindrical beaker having the same base as the cone, if the altitude of the cone is

1 foot, the radius of its base 6 inches and the base of the cone is 5 inches above the base of the cylinder? [Take $\pi=\frac{22}{7}$, s. g mercury=14.] Ans. $114\frac{7}{12}$ foot-pounds.

6. Show that if the table top in the illustrative example of Art. 319 weighs 20 pounds, each of the results is increased by $6\frac{2}{3}$.

7. A chain 20 feet long, weighing 15 pounds to a foot, hangs down from the deck of a ship into the hold; what work is done against gravity in hauling it up on deck? In hauling up the first 10 feet of its length?

Ans. 3000 foot-pounds, 2250 foot-pounds.

8. One end is turned off a cylinder of revolution 9 feet in length, so that a solid is left consisting of a cylinder 8 feet in length and a cone of altitude 1 foot. Find the center of gravity of the solid.

Ans. 3.83 feet from the common base of the cylinder and the cone.

9. One end is turned off a cylinder of revolution 9 feet in length, leaving a cylinder 8 feet in length capped by a hemisphere. Find the center of gravity of the solid.

Ans. 3.66 feet from the common base of the cylinder and the hemisphere.

10. Find the center of gravity of the part of the ellipse, $x=a\cos\phi$, $y=b\sin\phi$ in the first quadrant.

Ans. $x_0=\frac{4a}{3\pi}$, $y_0=\frac{4b}{3\pi}$.

11. Find the center of gravity of a circular arc of 90°.

Ans. $\frac{2\sqrt{2}}{\pi}a=0.9003a$ from the center.

12. Find the center of gravity of a broken line composed of a circular arc of 90° and its chord.

Ans. $\frac{1+\sqrt{2}}{\frac{1}{2}\pi+\sqrt{2}}a=0.8088a$ from the center.

13. Find the center of gravity of one of the halves into which a parabolic segment of altitude a, base $2b$, is divided by the altitude.

Ans. $\frac{2}{5}a$ from the base, $\frac{3}{8}b$ from the altitude.

14. Find the center of gravity of a paraboloid of revolution of altitude a, radius of base b.

Ans. $\frac{2}{3}a$ from the vertex.

15. Find the center of gravity of the solid formed by revolving one-half the parabolic segment of example 13 about the base.

Ans. $\frac{5}{16}b$ from its base.

16. Find the center of gravity of the solid left when a pyramid is cut from a cube of edge l by a plane passing through a diagonal of one face and a vertex of the opposite face.

Ans. It lies on the severed diagonal of the original cube, $\frac{l}{20}\sqrt{3}$ from the center of the cube.

17. Find the center of gravity of the area between one arch of the cycloid $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$ and the x-axis.

Ans. $y_0=\frac{5}{6}a$.

18. Find the center of gravity of the half above the initial line of the area bounded by the cardioid $r=2a\sin^2\frac{\theta}{2}$.

Ans. $x_0=-\frac{5a}{6}$, $y_0=\frac{16a}{9\pi}$.

19. Find the center of gravity of the sold formed by revolving the figure of example 18 about the initial line.

Ans. $x_0=-\frac{4}{5}a$.

20. A solid is formed of a hemisphere of radius a and a solid of revolution of height h, the two having a common base. What must be the value of h for the solid to stand with any point of its hemispherical surface in contact with a horizontal plane, the solid of revolution being (1) a cylinder, (2) a paraboloid?

Ans. (1) $h=\frac{a}{2}\sqrt{2}$, (2) $h=\frac{a}{2}\sqrt{6}$.

326. The Pappus-Cavalieri Theorems.—The first of the following very useful theorems was contained in a compilation of mathematical knowledge made by Pappus, of Alexandria, about 300 A. D.; it was proved by Cavalieri, an Italian, about 1629, by his famous "Method of Indivisibles"; it was announced at about the same time by a German, Guldin, who appropriated it from Pappus. It is commonly known as "Guldin's Theorem."

I. If a plane area is revolved about an axis in its own plane which does not pass through it, the solid generated has a volume equal to the product of the given area by the length of the path traced by its center of gravity.

II. If an arc of a plane curve is revolved about an axis in its own plane which does not pass through it, the surface generated has an area equal to the product of the length of the arc by the length of the path traced by its center of gravity.

In each case, let x_0 be the distance of the center of gravity G

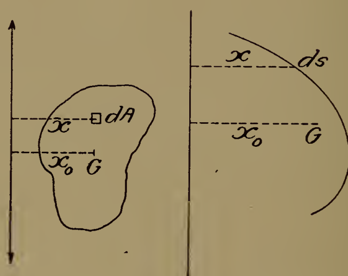


FIG. 109.

from the axis of rotation, and let x be the distance of any point P from the same axis. In I, let dA be the element of area at P; then if A is the revolved area, and V the volume it generates when its plane is turned through the angle α,

$$A = \int dA, \quad V = \int x\alpha dA = \alpha \int x dA,$$

and

$$x_0 = \frac{\int x dA}{\int dA}.$$

All these integrals are taken over the same area; hence $V = Ax_0\alpha$, which proves I, since $x_0\alpha$ is the length of the path traced by G.

In II, let ds be the element of arc at P; then if s is the length of the revolved arc, and S the surface it generates when its plane is turned through the angle α,

$$s = \int ds, \quad S = \int x\alpha ds = \alpha \int x ds,$$

and

$$x_0 = \frac{\int x ds}{\int ds}.$$

All these integrals are taken over the same arc; hence $S = s \cdot x_0 \alpha$, which proves II, since $x_0 \alpha$ is the length of the path traced by G.

For instance, if a circle of radius a is revolved about a tangent through 2π, its center describes a path $2\pi a$ in length, and the volume and surface of the resulting solid are $\pi a^2 \times 2\pi a = 2\pi^2 a^3$ and $2\pi a \times 2\pi a = 4\pi^2 a^2$.

327. *Examples.*

1. A rectangle of which the sides are 1 foot and 1 foot 8 inches in length is revolved about an axis in its plane parallel to the 1-foot sides and 1 foot 6 inches from the nearer one. Find the volume of the ring so formed.

Ans. $24\frac{4}{9}$ cubic feet, using $\pi = \frac{22}{7}$.

2. Given that the center of gravity of a half parabolic segment is $\frac{2}{5}a$ from the base, $\frac{3}{8}b$ from the altitude (example 13, Art. 325), find the volumes formed by revolving this area about the base and about the altitude.

Ans. $\frac{8}{15}\pi a^2 b$, $\frac{1}{2}\pi a b^2$.

3. From the known values of the surface and volume of a sphere find the positions of the centers of gravity of the arc and the area of a semi-circle.

4. A semi-circle is revolved about a tangent at its vertex; find the volume and the surface of the solid generated.

Ans. $a^3(\pi^2 - \frac{4}{3}\pi) = 5.6808a^3$; inner surface $2\pi a^2(\pi - 2) = 7.1728a^2$, outer surface $4\pi a^2$.

5. From the result of example 4, determine the center of gravity of the area enclosed by a circular arc and two perpendicu-

lar tangents, and find the volume of the solid generated as this area revolves about the chord of the quadrantal arc.

Ans. $\dfrac{10-3\pi}{3(4-\pi)}a=0.2234a$ from each tangent, volume $=\frac{1}{12}\pi\sqrt{2}(3\pi-8)a^3=0.5275a^3$.

6. From the results of examples 8, 12, 13, Art. 191, locate the center of gravity of an arch of the cycloid and that of the upper half of the arc of the cardioid.

Ans. Cycloid, $y_0=\frac{4}{3}a$; cardioid, $x_0=-\frac{4}{5}a$, $y_0=\frac{4}{5}a$.

328. Wind Pressure on a Plane Surface.—If the pressure exerted by the wind on a plane surface normal to the direction of the wind is uniform over the surface, and amounts to p pounds per square foot, the wind pressure on a plane surface whose normals make the angle θ with the direction of the wind will be $p\cos\theta$ pounds per square foot. The point where the line of action of the resultant wind-pressure meets the plane is called the center of effort of the pressure, or the center of wind-pressure. The process of finding this center of effort is precisely the same as that of finding the center of gravity of the area under pressure, for the forces acting are in both cases proportional to the area over which they act, the elements of pressure being $p\cos\theta dA$ in one case, $g\rho dA$ in the other. Consequently, the center of wind-pressure for a plane area is the same as the center of gravity.

The center of wind-pressure for a sail is determined by the method exemplified in the example of the trapezoidal area in Art. 324.

329. Total Fluid Pressure and Center of Fluid Pressure for a Plane Surface.—The pressure in a fluid is found to be the same in all directions at any one point, and to be exerted against any surface in a direction normal to the surface. In a fluid having no vertical motion, the upward pressure on the base of a vertical column h feet high and having a cross-section of 1 square foot, is therefore the weight of h cubic feet of the fluid plus whatever pressure there is on 1 square foot of the upper surface. If the fluid weighs w pounds to the cubic foot, the pressure due to the weight of the fluid alone is thus wh pounds per square foot at a depth of h feet. Fresh water weighs $62\frac{1}{2}$ pounds per cubic foot, sea water 64 pounds. At a depth of h feet in fresh water there is a pressure in every direction of $\left(p_0 + \frac{125h}{2}\right)$ pounds, the pressure on the upper surface being p_0 pounds per square foot. In an open body of water, p_0 is the atmospheric pressure, 15 pounds per square inch when the barometer stands at about $30\frac{1}{2}$ inches. In many cases, the atmospheric pressure acts on a surface equally in opposite directions, so that only the pressure due to the weight of water, called the water-pressure, need be considered.

330. Water-Pressure on a Vertical Plane.—Suppose a plane surface to be submerged vertically in still water; to find the total pressure on it, due to the weight of the water (called the *total water-pressure*), and the line of action of the resultant pressure (intersecting the plane of the surface in a point called the *center of pressure*).

Take as the axis of x the horizontal line in which the plane of the submerged surface cuts the surface of the water, and as the axis of y any convenient vertical line in the plane of the submerged surface. Divide the surface into infinitesimal elements of which dA at the point (x, y) is typical. Then the water-pressure on the element dA is $wydA$ in magnitude, and the resultant of all such elementary pressures has for its magnitude $\int wydA$ taken over the submerged surface, for the elementary pressures are parallel forces.

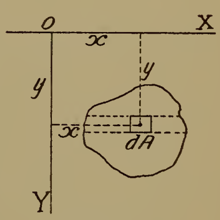


FIG. 110.

If the line of action of the total water-pressure intersects the plane of the submerged surface at (x_0, y_0), the moments of the total pressure about the axes are $x_0 \int wydA$ about the y-axis, and $y_0 \int wydA$ about the x-axis. The moments of the elementary pressure $wydA$ at (x, y) are wy^2dA about the x-axis, and $wxydA$ about the y-axis. Hence

$$x_0 \int wydA = \int wyxdA, \quad y_0 \int wydA = \int wy^2dA,$$

or

$$x_0 = \frac{\int wxydA}{\int wydA}, \quad y_0 = \frac{\int wy^2dA}{\int wydA}.$$

331. If (x_0', y_0') is the center of gravity of the submerged vertical plane surface considered homogeneous, $y_0' = \frac{\int ydA}{\int dA} = \frac{\int ydA}{A}$, A being the area of the submerged surface, and the total water-pressure,

$$P = \int wydA = w \int ydA = wy_0'A;$$

that is, the total pressure on a submerged vertical plane surface is the same as if the surface were horizontal and at a depth equal to the actual depth of its center of gravity; in other words, the mean water-pressure on the surface is the pressure at the center of gravity.

332. Water-Pressure on an Inclined Plane.—The total water-pressure on an inclined plane surface and the center of pressure can be found in essentially the same way by taking the axis of x as before and the axis of y in the inclined surface. If the inclination of the plane of the surface to the vertical is α, a point (x, y) of the surface is at a depth $y \cos \alpha$, so that the total water-pressure is

$$P = \int wy \cos \alpha dA = wy_0' \cos \alpha \cdot A,$$

and is again the submerged area multiplied by the pressure at its center of gravity, y_0' being the distance of the center of gravity from the x-axis, measured on the inclined plane.

Again, (x_0, y_0) being the coördinates (measured on the inclined plane) of the center of pressure, the equality of the moments about the coördinate axes of the total pressure and the sum of the elementary pressures gives

$$x_0 \int wy \cos \alpha dA = \int wxy \cos \alpha dA,$$
$$y_0 \int wy \cos \alpha dA = \int wy^2 \cos \alpha dA,$$

whence x_0 and y_0 have the same values independently of α.

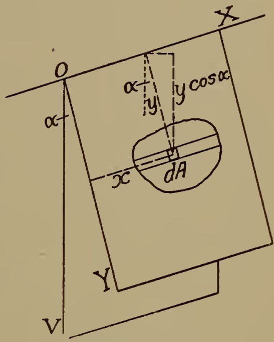


FIG. 111.

333. Computation of Centers of Pressure.—From Arts. 331 and 332 it appears that the center of pressure for a plane area is the same point of the area, however the plane may be revolved about its intersection with the free water-surface; and the total pressure is unchanged by any revolution of the area about a horizontal axis through its center of gravity.

The preceding discussion is typical of the process of finding centers of pressure, but the details of the method can often be altered to advantage. For convenience, we always consider the plane of the submerged area to be vertical, and take one of the axes, say the x-axis, horizontal; but the origin should be chosen so as to simplify as much as possible the relations between y and x on the boundary of the submerged area. For instance, if this area is a triangle, the origin should be taken at a vertex; if it is a circle, at the center.

In such a case, if the depth of the origin below the free surface is c, and that of the center of gravity of the area A is h, the total pressure P and the coördinates of the center of pressure are given by

$$P = whA, \quad Px_0 = \int wx(y+c)\,dA, \quad Py_0 = \int wy(y+c)\,dA;$$

and the *depth* of the center of pressure is $(y_0 + c)$.

If the plane of the submerged area is inclined at the angle α to the vertical, we have only to change the last result to $(y_0 + c)\cos\alpha$.

In evaluating y_0, it is always possible to take as the element of integration a strip of the area between two horizontal lines dy apart, and when rectangular coördinates are used, it is best to do so. If the area is symmetrical with regard to a vertical line, the center of pressure lies on that line; otherwise, x_0 may be evaluated by a double integration, or by using the horizontal strip as the element of integration and replacing x in the formula by the abscissa of the center of gravity of the element.

334. *Examples.*

Find the total fluid pressure and the depth below the free surface of its center of effort for each of the following vertical plane areas:

1. Rectangle; breadth b, height h; upper edge in surface.
Ans. $P = \frac{1}{2}wbh^2$, $y_0 = \frac{2}{3}h$.

2. Triangle; base b, horizontal; altitude h; vertex in surface.
Ans. $P = \frac{1}{3}wbh^2$, $y_0 = \frac{3}{4}h$, c. p. on median through upper vertex.

3. Triangle; base b, in surface; altitude h.
Ans. $P = \frac{1}{6}wbh^2$, $y_0 = \frac{1}{2}h$, c. p. on median through lower vertex.

4. Quadrant of circle; radius a, one edge in surface.
Ans. $P = \frac{1}{3}wa^3$, $y_0 = \dfrac{3\pi}{16}a$, $x_0 = \frac{3}{8}a$.

5. Circle; radius a, just submerged.
Ans. $P = \pi wa^3$, $y_0 = \frac{5}{4}a$.

6. Ellipse; semi-axis a vertical, center in surface.
Ans. $P = \frac{2}{3}wa^2b$, $y_0 = \dfrac{3\pi}{16}a$.

7. Parabolic segment; base $2b$, in surface; altitude h.

Ans. $P=\frac{8}{15}wh^2b,\ y_0=\frac{4}{7}h$.

8. Isosceles trapezoid; base 3 feet in surface, base 5 feet at a depth of 4 feet.

Ans. $P=2166\frac{2}{3}$ pounds, $y_0=\frac{36}{13}$feet$=$2feet 9.23 inches.

Find the total fluid pressure and the depth D of the center of pressure below the center of gravity for each of the following completely submerged vertical plane areas, h being the depth of the center of gravity below the free surface.

9. Square; side a. Ans. $P=wa^2h,\ D=\dfrac{a^2}{12h}$.

10. Circle; radius a. Ans. $P=\pi wa^2h,\ D=\dfrac{a^2}{4h}$.

11. Triangle; base $2b$, horizontal; altitude a; vertex up.

Ans. $P=wabh,\ D=\dfrac{a^2}{18h}$.

12. Parabolic segment; base $2b$, horizontal; altitude a; vertex up.

Ans. $P=\frac{4}{3}wabh,\ D=\dfrac{12a^2}{175h}$.

335. Kinetic Energy of an Extended Body.—If a material body moves so that all its points have the same velocity—a speed of v f/s in a given direction—the kinetic energy of an element dm of its mass is $\frac{1}{2}v^2dm$, and the kinetic energy of the whole body is $\int\frac{1}{2}v^2dm=\frac{1}{2}v^2\int dm=\frac{1}{2}mv^2$.

The motion just described is called *translation;* it is only in translation, when the paths of motion of the points of the body are parallel straight lines, and the body is rigid, that the kinetic energy is $\frac{1}{2}mv^2$.

Any motion of an extended body is the resultant of a rotation about an axis (which may itself be moving) and a translation along the axis; the kinetic energy of a body in any motion is the sum of the energies due to these two motions.

336. Kinetic Energy of a Rotating Body.—Suppose a body to be rotating about a fixed axis at the angular rate of ω radians a second. Imagine the body divided up into infinitesimal elements of volume by some three sets of surfaces, and consider each of the divisions of the body as a heavy particle. Let the volume of any one of these particles be dV, and the density ρ, so that its mass is $dm = \rho dV$, and let its distance from the axis of rotation be r feet. The particle is moving in a circle of radius r feet with an angular speed of ω radians a second, or with a speed in its path of $v = \omega r$ f/s. Its kinetic energy is therefore $dE = \frac{1}{2} dm v^2 = \frac{1}{2} dm \omega^2 r^2$. The kinetic energy of the whole body is the sum of the kinetic energies of all the particles, or is

$$E = \int \tfrac{1}{2} \omega^2 r^2 dm = \tfrac{1}{2} \omega^2 \int r^2 dm,$$

the integration being taken throughout the body.

If the body is homogeneous, ρ is constant, and $E = \frac{1}{2} \omega^2 \rho \int r^2 dV$.

337. Moment of Inertia and Radius of Gyration.—The quantity $\int r^2 dm$, which bears the same relation to the angular speed and kinetic energy of a rotating body that the mass bears to the linear speed and kinetic energy of a body in translation, is called the body's *moment of inertia* with reference to the given axis and is indicated by I.

The value $\dfrac{\int r^2 dm}{\int dm} = \dfrac{I}{m}$ is the mean value of the squared distance from the axis of rotation of points of the body, the distribution being proportional to the mass, or to both volume and density. This mean value is indicated by k^2; as $I = mk^2$, k is the distance from the axis at which a heavy particle of the same mass as the body would have the same moment of inertia, or the same kinetic energy when rotating about the given axis at the same angular rate. k is called the *radius of gyration* of the body with reference to the given axis.

338. Computation of I and k^2.—Moments of inertia are of fundamental importance, not only in the consideration of rotating bodies, but also in many other connections, such as the bending of beams and the stability of ships. The moments of inertia of homogeneous areas are used in the latter problems, and are convenient in the computation of moments of inertia of material bodies.

If ρ is the density of a body at an element of volume dV, the element of mass is $dm = \rho dV$, and

$$I = \int r^2 \rho dV; \quad k^2 = \frac{I}{m} = \frac{\int r^2 \rho dV}{\int \rho dV}.$$

If the body is homogeneous, ρ is constant, and

$$I = \rho \int r^2 dV; \quad k^2 = \frac{I}{m} = \frac{\int r^2 dV}{\int dV} = \frac{\int r^2 dV}{V}.$$

The transition to the conceptions of the moment of inertia and squared radius of gyration of an area or an arc, with dA or ds in place of dV, is the same as the corresponding process in the case of centers of gravity. In what follows, the density will be supposed uniform if not specified.

A few simple moments of inertia and certain general principles aid materially in the actual computations. It is a part of the definition that the same moment of inertia is obtained from a given mass at a given distance from a given axis whether the mass is concentrated at a point or distributed in any way over the circumference of a circle or the surface of a cylinder, and that the moment of inertia of a system of particles or bodies is the sum of the moments of inertia of its constituent parts.

Thus, for a circumference of radius a with reference to an axis perpendicular to its plane through its center, or for a cylindrical surface of revolution of radius a with reference to its geometric axis, $k = a$.

For a rectangle with reference to an axis in its plane parallel to one of its sides, the radius of gyration is the same as for one of the perpendicular sides; and for any right cylinder with reference to an axis parallel to its geometric axis, the radius of gyration is the same as for a section normal to its axis.

For, let the dimensions of the rectangle be l and b, and let the axis be parallel to b (Fig. 112). Then if lines are drawn across the rectangle parallel to the axis, the moment of inertia and the mass of the rectangle are the same as if the mass of every such line were concentrated at the point where the line crosses l, and so are what the corresponding values for the line l would become if the density of the line were b times that of the rectangle. As the radius of gyration is independent of this density, the first part of our proposition is proved.

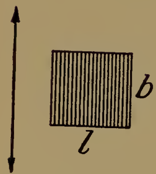


FIG. 112.

The second part of the proposition is proved in the same way by drawing cylindrical surfaces of revolution having the axis of reference as a common axis, and supposing the mass of every such

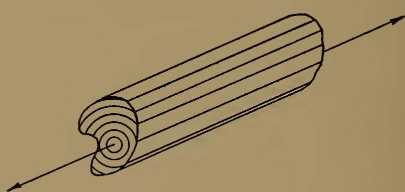


FIG. 113.

surface to be concentrated along the arc in which the surface cuts the normal section of the given cylinder.

339. *Examples.*

Find the values of I and k^2 for each of the following homogeneous figures, the first five by direct integration, the rest by combining values already found.

1. Straight line of length l; axis perpendicular to the line at one end. Ans. $I = \frac{l^3}{3}$, $k^2 = \frac{l^2}{3}$.

2. Straight line of length l; axis perpendicular to the line extended, and distant c from the nearer end.

Ans. $I = cl(c+l) + \frac{l^3}{3}$, $k^2 = c(c+l) + \frac{l^2}{3}$.

3. Rectangle of length l and breadth b; axis one of the sides of length b. Ans. $I = \frac{l^3b}{3}$, $k^2 = \frac{l^3}{3}$.

4. Rectangle of length l and breadth b; axis parallel to the sides of length b and distant c from the nearer one.

Ans. $I = clb(c+l) + \frac{l^3b}{3}$, $k^2 = c(c+l) + \frac{l^2}{3}$.

5. Triangle of altitude h and base b; axis parallel to the base and through the opposite vertex.

Ans. $I = \frac{1}{4}bh^3$, $k^2 = \frac{1}{2}h^2$.

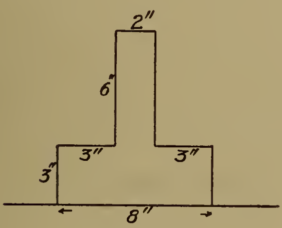


FIG. 114.

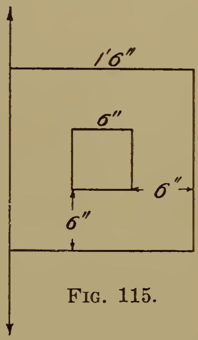


FIG. 115.

6. The accompanying T-beam section, axis as shown in Fig. 114. Ans. $I = 540$, $k^2 = 15$.

7. The hollow square in the accompanying figure (Fig. 115); axis shown. Ans. $I = 31{,}968$, $k^2 = 111$ (inch-units).

8. Two straight lines, each of length l, perpendicular to each other at their mid-points; axis parallel to one through an end of the other. Ans. $I = \frac{7}{12}l^3$, $k^2 = \frac{7}{24}l^2$.

9. A pole AB, standing upright with the end B on a horizontal plane, falls without sliding. Find the speed with which A hits the plane if $AB = 24$ feet (the kinetic energy is acquired through the work done by gravity during the fall). Ans. 48 f/s.

340. Perpendicular Axes for a Plane Area.—If k_x^2 and k_y^2 are the squared radii of gyration for a plane area or arc with reference to two perpendicular axes lying in its plane, the squared radius of gyration with reference to an axis perpendicular to the plane at the intersection of the given axes is

$$k_z^2 = k_x^2 + k_y^2.$$

Let the three mutually perpendicular axes be OX, OY, OZ, and take them as axes of coördinates. Then the moments of inertia of the given area with reference to these axes are:

$$I_x = \int y^2 dA, \quad I_y = \int x^2 dA, \quad \text{and} \quad I_z = \int r^2 dA,$$

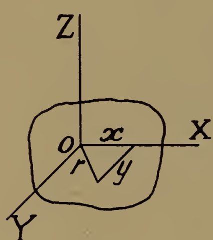


FIG. 116.

where, since $r^2 = x^2 + y^2$,

$$I_z = \int x^2 dA + \int y^2 dA = I_y + I_x.$$

Then

$$Ak_z^2 = Ak_x^2 + Ak_y^2,$$

or

$$k_z^2 = k_x^2 + k_y^2.$$

An exactly similar proof establishes the proposition in the case of an arc.

341. Parallel Axes for any Body.—If k_a^2 and k_g^2 are the squared radii of gyration for any body with reference to any axis a and an axis g, parallel to a through the center of gravity G; and if R is the distance of G from a, then

$$k_a^2 = k_g^2 + R^2.$$

Choose a set of perpendicular axes, taking g as the axis of z, and let a cut the plane of xy at (h, k) (see Fig. 117).

Then $I_g = mk_g^2 = \int (x^2 + y^2)\, dm$, and $I_a = mk_a^2 = \int r^2 dm$, where $r^2 = (x-h)^2 + (y-k)^2$ and $h^2 + k^2 = R^2$; *i. e.*, $r^2 = (x^2 + y^2) + R^2 - 2hx - 2ky$.

From the formulas for the coördinates of the center of gravity,

$$0 = \frac{\int x dm}{\int dm}, \quad 0 = \frac{\int y dm}{\int dm}.$$

Hence, from

$$mk_a^2 = \int r^2 dm = \int (x^2 + y^2)\, dm + \int R^2 dm - 2h \int x dm - 2k \int y dm$$

we find

$$mk_a^2 = mk_g^2 + mR^2,$$

or

$$k_a^2 = k_g^2 + R^2.$$

It is readily seen that if k_a^2 and k_b^2 are the values of k^2 for any body with reference to two parallel axes, distant R_a and R_b respectively from the center of gravity,

$$k_a^2 - k_b^2 = R_a^2 - R_b^2.$$

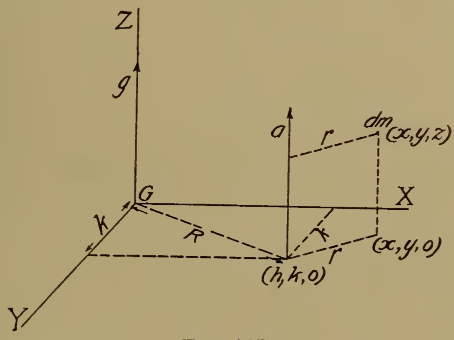


FIG. 117.

342. Fundamental Radii of Gyration. Straight Line and Rectangle.—The squared radius of gyration for a straight line of length l with reference to an axis, a, perpendicular to l at one end, is $\frac{l^2}{3}$, and with reference to a parallel to a through the middle point of l is $\frac{l^2}{12}$, for, from the theorem of parallel axes, we have $\frac{l^2}{3} = k_g^2 + \frac{l^2}{4}$. The same theorem will give k^2 for the line l with reference to a perpendicular to l anywhere in space.

These are also the values of k^2 with reference to the same axes for any rectangle having l as one side, and parallels to the axis of reference as the perpendicular sides.

The squared radius of gyration for a rectangle having the dimensions a and b, with reference to an axis perpendicular to its plane through its center, is found from the theorem of perpendicular axes to be $\frac{a^2+b^2}{12}$.

The theorem of parallel axes will now give the value of k^2 for any axis perpendicular to the plane of the rectangle.

These will also be the values of k^2, with reference to the same axes, for any right parallelopiped having the rectangle as a cross-section.

343. *Examples.*

Find I and k^2 for each of the following homogeneous figures with reference to three mutually perpendicular axes through the center of gravity, one of them being an axis of symmetry.

1.

$$k_x^2=6, \quad k_y^2=\tfrac{11}{3}, \quad k_z^2=\tfrac{29}{3}.$$
$$I_x^2=180, \quad I_y^2=132, \quad I_z^2=312 \text{ (Fig. 118).}$$

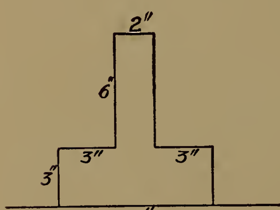


FIG. 118.

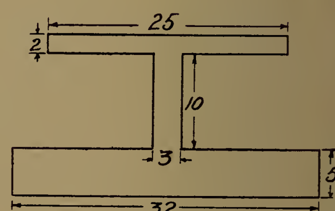


FIG. 119.

2.

$$I_x=8025, \quad I_y=16{,}280, \quad I_z=24{,}305,$$
$$k_x^2=\tfrac{535}{6}, \quad k_y^2=\tfrac{407}{6}, \quad k_z^2=\tfrac{4861}{48} \text{(Fig. 119).}$$

3.

$$I_x = \frac{hb^3}{48}, \quad I_y = \frac{h^3b}{36},$$

$$I_z = \frac{hb}{144}(3b^2+4h^2),$$

$$k_x{}^2 = \frac{b^2}{24}, \quad k_y{}^2 = \frac{h^2}{18},$$

$k_z{}^2 = \frac{1}{72}(3b^2+4h^2)$ (Fig. 120).

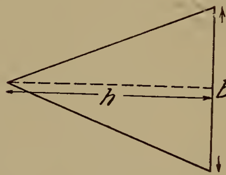


FIG. 120.

4. Find k^2 for these figures with reference to axes perpendicular to their planes, the axis passing through one of the lower corners in the figures of examples 1 and 2, through the left vertex in the figure of example 3.

Ans. (1) $k^2 = \frac{104}{3}$, (2) $k^2 = \frac{1189}{3}$, (3) $k^2 = \frac{12h^2+b^2}{24}$.

5. One diagonal of a rhombus is equal to a side; find k^2 with reference to each diagonal. Ans. $\frac{a^2}{24}$ and $\frac{a^2}{8}$.

6. The dimensions of a rectangular parallelopiped are l, a and b. Find its moment of inertia with reference to an edge of length l. Ans. $\frac{abl}{3}(a^2+b^2)$.

7. Find the moment of inertia with reference to a lateral edge for a right prism of which the cross-section is an equilateral triangle.

Ans. Lateral edge being l, edge of base a, $I = \frac{5}{48} la^4\sqrt{3}$.

8. A cube balanced on one edge on a horizontal plane is slightly disturbed and falls without sliding; what is its angular rate of rotation when it hits the plane, and what is then the speed of a point on the edge opposite to the stationary edge?

Ans. $\omega^2 = \frac{3g}{2l}\sqrt{2}$; $v^2 = 3gl\sqrt{2}$.

344. k^2 for a Circumference or for a Circle.—For a circumference of radius a, with reference to an axis perpendicular to its plane at its center, $k=a$; and with reference to any diameter, k must be the same as with reference to any other diameter. The value of k^2 for a circumference with reference to any diameter is therefore determined by the theorem of perpendicular axes, $a^2 = k^2 + k^2$, to be $k^2 = \frac{a^2}{2}$.

To find k^2 for a circle of radius a with reference to a perpendicular to its plane at its center O, divide the circle into elementary rings by concentric circumferences dr apart; then, assuming the density $=1$,

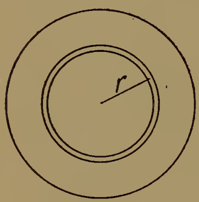


FIG. 121.

$$dm = 2\pi r dr,$$

$$k^2 = \frac{\int_0^a 2\pi r^3 dr}{\int_0^a 2\pi r dr} = \frac{\frac{1}{2}\pi a^4}{\pi a^2} = \frac{a^2}{2}.$$

Using the theorem of perpendicular axes, as we did for the circumference, we find k^2 for the circle with reference to any diameter to be $\frac{a^2}{4}$.

We might have obtained the last result by observing that k^2 for the element $2\pi r dr$ is $\frac{r^2}{2}$, so that its moment of inertia is $\pi r^3 dr$, and for the circle,

$$k^2 = \frac{\int_0^a \pi r^3 dr}{\int_0^a 2\pi r dr} = \frac{\frac{1}{4}\pi a^4}{\pi a^2} = \frac{a^2}{4}.$$

Again, we might have used rectangular coördinates, taking the diameter in question as the axis of x, and using either the element xdy, for which $k = y$, or the element ydx, for which $k^2 = \frac{y^2}{3}$. The corresponding elements of the moment of inertia would have been $xy^2 dy$ and $\frac{1}{3} y^3 dx$.

The theorem of parallel axes will now give k^2 for any circumference or circle with reference to any axis in or perpendicular to its plane.

The value of k^2 for a right circular cylinder with reference to its geometric axis or to any parallel axis is the same as for its right section.

345. k^2 for any Area or Arc.—The methods suggested for the circle will give k^2 for any area with reference to the coördinate axes, and, thence, by the theorem of perpendicular axes, with reference to a perpendicular to the plane at the origin. The theorem of parallel axes will then give k^2 with reference to any axis parallel to either coördinate axis or perpendicular to both of them. These results will include the values of k^2 for any right cylinder with reference to any parallel to its elements.

If the boundary of an area is given by a polar equation, it is best to take $dm=rd\theta dr$ and perform a double integration, and if k^2 is wanted for an axis perpendicular to the plane at the pole, to determine it directly.

346. *Examples.*

1. Find k^2 for an ellipse having the semi-axes a and b with reference to each principal diameter and to the perpendicular to the plane of the ellipse at the center.

Ans. $k_x^2=\dfrac{b^2}{4}, \quad k_y^2=\dfrac{a^2}{4}, \quad k_z^2=\dfrac{a^2+b^2}{4}.$

2. Find I for an elliptic right cylinder with reference to an element through an extremity of the major axis of a right section. Length l, semi-axes a (major) and b.

Ans. $\frac{1}{4}\pi abl(5a^2+b^2)$.

3. Find k^2 for a parabolic segment, height h, base $2b$, with reference to its axis of symmetry and to a perpendicular to its plane through its vertex.

Ans. $k_x^2=\frac{1}{5}b^2$, $k_z^2=\frac{1}{35}(7b^2+15a^2)$.

4. Find k^2 for the area bounded by the x-axis, and an arch of the cycloid, $x=a(\phi-\sin\phi)$, $y=a(1-\cos\phi)$, with reference to the x-axis. Ans. $k_x^2=\frac{35}{36}a^2$.

5. Find k^2 for the area bounded by the cardioid $r=2a\sin^2\frac{\theta}{2}$. with reference to the initial line and to the perpendicular to its plane at the pole. Ans. $k_x^2=\frac{7}{16}a^2$, $k_z^2=\frac{35}{24}a^2$.

6. Find k^2 for the arc of a cycloidal arch with reference to its base. Ans. $\frac{32}{15}a^2$.

7. Find k^2 for the arc of a cardioid with reference to a perpendicular to its plane at the cusp, and with reference to a parallel axis through the center of gravity.

Ans. $k^2 = \frac{32}{15}a^2$, $k_g{}^2 = \frac{112}{75}a^2$.

8. Show that if a vertical plane area is submerged in a fluid so that its center of gravity is h below the level free surface, its center of fluid pressure will be at a depth D below its center of gravity, where $hD = k_g{}^2$, k_g being the radius of gyration for the submerged area with reference to a horizontal axis through its center of gravity.

347. k^2 for a Solid of Revolution with Reference to the Geometric Axis.—In finding k^2 for a solid of revolution with reference to the axis of revolution, the most convenient element of volume is that generated by the revolution of some element of the generating area. For example, to find k^2 for a sphere of radius a with reference to a diameter, take the diameter as the axis of x; the sphere is generated by the revolution about the axis of x of the circle $x^2 + y^2 = a^2$, or, in polar coördinates, $r = a$.

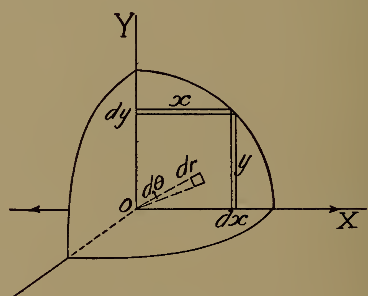


FIG. 122.

The element ydx generates a disc, for which

$$dm = dV = \pi y^2 dx, \quad k^2 = \frac{y^2}{2}, \quad dI = \tfrac{1}{2}\pi y^4 dx;$$

$$k^2 = \frac{\frac{1}{2}I}{\frac{1}{2}m} = \frac{\int_0^a \frac{1}{2}\pi y^4 dx}{\int_0^a \pi y^2 dx} = \frac{\frac{1}{2}\pi \int_0^a (a^2 - x^2)^2 dx}{\pi \int_0^a (a^2 - x^2)\, dx} = \frac{\frac{1}{2}\pi a^5 (1 - \frac{2}{3} + \frac{1}{5})}{\pi a^3 (1 - \frac{1}{3})} = \frac{2a^2}{5}.$$

The element xdy generates a cylindrical shell, for which

$$dm = dV = 2\pi yxdy, \quad k^2 = y^2, \quad dI = 2\pi y^3 xdy;$$

and again, the element $rd\theta dr$ generates a ring, for which

$$dm = dV = 2\pi r \sin\theta r d\theta dr, \quad k^2 = r^2 \sin^2\theta, \quad dI = 2\pi r^4 \sin^3\theta d\theta dr;$$

each of these, the first by a single integration, the second by a double integration, gives in the same way $k^2 = \frac{2}{5}a^2$.

k^2 for an axis parallel to the axis of revolution is found directly by the theorem of parallel axes.

348. *Examples.*

1. Find k^2 for an ellipsoid of revolution with reference to the axis of revolution.

Ans. Length being $2a$, greatest transverse diameter $2b$, $k^2 = \dfrac{2b^2}{5}$.

2. Find k^2 for a cone of revolution with reference to the geometric axis.

Ans. Height being h, radius of base b, $k^2 = \frac{3}{10}b^2$.

3. Find k^2 for a paraboloid of revolution with reference to the geometric axis.

Ans. Height being h, radius of base b, $k^2 = \dfrac{b^2}{3}$.

4. Find k^2 with reference to the geometric axis for the solid generated by the cycloid, $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$, in revolving about its base. Ans. $k^2 = 1.575a^2$.

5. Find k^2 with reference to the geometric axis for the solid generated by revolving the cardioid $r = 2a\sin^2\frac{\theta}{2}$ about the initial line. Ans. $k^2 = \frac{24}{35}a^2$.

6. If all of the solids in examples 1-4 are of the same length, and if each of them weighs 480 pounds to the cubic foot and has a maximum transverse diameter of 4 feet, find their respective

kinetic energies in foot-pounds when each is making 2 revolutions a second.

Ans. $I_1=1024\pi^4$, $I_2=384\pi^4$, $I_3=640\pi^4$, $I_4=945\pi^4$.

7. Find k^2 with reference to the geometric axis for the surface formed by revolving an arch of the cycloid about its base.

Ans. $k^2=\frac{96}{35}a^2$.

8. Find k^2 with reference to the geometric axis for the surface formed by revolving the cardioid $r=2a\sin^2\frac{\theta}{2}$ about the initial line.

Ans. $k^2=\frac{160}{43}a^2$.

349. Axis of Reference not an Axis of Symmetry.—In other cases of finding k^2 for a solid, it is generally most convenient to

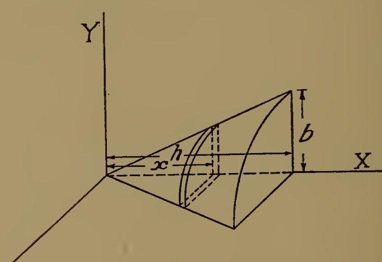


FIG. 123.

divide the solid into elements of volume by planes that are either parallel or perpendicular to the axis of reference. For example, to find k^2 for a cone with reference to a perpendicular to the geometric axis through the vertex, divide the cone by planes parallel to the axis of reference and perpendicular to the geometric axis (see Fig. 123). Let x be the distance of any one of these planes from the axis of reference; then the corresponding element of volume is $dV=\pi\left(\frac{bx}{h}\right)^2 dx$, and its k^2 is $x^2+\frac{1}{4}\left(\frac{bx}{h}\right)^2$.

The moment of inertia is therefore, if we take $\rho=1$,

$$I=\int k^2dV=\frac{\pi b^2}{4h^4}(4h^2+b^2)\int_0^h x^4dx=\frac{\pi b^2h}{20}(4h^2+b^2),$$

$V = \frac{\pi b^2 h}{3}$; hence $k^2 = \frac{3}{20}(4h^2 + b^2)$.

With reference to a parallel axis through the center of gravity,

$$k_g^2 = \tfrac{3}{5}h^2 + \tfrac{3}{20}b^2 - \tfrac{9}{16}h^2 = \tfrac{3}{80}(h^2 + 4b^2).$$

350. *Examples.*

1. Find k^2 for a cylinder of altitude h, radius of base b, with reference to a diameter of its base, and to a parallel axis through the center of gravity. Ans. $k^2 = \frac{b^2}{4} + \frac{h^2}{3}$, $k_g^2 = \frac{b^2}{4} + \frac{h^2}{12}$.

2. Find k^2 for a paraboloid of revolution of altitude h, radius of base b, with reference to a tangent at the vertex, and to a parallel axis through the center of gravity.

Ans. $k^2 = \frac{b^2}{6} + \frac{h^2}{2}$, $k_g^2 = \frac{b^2}{6} + \frac{h^2}{18}$.

3. Find k^2 for an ellipsoid of semi-axes a, b, c with reference to each of its principal diameters.

Ans. $k_x^2 = \frac{b^2 + c^2}{5}$, $k_y^2 = \frac{c^2 + a^2}{5}$, $k_z^2 = \frac{a^2 + b^2}{5}$.

4. A circle of radius a is revolved about a line in its plane distant $na (n > 1)$ from its center. Find k^2 with reference to the geometric axis and with reference to a diameter perpendicular to the geometric axis for the solid so formed, and for its surface.

Ans. Geom. axis: Solid, $k^2 = \frac{a^2}{4}(4n^2 + 3)$; surface, $k^2 = \frac{a^2}{2}(2n^2 + 3)$. Perp. axis: Solid, $k^2 = \frac{a^2}{8}(4n^2 + 5)$; surface, $k^2 = \frac{a^2}{4}(2n^2 + 5)$.

5. A parabolic segment, height h, base $2b$, revolves about the base; find k^2 for the solid so generated, with reference to a diameter of the maximum circular section and with reference to the geometric axis. Ans. $k_x^2 = \frac{1}{21}(4h^2 + 3b^2)$, $k_z^2 = \frac{8}{21}h^2$.

351. D'Alembert's Principle.—For convenience of statement we divide the forces acting on a body or system of bodies into two classes: *external* or *impressed forces*, the source of which is outside of the body, and *internal forces,* the reactions between the particles of the body or system.

Consider a system composed of two connected particles, P_1 and P_2, and suppose the total external force acting on P_1 to be f_1, and that acting on P_2 to be f_2. Let the reaction of P_2 upon P_1 be r_1, then the reaction of P_1 upon P_2 is $r_2 \equiv -r_1$, a force having the same line of action as r_1. Let the accelerations of P_1 and P_2 be a_1 and a_2, their masses m_1 and m_2. Then m_1a_1 is the single force which would give P_1 the motion it actually has; m_1a_1 is therefore called the *effective* force for P_1. m_2a_2 is the effective force for P_2. Now the equations of motion for P_1 and P_2 are:

$$f_1 + r_1 \equiv m_1a_1, \quad f_2 + r_2 \equiv m_2a_2,$$

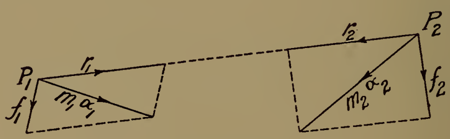


FIG. 124.

where the $+$ indicates the process of finding a resultant, and the equality of $(f+r)$ with ma holds for magnitude, direction and line of action; *i. e.,* is an identity. If we now add these identities, we have

$$f_1 + r_1 + f_2 + r_2 \equiv m_1a_1 + m_2a_2,$$

which reduces, on account of the relation between r_1 and r_2, to

$$f_1 + f_2 \equiv m_1a_1 + m_2a_2,$$

another identity.

This reasoning can evidently be applied to any number of particles, for all the internal forces occur in equal, directly opposed pairs, and drop out in the summation. If the particles form a continuous body or system of such bodies, the summation becomes an integration. Then *D'Alembert's Principle* may be stated as follows:

The resultant of the impressed forces acting on a material body or system is identical with the resultant of the effective forces for all its particles.

It follows from this that the sum of the resolved parts of the impressed forces in any given direction is equal to the sum of the resolved parts of the effective forces in the same direction.

For instance, if a body weighing W pounds is falling under the action of gravity alone, we know that the resultant of the impressed forces is a vertical force of W pounds acting through its center of gravity; hence the resultant of the effective forces for all its particles, which we may indicate by $\int adm$ taken throughout the body, is this same force; then if the body is rotating, so that a is not the same for all the particles, the horizontal components of adm must balance one another when summed up for the whole body.

352. D'Alembert's Principle Applied to Rotation.—The application to linear motion is only half the use of D'Alembert's Principle; the other half is its application to angular motion, or rotation about an axis. The kinematics and dynamics of angular motion can be founded theoretically on the corresponding sciences of linear motion, or may be established independently.

It is found that the moment of a force plays the same part in angular motion that the force itself plays in linear motion. In Art. 314 we defined the moment of a force about an axis perpendicular to its line of action. If we have a force F and an axis l in any position, we can draw a plane parallel to l through F's line of action, and resolve F in this plane into f' parallel to l and f perpendicular to l. Then the moment of F about l is defined to be the same as the moment of f about l.

It can be shown that the moment about any axis of the resultant of any set of forces is the sum of the moments about the same axis of the component forces. Consequently, D'Alembert's Principle tells us that the sum of the moments of the impressed forces about any axis is equal to the sum of the moments of the effective forces about the same axis.

For instance, if a falling body is acted upon by both gravity and atmospheric resistance and descends without rotation, all of its particles having the same acceleration α, the sum of the moments of the effective forces about an axis through its center of gravity G, if x is the distance of any particle of mass dm from G, is $\int x\alpha dm = \alpha \int x dm$ taken throughout the body, and is therefore zero. Moreover, as the sum of the forces of gravity acting on the particles passes through G, gravity has a zero moment about the same axis; therefore the total atmospheric resistance has the moment zero about any axis through G, and so is a single force acting through G.

If, on the other hand, the falling body starts at rest and rotates as it falls, the total atmospheric resistance does not pass through G.

353. The Equation of Rotation.—If a body is capable of turning about a fixed axis, it will turn if acted upon by a force that has a moment about that axis. The numerical relation between the moment and the rotary motion is obtained as follows: Let Fig. 125 represent a section of the body by a plane perpendicular to the axis at O, and let dm at P be an element of mass. Let ω be the angular rate at which P moves about O when the body rotates, and let $OP = r$; then the acceleration of P has two resolved parts: $\frac{v^2}{r} = r\omega^2$ in the direction PO, and $\frac{dv}{dt} = r\frac{d\omega}{dt}$ in the direction of the tangent PT.

FIG. 125.

The effective force for the particle at P is therefore the resultant of $r\omega^2 dm$ along PO, which has no moment about the axis, and $r\frac{d\omega}{dt}dm$, which has the moment $r^2\frac{d\omega}{dt}dm$.

The sum of the moments of all the effective forces about the axis of rotation is therefore $\int r^2\frac{d\omega}{dt}dm$ taken throughout the body.

If the body is rigid, $\frac{d\omega}{dt}$ is the same for all its points, and the sum becomes

$$\frac{d\omega}{dt}\int r^2 dm = I\frac{d\omega}{dt} = mk^2\frac{d\omega}{dt}.$$

If M is the sum of the moments about the axis of rotation of all the external forces, we have from D'Alembert's Principle,

$$M = I\frac{d\omega}{dt}.$$

This is a second equation of motion; it is distinguished from the equation of linear motion, $f=ma$, by being called the *equation of rotary motion.* The two equations are also called the equations of *translation* and *rotation.*

There is an evident correspondence in the two equations between momentum and force, moment of inertia and mass, angular acceleration and linear acceleration. The moment of inertia, however, is not, like the mass, a fixed characteristic of the body, for it varies with the relative position of the body and the axis.

354. The Compound Pendulum.—The compound pendulum in its simplest form consists of a rigid body capable of turning freely on a fixed horizontal axis; this is the pendulum used in clocks and in physical apparatus for determining local values of the acceleration due to gravity. The devices used to minimize friction at the axis are so successful that the action of this force may be neglected in considering the motion. The only impressed forces, then, are the forces of gravity on the particles of the pendulum, the resultant of which acts vertically downward through the center of gravity G of the pendulum.

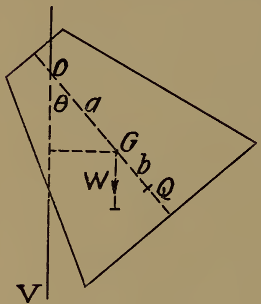


FIG. 126.

Let Fig. 126 represent a section of the pendulum by a plane perpendicular to the axis at O and passing through G; OV and OG are the traces of planes containing the axis, the first of them vertical. Let $OG=a$; then the moment of the impressed forces is $Wa\sin\theta$, if W is the weight of the pendulum and the angle VOG is called θ.

According to Art. 353,

$$Wa \sin \theta = -I_a \frac{d\omega}{dt} = -\frac{W}{g} k_a^2 \frac{d^2\theta}{dt^2},$$

and I_a and k_a^2 are the moment of inertia and squared radius of gyration for the pendulum with reference to the axis of rotation.

Hence

$$\frac{d^2\theta}{dt^2} = -\frac{ga}{k_a^2} \sin \theta.$$

This is precisely the equation for the motion of a simple pendulum of which the length is $l=\frac{k_a^2}{a}$; that is, the compound pendulum vibrates as if all its mass were concentrated at a point Q of the line OG, called the *center of oscillation*, at a distance l from O. l is called the length of the equivalent simple pendulum.

If the squared radius of gyration for the pendulum with reference to a parallel to the axis of rotation through G is k_g^2, $k_a^2=k_g^2+a^2$, $l=\frac{k_g^2}{a}+a$. If we let $GQ=b$, $OQ=a+b$ and $ab=k_g^2$.

355. *Examples.*

1. A cone of altitude $4a$, base of radius $2a$, is free to rotate about a small smooth axis, perpendicular to the geometric axis, a from the vertex. Find the length of the equivalent simple pendulum. Ans. $l=\frac{13}{5}a$.

2. A rod of a centrifugal governor is $\frac{1}{4}$ inch in diameter and 6 inches long and carries a sphere 1 inch in diameter at the lower end. What is the length of the equivalent simple pendulum, supposing the rod suspended from its extreme upper end?
Ans. Very nearly 6 inches.

356. Moments of Inertia Determined by Experiment.—Suppose we wish to find the moment of inertia of a body with reference to a given axis, and that the shape is inconvenient for computation. An apparatus may be set up similar to a lathe, but driven by a weight hung from a light cord wound on a drum carried on the axle and concentric with its axis of revolution. The body may be fastened to both head-stocks in such a way that the axis with reference to which the moment of inertia is desired will coincide with the axis of revolution. Let the radius of the drum be $\frac{1}{2}$ foot and the driving weight 10 pounds, and suppose the weight is observed to fall $1\frac{1}{2}$ feet in the first 2 seconds. Suppose the moment of inertia of the moving parts of the apparatus itself to be $\frac{1}{20}$ (engineer's units) and assume the effect of friction negligible. Let $g=32$.

Let the weight descend s feet, and the apparatus rotate through θ radians, in t seconds, and let T be the tension in the cord. Then the equation of translation for the weight and the equation of rotation for the revolving mass give, if I is the moment of inertia of the body and the apparatus together,

$$10-T=\frac{10}{32}\frac{d^2s}{dt^2}, \qquad (1)$$

$$T\times\frac{1}{2}=I\frac{d^2\theta}{dt^2}. \qquad (2)$$

Evidently $s=\frac{1}{2}\theta$, $\dfrac{d^2\theta}{dt^2}=2\dfrac{d^2s}{dt^2}$, so, from (2),

$$T=4\frac{d^2s}{dt^2}\cdot I=4aI,$$

and from (1),

$$10-4aI=\tfrac{10}{32}a, \quad I=\frac{5}{64a}(32-a).$$

The constant acceleration a is readily seen to be $\frac{3}{4}$ f/s²; hence $I=\frac{625}{192}$. The moment of inertia of the body alone is $I'=\frac{625}{192}-\frac{1}{20}$. If the body weighs 150 pounds, the corresponding radius of gyration is k, where

$$k^2=\frac{32I'}{150}=\tfrac{25}{36}-\tfrac{32}{3000}=\tfrac{25}{36}(1-0.01536),$$

$$k=\tfrac{5}{6}(1-0.008)\text{ feet}=4.96\text{ inches}.$$

If the effect of friction cannot be neglected, it is best allowed for by making two tests with different driving weights and eliminating the moment of friction and the two tensions from the four equations of motion.

If a body is suspended from an axis and allowed to vibrate through a small angle, its time of vibration, $T=\pi\sqrt{\dfrac{l}{g}}$, will give the length of the equivalent simple pendulum, $l=\dfrac{gT^2}{\pi^2}=\dfrac{k^2}{a}$, where k is the radius of gyration with reference to the axis of suspension, and a is the distance of this axis from the center of gravity. Then if a is known, $k=\dfrac{T}{\pi}\sqrt{ga}$.

357. Combined Translation and Rotation.—It can be shown that when a body moves in any way, its center of gravity moves as if it were a heavy particle having the same mass as the body and acted upon by forces equal to the forces acting on the body. At the same time, the body if rigid revolves about any axis through its center of gravity precisely as it would if the center of gravity were fixed.

Consider, for instance, the motion of a sphere down a rough inclined plane. Let the weight of the sphere be W pounds, its radius a feet, the coefficients of friction μ and μ' and the inclination of the plane to the horizontal ϕ. In t seconds, let the center of the sphere move s feet down the plane, and let the radius to the point initially in contact with the plane rotate through θ radians. Then if the sphere rolls, $s=a\theta$; if it also slides, $s>a\theta$. The rolling is due to rotation about the horizontal axis, and is caused by friction alone; the friction may not be sufficient to cause rotation rapid enough to keep up with the translation.

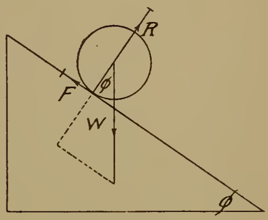


FIG. 127.

The equation for translation down the plane gives

$$W \sin \phi - F = \frac{W}{g}\frac{d^2s}{dt^2}; \quad \frac{d^2s}{dt^2} = g\left(\sin \phi - \frac{F}{W}\right). \qquad (1)$$

The equation for rotation about the horizontal diameter gives

$$aF = I\frac{d\omega}{dt} = \tfrac{2}{5}a^2\frac{W}{g}\frac{d^2\theta}{dt^2}; \quad \frac{d^2\theta}{dt^2} = \frac{5g}{2a}\frac{F}{W}. \qquad (2)$$

If the sphere merely rolls, $s=a\theta, \dfrac{d^2s}{dt^2} = a\dfrac{d^2\theta}{dt^2}$,

$$g\left(\sin \phi - \frac{F}{W}\right) = g\tfrac{5}{2}\frac{F}{W}; \quad \frac{F}{W} = \tfrac{2}{7}\sin \phi. \qquad (3)$$

Thence

$$\frac{d^2s}{dt^2} = g\tfrac{5}{7}\sin \phi. \qquad (4)$$

From this constant acceleration, the motion of the center of the sphere down the plane can be determined directly.

To determine whether pure rolling is possible or not, we have the equation for translation normal to the plane:

$$R = W\cos\phi, \qquad (5)$$

from which $\frac{F}{R} = \frac{2}{7}\tan\phi$, by (3). The greatest value possible for $\frac{F}{R}$ is μ, the coefficient of statical friction; if μ is greater than $\frac{2}{7}\tan\phi$, the sphere will merely roll; if μ is less than $\frac{2}{7}\tan\phi$, the sphere will also slide.

In case the sphere both rolls and slides, we have the equations of motion, 1, 2 and 5, and we also know that $\frac{F}{R} = \mu'$ constantly. Thus we have

$$\left.\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{5g}{2a}\mu'\cos\phi \\ \omega = \frac{d\theta}{dt} &= \left(\frac{5g}{2a}\mu'\cos\phi\right)t. \\ \theta &= \left(\frac{5g}{4a}\mu'\cos\phi\right)t^2. \end{aligned}\right\} \qquad \begin{aligned} \frac{d^2s}{dt^2} &= g(\sin\phi - \mu'\cos\phi). \\ v = \frac{ds}{dt} &= g(\sin\phi - \mu'\cos\phi)t. \\ s &= \tfrac{1}{2}g(\sin\phi - \mu'\cos\phi)t^2. \end{aligned}$$

The amount of slipping at any time is $(s - a\theta)$, and the rate of slipping is $v - a\omega = gt(\sin\phi - \frac{7}{2}\mu'\cos\phi)$ f/s.

358. *Examples.*

1. Show that if a cylinder rolls directly down an inclined plane without sliding, μ must be greater than $\frac{1}{3}\tan\phi$, and the acceleration down the plane is $\frac{2}{3}g\sin\phi$ f/s². Show that if the cylinder slides as well as rolls, the acceleration of the center down the plane is $g(\sin\phi - \mu'\cos\phi)$ f/s², and the rate at which the point of contact slips on the surface is $gt(\sin\phi - 3\mu'\cos\phi)$ f/s.

2. Two spheres, each of radius a and weighing W pounds, look alike, and will stand in any position on a horizontal plane, but one is said to be solid, the other hollow. They are rolled (without sliding) down an inclined plane; sphere A goes 10 feet

and sphere B 9 feet in the same time. Which is hollow, and what is its radius of gyration?

Ans. $\dfrac{d^2s}{dt^2} = \dfrac{a^2g\sin\phi}{a^2+k^2}$ in general, B is hollow, with $k^2 = \frac{5}{9}a^2$.

3. A hollow cylinder of mean radius a, of which the thickness may be neglected, and a homogeneous cylinder of radius a having the same weight are started together down an inclined plane; show that their accelerations are in the ratio $\frac{3}{4}$ if they roll without sliding.