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BY

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CARNEGIE TECHNICAL SCHOOLS

ALGEBRA AND TRIGONOMETRY

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PREFACE.

ALTHOUGH this book has been designed to meet the specific needs of the Carnegie Technical Schools, the growing demand in the field of technical education for a form of mathematical instruction that will eliminate the purely speculative and concentrate the more utilitarian features of mathematical science, leads the author to believe that such a work as this will not be entirely inept outside of the Carnegie Technical Schools.

It is believed that the intellectual stimulus and discipline that is usually attributed to mathematical studies can be as readily conveyed by those things that are at the same time practically useful, as by those that are merely speculative; perhaps much more readily.

The child can be taught to read as successfully by giving it exercises that contain useful information, as by requiring it to drone over masterpieces of prose and verse that leave no valuable residue whatever in the childish mind.

The subjects discussed and the problems illustrative of them have been selected after a careful gleaning of the author's experience with students of varying tastes and mentality, with the end in view of making the subjects vital and pertinent to the special training they are seeking, and at the same time of developing their powers of independent and accurate thinking.

The vital thing in the art of instruction, in the author's opinion, is to retain for the subject under investigation, not only the students interest but his respect and confidence. To that end he must feel that he is not simply grinding thin air to make, too often, an intellectual fog.

An effort has been made to avoid the extreme of pruning too closely, and, in consequence, everything that in the author's judgment can have even a remote bearing upon a student's usefulness in technical pursuits has been inserted more or less briefly.

The writer wishes to acknowledge his indebtedness to Mr. W. A. Bassett and Mr. Lightcap, instructors in the Carnegie Technical Schools, and especially to Professor Walter F. Knox, for valuable assistance and suggestions.

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ALGEBRA.

CHAPTER I.

ALGEBRA AND ARITHMETIC.

ARTICLE 1. Algebra is merely an extension of the field of Arithmetic, the primary difference being the use of letters as symbols, in addition to that of the Arabic characters (1, 2, 3, 4, etc.) and the general employment of *equations*.

Fundamental Operations.

ART. 2. The arithmetical operations called addition, subtraction, multiplication, and division, have the same meaning in algebra. For instance, in arithmetic we would understand by the symbols $5 + 4$, that 5 and 4 are added; so in algebra by the symbols $a + b$, we would understand that a quantity represented by a was added to another quantity represented by b. Also $a - b$ would mean, as in arithmetic, that b was to be subtracted from a.

Negative Numbers.

ART. 3. In arithmetic it is always necessary in subtraction that the quantity subtracted (called the subtrahend) be less than the quantity from which it is taken (called the minuend).

The use of letters, especially when unknown quantities are thus represented, makes desirable an extension of the operation of subtraction, making it *always* possible.

The desirability of representing in symbols such balancing conditions as debit and credit, profit and loss, above zero and below zero, etc., has suggested the idea of *negative* numbers.

ILLUSTRATION. A business man whose assets are $9000 loses $10,000; he not only has nothing left, but is $1000 in debt.

To express this condition the idea of negative numbers must be introduced, and it is said that he is worth —$1000.

Again, *A* travels from *C* to *D* 5 miles. At *D*, his distance from *C* is 5 miles then. He returns from *D* to *C*, and each mile he travels toward *C* reduces his distance from *C* one mile, and he is successively 4, 3, 2, 1, and 0 miles from *C*; the last symbol (zero) indicating that he has arrived at *C*. Suppose he is carried through *C* to *E* 2 miles, then we may say he is -2 miles from *C*.

The idea of positive and negative quality may thus be represented by opposite directions, from a fixed point. Let this idea be applied to the series of numbers, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, etc. It is plain that this series can be extended to the right indefinitely by adding 1 to each successive number, but if, starting from any number in this series, we return toward the left, by subtracting 1 each time we cannot extend this process indefinitely, because the series of arithmetical numbers, here represented, ends on the left with what we call zero.

Now suppose we agree to extend the series to the left of zero, still subtracting 1 each time, we must have some new designation for the resulting numbers, and we agree to call them negative numbers and to represent the series thus:

As $-8, -7, -6, -5, -4, -3, -2, -1$
$0, 1, 2, 3, 4, 5, 6, 7, 8,$ etc.

We can now express the condition of the business man referred to above. He has lost not only his $ 9000, which would stand in the series to the right of 0, by successive subtractions, but these subtractions still continued until the number representing the final condition stood far down in the series to the left, that is, was -1000.

From the method of obtaining this idea of negative numbers it is plain that they are all less than zero.

A practical illustration in point is the thermometer scale; below zero readings being represented as negative numbers; thus, $-5°$, $-10°$, etc., mean 5 degrees and 10 degrees below zero, respectively, etc.

SYMBOLS.

ART. 4. The signs of addition, subtraction, multiplication, and division ($+ - \times \div$) are the same as in arithmetic, and have the same general significance.

ART. 5. The greater necessity for indicating operations that cannot be completely performed in algebra makes frequent use of the parenthesis or bracket necessary [(. . .) [. . .] {. . .}]. It is often desirable to treat a polynomial, for instance, as if it were a monomial, and one of the important functions of the parenthesis is to bind together several terms where they are to be treated like a single term. For instance, if we wanted to show that the expression $x^3 y^3 + 3x^2 y^2 + 6xy + 7$ was like the expression $a^3 + 3a^2 + 6a + 7$ in form, it can be written thus, $(xy)^3 + 3(xy)^2 + 6(xy) + 7$. Or, that $x + y + z + ax + ay + az$ was like $m + am$, the first expression can be written $(x + y + z) + a(x + y + z)$.

ART. 6. The expression $4a^2 - (5a + 2b - c)$ means that the expressions $5a$, $2b$, and $-c$ are all to be subtracted from $4a^2$. The subtraction is merely deferred by the use of the parenthesis until it is desirable to actually perform it. As long as the parenthesis embraces an expression, its parts are inseparable and must be taken together.

ADDITION OF NEGATIVE QUANTITIES.

ART. 7. The addition of negative quantities adds a new concept to the theory of addition.

If we are required to find the value of the expression $7+(-2)$ we may have recourse to the series of positive and negative numbers previously referred to; thus:

$$\rightarrow$$

$$+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1$$
$$-9-8-7-6-5-4-3-2-1 \quad 0+1+2+3+4+5+6+7+8,$$
$$-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1$$

$$\leftarrow$$

It will be observed that the numbers increase by unity to the right as indicated by the upper row of 1's and decrease to the left, as indicated by the lower row.

Now, suppose that it was required to find $7+(-2)$. This indicates that we must count two to the left by the lower row from $+7$, which gives $+5$. If it was required to find $7+(-9)$ we must count 9 units to the left, which brings us to -2, etc., hence we say $7+(-2)=5$, and $7+(-9)=-2$; but $7+(-2)=5$ is the same as $7-2=5$, which is simple arithmetical subtraction, but for $7+(-9)=-2$ we have no arithmetical equivalent. From an observation of the example and others like it, using the series, we are enabled to evolve this rule.

To add negative numbers subtract the less number from the greater, arithmetically, and prefix the sign of the greater number.

EXAMPLES. $5-8=-3$,

that is, 5 is subtracted from 8 and the sign of 8 is prefixed.

$$+9-2=+7, \text{ etc.}$$

This rule may be extended to a series of quantities by combining the positive quantities into one group and the negative into another, thus:

$$7 - 5 + 11 - 3 - 10 + 13 - 1 = (7 + 11 + 13) + (-5 - 3 - 10 - 1) = 31 + (-19) = +31 - 19 = +12, \text{ etc.}$$

Subtraction of Negative Numbers.

ART. 8. In the expression $a - b = c$, b is evidently the number which added to c will give a, and in general we understand that the subtrahend added to the remainder must always give the minuend. Applying this principle to negative numbers $7 - (-2) = 9$, for -2 is the quantity added to 9 gives 7, according to the rule for addition already stated. Again, $-5 - (-3) = -2$, for -3 added to -2 gives -5. But $7 + 2$ also equals 9 and $-5 + 3$ also equals -2; hence we may express the rule:

To subtract a negative (or positive) quantity change its sign and add algebraically.

ADDITION.

Definitions.

ART. 9. Since the use of letters makes it necessary to merely indicate the operations of addition, subtraction, multiplication, etc., it is necessary to add to the language of arithmetic certain names for these new relations. For instance, in the expression $5ax$, 5 is called the *coefficient* of ax; the expression might have been written $a5x$, then a would have been called the coefficient of the expression; likewise in $x5a$, x is the coefficient.*

It is customary, however, when a number is present in an expression to regard it as the coefficient.

* It is sometimes necessary to extend this idea of a coefficient to include any group of letters or numbers or both, in an expression. For instance, $5a$ might be called the coefficient of x, or in $3a^2bc$, $3a^2b$ might be called the coefficient of c, etc.

From this description, formulate a definition for *coefficient.*

ART. 10. A single expression, involving letters or numbers in any amount, whose parts are not connected by plus or minus signs is called a monomial.

ART. 11. Two or more monomials joined by plus or minus signs form a *polynomial.* Each monomial is called a *term* of the polynomial.

A polynomial of two terms is called a binomial; one of three terms, a trinomial, etc.

ART. 12. Monomials are said to be *like* when they differ, if at all, only in their numerical coefficients. For example, $5a^2xy$, $6a^2xy$, $10a^2xy$, etc., are like.

Give an example of a monomial, of a binomial, of a trinomial.

Addition of Monomials.

ART. 13. When monomials are like, each consists of the same letters, affected in exactly the same way; the numerical coefficients simply indicate how many of these same groups of letters each monomial contains. Plainly then to add monomials; *if they are like, add the coefficients with their proper signs and attach the sum as coefficient to the common literal part; if unlike, connect them by their proper signs.*

It is clearly impossible to collect unlike monomials into one expression, as it would be arithmetically impossible to add pounds, inches, and pints. Unlike monomials are essentially different things, and their addition or subtraction can only be indicated, not actually performed.

Addition of Polynomials.

ART. 14. As polynomials are made up of monomials, their addition resolves itself into an addition of the monomials that compose them. To facilitate this, it is desirable to arrange the like monomials, which occur in the polynomials to be added, in columns. *Then add the like monomials according to their rule of addition, and to these sums join all the unlike monomials occurring in all the polynomials, with their proper signs.*

For example:

Add $3a^2x - 2aby + 7xy^2 - 6a^2$ and $a^2x + 5xy^2 + 3b^2$.

Arrange,
$$\begin{array}{rrrrr} 3a^2x & -\ 2aby & +\ \ 7xy^2 & -\ 6a^2 & \\ a^2x & & +\ \ 5xy^2 & & +\ 3b^2 \\ \hline 4a^2x & -\ 2aby & +\ 12xy^2 & -\ 6a^2 & +\ 3b^2 \end{array}$$

Subtraction of Monomials.

ART. 15. From what has been said under addition it follows; to subtract like monomials: *Subtract algebraically their numerical coefficients and attach the common literal part to the remainder.*

If the monomials are unlike: *Change the sign of the subtrahend and join it to the minuend.*

To Subtract Polynomials.

ART. 16. *Arrange like terms under each other and subtract according to rule for monomials.*

For example: From $2x + 17x^3 - 20$ take $-3x^2 - 15 + 39x^3$.

Arrange, $\quad 2x + 17x^3 - 20$

Subtract, $\quad -3x^2 \quad + 39x^3 - 15$

$$3x^2 + 2x - 22x^3 - 5$$

EXERCISE I.

Addition.

Find sum in each following example:

1. $2a - 3x^2$, $5x^2 - 7a$, $-3a + x^2$, and $a - 3x^2$.
2. $m^2 - n^2 + 3m^2n - 5mn^2$, $3m^2 - 4m^2n + 3n^3 - 3mn^2$, $m^3 + n^3 + 3m^2n$, $2m^3 - 4n^3 - 5mn^2$, $6m^2n + 10mn^2$, and $-6m^3 - 7m^2n + 4mn^2 + 2n^3$.

3. $3y^{\frac{1}{2}} - 4y^{\frac{1}{3}} + 2y^2$, $5y^2a - mn + y^{\frac{1}{3}}$, $4y^2 - 3y^3$, and $5y^{\frac{1}{3}} - 4 + y^{\frac{1}{2}}$.

4. $b - 7 - 6b^2 + 14b^3$, $6 + 16b^3 - 9b - 2b^2$, and $4b^2 - 9b^3 + 13b + 10$.

5. $\frac{3}{8}x - \frac{1}{5}y + \frac{11}{14}z$ and $-\frac{1}{4}x + \frac{1}{3}y - \frac{5}{7}z$.

6. $13(a + 2b) - 15(2b + c)$, $6(2b + c) - 7(c + a)$, and $2(c + a) - 5(a + 2b)$.

7. $2y^2z - 3yz^2 + 5y^3$, $\frac{3}{2}yz^2 - 4z^3 + 7y^2z$, $9y^3 - 5z^3 - \frac{5}{4}yz^2$, and $-6z^3 - 3y^2z + 7y^3$.

8. $8a^3 - 11a - 7a^2$, $2a - 6a^2 + 10$, $-5 + 4a^3 + 9a$, and $13a^2 - 5 - 12a$.

9. $5m^2 - 13m - 4 + 5m^3$, $7m^3 - 7m^2 + 2m - 14$, $6m^3 + 8 - 10m - 8m^2$, and $15 - 16m^2 + 15m - m^3$.

10. $3x^2yz - xy^2z + 7z^2$, $9z^2y - x^2yz + 7xy^2z$, $2z^2y - 3z^2 + 5xyz$, and $z^2 - 4xyz$.

11. $\frac{3}{5}x^2 - \frac{2}{3}x - \frac{2}{7}y + \frac{1}{3}$, $\frac{1}{2}x^2 - \frac{1}{4} + \frac{1}{3}x + \frac{5}{7}y$, and $\frac{4}{7}y - \frac{5}{3}x + \frac{11}{12} - \frac{1}{10}x^2$.

12. $.2ay^2 - .3a^2 + .06y^3m + n^3$, $.5ay^2 + .25y^3m - .4a^2 - .35n^3$, $.3ay^2 + .05n^3 - a^2 - .04y^3m$, and $-.6n^3 + .3a^2 - .15y^3m - ay^2$.

13. Simplify, $8mx - 5x^2 + 3m^2 + 2x^2 - 8m^2 + 13m^2 - 18mx + 6x^2 - 9m^2$.

14. Add, $2n^3 + 7an^2 - 4a^2n + 3a^3$, $8a^2n - 15an^2 - 5a^3 - 10n^3$, $3n^3 - 6an^2$, and $-an^2 + n^3 - 4a^3$.

EXERCISE II.

Subtraction.

Subtract:

1. $3a^2y - 2bx^2 - x^3$ from $2a^2y + 3x^3 - 4bx^2$.

2. $x^3 - 3x^2y + 3xy^2 - y^3$ from $2x^3 - 2x^2y + 4xy^2$.

3. $m^3 + 1$ from $3m^2 + 2m - 6$.

4. $3x^3 - 4x^2 + 2x$ from $2x^3 - x^2$.

5. $a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ from $3a^{\frac{2}{3}} + 6a^{\frac{1}{3}}b^{\frac{1}{3}} - 5b^{\frac{2}{3}}$.

6. $3(m+n)+2(x+y)$ from $5(m+n)-7(x+y)$.
7. $ax+by+c$ from $mx+ny-d$.
8. What must be added to $2x^2-3xy-y^2$ to make $x^2+2xy+y^2$?
9. Perform the indicated operations; $4x-3y-[2x-5-(3y-x-2)]$.
10. $3a^2-[7a-(a^2-2a+9)]$.
11. Subtract $7xy+3x-4yx+2y$ from $9xy-5x+\frac{5}{4}y$ and add to the remainder $xy-x+\frac{1}{4}y$.
12. Simplify, $3x+2y-(9y-7x)-(-x-y)$.

Multiplication.

ART. 17. In the series of numbers it may be conceived that a number may be revolved from the positive side of zero to the negative side, or vice versa. Thus in the series,

$$-9-8-7-6-5-4-3-2-1,\ 0,$$
$$+1+2+3+4+5+6+7+8++$$

$+6$ may be carried over as if revolved on an arm pivoted at 0, to -6 on the other side, but that is equivalent to multiplying 6 by -1; or $+3$ may be revolved to -6, which corresponds to multiplying 3 by -2, etc. Hence, multiplying by a negative quantity carries the number over to the other side of zero.

Suppose -6 is multiplied by -1, then by this rule the product would be $+6$, for -6 would be revolved to $+6$ on the other side, or -3 multiplied by -2 would give us $+6$; hence, *two negative quantities multiplied together produce a positive quantity.*

A general rule may be stated thus:

To multiply any two quantities, multiply the quantities independent of their signs, then prefix a sign determined by the rule that like signs give + and unlike signs −.

Division.

ART. 18. Division being the reverse of multiplication, the rules governing its application are derived from the rules governing multiplication. If it is required to find $16 \div 2$, it is necessary only to find the number which multiplied by 2, gives 16, which is 8. Likewise, $28 \div -4 = -7$, because -7 is the number by which -4 must be multiplied to equal 28.

Again, $-32 \div -4 = 8$, for like reason; also $-ab \div b = -a$, or $-abc \div -bc = +a$, etc. Hence, *to divide, find the quotient of the quantities independent of sign, and prefix the sign determined by the rule that* LIKE SIGNS GIVE + AND UNLIKE SIGNS, −.

Exponents.

ART. 19. When a quantity is multiplied by itself two or more times, as $a \times a \times a \times a$, the result is represented by an abbreviated notation, as a^4 in the above instance. The 4 is called an exponent and indicates the number of times the quantity (as a) called the *base*, is repeated, thus: $(ab)^5$ means that (ab) has been multiplied by itself 5 times, 5 being called the exponent, and ab, the base. In general such an expression is called a *power* of the quantity, and the power is said to be an even power when the exponent is an even number, and odd, when it is an odd number.

Laws of Exponents.

ART. 20. If it is required to find the product $a^4 \times a^5$, it is understood that we have a product of "a" multiplied by itself 4 times, multiplied by another product of a multiplied by itself 5 times, which is plainly the same thing as a single product of a multiplied by itself 9 times. But $9 = 4 + 5$; that is, the exponents are added.

Hence, IN MULTIPLYING EXPRESSIONS CONTAINING THE SAME QUANTITY AFFECTED BY DIFFERENT EXPONENTS, ADD THE EXPONENTS OF THAT QUANTITY IN THE SEVERAL EXPRESSIONS TO OBTAIN ITS EXPONENT IN THE PRODUCT.

For example,

$$a^2 b^3 c \times ab^2 c^4 = a^{2+1} b^{3+2} c^{1+4} = a^3b^5c^5.$$

Let it be required to find $a^8 \div a^3$. This expression means that a quantity is to be found that multiplied by $a^3 = a^8$. By the law of exponents for multiplication, 3 + some number = 8, from which that number required = 8 − 3 = 5. But 8 − 3 = the difference of the exponents involved, hence, *to divide two expressions containing the same quantities, subtract the exponents of the quantities in the divisor from the exponents of the same quantities in the dividend, and the remainders will be the exponents of those quantities in the quotient.*

Thus, $a^4b^3c^2d^5 \div a^2bd^3 = a^{4-2}b^{3-1}c^2d^{5-3} = a^2b^2c^2d^2$.

Again, $a^nb^mc^3d^4 \div a^2b^xc^yd^2 = a^{n-2}b^{m-x}c^{3-y}d^2$, etc. By this law of division $a^5 \div a^5 = a^0$, but $a^5 \div a^5 = 1$ because any quantity divided by itself is 1; therefore, $a^0 = 1$, but a is any quantity whatever; hence *any quantity* with exponent 0 equals 1.

To Multiply a Polynomial by a Monomial.

ART. 21. *To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and take the algebraic sum of the products*, thus:

$$\begin{array}{r} 3x^2y^3 - 2xyz + y^2z^3 - 5x^2y \\ 2x^3y \\ \hline 6x^5y^4 - 4x^4y^2z + 2x^3y^3z^3 - 10x^5y^2 \end{array}$$

To multiply one polynomial by another, multiply each term of one polynomial by each term of the other, and take the algebraic sum of the products, observing the laws of signs and of exponents, thus:

$$\begin{array}{r} 2a^3xy + aby - 3x^2y - 4xyz \\ 2ax - x + 3x^2yz \\ \hline \end{array}$$

$$4a^4x^2y + 2a^2bxy - 6ax^3y - 8ax^2yz - 2a^3x^2y - abxy + 3x^3y + 4x^2yz + 6a^3x^3y^2z + 3abx^2y^2z - 9x^4y^2z - 12x^3y^2z^2.$$

Again,

$$\begin{array}{l} a^2 - ab + b^2 \\ \qquad\;\; a + b \\ \hline a^3 - a^2b + ab^2 \\ \qquad + a^2b - ab^2 + b^3 \\ \hline a^3 + b^3 \end{array}$$

These rules are plain, since a polynomial is the algebraic sum of a number of terms.

Division of a Polynomial by a Monomial.

ART. 22. *To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, dividing the numerical factor in each term of the dividend by the numerical factor in the monomial divisor, subtract exponents of literal factors, and take the algebraic sum of the quotients.* Thus, $4x^2yz - 6x^3y - 8x^2y^2 + 10xy^3 \div 2xy = 2xz - 3x^2 - 4xy + 5y^2$.

Division of one Polynomial by Another.

ART. 23. Since the quotient in division is always the quantity by which the divisor must be multiplied to equal the dividend, it follows that if the divisor and dividend are both arranged with the terms having the highest power of the same letter standing first, then the first term of the quotient will be that quantity by which the first term of the divisor must be multiplied to equal the first term of the dividend, and hence it must be one term of the quotient, because this highest degree term of the dividend could only be gotten by multiplying the highest degree term of the divisor by this term of the quotient. Now if every term of the divisor be multiplied by this first term of the quotient, the product will be at least a part of the dividend. The remainder, obtained by subtracting this product from the whole dividend, will represent what is left undivided of the dividend. If this undivided part be treated in the same way, the next remainder (if there be one) will be another and smaller undivided part of the quotient. Plainly each step reduces the remainder to a smaller and simpler expression, and eventually, if the division can be exactly performed, there will be no remainder. The following illustration, although not exactly analogous, may help to throw light on the reason why division is performed as above. Suppose a barrel of apples of unequal size is to be divided into three parts as equally as possible. The natural procedure would be to divide the largest apples into three parts, then the next size into three parts, and so on until the contents of the barrel is exhausted, that is, there is no remainder.

EXAMPLE. Divide $3n^4 - 25n^2 - 13n - 11n^3 - 2$ by $1 + 4n + 3n^2$.

Rearranging according to powers:

$$\begin{array}{r|l} 3n^4 - 11n^3 - 25n^2 - 13n - 2 & 3n^2 + 4n + 1 \\ \underline{3n^4 + 4n^3 + n^2 \qquad\qquad\qquad} & \overline{n^2 - 5n - 2} \\ -15n^3 - 26n^2 - 13n \qquad & \\ \underline{-15n^3 - 20n^2 - 5n \qquad} & \\ -6n^2 - 8n - 2 & \\ \underline{-6n^2 - 8n - 2} & \end{array}$$

Dividing first term of dividend by first term of divisor, multiplying, and subtracting:

Again, divide $-25x^3y^2 + 12x^4y + 12x^2y^3$ by $3x^2y - 4xy^2$.

Rearranging according to descending powers of x, [according to powers of y would serve as well] and dividing:

$$\begin{array}{r|l} 12\,x^4y - 25\,x^3y^2 + 12\,x^2y^3 & 3\,x^2y - 4\,xy^2 \\ \cline{2-2} \underline{12\,x^4y - 16\,x^3y^2 \qquad\qquad} & 4\,x^2 - 3\,xy \\ -\,9\,x^3y^2 + 12\,x^2y^3 & \\ \underline{-\,9\,x^3y^2 + 12\,x^2y^3} & \end{array}$$

EXERCISE III.

Multiplication.

Multiply:

1. $a - b + 2$ by $3\,a + b$.
2. $a^2 - ab + b^2$ by $a + b$.
3. $x^2 - xy + y^2$ by $x^2 + xy + y^2$
4. $3\,mn - 2\,m^2 + n^2 - 1$ by $m^3 - n^3$.
5. $\frac{3}{2}\,a^2b - \frac{1}{3}\,a + \frac{2}{5}\,ab^2$ by $\frac{1}{2}\,a - \frac{1}{3}\,b$.
6. $(x + a)\,(x + b)$ by $(x + c)$.
7. $x^2 - 2\,x + 1$ by $x - 1$.
8. $a^5 - 5\,a^3 - 4\,a + 8$ by $a^3 + 2\,a - 3$.
9. $-\,2\,a^3m + m^2 - 3\,m^3 + 5$ by $3 - 2\,a$.
10. $3\,x^4 - 4\,x^2y^2 - 2\,xy^3 + 5\,y^4$ by $2\,x^2 - xy + 3\,y^2$.
11. $x^m + 2\,x^my^n + y^n$ by $x^a + y^2$.
12. $x^{n-1} + x^{n-2} + x^{n-3} + x^{n-4}$ by $x - 1$.
13. $y^{\frac{2}{3}} - 2\,y^{\frac{1}{2}} + 5\,xy - x^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + 2\,x^{\frac{2}{3}}y^{\frac{1}{2}} - y^{\frac{1}{3}}$.
14. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
15. $a - a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$.
16. $3\,(m + n)^{\frac{1}{2}} - 2\,(a + b)^{\frac{1}{3}}$ by $(m + n)^{\frac{1}{4}} + (a + b)^{\frac{2}{3}}$.
17. $a^{-\frac{1}{2}} + 3\,a^{-\frac{1}{4}}\,c^{-\frac{1}{3}} - 2\,c^{-1}$ by $a^{\frac{1}{2}} - c^{-\frac{2}{3}}$.
18. $15\,x^{4m} - 19\,x^{3m}y^m - 30\,x^{2m}y^{2n} + 42\,x^my^{3n} + 75\,y^{4n}$ by $2\,x^m + y^n$.

19. Simplify $(x^4 + 3x^3 - 2x + 5)(x^3 - 2x^2 + 7x - 3)$.

20. Simplify $[(a + b) + 2c][(a + b) - 2c]$.

EXERCISE IV.

Division.

Divide:

1. $x^4 - x^2y^2 + y^4$ by $x^2 + xy + y^2$.

2. $x^3 - y^3$ by $x - y$.

3. $m^5 + n^5$ by $m + n$.

4. $6(a - b)^{n-2} - 9(a - b)^{n-1} + 12(a - b)^n$ by $3(a - b)^{n-3}$.

5. $a^2x^4 + (2ac - b^2)x^2y^2 + c^2y^4$ by $ax^2 + bxy + cy^2$.

6. $8m^4 - 22m^3n + 43m^2n^2 - 38mn^3 + 24n^4$ by $2m^2 - 3mn + 4n^2$.

7. $a^{3m} + b^{3m}$ by $a^m + b^m$.

8. $9x^{n-4} + 19x^{n-1} + 5x^{n-2} - 30x^n + 4x^{n+1}$ by $2x^{n-5} - 7x^{n-4} - 3x^{n-6} + x^{n-3}$.

9. $8a^3b^3 - 64x^6y^6$ by $2ab - 4x^2y^2$.

10. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ by $x + y + z$.

11. $1 - a^6$ by $1 + 2a + 2a^2 + a^3$.

12. $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

13. $x + 3x^{\frac{3}{4}}y^{\frac{1}{4}} - 24x^{\frac{1}{2}}y^{\frac{1}{2}} - 3x^{\frac{1}{4}}y^{\frac{3}{4}} - y$ by $x^{\frac{1}{2}} + 3x^{\frac{1}{4}}y^{\frac{1}{4}} - y^{\frac{1}{2}}$.

14. What number must be multiplied by $x + 2y + 3z$ to give $9x^2 + 24xy + 12y^2 + 30xz + 24yz + 9z^2$?

15. Divide $5r^4s^4 - 26r^3s^5 + 2r^7s - 5r^6s^2 - 11r^5s^3 + 7r^2s^6 - 12rs^7$ by $r^4 - 4r^3s + r^2s^2 - 3rs^3$.

CHAPTER II.

FACTORING.

ARTICLE 24. A *factor* of a quantity is a divisor of that quantity.

A quantity has as many factors as it has distinct divisors.*

ART. 25. A factor of a factor of a quantity is a factor of the quantity itself.

ILLUSTRATION: $a^4 - b^4$ has the factor $a^2 - b^2$, and $a^2 - b^2$ has the factor $a - b$; hence, $a - b$ is factor of $a^4 - b^4$, hence:

RULE — *Find the most evident factors of a quantity and examine these factors for factors.* Every step simplifies the process.

ART. 26. There are several general types of quantities with respect to factoring. First: The difference of the squares of two quantities, which is always factorable into the sum and the difference of the two quantities, thus:

$$a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 - b^2)(a^2 + b^2);$$
$$x^6y^2 - a^2b^4 = (x^3y)^2 - (ab^2)^2 = (x^3y - ab^2)(x^3y + ab^2);$$
$$(m^2 + 2n)^2 - (rs - c)^2 = (m^2 + 2n + rs - c)(m^2 + 2n - rs + c), \text{ etc.}$$

Second: Trinomials of the type $x^2 + ax + b$ when b is the product of two factors whose *algebraic* sum is a, (the

* Factors (or quantities in general) are called *prime* when they have no factors except themselves and unity.

signs of the factors being taken into consideration), thus: $x^2 - x - 6 = (x - 3)\ (x + 2)$ for $b = -6 = (-3) \times (+2)$ and $a = -1 = -3 + 2$, hence:

Rule for factoring trinomials of the form $x^2 + ax + b$:

SEPARATE THE LAST TERM INTO TWO FACTORS WHOSE ALGEBRAIC SUM WILL BE EQUAL TO THE COEFFICIENT OF THE MIDDLE TERM WITH PROPER SIGN; EACH OF THESE FACTORS WITH ITS PROPER SIGN ATTACHED TO x, WILL FORM A FACTOR OF THE TRINOMIAL.

ART. 27. If the trinomial is of the form $mx^2 + nx + p$, it may be factored if m and p are each divisible into two factors, such that the sum of the products of these factors multiplied diagonally equals n, thus:

$$\begin{array}{ll} & 6 = \quad 3 \times 2 \\ 6x^2 - 5x - 4 = (3x - 4)\ (2x + 1) \text{ for} & -4 = -4 \times 1 \text{ and} \end{array}$$
$(3 \times 1) + (-4 \times 2) = -5$.

Article 27 may be reduced to the form of Article 26 by a simple transformation, thus:

Multiply $6x^2 - 5x - 4$ by $6 = 36x^2 - 30x - 24 = (6x)^2 - 5(6x) - 24$. Let $6x = y$, then $36x^2 - 30x - 24 = y^2 - 5y - 24 = (y - 8)\ (y + 3)$ [by Article 26] $= (6x - 8)\ (6x + 3)$.

$$\therefore\ 6x^2 - 5x - 4 = \frac{(6x - 8)\ (6x + 3)}{6} = \frac{(6x - 8)}{2} \times \frac{(6x + 3)}{3} = (3x - 4)\ (2x + 1).$$

ART. 28. The trinomial that is a perfect square is evidently a special case of Article 26.

ART. 29. It is often possible to factor an expression by grouping the terms and removing a factor from similar groups, thus revealing a common factor in each group; thus:

$x^3 - y^3 - x^2 + y^2 - x^2 + 2\,xy - y^2 = x^3 - y^3 - (x^2 - y^2) - (x - y)^2 = (x - y)\,(x^2 + xy + y^2 - x - y - x + y) = (x - y)\,[x^2 + xy + y^2 - 2\,x]$, or $x^3z^2 - 8\,y^3z^2 - 4\,x^3n^2 + 32\,y^3n^2 = z^2\,(x^3 - 8\,y^3) - 4\,n^2\,(x^3 - 8\,y^3) = z^2\,(x^3 - (2\,y)^3) - (2\,n)^2\,(x^3 - (2\,y)^3) = [z^2 - (2\,n)^2]\,[x^3 - (2\,y)^3] = (z - 2\,n)\,(z + 2\,n)\,(x - 2\,y)\,(x^2 + 2\,xy + 4\,y^2)$.

A little ingenuity in arrangement and grouping often reveals concealed factors, thus:

$x^4 + 2\,abx^2 - a^4 + 3\,a^2b^2 - b^4$ may be written $(x^4 + 2\,abx^2 + a^2b^2) - (a^4 - 2\,a^2b^2 + b^4) = (x^2 + ab)^2 - (a^2 - b^2)^2 = (x^2 + ab + a^2 - b^2)\,(x^2 + ab - a^2 + b^2)$.

Art. 30. The difference of the *even* powers of two quantities is always divisible by the sum or the difference of those quantities; as,

$x^4 - y^4 = (x + y)\,(x - y)\,(x^2 + y^2)$; or $(m^2 + mn)^2 - (x - y)^2 = [(m^2 + mn) + (x - y)]\,[(m^2 + mn) - (x - y)]$.

Art. 31. The difference of the *odd* powers of two quantities is always divisible by the difference of the quantities; as,

$x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$ or $(a + 2)^3 - (b - 1)^3 = (a + 2 - b + 1)\,[(a + 2)^2 + (a + 2)\,(b - 1) + (b - 1)^2]$, etc.

Art. 32. The sum of the *odd* powers of two quantities is always divisible by the *sum* of the quantities; thus:

$a^5 + b^5 = (a + b)\,(a^4 - a^3\,b + a^2\,b^2 - ab^3 + b^4)$ or $(2x - y)^3 + (3z + 1)^3 = (2x - y + 3z + 1)\,[(2x - y)^2 - (2x - y)\,(3z + 1) + (3z + 1)^2]$.

Art. 33. The sum of the *even* powers of two quantities is never divisible by either the sum or difference of the quantities.

Such quantities may be sometimes factored by adding and subtracting the same quantity; thus,

$x^4 + 4y^4 = x^4 + 4y^2x^2 + 4y^4 - 4y^2x^2 = (x^2 + 2y^2) - (2xy)^2 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$ etc.

EXERCISE V.

Factor the following:

1. $x^8 - x^4y^4$.

2. $b^6 - b^3 - 110$.

3. $6u^2 - 23u + 20$.

4. $(a + b)^3 + (a - b)^3$.

5. $x^6 + y^6$.

6. $x^8 + y^8 + x^4y^4$.

7. $mb + z^2 - mz - bz$.

8. $m^2 - 14m - 176$.

9. $y^2 - z^2 + 2z - 1$.

10. $y^2 + y - 72$.

11. $(2x - 3y)^2 - (x - 2y)^2$.

12. $n^2 - 2mn + m^2 - x^2$.

13. $1 + \dfrac{a^2 + b^2 - c^2}{2ab}$.

14. $x^3 - y^3 - (x^2 - y^2) - (x - y)^2$.

15. $\dfrac{25}{m^2} - \dfrac{40}{mx^2} + \dfrac{16}{x^4}$.

16. $x^{-6} - y^{-6}$.

17. $15t + 5ts - s - 3$.

18. $a^{\frac{2}{3}} - m^{\frac{2}{3}}$.

19. $6x^3 - 7ax^2 - 20a^2x$.

20. $x^2 + ax + x + a$.

21. $(x + 1)^2 - 5x - 29$.

22. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

23. $z^5 + 7z^3 - 5z^2 - 35$.

24. $m^3 + m^2 - 7m - 3$.

25. $4(u - v)^2 - (a + b)^2$.

26. $y^2 + 3y^3 - y^4 - 3y$.

27. $72t^2 + 41t - 45$.

28. $a^2 - (b - c)^2$.

GREATEST COMMON DIVISOR.

ART. 34. Definition: The greatest common divisor or greatest common factor (abbreviated G.C.D. or G.C.F.) of two or more quantities is the *greatest* quantity that will divide them all.

Hence, to find the G.C.D., separate the quantities into their *prime* factors (what are prime factors?), select the factors that are common to all, repeating each factor the *least* number of times it is contained in any one of the quantities. The product of these common factors thus repeated is the G.C.D.

For example, find the G.C.D. of

$16\,x^2\,y^3\,z^3\,m^3$, $169\,y^4\,z^6\,m$, and $39\,x^7\,y^8\,m^4$.

$16\,x^2\,y^3z^3m^3 = 2.2.2.2.\ x.x.\ y.y.y.\ z.z.z.\ m.m.m.$

$169\,y^4\,z^6\,m = 13.13.\ y.y.y.y.\ z.z.z.z.z.z.\ m.$

$39\,x^7\,y^8\,m^4 = 3.13.$
$x.x.x.x.x.x.x.\ y.y.y.y.y.y.y.y.\ m.m.m.m.$

The common factors are y and m, y being repeated three times as the least number, and m occurring but once hence, G.C.D. $= y^3m$.

Again, find the G.C.D. of

$x^2 + 5x + 6$, $x^2 + 7x + 10$, and $x^2 + 12x + 20$.

$x^2 + 5x + 6 = (x + 2)(x + 3)$

$x^2 + 7x + 10 = (x + 5)(x + 2)$

$x^2 + 12x + 20 = (x + 2)(x + 10)$

$x + 2$ is the only common factor, therefore, $x + 2 =$ G.C.D.

G.C.D. Without Factoring.

ART. 35. Let A and B be any two quantities of which, say, A is the greater, and let B be contained in A, Q times with a remainder R, then,

$$A = QB + R \text{ or } R = A - QB.$$

Since the sum of a fraction and a whole number can never equal a fraction, any factor common to A and B must be contained in R; otherwise, if we were to divide the above equation by such a factor, the quotient on the right of the equality sign would be a whole number (since the factor exactly divides A and B), and on the left it would be a fraction, which is impossible; hence, R contains all the factors common to A and B. But it may also contain other factors. If we divide B by R and say R is contained in B, M times, with a remainder R', then $B = MR + R'$. By the same reasoning R' contains all the factors common to R and B, and hence all factors common to A and B.

Suppose finally that a remainder obtained by such successive division is contained exactly in the previous remainder, then it represents all the factors common to A and B, and no others; hence, it is the G.C.D. of A and B.

Hence the rule: DIVIDE THE GREATER QUANTITY BY THE LESS AND IF THERE IS A REMAINDER DIVIDE THE FIRST DIVISOR BY THIS REMAINDER, AND THEN THIS REMAINDER BY THE REMAINDER RESULTING FROM THIS LAST DIVISION, AND SO ON UNTIL A REMAINDER IS EXACTLY CONTAINED IN THE PREVIOUS REMAINDER. THIS LAST REMAINDER IS THE G.C.D.

EXAMPLE. Find the G.C.D. of

$x^4 - 2x^3 + 2x^2 - 2x + 1$ and

$x^3 - 3x^2 + 3x - 1$

$$
\begin{array}{l|l}
x^4 - 2x^3 + 2x^2 - 2x + 1 & x^3 - 3x^2 + 3x - 1 \\
\underline{x^4 - 3x^3 + 3x^2 - x } & x + 1 \\
x^3 - x^2 - x + 1 & \\
\underline{x^3 - 3x^2 + 3x - 1} & \\
2 \;\; \underline{\big| 2x^2 - 4x + 2} & \\
\qquad x^2 - 2x + 1 &
\end{array}
$$

Since the G.C.D. contains *only common* factors, the removal from or introduction of any factor into either one, if it is not also a factor of the other, does not affect the *common* factors, and hence does not affect the G.C.D. Hence, we may take out the factor 2 from above remainder, without affecting the result.

To continue:

$$\begin{array}{l|l}
x^3 - 3x^2 + 3x - 1 & x^2 - 2x + 1 \\
\underline{x^3 - 2x^2 + x} & x - 1 \\
\quad - x^2 + 2x - 1 & \\
\quad \underline{- x^2 + 2x - 1} &
\end{array}$$

Hence, $x^2 - 2x + 1 =$ G.C.D.

Again, find the G.C.D. of

$$x^5 - 3x^4 - 3x^3 - 15 - 19x \text{ and}$$
$$3x^4 - 3x^3 + x^5 - 15 + 9x^2 - x.$$

To facilitate the process, both quantities should first be arranged according to the ascending or descending powers of one of the quantities involved.

Rearranging the above and dividing:

$$\begin{array}{l|l}
x^5 - 3x^4 - 3x^3 - 19x - 15 & x^5 + 3x^4 - 3x^3 + 9x^2 - x - 15 \\
\underline{x^5 + 3x^4 - 3x^3 + 9x^2 - x - 15} & 1 \\
-3x \;\underline{|-6x^4 + 9x^2 - 18x} & \text{(Remove factor } - 3x \text{ which} \\
\qquad 2x^3 - 3x + 6 & \text{does not occur in the divisor.)}
\end{array}$$

Multiply first divisor by 2, since 2 is not a factor of $2x^3 - 3x + 6$.

$$\begin{array}{l|l}
2x^5 + 6x^4 - 6x^3 + 18x^2 - 2x - 30 & 2x^3 - 3x + 6 \\
\underline{2x^5 \qquad - 3x^3 + 6x^2} & x^2 + 3x - 3 \\
6x^4 - 3x^3 + 12x^2 - 2x - 30 & \\
\underline{6x^4 \qquad - 9x^2 + 18x} & \\
- 3x^3 + 21x^2 - 20x - 30 & \text{(Multiply by 2 to} \\
\underline{\qquad\qquad\qquad\qquad 2} & \text{make divisible.)} \\
- 6x^3 + 42x^2 - 40x - 60 & \\
\underline{- 6x^3 \qquad + 9x - 18} & \\
7 \;\underline{|42x^2 - 49x - 42} & \text{(Divide by 7.)} \\
\qquad 6x^2 - 7x - 6 &
\end{array}$$

Multiply $(2x^3 - 3x + 6)$ by 3.

$$
\begin{array}{rl}
6x^3 - 9x + 18 & \quad \underline{|6x^2 - 7x - 6} \\
\underline{6x^3 - 7x^2 - 6x} & \quad x + 7 \\
7x^2 - 3x + 18 & \\
\underline{6} & \\
42x^2 - 18x + 108 & \\
\underline{42x^2 - 49x - 42} & \\
31x + 150 &
\end{array}
$$

which is plainly not contained in $6x^2 - 7x - 6$ and hence the original numbers had no G.C.D.

It is unnecessary to carry the process further, as the next division would leave a remainder containing only a number. If it were the G.C.D. it could have been seen by inspection at first.

To Find the G.C.D. of More than Two Quantities.

ART. 36. Rule. Find G.C.D. of any two of the quantities and then the G.C.D. of this G.C.D. and a third quantity, then the G.C.D. of this last G.C.D. and a fourth and so on until all the quantities have been used. The last G.C.D. will be the G.C.D. of all the original quantities.

EXERCISE VI.

Find the G.C.D.

1. $8 - a^3$; $a^2 - 4$.

2. $x^6 - y^6$; $x^4 + xy^3$; $x^6 + 2x^3y^3 + y^6$.

3. $y^2 - 3y + 2$; $y^4 - 6y^2 + 8y - 3$.

4. $u^3 - 3u^2 + 4$; $3u^3 - 18u^2 + 36u - 24$.

5. $(a + b)^2 - (c + d)^2$; $ax + bx + cx + dx$.

6. $x^2 + 5x + 6$; $x^2 + 7x + 10$; $x^2 - x - 6$.

7. $a^3 - b^3$, $(a - b)^3$; $a^2 - 2ab + b^2$.

8. $z^3 + 2z^2 + 2z + 1$; $z^3 - 4z^2 - 4z - 5$.

9. $m^3 - 19m - 30$; $m^3 + 10m^2 + 31m + 30$.

10. $x^4 - 4x^3 - 16x^2 + 7x + 24$; $2x^3 - 15x^2 + 9x + 40$.

11. $6x^2 + x - 2$; $9x^3 + 48x^2 + 52x + 16$.

12. $2t^4 - 3t^3 - 9t^2 + 9t - 2$; $3t^5 - 4t^4 - 23t^2 + 41t^2 - 20t + 3$.

13. $x^4 - 9x^2 - 30x - 25$; $x^5 + x^4 - 7x^2 + 5x$.

14. $2x^2 - 7x + 3$; $3x^2 - 7x - 6$.

15. $y^4 - 2y^3 - 13y^2 + 38y - 24$; $y^4 - 4y^3 - 7y^2 + 34y - 24$.

16. $3x^3 + 9x^2y - 6xy^2 - 6y^3$; $24x^3 + 6x^2y - 12xy^2 - 18y^3$.

17. $10n^3 + n^2 - 9n + 24$; $20n^4 - 17n^2 + 48n - 3$.

18. $12(a^4 - b^4)$; $10(a^6 - b^6)$; $8(a^4b + ab^4)$.

19. $x^3 - 3x^2 - 4x + 12$; $x^3 - 7x^2 + 16x - 12$; $2x^3 - 9x^2 + 7x + 6$.

20. $x^2 + 11x + 30$; $2x^2 + 21x + 54$; $9x^3 + 53x^2 - 9x - 18$.

21. $z^2 + 11z + 30$; $z^3 - 12z^2 + 41z - 30$; $z^4 - 12z^3 + 47z^2 - 72z + 36$.

22. $6x^2 + x - 2$; $9x^3 + 48x^2 + 52x + 16$.

23. $2n^4 - 5n^3 - 3n^2p^2 + 7np^3 + 3p^4$; $8n^3 - 4n^2p - 8np^2 - 6p^3$.

24. $x^4 + x^2y^2 + y^4$; $x^8 + x^4y^4 + y^8$; $x^{16} + x^8y^8 + y^{16}$.

LEAST COMMON MULTIPLE.

ART. 37. The least common multiple of any number of quantities (abbreviated L.C.M.) is the least number that contains them all.

From this definition, the following rule is immediately inferred:

TO FIND THE LEAST COMMON MULTIPLE OF ANY NUMBER OF QUANTITIES, SEPARATE THESE QUANTITIES INTO THEIR PRIME FACTORS, AND SELECT ALL THE SEPARATE FACTORS THAT OCCUR IN ALL THE QUANTITIES, REPEATING EACH ONE THE **greatest** NUMBER OF TIMES IT APPEARS IN ANY ONE OF THE QUANTITIES. THE PRODUCT OF THESE FACTORS WILL BE THE L.C.M.

EXAMPLE: find the L.C.M. of

$$27x^7y,\ 6x^2y^4,\ 4xy^5,\ 3x^4y^4$$
$$27x^7y = 3 \cdot 3 \cdot 3 \cdot x.x.x.x.x.x.x.\ y.$$
$$6x^2y^4 = 3.2.\ x.x.\ y.y.y.y.$$
$$4xy^5 = 2.2.\ x.\ y.y.y.y.y.$$
$$3x^4y^4 = 3.\ x.x.x.x.\ y.y.y.y.$$

The separate factors are 3, 2, x, and y.

3	occurs	3	times	in $27x^7y$.
2	"	2	"	" $4xy^5$.
x	"	7	"	" $27x^7y$.
y	"	5	"	" $4xy^5$.

Hence, L.C.M. is $3.^3 2.^2 x^7 y^5 = 108x^7y^5$.

To Find L.C.M. Without Factoring.

ART. 38. Since the product of two quantities contains all the factors of both quantities with the common factors repeated as many times as they occur in both together, and since the G.C.D. of these two quantities contains only the *common* factors, if the product of the two quantities be divided by their G.C.D. the quotient will contain all the factors of both and their common factors once only; hence, this quotient is their L.C.M. Put this into rule.

If there are more than two quantities, find the L.C.M. of any two of them, then the L.C.M. of this L.C.M. of two and a third quantity, then the L.C.M. of this last L.C.M. and a fourth, and so on, until the quantities have all been used. The last L.C.M. will be the L.C.M. of all the quantities.

EXERCISE VII.

Find the L.C.M. of

1. $16\,(x^3 - y^3)$; $24\,(x^4 - y^4)$; $36\,(x^3 + y^3)$.
2. $x^2 + 7x + 12$; $x^2 + x - 12$; $x^2 + 3x - 4$.
3. $(a + b)^2 - c^2$; $(a - b)^2 - c^2$; $(a + c)^2$; $(a - c)^2 - b^2$.
4. $x^2 - 3x - 70$; $x^3 - 39x - 70$.
5. $4s^2 - 7st + 3t^2$; $3s^3 - 4s^2t + 3st^2 - 2t^3$.
6. $a^2 - 3a - 4$; $a^2 - a - 12$; $a^2 + 5a + 4$.
7. $2y^3 + 5 - 8y + y^2$; $42y^2 + 30 - 72y$.
8. $3x^4 - x^3 - 2x^2 + 2x - 8$; $6x^3 + 13x^2 + 3x + 20$.
9. $u^5 - 2u^4 + u^2$; $2u^4 - 4u^3 - 4u - 4$.
10. $x^3 - 6x^2 - 5x + 12$; $x^3 - 5x^2 + 2x + 8$; $x^3 - 4x^2 + x + 6$.
11. $3z^2 - 5z + 2$; $4z^3 - 4z^2 - z + 1$.
12. $y^2 - 4b^2$; $y^3 + 2by^2 + 4b^2y + 8b^3$; $y^3 - 2by^2 + 4b^2y - 8b^3$
13. $a^2 + a - 2$; $a^3 + 2a^2 + 2a + 1$.
14. $x^3 - x^2 - 9x + 9$; $x^4 - 4x^2 + 12x - 9$.
15. $4z^3 - 8z^2 + 5z - 3$; $2z^4 - 3z^3 + 6z^2 - 3z + 2$.
16. $x^2 + 5x + 6$; $x^2 + 6x + 8$; $x^2 - 3x - 10$.

17. Find the least quantity, which when divided by $y^2 + y - 2$, $y^2 - y - 6$, and $y^2 - 4y + 3$, leaves in each case the remainder, $2y^2 - 3y + 1$.

18. The G.C.D. of two quantities is $x^2 - xy + y^2$, and their L.C.M. is $x^5 + x^4y + x^3y^2 + x^2y^3 + x^2y^3 + xy^4 + y^5$. One of the quantities is $x^3 + y^3$. Find the other one.

19. Four pendulums beat, respectively, every second, every $1\frac{1}{4}$ seconds, every $1\frac{1}{2}$ seconds, every $1\frac{1}{6}$ seconds. They are started together; after what time will they all beat together again, and how many beats will each make in the interval?

FRACTIONS.

ART. 39. The rules which apply to operations with fractions in arithmetic, apply equally to algebraic fractions, if the slight modifications of the four fundamental operations, addition, subtraction, multiplication, and division, applied to algebraic quantities, are observed.

It must be remembered that the use of letters in algebra often makes it impossible to condense expressions, as in the use of numbers exclusively. Hence the operations are more frequently indicated than actually performed.

Although, for example, $1\frac{2}{5}$ really means $1 + \frac{2}{5}$, the structure is not so apparent as in the exactly similar algebraic expression, $a + \frac{b}{c}$.

Both are mixed numbers and both are reduced to fractions in exactly the same way; viz.

$$1\tfrac{2}{5} = \frac{1 \times 5 + 2}{5} = \tfrac{7}{5}$$

$$a + \frac{b}{c} = \frac{a \times c + b}{c} = \frac{ac + b.}{c}$$

ART. 40. Reduction to lowest terms, reduction to common denominator, addition, subtraction, multiplication, and division, are accomplished in ways exactly analogous to the similar arithmetical operations, the algebraic concept of sign, and the use of letters, introducing a purely superficial difference.

A general rule may be formulated thus:

Perform the required operation as indicated in arithmetic, observing the laws of algebraic addition, subtraction, multiplication, and division.

It is to be observed that in reducing an algebraic fraction to its lowest terms, *every* term of both numerator and denominator must be divisible by the factor removed, as,

$$\frac{a^2x + aby}{ac^2 - a^3d} = \frac{ax + by}{c^2 - a^2d} \text{ not } \frac{a^2x + aby}{ac^2 - a^3d} = \frac{ax + aby}{c^2 - a^3d}.$$

Signs.

ART. 41. Since changing *all* the signs of both numerator and denominator is equivalent to multiplying (or dividing) by -1, the value of the fraction is not changed thereby. But if the signs of either numerator or denominator (but not of both) be changed, or if the signs of any parts (not all) of either or both be changed, the value of the fraction is changed. Thus:

$$\frac{x^2y - xy^2}{-abc + mn} = \frac{xy^2 - x^2y}{abc - mn} \quad \text{but} \quad \frac{x^2y - xy^2}{abc + mn}$$

is not equal to $\dfrac{x^2y + xy^2}{abc - mn}$.

Also when a fraction has a minus sign in front of it, the sign of each term of the numerator must be changed, when the denominator is removed or when the numerator is combined with any other expression (as in adding or subtracting). The effect being the same as removing a parenthesis. Thus:

$$\frac{ax + b}{c} - \frac{cx + d}{a} = \frac{a^2x + ab - c^2x - cd}{ac}.$$

EXERCISE VIII.

Reduce to lowest terms:

1. $\dfrac{x^4 + (2b^2 - a^2)x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$.

2. $\dfrac{a^2 - a - 12}{(a + 3)^3}$.

3. $\dfrac{y^4 + y^2 - 2}{y^4 + 5y^2 + 6}$.

4. $\dfrac{b^2 - (c - d)^2}{(b + d)^2} - c^2$.

5. $\dfrac{6x^2 + 13x + 6}{10x^2 + 13x - 3}$.

Reduce to mixed numbers:

6. $\dfrac{12u^2 - 5u - 6}{3u^2 + 2}$.

7. $\dfrac{x^3 + x^2 - 1}{x^2}$.

8. $\dfrac{z^3 - 3z^2 + 2z - 3}{z - 1}$.

9. $\dfrac{m^3 - n^3 - 1}{m - n}$.

Simplify the following:

10. $\dfrac{2}{m} + \dfrac{m - 6}{3m + 6} - \dfrac{1}{m^2 + 2m}$.

11. $\dfrac{-1}{(a - 1)^2} + \dfrac{2}{x - 1} - \dfrac{2x}{x^2 + 1}$.

12. $\dfrac{5}{x^2 - 2x - 3} - \dfrac{4}{x^2 - 9} - \dfrac{7}{x^2 + 4x + 3}$.

13. $\dfrac{3}{x - 2} + \dfrac{4}{(x - 2)^2} - \dfrac{5}{(x - 2)^3}$.

14. $\dfrac{ax + b}{x^2 + 1} - \dfrac{c}{2x - 3}$.

15. $\dfrac{3}{x - 1} + \dfrac{7}{x - 3} - \dfrac{9x - 13}{x^2 - 4x + 3}$.

16. $\left(\dfrac{m^4 - n^4}{m^2 - n^2} \div \dfrac{m + n}{m^2 - mn}\right) \div \left(\dfrac{m^2 + n^2}{m - n} \div \dfrac{m + n}{mn - n^2}\right)$.

17. $\dfrac{a + b}{(b - c)(c - a)} + \dfrac{b + c}{(c - a)(a - b)} - \dfrac{a - c}{(a - b)(b - c)}$.

18. $\dfrac{\dfrac{2x + y}{y} - \dfrac{y}{2x + y}}{\dfrac{x}{x + y} - \dfrac{x + y}{x}}$.

19. $\left(\dfrac{x^5}{y^5} - \dfrac{y^5}{x^5}\right) \div \left(\dfrac{x}{y} - \dfrac{y}{x}\right)$.

20. $1 + \dfrac{1}{x + \dfrac{1}{x}}$.

CHAPTER III.

EQUATIONS.

ARTICLE 42. An equation is a statement of equality between two equal expressions, thus:

$$ax^2 + bx + c = d, \quad \text{or} \quad 3x + 2y + z = 24, \text{ etc.}$$

ART. 43. An equation usually consists of letters and figures, although it may contain either exclusively; as,

$$5x + y = 11, \quad \text{or} \quad ax + b = c, \quad \text{or} \quad 5 + 2 = 7.$$

The last is called an arithmetical equation, the others algebraic equations.

Of the quantities involved in an algebraic equation, part are usually known, that is, their values are known, and part are unknown. It is customary to represent the latter by the last letters of the alphabet; the former by the first letters of the alphabet or by figures.

Thus: x, y, z, etc., are the symbols for unknown value, and a, b, c, d, etc., 1, 2, 3, 4, etc., for the known.

ART. 44. Equations involving only letters are called literal equations. Those involving unknowns and figures, numerical equations.

ART. 45. Solving an equation is the finding of the value or values of the unknown quantities that make it true, or, technically, satisfy it. That is, the finding of the values which when substituted for the unknowns to which they belong, make the two sides of the equation the same. The equation then becomes an identity.

ART. 46. An equation is usually satisfied only for a limited number of values of the unknown quantities. Thus: $3x + 6 = 12$ is satisfied only when $x = 2$, for then, and then only, we have $3 (2) + 6 = 6 + 6 = 12$, an identity, or, $ax + b = c$ is satisfisd only by $x = \frac{c - b}{a}$, for then we have $a \frac{(c - b)}{a} + b = c$ or $c - \not{b} + \not{b} = c$.

$x^2 - 5x + 6 = 0$ is satisfied only when $x = 2$ or 3 for $(2)^2 - 5 \ (2) + 6 = 4 - 10 + 6 = 0$, or $(3)^2 - 5 \ (3) + 6 = 9 - 15 + 6 = 0$.

When an equation is of such a nature that any value whatever of the value unknown satisfies it, it is called an *identity* and is written thus:

$$ax + b \equiv 2x + 3$$

Uses of Literal Equations.

ART. 47. A literal equation is of special service, because the value (or values) of the unknown obtained from it is general for all cases of the same kind. For instance, it is required to find the mean of any two quantities (that is the quantity midway between them).

Let x = the mean and let a and b represent any two quantities; then $a - x = x - b$, or $2 \ x = a + b$; $x = \frac{a + b}{2}$.

Such a result is called a formula, because it is true for all quantities, and may be stated as a general rule. For instance, in the above case, the result may be expressed thus:

The mean of any two quantities is half their sum.

Uses of Letters in General.

ART. 48. The use of letters enlarges the value of results and gives algebra a notable advantage over arithmetic.

Suppose it is required to prove that half the sum of two numbers plus half their difference equals the greater number.

If we use a and b as the numbers, a being greater, we have $\frac{a+b}{2} + \frac{a-b}{2} = \frac{2a}{2} = a$; which gives a general result, as a and b may stand for any numbers.

Art. 49. The use of letters is valuable again because all the operations performed must be indicated and the anatomy of the result is evident, whereas the ability to combine simple figures covers up the traces of the operations where they alone are employed. Thus, if $(x + a)$ is multiplied by $(x + b)$, we get $x^2 + (a + b)x + ab$, which reveals the well-known rule for factoring trinomials. What is it?

While if $(x + 2)$ is multiplied by $(x - 5)$ we get $x^2 - 3x - 10$ wherein there is no evidence of the relation between -3 and -10, and the coefficients, 2 and -5.

Degree of an Equation.

Art. 50. The degree of an equation is the largest sum of the exponents of the unknown quantities in any one term; for example, $x^2y + 2x + 3y^2 + xy = 0$ is of the third degree, for the first term contains x^2 and y^1 making sum of exponents 3, and no other term is higher. But $a^2x + aby + cx^2$ is of the second degree, for the exponents of a, b, and c do not count, as they are regarded as known quantities.

Equations of the first degree like $ax + b = c$, or $5x + 3y = 7$, etc., are called *linear* equations; those of the second degree like $ax^2 + bx + c = 0$, or $3x^2 + 2xy = y^2$, are called quadratics; those of third degree, cubics; of fourth degree, bi-quadratics, etc.

Art. 51. An equation may plainly contain any number of unknown quantities. In order to find the value or values of the unknowns, the number of independent equations must equal the number of unknown quantities involved, because each unknown requires a separate condition to distinguish it from the others.

ART. 52. When two or more equations involving two or more unknown quantities, are all satisfied for the same values of those unknowns, the equations are called *simultaneous* equations.

They are said to be independent when no one of them can be derived from any other. They must represent entirely distinct conditions.

Thus: $3x + 2y = 8$
$6x + 4y = 16$ are simultaneous but not independent.

Which of the following are independent?

Are they all simultaneous?

1. $\begin{cases} 15x + 77y = 92. \\ 5x - 3y = 2. \end{cases}$

2. $\begin{cases} 6y - 5x = 18. \\ 12x - 9y = 0. \end{cases}$

3. $\begin{cases} 5x + 6y = 17. \\ 10x + 12y = 34. \end{cases}$

4. $\begin{cases} 5p + 3q = 68. \\ 2p + 5q = 69. \end{cases}$

5. $\begin{cases} \dfrac{x}{4} + \dfrac{y}{2} = 12. \\ \dfrac{x}{2} + y = 27. \end{cases}$

6. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{3} = 7. \\ \dfrac{x}{6} + \dfrac{y}{2} = 6\frac{1}{2}. \end{cases}$

Clearing of Fractions.

ART. 53. Since the equality of two quantities is not affected by multiplying both by the same or equal quantities, the fractions may be removed from an equation by multiplying by the L.C.M. of the denominators involved. Thus:

$\frac{3}{2}x^2 + \frac{2}{7}x + \frac{3}{5} = \frac{1}{2}$ may be cleared of fractions by multiplying both sides by 70, the L.C.M. of the denominators, and the result, $105x^2 + 20x + 42 = 35$, is free of fractions.

The L.C.M. of all the denominators is called the *Least Common Denominator*, abbreviated to L.C.D.

NOTE. Of course *any* multiple of the denominators will remove fractions, but the L.C.D. gives simplest result.

Transposition.

ART. 54. **Rule.** ANY TERM MAY BE TRANSFERRED FROM ONE TERM OF AN EQUATION TO THE OTHER IF ITS SIGN BE CHANGED.

Let $x + b = c$ be any equation; if $-b$ be added to both terms the equality is not affected, then $x + \not{b} - \not{b} = c - b$ or $x = c - b$, which justifies the rule. Again, let the equation be $x - b = c$, and add $+b$ to both sides, $x - \not{b} + \not{b} = c + b$ or $x = c + b$, which again verifies the rule.

To Solve a Linear Equation of One Unknown.

ART. 55. Since to solve an equation is to find the value of the unknown that satisfies it, the requisite steps are to the end of simplifying the equation to the utmost. Hence:

CLEAR THE EQUATION OF FRACTIONS, IF NECESSARY; TRANSPOSE ALL THE TERMS CONTAINING THE UNKNOWN TO ONE SIDE OF THE EQUATION (PREFERABLY TO THE LEFT SIDE), AND ALL OTHER TERMS TO THE OTHER SIDE, COLLECTING THEM; COMBINE THE TERMS CONTAINING THE UNKNOWN BY COLLECTING WHERE POSSIBLE AND BY EXPRESSING THE RESULT AS THE PRODUCT OF TWO FACTORS, ONE OF WHICH IS THE UNKNOWN ITSELF; DIVIDE THROUGH BY THE OTHER FACTOR, AND THE RESULT IS THE VALUE OF THE UNKNOWN, OR THE SOLUTION OF THE EQUATION.

EXAMPLE. $\frac{5x-6}{5} - \frac{3x}{4} = \frac{x-9}{10}$ (L.C.D. = 20).

Clearing of fractions: $20x - 24 - 15x = 2x - 18.$

Transposing: $20x - 15x - 2x = 24 - 18.$

Collecting: $3x = 6$ or $3(x) = 6.$

Dividing by 3: $x = 2.$

Again: $\dfrac{x - 2a}{x + 3a} - \dfrac{13a^2 - 2x^2}{x^2 - 9a^2} = 3$ (L.C.D. $= x^2 - 9a^2$).

Clearing: $x^2 - 5a + 6a^2 - (13a^2 - 2x^2) = 3x^2 - 27a^2$.

Removing parentheses: *

$$x^2 - 5ax + 6a^2 - 13a^2 + 2x^2 = 3x^2 - 27a^2.$$

Transposing:

$$x^2 + 2x^2 - 3x^2 - 5ax = -6a^2 + 13a^2 - 27a^2.$$

Collecting:

$$-5ax = -20a^2, \text{ or } (-5a)x = -20a^2; x = 4a.$$

Hence the rule: *When a parenthesis, before which is a minus sign, is removed, or the denominator of a fraction, before which is the minus sign, is removed, the signs of all terms included in the parenthesis or in the numerator must change.*

Simultaneous Linear Equations.

Art. 56. Such equations must be solved by finding a single equation, involving but one unknown quantity, which includes all the conditions. Just as combining any two expressions (by multiplication, addition, or otherwise) introduces into the result all the factors and other characteristics contained by both, so the combination of two or more equations gives an equation which contains the qualities and conditions of all. Upon this fact is based the rule for solving simultaneous equations.

* Since a minus sign always means a subtraction, actual or indicated, the minus sign before the parenthesis in the above example indicates that the quantities within are to be subtracted, and since removing the parenthesis partially performs the operation, according to the law of negative numbers, the signs of the quantities inclosed must all be changed. The same thing applies to the numerators of fractions, when the denominators are removed by clearing.

EXAMPLE. $4x+9y=79$ (Are these independent?) (1)

$7x-17y=40$ . . . . (2)

Multiply (1) by 7 and (2) by 4; then (1) by 17 and (2) by 9.

$$\begin{array}{lrl} & 28x + 63y = 553 & \quad 68x + 153y = 1343 \\ & 28x - 68y = 160 & \quad 63x - 153y = 360 \\ \text{Subtract,} & 131y = 393 & \quad \text{Add} \quad 131x = 1703 \\ \text{Divide by } 131, & y = 3 & \quad x = 13 \end{array}$$

The equation $131y = 393$, while it contains only the unknown y, by the above principle contains all the conditions of both original equations; hence, the value 3 of y, found from it, will agree with the value of x (13), similarly found, and will satisfy with it both equations.

Verification:

$4(13) + 9(3) = 79$ or $52 + 27 = 79$, that is $79 = 79$.
$7(13) - 17(3) = 40$ or $91 - 51 = 40$, that is $40 = 40$.

ART. 57. The process of combining simultaneous equations, so as to reduce the number of unknown quantities in the result, is called *elimination.*

There are several devices employed to effect elimination, one of which is illustrated in the example solved above. This method may be thus described:

MULTIPLY EACH EQUATION IF NECESSARY BY A FACTOR (PREFERABLY THE LEAST ONE), THAT WILL MAKE THE COEFFICIENTS OF ONE OF THE UNKNOWNS THE SAME IN BOTH. IF THE TERMS CONTAINING THESE UNKNOWNS HAVE THE SAME SIGN, SUBTRACT ONE EQUATION FROM THE OTHER; IF THEY HAVE OPPOSITE SIGNS, ADD.

If there are more than two equations, say three, combine the equations in pairs (say 1st and 2d, and 2d and 3d), thus acquiring two simultaneous equations containing only two unknowns. The combination of these two will eliminate one of the two remaining unknowns, leaving but one. Then divide by the coefficient of this remaining unknown, and the result will be its value for the set of simultaneous equations.

EXAMPLE.

$$\tfrac{1}{2}x + \tfrac{1}{3}y = 12 - \tfrac{1}{6}z \text{ or } \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{6}z = 12 \qquad (1)$$

$$\tfrac{1}{2}y + \tfrac{1}{3}z = 8 + \tfrac{1}{6}x \text{ or } -\tfrac{1}{6}x + \tfrac{1}{2}y + \tfrac{1}{3}z = 8 \qquad (2)$$

$$\tfrac{1}{2}x + \tfrac{1}{3}z = 10 \qquad \tfrac{1}{2}x \quad + \tfrac{1}{3}z = 10 \qquad (3)$$

$$\tfrac{1}{4}x + \tfrac{1}{6}y + \tfrac{1}{12}z = 6$$

Subtract $\frac{1}{3}$ of (2) from $\frac{1}{2}$ of (1) $-\frac{1}{18}x + \frac{1}{6}y + \frac{1}{9}z = \frac{8}{3}$

$$\tfrac{11}{36}x - \tfrac{1}{36}z = \tfrac{10}{3} \qquad (4)$$

Multiply (4) by 12 add (3) to (4) $\quad \frac{11}{3}x - \frac{1}{3}z = 40$

$$\tfrac{1}{2}x + \tfrac{1}{3}z = 10$$

$$\tfrac{25}{6}x \quad = 50$$

$$x \quad = 12$$

Substituting $x = 12$ in (3) $\quad 6 + \frac{1}{3}z = 10, \; z = 12$

Substituting $x = 12, z = 12$ in (1) $\quad 6 + \frac{1}{3}y + 2 = 12, \; y = 12$

Elimination by Substitution.

ART. 58.

EXAMPLE.

$$\frac{3}{u} = \frac{1}{2 - v} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$\frac{1}{u - 1} = \frac{2}{v + 3} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

From (1) $\quad 6 - 3v = u.$

$$3v = 6 - u$$

$$v = \frac{6 - u}{3}.$$

Substituting value of

$$v = \frac{6 - u}{3} \text{ in } (2); \frac{1}{u - 1} = \frac{2}{\frac{6 - u}{3} + 3} \quad . \; . \; . \; . \; . \quad (3)$$

Simplifying (3) $\frac{1}{u-1} = \frac{2}{\frac{6-u+9}{3}}$ or $\frac{1}{u-1} = \frac{6}{15-u}$.

Clearing of fractions,

$$15 - u = 6u - 6,$$

whence $\quad -7u = -21, u = 3.$

From (3) if $u = 3, v = \frac{6-3}{3} = 1.$

Rule. SOLVE ONE OF THE EQUATIONS FOR ONE OF THE UNKNOWNS IN TERMS OF THE OTHER AND KNOWN TERMS; SUBSTITUTE THIS VALUE FOR THE SAME UNKNOWN IN THE OTHER EQUATION; CLEAR OF FRACTIONS (IF NECESSARY); COLLECT AND DIVIDE THROUGH BY THE COEFFICIENT OF THE UNKNOWN INVOLVED. THE RESULT IS A ROOT OF THE ORIGINAL EQUATIONS. THIS ROOT SUBSTITUTED IN EITHER ORIGINAL EQUATION GIVES THE VALUE OF THE OTHER UNKNOWN.

Elimination by Indeterminate Coefficients.

ART. 59.

EXAMPLE: $3x + 5y = 19$ . . . . (1)

$5x - 4y = 7$ . . . . (2)

Since either equation may be multiplied by any factor whatever, we will multiply (2) by, say, m, and add.

$3x + 5y = 19$ . . . . (1)

$5mx - 4my = 7m$ . . . . (2)

$$3x + 5mx + 5y - 4my = 19 + 7m$$

or $\quad x(3 + 5m) + y(5 - 4m) = 19 + 7m$ . . (3)

Equation (3) plainly contains both (1) and (2) and the conditions that belong to both, hence, whatever is true of (3) is true of both (1) and (2).

Since m may have any value, let it be $-\frac{3}{5}$, so that the coefficient of x shall be zero, which is plainly equivalent to eliminating x from (1) and (2), as previously explained. Then:

Substituting $m = -\frac{3}{5}$ in (3)

$$x(3-3) + y\left(5 + \frac{12}{5}\right) = 19 - \frac{21}{5}$$

or

$$\frac{37}{5}y = \frac{74}{5}$$

$$y = 2.$$

Again, let $m = \frac{5}{4}$ in (3), then:

$$x\left(3 + \frac{25}{4}\right) + y(5-5) = 19 + \frac{35}{4}$$

or

$$\frac{37x}{4} = \frac{111}{4}$$

$$x = 3.$$

Rule. MULTIPLY EITHER EQUATION BY ANY LITERAL FACTOR AS m; ADD (OR SUBTRACT) THIS RESULT TO THE OTHER EQUATION, AND IN THE RESULTING COMPOUND EQUATION GIVE m SUCCESSIVE VALUES THAT WILL CAUSE THE COEFFICIENTS OF THE UNKNOWNS TO SUCCESSIVELY VANISH; THE VALUE OF THE REMAINING UNKNOWN WILL BE THE VALUE THAT SATISFIES THE SIMULTANEOUS EQUATIONS.

More than Two Unknown Quantities.

ART. 60.

EXAMPLE.
$$\begin{cases} 7x + 3y - 2z = 16 \quad \ldots \quad (1) \\ 2x + 5y + 3z = 39 \quad \ldots \quad (2) \\ 5x - y + 5z = 31 \quad \ldots \quad (3) \end{cases}$$

Multiply 2 by (a) and 3 by (b) and add to (1)

$$\begin{aligned} 7x + 3y - 2z &= 16 \\ 2ax + 5ay + 3ax &= 39a \\ 5bx - by + 5bz &= 31b \end{aligned}$$

$$(7 + 2a + 5b)x + (3 + 5a - b)y + (3a + 5b - 2)z = 16 + 39a + 31b \quad \ldots \quad (4)$$

We must evidently determine the values of a and b that will simultaneously reduce two of the terms of (4) to zero, say the x and y terms. That means that we must solve the two equations.

$$7 + 2a + 5b = 0 \text{ for } a \text{ and } b \quad . \quad . \quad . \quad (5)$$

and

$$3 + 5a - b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Transpose and multiply (6) by 5

$$2a + 5b = -7$$
$$25a - 5b = -15.$$

Add

$$27a = -22$$

$$a = -\frac{22}{27}$$

Whence substituting

$$a = -\frac{22}{27} \text{ in } (5),\ 7 - \frac{44}{27} + 5b = 0.$$

Whence

$$5b = -\frac{145}{27} \qquad b = -\frac{29}{27}.$$

Substituting this value of a and b in (4)

$$\left(7 - \frac{44}{27} - \frac{145}{27}\right)x + \left(3 - \frac{110}{27} + \frac{29}{27}\right)y +$$
$$\left(-\frac{22}{9} - \frac{145}{27} - 2\right)z = 16 - \frac{858}{27} - \frac{899}{27}.$$
$$-\frac{265}{27}z = -\frac{1325}{27},\ z = 5 \text{ etc.}$$

ART. 61. State the rule for elimination of any number of unknown quantities.

General Rule. COMBINE THE EQUATIONS TWO AND TWO, ELIMINATING ONE OF THE UNKNOWN QUANTITIES; THEN COMBINE THE RESULTING EQUATIONS TWO AND TWO, ELIMINATING ANOTHER UNKNOWN, AND CONTINUE UNTIL THERE REMAINS BUT ONE EQUATION CONTAINING ONE UNKNOWN, THE VALUE OF THIS LAST UNKNOWN WILL BE ONE OF THE VALUES SOUGHT. SUBSTITUTION WILL GIVE THE OTHERS.

EXERCISE IX.

1. $(y+4)(y-2)=(y+3)(3y+4)-(2y+1)(y-6)$.

2. $(x+2)^2+3x=(x-2)^2+5(16-x)$.

3. $\frac{x-2}{2x-5}=\frac{x-5}{2x-2}$.

4. $\frac{25}{z-\frac{7}{2}}-\frac{10}{3z-4}=0$.

5. $\frac{x-3}{x^2-9}-\frac{12-2x}{x^2-36}=\frac{3x-27}{x^2-81}$.

6. $\frac{x^3-b^2x}{x^2+2bx+b^2}=x$.

7. $\left(y+\frac{2y}{y-a}\right)-\left(y-\frac{2y}{y-a}\right)=a$.

8. $\frac{7z-9}{z-3}=\frac{9z+4}{36}-\frac{6z-5}{24}$.

9. $2x-\frac{3x+7}{11}=\frac{x}{2}+1$.

10. $\frac{a(b^2+x^2)}{bx}=ac+\frac{ax}{b}$.

11. $\frac{6y+7}{9}+\frac{7y-13}{6y+3}=\frac{2y+4}{3}$.

12. $\frac{2x+1}{2x-16} - \frac{2x-1}{2x+12} = \frac{9x+17}{x^2-2x-48}.$

13. $\frac{.3}{.01-.02x} - \frac{.7}{.01+.02x} - \frac{.2-x}{.01(4x^2-1)} = 0$

$\frac{3}{\qquad\qquad}$

14. $1 + \frac{3}{1+\frac{3}{1-x}} = 10.$

15. $\frac{3(5y-3)}{2(4y+3)} = \frac{6}{5}.$

16. $\frac{x-3}{2x+1} + \frac{2x-1}{4x-3} = 1.$

17. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}.$

18. $\frac{2}{z-2} - \frac{5}{z+2} = \frac{2}{z^2-4}.$

19. $\frac{ax-b}{c} - \frac{bx+c}{a} = abc.$

20. $\frac{6y+7}{3} - \frac{3}{y+2} = 2y + \frac{1}{2}.$

21. $\frac{3}{4} - \frac{\frac{3}{4}x+\frac{3}{4}}{\frac{3}{4}+x} = \frac{\frac{3}{4}}{\frac{3}{4}+x} - \frac{3}{4}.$

22. $\frac{1}{2} + \frac{2}{y+2} = \frac{13}{8} - \frac{5y}{4y+8}.$

23. $\frac{x-7}{x+7} - \frac{2x-15}{2x-6} = -\frac{1}{2(x+7)}.$

24. $(3a-x)(a-b) + 2ax = 4b(a+x).$

EXERCISE X.

Equations containing more than one unknown.

1. $\begin{cases} 5x = 2y + 78 \\ 3y = x + 104. \end{cases}$

2. $\begin{cases} 4 + z = \dfrac{3y}{4} \\ y - 8 = \dfrac{4z}{5}. \end{cases}$

3. $\begin{cases} \dfrac{u+v}{3} + u = 15 \\ \dfrac{u-v}{5} + v = 6. \end{cases}$

4. $\begin{cases} x - \dfrac{2y - x}{23 - x} = 20 + \dfrac{2x - 59}{2} \\ y - \dfrac{y - 3}{x - 18} = 30 - \dfrac{73 - 3y}{3}. \end{cases}$

5. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1 - \dfrac{x}{c} \\ \dfrac{x}{b} + \dfrac{y}{a} = 1 + \dfrac{y}{c}. \end{cases}$

7. $\begin{cases} \dfrac{2x - y}{2} + 14 = 18 \\ \dfrac{2y + x}{3} + 16 = 19. \end{cases}$

6. $\begin{cases} 5t + 4s = 49\frac{1}{2} \\ 2t + 7s = 63. \end{cases}$

8. $\begin{cases} 3x + 16y = 5 \\ -5x + 28y = 19. \end{cases}$

9. $\begin{cases} \dfrac{7 + x}{5} - \dfrac{2x - y}{4} = 3y - 5 \\ \dfrac{5y - 7}{2} + \dfrac{4x - 3}{6} = 18 - 5x. \end{cases}$

10. $\begin{cases} \frac{1}{2}y + \frac{1}{3}z = 7 \\ 2x + 3y = 43. \end{cases}$

11. $\begin{cases} \dfrac{3}{u} + \dfrac{5}{v} = 2. \\ \dfrac{9}{u} - \dfrac{10}{v} = 1. \end{cases}$

12. $\begin{cases} \dfrac{3}{2x - 3y} + \dfrac{5}{y - 2} = -4 \\ \dfrac{7}{2x - 3y} + \dfrac{3}{y - 2} = -\frac{2}{3}. \end{cases}$

13. $\begin{cases} \dfrac{6 + x - y}{1 - x - y} = -\dfrac{7}{4} \\ 2x + 3y = -1. \end{cases}$

14. $\begin{cases} ay - bz = 2ab \\ 2by + 2az = 3b^2 - a^2. \end{cases}$

15. $\begin{cases} \dfrac{x - 3y}{2} - \dfrac{y - 3x}{2} = 8 \\ \dfrac{\frac{1}{5}x + \frac{3}{4}y}{\frac{1}{2}x - \frac{11}{3}y} = -\dfrac{7}{30}. \end{cases}$

16. $\begin{cases} 2x + 3y - 4z = 8 \\ 3x - 4y + 2z = 3 \\ 4x - 2y - 3z = 5. \end{cases}$

17. $\begin{cases} 2x - 3y = 4 \\ 4x - 3z = 2 \\ 4y + 2z = -3. \end{cases}$

18. $\begin{cases} 1.4x + .32y = 3.76 \\ .28x + 9.6y = 29.36. \end{cases}$

19. $\begin{cases} \dfrac{4y}{5 + y} = \dfrac{5y}{12 + x} \\ \dfrac{2x}{5} + y = 7. \end{cases}$

20. $\begin{cases} x - y + z = 10 \\ 3x - 8y + 10z = 50 \\ 5x + 2y - 3z = 40. \end{cases}$

21. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 9 \\ \dfrac{2}{x} - \dfrac{3}{y} + \dfrac{4}{z} = 11 \\ \dfrac{5}{x} + \dfrac{2}{y} - \dfrac{3}{z} = 4. \end{cases}$

EXERCISE XI.

Problems — Simple Equations.

1. The sum of two numbers is 18, and the quotient of the less divided by the greater is $\frac{1}{5}$. What are the numbers?

2. A steamer can run 20 miles an hour in still water. If it can run 72 miles with the current in the same time that it can run 48 miles against the current, what is the rate of the current?

3. A cistern has three pipes. The first pipe can fill it in half the time required by the second and the second takes two-thirds as long as the third. If the three pipes be open together the cistern will be filled in six hours. In what time can each pipe fill it?

4. The diameter of a driving wheel on a shaft is six inches greater than that of the driven wheel. While the driving wheel makes 64 revolutions, the driven wheel makes 122. What is the circumference of each?

5. The perimeter of a triangle is 82 in. The shortest side is respectively 6 in. and 19 in. less than the other two sides. What is the length of each?

6. A ball is shot into the air with a certain velocity; if its initial velocity had been 380 ft. per second greater, it would have risen $\frac{1}{3}$ higher in 5 seconds. What was its initial velocity and how high did it rise?

7. Jupiter is approximately 370,000,000 miles farther from the sun than the earth. If it takes light, traveling about 186,000 miles per second, 998 seconds longer to reach the earth from Jupiter when he is on the other side of the sun from the earth, what are the distances of the earth and Jupiter from the sun?

8. The distance between two points in a straight line is 21 miles. A train traveling on the circumference of a circle of which this distance is the diameter takes 13 minutes longer to arrive at the second point than another train, which traveled in a straight line. If the two trains had exchanged tracks it would have taken the second train 31 minutes longer to reach the second point. Find the rate of each.

9. If 1 be added to the numerator of a fraction, the resulting fraction will be equal to $\frac{1}{4}$; but if 1 be added to the denominator, the result will be $\frac{1}{5}$. What is the fraction?

10. A number has two digits, the tens' digit being twice the units' digit. If the digits be interchanged the resulting number is 18 less than the original. What is the number?

11. The sum of the lengths of two conductors is 3050 centimeters. The resistivity (that is, the resistance per centimeter) of one is .0016 ohms and of the other .00972 ohms. If each is increased 500 centimeters in length, the sum of their resistances is 29.66 ohms. Find their lengths.

12. A boat must return to its landing in 5 hours. How far may it steam down stream at the rate of 15 miles per hour and return at the rate of 10 miles per hour?

13. *A* and *B* can do a piece of work in 20 days, *A* and *C* in 30 days, and *B* and *C* in 40 days. How long would it take each alone?

14. A man invested $2610 in 5% and 6% bonds, paying 95 for the former and 105 for the latter. His income from both together was $144. How much did he invest in each?

15. A train 300 ft. long passes a train 360 ft. long in 56¼ seconds when both run in the same direction; but if they run in opposite directions they pass in 6¼ seconds. What are the rates of each in miles per hour?

16. A mass of tin and lead, weighing 240 lbs., loses 28 lbs. weighed in water. If 74 lbs. of tin lose 10 lbs., and 115 lbs. of lead lose 10 lbs. in water, how many lbs. of each in the mass?

17. Find the time between 2 and 3 o'clock when the hands of a watch are together.

18. The rear wheels and fore wheels of a carriage are respectively 16 and 14 ft. in circumference. How far has the carriage traveled when the fore wheels have made 51 more revolutions than the rear wheels?

CHAPTER IV.

GRAPHICS.

Article 62. A point is said to have position, but the statement means nothing unless its position is determined with respect to some fixed point or lines. Given a fixed point, or better intersecting straight lines, and the position of any point can be definitely determined by its distances and directions from them. For instance, the position of any building in a city whose streets are symmetrically laid out may be accurately stated by indicating its distances from two principal intersecting streets.

Further, the relative positions of several points and their

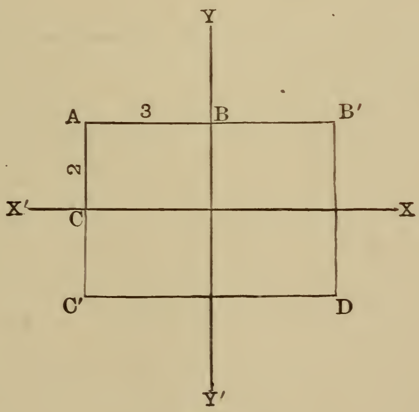

Fig. 1.

distances apart may be easily obtained if their distances from two intersecting straight lines are known. For instance, the position of the point A is accurately defined if we say its distance from the line XX' is 2 units upward and its distance from YY' is 3 units to the left.

The distance AB to YY' is called the *abscissa* of A and the distance AC is called its *ordinate* for convenience. The abscissa and ordinate together are known as the *co-ordinate* of the point.

ART. 63. To avoid the cumbrous expressions, "to the right," "to the left," etc., it is agreed that distance measured to the right of YY' shall be called plus, to the left, negative; and that distance measured upward from XX' shall be called positive, and downward, negative.

ART. 64. The reference lines XX' and YY' usually intersect at right angles for simplicity. They clearly divide space into four quadrants numbered 1, 2, 3, 4, around to the left from X to X again. Thus XOY is quadrant 1, YOX' is 2, $X'OY'$ is 3, $Y'OX$ is 4. The abscissas are universally designated by x and the ordinates by y.

The point A in the last illustration is $x = -3$. $y = 2$ or more briefly $(-3, 2)$, x always being written first.

ART. 65. The values of x and y remaining 3 and 2 respectively, a point in the first quadrant would be $(3, 2)$, in the second, $(-3, 2)$, in the third $(-3, -2)$, and in the fourth $(3, -2)$, represented in the figure as B', A, C', and D, respectively.

ART. 66. Since a line may be regarded as made up of points, if the position of every point in the line with respect to the axes XX' and YY' is known, the position of the entire line is known.

ART. 67. Whenever the relation between the abscissa and ordinate of *every* point on a line is the same, the expression of this relation in the form of an equation is said to give the equation of the line. For example, if the ordinate is always 4 times the abscissa for every point on a line, $y = 4x$ is called the equation of the line.

Again, if 3 times the abscissa is equal to 5 times the ordinate plus 2, for every point on a line, then $3x = 5y + 2$ is the line's equation.

ART. 68. Clearly since an equation represents the relation between the abscissa x and the ordinate y for every point on a line, if either coördinate is known for any point on the line, the other one may be found by substituting the known one in the equation and solving it for the unknown.

For example, let $2y = 7x - 1$ be the equation for a line, and a point is known to have the abscissa $x = 2$. To find its ordinate substitute $x = 2$ in the equation; $2y = 7(2) - 1 = 14 - 1 = 13$; $y - 6\frac{1}{2}$ Therefore the ordinate corresponding to the abscissa $x = 2$ is $6\frac{1}{2}$.

ART. 69. Further, if the equation is given, the whole line may be reproduced by locating its points. If x for example be given a series of values from 0 to 10 inclusive, by substituting these values in the equation, the corresponding values of y are found, and 11 points are thus located on the desired line. If more points are needed the range of values for x may be indefinitely extended, and if these points are joined we have the line. For example, let the equation of a line be $x^2 + y^2 = 9$, to reproduce the curve represented. For convenience in calculating solve for y;

$$y = \pm\sqrt{9 - x^2}.$$

Then give x a series of values to locate points on this line.

If $x = 0$ $y = \pm\sqrt{9 - 0} = \pm 3$.
If $x = 1$ $y = \pm\sqrt{9 - 1} = +\sqrt{8} = \pm 2.83$.
If $x = 2$ $y = \pm\sqrt{9 - 4} = \pm\sqrt{5} = \pm 2.24$.
If $x = 3$ $y = \pm\sqrt{9 - 9} = \pm\sqrt{0} = 0$.
If $x = 4$ $y = \pm\sqrt{9 - 16} = \pm\sqrt{-7} =$ an imaginary.

The last value for y shows that the point whose abscissa is 4 is not on the curve at all; and since any larger values of x would continue to give imaginary values for y, the curve does not extend beyond $x = 3$.

Since we have given x only positive values so far, all our points so determined lie to the right of the YY' axis. To make the examination complete, let x take a series of negative values, thus:

$$\text{If } x = -1 \quad y = \pm\sqrt{9-1} = \pm\sqrt{8} = \pm\ 2.83.$$
$$\text{If } x = -2 \quad y = \pm\sqrt{9-4} = \pm\sqrt{5} = \pm\ 2.24.$$
$$\text{If } x = -3 \quad y = \pm\sqrt{9-9} = \sqrt{0} = 0.$$

The similarity of these results shows that the curve is symmetrical with respect to the axes, that is, it is alike on both sides of the axes.

If now these points are located with respect to the axes XX' and YY' and are joined the result is an approximation to the curve; it is only an approximation because the points are few and not close enough together.

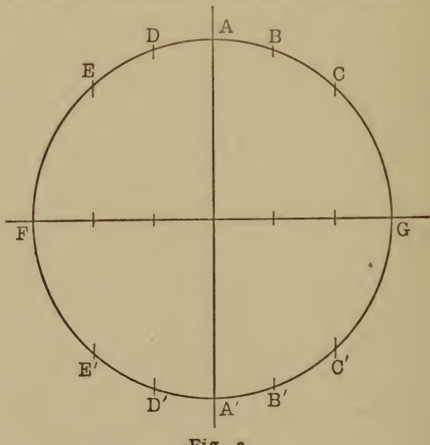

Fig. 2.

The result is shown in Fig. 2, using $\frac{1}{4}$ in. as a unit for scale. The points are (0, + 3), (0, − 3), [being A and A'

in the figure] $(1, \sqrt{8})$, $(1, -\sqrt{8})$ [being B and B'] $(2, \sqrt{5})$, $(2, -\sqrt{5})$ [being C and C'], $(3, 0)$ [G], $(-1, \sqrt{8})$, $(-1, -\sqrt{8})$ [D and D'], $(-2, \sqrt{5})$ $(-2, -\sqrt{5})$ [E and E'] and $(-3, 0)$ [F].

Clearly if more points are needed to trace the curve accurately through them (as is the case here), it is necessary to take more values of x between -3 and $+3$, for example:

$$x = 0 \quad y = \pm\sqrt{9} = \pm 3.$$
$$x = .2 \quad y = \pm\sqrt{9 - .04} = \pm\sqrt{8.96} = \pm 2.99.$$
$$x = .4 \quad y = \pm\sqrt{9 - .16} = \pm\sqrt{8.84} = \pm 2.97.$$
$$x = .6 \quad y = \pm\sqrt{9 - .36} = \pm\sqrt{8.64} = \pm 2.94.$$
$$x = .8 \quad y = \pm\sqrt{9 - .64} = \pm\sqrt{8.36} = \pm 2.89.$$
$$x = 1 \quad y = \pm\sqrt{9 - 1} = \pm\sqrt{8} = \pm 2.83, \text{ etc.}$$

Making a similar table for the corresponding negative values of x, the result is three times as many points on the curve as before, and as they are closer together the curve is much more readily drawn through them, and it will be much more accurate.

Take another example: $9x^2 + 16y^2 = 144$.

Solving for y; $y = \pm\frac{3}{4}\sqrt{16 - x^2}$.

Then if $x = 0 \quad y = \pm\frac{3}{4}\sqrt{16} = \pm 3.$

" $x = \pm .2 \quad y = \pm\frac{3}{4}\sqrt{16 - .04} = \pm\sqrt{15.96} = \pm 2.99 +.$

" $x = \pm .4 \quad y = \pm\frac{3}{4}\sqrt{16 - .16} = \pm\sqrt{15.84} = \pm 2.98 +.$

" $x = \pm .6 \quad y = \pm\frac{3}{4}\sqrt{16 - .36} = \pm\sqrt{15.64} = \pm 2.96 +.$

" $x = \pm .8 \quad y = \pm\frac{3}{4}\sqrt{16 - .64} = \pm\sqrt{15.36} = \pm 2.94.$

" $x = \pm 1 \quad y - \pm\frac{3}{4}\sqrt{16 - 1} = \pm\frac{3}{4}\sqrt{15} = \pm 2.9$, etc.

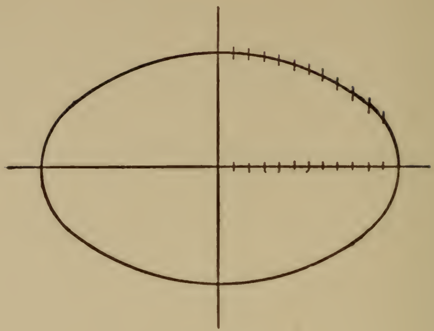

Fig. 3.

The result is indicated in Fig. 3, same scale as before.

ART. 70. Clearly a curve can be traced thus representing almost any form of equation.

Suppose the equation $x^3 - 7x^2 + 7x + 15 = y$ is given. The location of a number of points by giving x a series of values and calculating corresponding values of y from the equation, will enable us to draw through them the curve represented by the equation. In most cases there will be certain values of x which will make the value of y zero, such values of x will be roots of the equation $x^3 - 7x^2 + 7x + 15 = 0$, that is these values of x identically satisfy this equation.

But if y is zero for a point, the point must be on the x axis, for by definition the value of y is the distance from the x axis to the point, hence the curve must cross the x axis at these points where y is zero. If then none of the values given to x make y exactly zero, but do make y change from a positive value for one value of x, to a negative value for the next, or vice versa, it must pass through zero to change from one sign to the other, and hence the curve must cross the x axis.

As an illustration, take the equation

$$x^3 - 5x^2 + x + 11 = y.$$

As before, make a table of values of x and y, and locate the points as follows:

If $x = 0$	$y = 11$	If $x = 3$	$y = -4$
If $x = .5$	$y = 10.375$	If $x = 3.5$	$y = -3.875$
If $x = 1$	$y = 8$	If $x = 4$	$y = -1$
If $x = 1.5$	$y = 4.625$	If $x = 4.5$	$y = 5.375$
If $x = 2$	$y = 1$	If $x = -1$	$y = 4$
If $x = 2.5$	$y = 2.125$	If $x = 1.5$	$y = 5.125$

The curve connecting these points crosses the x axis at three points; one between 2 and 2.5, one between 4 and 4.5, and one between -1 and -1.5. Hence the three roots of the equation $x^3 - 5x^2 + x + 11 = 0$ are between 2 and 2.5; between 4 and 4.5, and between -1 and -1.5.

If the values of x in the above table had been taken closer together, the points of crossing would have been more accurately known.

EXERCISE XII.

Graphs.

Construct the graph of:

1. $2x - 5 = y$.

2. $\frac{2}{5}(x - 3) = y$.

3. $3x - 5y = 7$.

4. $5 - 2x = 2y$.

5. $\frac{3}{4}(4 + 5x) = \frac{1}{2}y$.

6. $9x^2 + 16y^2 = 144$.

7. $x^2 - 4y^2 = 4$.

8. $x^2 = 4y$.

9. $y^2 = \frac{9}{16}x$.

10. $x^2 + y^2 = 25$.

11. $x^3 - 3x^2 + 1 = y$.

12. $\frac{2}{3}x^2 + \frac{3}{2}y^2 = 6$.

13. $y^2 = 4(4 - x)$.

14. $x^2 = 9(9 - y)$.

15. State the equation for the line $\frac{2}{3}$ of whose abscissas minus 1 equals $\frac{3}{4}$ of its ordinates.

16. State the equation for the line, the sum of the squares of the abscissas and ordinates of its points being equal to 16.

CHAPTER V.

THE BINOMIAL THEOREM.

ARTICLE 71. Let us consider the product $(a + b)(a + b)(a + b)$ or $(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$.

A very casual examination will show that the first term of this product, which is the cube of $(a + b)$, is the first letter raised to the power 3, and that the exponent of this first letter *decreases* by one in each successive term to the right until it disappears in the last term; also that the second letter appears first in the second term and *increases* by one in each term to the right until it reaches 3 in the last term.

Also the coefficient of the first term is 1, and of the second term is the same as the exponent of the binomial, that is, 3. The coefficient of the third term may be gotten by multiplying the coefficient of the preceding term (the second) by the exponent of the first letter and dividing the product by the number of this term; thus, $\frac{3 \times 2}{2} = 3$.
The coefficient of the fourth term (which in this case is the last) may be gotten in the same way from the third;

thus, $$\frac{3 \times 1}{3} = 1.$$

Now let us see if these rules hold for the fourth power of $(a + b)$,

$$(a + b)(a + b)(a + b)(a + b) = (a + b)^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Clearly the rules for exponents apply; for the exponents of a begin with 4 and decrease by one to the right, and the exponents of b begin with one in the second term and increase by one, to 4 in the last term.

Also the coefficient of the second term is 4, the exponent of the binomial; the coefficient of the third term is gotten from the second by the rule already indicated, namely,

$$\frac{4 \times 3}{2} = 6.$$

For the fourth term also, $\frac{6 \times 2}{3} = 4$; and for the last term,

$$\frac{4 \times 1}{4} = 1.$$

And so an examination of higher powers will in every case verify these rules, which are called empirical rules, because they are derived by inspection and verified by trial, although they have a good mathematical foundation. These rules constitute what is called the *Binomial Theorem.*

EXAMPLE. Develope the expression $(x + y)^7$ by the binomial theorem.

$$(x + y)^7 = x^7 + 7\,x^{(7-1)}\,y + \frac{7 \times 6}{2}\,x^{7-2}\,y^2 + \frac{21 \times 5}{3}\,x^{7-3}\,y^3 + \frac{35 \times 4}{4}\,x^{7-4}\,y^4 + \frac{35 \times 3}{5}\,x^{7-5}\,y^5 + \frac{21 \times 2}{6}\,x^{7-6}\,y^6 + \frac{7 \times 1}{7}\,y^7$$ or $$(x + y)^7 = x^7 + 7\,x^6y + 21\,x^5y^2 + 35\,x^4y^3 + 35\,x^3y^4 + 21\,x^2y^5 + 7\,xy^6 + y^7.$$

It is to be also observed that the number of terms is always one greater than the power to which the binomial is raised; for example, in the above problem there are 8 terms = 7 + 1, etc. Also the coefficients of the terms equally distant from the first and last terms are the same, as may be readily seen by writing the series of coefficients in the last problem as follows:

$$1,\ 7,\ 21,\ 35,\ 35,\ 21,\ 7,\ 1.$$

A simple examination of the product of $(a-b)$ $(a-b)$ $(a-b)$, $(a-b)$ $(a-b)$ $(a-b)$ $(a-b)$ etc., will show that the terms are alternately plus and minus, the first term always being plus, and that the developement in other respects is exactly like that for $(a+b)$.

Art. 72. These rules may be applied to trinomials by grouping two of the terms together, thus, $[(a+b)+c]^4 = (a+b)^4 + 4\ (a+b)^3\ c + 6\ (a+b)^2\ c^2 + 4\ (a+b)\ c^3 + c^4$; then $(a+b)^3$ and $(a+b)^2$ may be developed separately by the theorem and the results substituted. Likewise $(a+b+c+d)^5$ could be grouped thus, $[(a+b)+(c+d)]^5$, and so on by like process for any polynomial.

Art. 73. It is to be observed that these rules for the expansion of a binomial do not require that the terms of the binomial should be single letters; for example. $(2x+3y)^4$ can be expanded in exactly the same way as $(a+b)^4$, by treating the $2x$ as a and $3y$ as b; thus $(2x+3y)^4 = (2x)^4 + 4\ (2x)^3\ (3y) + 6\ (2x)^2\ (3y)^2 + 4\ (2x)\ (3y)^3 + (3y)^4$

This last expression can then be simplified by performing the indicated operations as follows:

$(2x)^4 = (2)^4\ (x^4) = 16\ x^4$	$(3y)^2 = (3)^2\ (y^2) = 9\ y^2$
$(2x)^3 = (2)^3\ (x^3) = 8x^3$	$(3y)^3 = (3)^3\ (y^3) = 27\ y^3$
$(2x)^2 = (2)^2\ (x^2) = 4x^2$	$(3y)^4 = (3)^4\ (y^4) = 81\ y^4$

Making these substitutions, the first expression becomes $(2x+3y)^4 = 16\ x^4 + 96\ x^3\ y + 216\ x^2\ y^2 + 216\ x\ y^3 + 81\ y^4$.

Clearly in the binomial $(a + b)$, a or b can be made to represent any expressions whatever, and hence any polynomial, however complex, may be expanded by the theorem.

The binomial theorem may be employed to abbreviate the process of squaring numbers, especially those of two digits, as 47, say. The process is as follows:

$(47^2 = (40 + 7)^2 = (40)^2 + 2 \times 40 \times 7 + 7^2$ by the theorem.

This may be collected thus, $\begin{array}{r} 1600 \\ 560. \\ \underline{49} \\ 2209 \end{array}$ Since the first digit is of necessity in ten's place its square will be always so many hundreds, and hence there will be always two zeros with the square of the first digit; twice the product of the two digits will always have one zero with it as the second term of the development. The addition can be simplified then by neglecting the zeros and keeping the figures in their proper position in the sum by writing each succeeding term so that one figure projects to the right, thus,

$$\begin{array}{rrl} (47)^2 = & 16 & = 4^2 \\ & 56 & = 2 \times 7 \times 4 \\ & \underline{49} & = 7^2 \\ & 2209 & = (47)^2 \end{array}$$

EXERCISE XIII.

Expand by the Binomial Theorem.

1. $(a + b)^4$.
2. $(2x + 3y)^5$.
3. $(x - 2z)^3$.
4. $\left(\frac{x}{2} + \frac{y}{3}\right)^4$.
5. $(3c - 5d)^4$.
6. $(2 - x)^7$.
7. $(y - 1)^6$.
8. $(xy - z)^3$.

9. $(3 - c^2)^5$.
10. $(1 - x^2)^6$.
11. $(x + \frac{3}{2})^5$.
12. $(m - \frac{2}{3} n)^4$.
13. $(x^3 - y^2z)^3$.
14. $(r^{-\frac{1}{2}} + s^{-\frac{1}{3}})^5$
15. $(2x^{\frac{1}{3}} + y^{-\frac{3}{2}})^6$.
16. $(3\sqrt{x} - 2\sqrt{y})^4$.
17. $(a^{\frac{2}{3}} + 5)^5$.
18. $\left(x + \frac{1}{x}\right)^4$
19. $(y^2 + y - 1)^3$.
20. $(2 + x + x^2)^4$.
21. $(\sqrt{x} + 1)^5$.
22. $(xy - 2)^4$.
23. $(R - 1)^6$.
24. $(a - x)^7$.
25. $(1 + x - y - z)^3$.
26. $(3y - 1)^5$.
27. $(\frac{2}{3} + \frac{3}{5} a)^4$.
28. $\left(x + \frac{1}{2x}\right)^5$.

INVOLUTION AND EVOLUTION.

ART. 74. The process of multiplying any quantity by itself any number of times is called raising it to the *power* indicated by the number of times it is used as a factor. For example $a \times a \times a = a^3$ is the third power or cube of a. Power in algebra differs from power in arithmetic chiefly in the matter of sign. From the rule for signs in multiplication, evidently all even powers of either positive or negative quantities will be positive, while odd powers of negative quantities are negative. Raising to powers is called *Involution.*

ART. 75. To raise a monomial to any power, raise the numerical part arithmetically to the required power, and multiply the exponents of the literal factors by the power, as directed in the rule for exponents.

EXAMPLE.

$$(5a^2bc^3)^3 = (5)^3 a^{2\times3} b^{1\times3} c^{3\times3} = 125a^6b^3c^9; \; (-3x^{-1}y^2z)^5$$
$$= (-3)^5 x^{-1\times5} y^{2\times5} z^{1\times5} = -243x^{-5}y^{10}z^5.$$

ART. 76. To raise a polynomial to any power, apply the binomial theorem as indicated in Article 72.

EXTRACTION OF ROOTS.

ART. 77. The *root* of a quantity is one of its *equal* factors. Evidently the finding of any root is the exact reverse of raising it to a power.

Hence, to extract a root of a monomial, extract arithmetically the required root of the numerical factor, and divide the exponents of the literal factors by the index of the root.

Roots are indicated as in arithmetic by the radical sign, $\sqrt{}$ and the figure indicating the root called its index, above the radical sign to the left. Thus, $\sqrt[5]{65a^2b}$ indicates the fifth root of $65a^2b$. This is often expressed as a fractional power, thus $(65a^2b)^{\frac{1}{5}}$. Extraction of roots is known as *Evolution.*

To Extract the Square Root of a Polynomial.

ART. 78. Since by making use of parentheses, and thus grouping terms, any quantity may be put into the form of a binomial, its square may be regarded as the square of a binomial, and the simple binomial form $(a+b)^2 = a^2 + 2ab + b^2$ may serve as a model. In this expression, a and b are to stand for any two quantities, monomial or polynomial. For example,

$$(2x^2 + 3cy + m^3 + n)^2 = [(2x^2 + 3cy) + (m^3 + n)]^2$$
$$= (2x^2 + 3cy)^2 + 2(2x^2 + 3cy)(m^3 + n) + (m^3 + n)^2.$$

The expression $(a+b)^2 = a^2 + 2ab + b^2$ may be put in the form $(a+b)^2 = a^2 + b(2a+b)$. Since as we have shown any quantity may be expressed in this general form, the square root of the first term of such quantity is the first term of its complete square root, for $\sqrt{a^2} = a$ in the form $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b(2a+b)$. If now a^2 be subtracted from the square of the binomial, the remainder $2ab + b^2 = b(2a + b)$ will contain the rest of the root. Since the first term of this remainder is made up of $2a$ and b, if we divide it by twice the first term of the root already found (a), the quotient will be b, the second term of the root.

The form of the remainder, $b(2a + b)$, shows that to make the process complete, b must be added to $2a$, and the sum multiplied by b.

If this when subtracted still does not exhaust the original quantity, the same process repeated using the two terms of the root already found, $a + b$, as a single term, will add another term to the root. This would show that the original expression was in the form $(a + b + c)^2$ instead of $(a + b)^2$, thus,

$$[(a + b) + c]^2 = (a + b)^2 + 2(a + b)c + c^2.$$

Now $(a + b)^2$ has already been subtracted from the above expression in the process just described, hence the remainder consists of an expression of the form

$$2(a + b)c + c^2 = c[2(a + b) + c],$$

which is exactly like the first remainder, if $a + b$ is regarded as a single quantity, indicated by the parenthesis.

This process may be repeated until there is no remainder, or one too small to further contain the terms of the root already found. In the latter case there is no exact root.

ART. 79. All this may be put into a rule as follows:

To extract the square root of a polynomial, extract the square root of the first term, this will be the first term of the root. Double this first term of the root, and as a trial divisor, divide the product into the remainder left after dropping the first term of the polynomial; the quotient is the second term of the root. Complete the trial divisor by add-

ing this second term of the root, and multiply this sum by the second term of the root. If, after subtracting this product, there is still a remainder, double all of the root already found and divide the first term of this product into the first term of the remainder to get the third term of the root. Complete the divisor as before and multiply the completed divisor by the third term of the root. Continue this process until there is no remainder. The polynomial should be arranged according to ascending (or descending) powers of one letter.

EXAMPLE. Extract the square root of

$25x^4 + 20x^2y - 4yz^2 + 4y^2 + z^4 - 10x^2z^2$

$25x^4 + 20x^2y + 4y^2 - 10x^2z^2 - 4yz^2 + z^4 \;|\; 5x^2 + 2y - z^2$

$25x^4$

Trial divisor: $10x^2$ $\qquad\qquad$ $20x^2y + 4y^2 - 10x^2z^2 - 4yz^2 + z^4$

$\qquad\qquad\quad + 2y$

Complete div. $10x^2 + 2y$

$(10x^2 + 2y)\,2y =$ $\qquad\qquad$ $20x^2y + 4y^2$

$2(5x^2 + 2y) = 10x^2 + 4y$ (trial div.) $\qquad$ $-10x^2z^2 - 4yz^2 + z^4$

$\qquad\qquad\qquad - z^2$

$10x^2 + 4y - z^2$ (complete)

$(10x^2 + 4y - z^2)(-z^2) =$ $\qquad\qquad$ $-10x^2z^2 - 4yz^2 + z^4$

This expression was of the form $(a + b + c)^2$, for $[(5x^2 + 2y) - z^2]^2 = (5x^2 + 2y)^2 + 2(5x^2 + 2y)(-z^2) + (-z^2)^2 = 25x^4 + 20x^2y - 4yz^2 + 4y^2 + z^4 - 10x^2z^2$.

Extraction of Cube Root of Polynomials.

ART. 80. A similar consideration of the cube of the binomial $(a + b)$, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b(3a^2 + 3ab + b^2)$, will make the following rule clear:

To extract the cube root of a polynomial, arrange, for convenience, the polynomial according to the ascending or descending power of one letter. Extract the cube root of *the first term; this will be the first term of the root. Multiply the square of this first term of the root by 3 ($3a^2$ in the above formula) for a trial divisor; this product divided into the first term of the remainder left by dropping the first term of the polynomial, will give the second term of the root ($3a^2b \div 3a^2 = b$), complete the divisor by adding to the trial divisor three times the product of the first and second terms of the root and the square of the second term of the root ($3a^2 + 3ab + b^2$); multiply this sum by the second term of the root ($[3a^2 + 3ab + b^2] \times b$) and subtract from the remainder of the polynomial. If there is still a remainder large enough to contain the trial divisor, repeat the process, taking the two terms of the root already found together as a single term.*

EXAMPLE. Extract the cube root of

$$63x^4 + 27x^6 + 21x^2 - 44x^3 - 54x^5 - 6x + 1.$$

ARRANGING:

$$27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1 \;\Big|\; \underbrace{3x^2}_{=a} \underbrace{-2x}_{=b} \underbrace{+1}_{=c}$$

$$27x^6$$

Trial Divisor:	
$(3x^2)^2\,3 = 27x^4$	$-54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1$
$(3x^2)(-2x)3 = -18x^3$	$-54x^5 + 36x^4 - 8x^3$
$(-2x)^2 = +4x^2$	$= (27^4 - 18x^3 + 4x^2)(-2x)$
$27^4 - 18x^3 + 4x^2$ Complete.	
Trial Divisor:	
$(3x^2 - 2x)^2\,3 = 27x^4 - 36x^3 + 12x^2$	$27x^4 - 36x^3 + 21x^2 - 6x + 1$
$(3x^2 - 2x)(+1)(3) = 9x^2 - 6x$	
$(1)^2 = +1$	
$27x^4 - 36x^3 + 21x^2 - 6x + 1$ Complete.	$27x^4 - 36x^3 + 21x^2 - 6x + 1 = (27x^4 - 36x^3 + 21x^2 - 6x + 1)(+1)$

ART. 81. Evidently similar rules for higher powers may be readily stated by referring to the corresponding developments of $(a + b)$, but they become rather complicated and are rarely needed.

The fourth power being the square of the square, the fourth root may be gotten by extracting the square root twice. Likewise, the sixth root by extracting successively the square and the cube root, or vice versa, etc.

If it be remembered that numbers may be expressed as binomials (as $67 = 60 + 7$ or $139 = 130 + 9$, etc.) the arithmetical application of these rules will be readily understood.

EXERCISE XIV.

Extract the following roots:

1. $\sqrt{a^2b^2(a+b)^4}$. **5.** $\sqrt[6]{64x^6y^{18}z^{30}}$.

2. $\sqrt[4]{25bx^8y^{12}a^{20}}$. **6.** $\sqrt[3]{-729(a+b)^3c^6}$.

3. $\sqrt[5]{-32m^{10}n^5r^{15}}$. **7.** $\sqrt[4]{x^{4n}y^{8m}z^{12}}$.

4. $\sqrt[3]{\frac{27}{8}a^6b^9c^{-3}}$. **8.** $\sqrt{16(r+s)(r+s)^3}$.

Extract the square root of the following:

9. $13x^2y^2 - 12x^3y + 4x^4 - 6xy^3 + y^4$.

10. $4m^4 - 12m^2n^2 + 9n^4 + 16m^2p^2 - 24n^2p^2 + 16p^4$.

11. $9 - 24y - 68y^2 + 112y^3 + 196y^4$.

12. $4 + 9b^2 - 20a + 25a^2 + 30ab - 12b$.

13. $\frac{x^4}{4} + \frac{y^6}{9} + \frac{x^2}{4} - \frac{x^2y^3}{3} + \frac{1}{16} - \frac{y^3}{6}$.

14. $\frac{m^4}{n^4} - \frac{4m^3}{n} + 4m^2n^2 + 6m - 12n^3 + \frac{9m^4}{n^2}$.

15. $y^6 - 6y^5z + 15z^2y^4 - 20z^3y^3 + 15z^4y^2 - 6z^5y + z^6$.

16. $x^2 + 4x^3 + 4x^2 + 4x + 4 + \frac{1}{x^2}$.

Extract the cube root of the following:

17. $a^6 + 3a^5 + 6a^4 + 7a^3 + 6a^2 + 6a + 1$.

18. $x^3 - 6x^2y + 12xy^2 - 8y^3 - 3x^2z + 3xz^2 + 12xyz - 12y^2z - 6yz^2 - z^3$.

19. $8y^{6m} - 36y^{5m} + 66y^{4m} - 63y^{3m} + 33y^{2m} - 9y^m + 1$.

20. $x^3 - x^2y + \frac{xy^2}{3} - \frac{y^3}{27}$.

21. $12a^2 - \frac{125}{a^3} - 54a - 59 + \frac{135}{a} + 8a^3 + \frac{75}{a^2}$.

22. $1 - 6y + 9y^2 + 9y^3 - 9y^4 - 6y^5 - y^6$.

23. $\frac{3}{x} + \frac{1}{8x^3} - \frac{3}{4x^2} + 12x - 7 - 12x^2 + 8x^3$.

24. $27m^6n^6 + 54m^5n^5 + 9m^4n^4 - 28m^3n^3 - 3m^2n^2 + 6mn - 1$.

CHAPTER VI.

SURDS.

ARTICLE. 82. It is understood that the expression x^5 means that x is to be taken five times as a factor; that is,

$$x.x.x.x.x. = x^5.$$

It is found convenient to represent in somewhat similar symbols other operations involving factors. These symbols have been suggested by the simpler operations with integral exponents, thus:

Each of the successive expressions a^6, a^5, a^4, a^3, a^2, a^1, can be derived from the next preceeding by dividing by a, that is, $a^5 = a^6 \div a$: $a^4 = a^5 \div a$, etc., so that each division reduces the exponent by unity. If we continue this process in above series, we get a^6, a^5, a^4, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4}, etc.

By the primary laws of division

$$a^1 \div a^1 = 1 = a^0 \text{ by above series.}$$

$$a \div a = \frac{1}{a} = a^{-1} \text{ by above series.}$$

$$a^{-1} \div a = \frac{a^{-1}}{a} = \frac{\frac{1}{a}}{a} = \frac{1}{a^2} = a^{-2} \text{ by above series.}$$

Hence, we may adopt the symbols:

$$a^{-1} \text{ for } \frac{1}{a}, \; a^{-2} \text{ for } \frac{1}{a^2}, \; a^{-3} \text{ for } \frac{1}{a^3}, \text{ etc.}$$

Also $a^0 = 1$.

Since $a^{-1} = \frac{a^{-1}}{1} = \frac{1}{a}$ and $a^{-2} = \frac{a^{-2}}{1} = \frac{1}{a^2}$, etc.

We may state the rule thus:

ANY QUANTITY MAY BE MOVED FROM DENOMINATOR TO NUMERATOR OF A FRACTION, OR VICE VERSA, BY CHANGING THE SIGN OF ITS EXPONENT, OR:

A QUANTITY WITH A NEGATIVE EXPONENT IS THE RECIPROCAL OF THE SAME QUANTITY WITH THE EXPONENT MADE POSITIVE.

FRACTIONAL EXPONENTS.

ART. 83. Let us consider a series of expressions like the following:

$$a^{16},\ a^{8},\ a^{4},\ a^{2},\ a^{1}.$$

Each one obtained from the preceeding, by dividing its exponent by 2, is called its square root. If this process of division is continued, we get,

$$a^{16},\ a^{8},\ a^{4},\ a^{2},\ a^{1},\ a^{\frac{1}{2}},\ a^{\frac{1}{4}},\ a^{\frac{1}{8}},\ \text{etc.}$$

and by analogy, $a^{\frac{1}{2}}$ may be called the square root of a; $a^{\frac{1}{4}}$, the square root of $a^{\frac{1}{2}}$ or the fourth root of a, etc.

Likewise, the series a^{27}, a^{9}, a^{3}, a^{1}, $a^{\frac{1}{3}}$, $a^{\frac{1}{9}}$, $a^{\frac{1}{27}}$, etc., indicates a like relation with third, ninth, twenty-seventh root, and in general, the symbol $a^{\frac{1}{n}}$ means an nth root of a.

RADICAL SIGNS.

ART. 84. There is another symbolism which arises from the older method of indicating square root by prefixing the Latin word *radix*, meaning root, to the quantity involved; for instance, radix 2 meant, in modern phraseology, the square root of 2, etc.

Eventually radix was abbreviated to r, and finally the r was extended to cover the quantity whose root was required; thus, $\sqrt{2}$.

Hence $a^{\frac{1}{2}}$ is equivalent to $\sqrt{a}$.

To complete this symbolism, a small index is superposed upon this radical sign, to indicate other roots, thus:

$$a^{\frac{1}{4}} = \sqrt[4]{a},\ a^{\frac{1}{5}} = \sqrt[5]{a},\ \text{and in general } a^{\frac{1}{n}} = \sqrt[n]{a}.$$

ART. 85. Any root thus indicated, except even roots of negative numbers (discussed later), is called a *surd.*

ART. 86. Surds are said to be similar, when they have the same quantity under the radical sign, when in there simplest form.

$a\sqrt{3}$, $3\sqrt{3}$, $2\sqrt{3}$, etc., are similar.

ART. 87. When an expression is wholly under the radical sign it is said to be a *pure surd;* otherwise, a *mixed surd.*

$\sqrt{7}$, $\sqrt{9}$, $\sqrt{13}$, etc., are pure surds.

$3\sqrt{2}$, $5\sqrt{7}$, $4\sqrt{3}$, etc., are mixed surds.

To Simplify Surds.

ART. 88. Example: simplify $\sqrt{27a^3b^2c^5}$.

$$\sqrt{27a^3b^2c^5} = \sqrt{(9a^2b^2c^4)(3ac)} = 3abc^2\sqrt{3ac},$$

since the square root of $9a^2b^2c^4$ is $3abc^2$.

Again, $\sqrt{216} = \sqrt{36 \times 6} = 6\sqrt{6}$,

or $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt[3]{3}$. Then,

Rule: SEPARATE THE EXPRESSION UNDER THE RADICAL INTO TWO FACTORS ONE OF WHICH IS A PERFECT POWER, AND THE LARGEST FACTOR OF THIS KIND. EXTRACT THE ROOT (INDICATED) OF THIS FACTOR, PLACING IT OUTSIDE OF THE RADICAL SIGN, LEAVING THE OTHER FACTOR UNDER THE RADICAL. MULTIPLY THE ROOT THAT HAS BEEN REMOVED BY ANY FACTOR ALREADY OUTSIDE OF THE RADICAL SIGN.

EXAMPLE. Simplify, $3\sqrt{98}$

$$3\sqrt{98} = 3\sqrt{49 \times 2} = 3\,(7)\,\sqrt{2} = 21\sqrt{2}.$$

Again, $ab\sqrt{a^3b^5x^2} = ab\sqrt{(a^2b^4x^2)\,(ab)} = a^2b^3x\sqrt{ab}.$

Order of a Surd.

ART. 89. The index of the radical indicates the order of the surd. For instance, $\sqrt{5}$ is a surd of 2d order, or a quadratic surd; $\sqrt[3]{5}$ is a surd of 3d order, or a cubic surd; $\sqrt[4]{5}$ is a surd of 4th order, or a bi-quadratic, etc.

Reduction of Mixed to Entire Surds.

ART. 90. The reduction of mixed surds to entire surds is the exact reverse of the simplification of surds.

EXAMPLE. $2\sqrt{3} = \sqrt{(2)^2 \times 3} = \sqrt{4 \times 3} = \sqrt{12}$

Again, $3\sqrt[3]{7} = \sqrt[3]{(3)^3 \times 7} = \sqrt[3]{27 \times 7} = \sqrt[3]{189}$, etc.

Rule: RAISE THE ENTIRE EXPRESSION OUTSIDE THE RADICAL SIGN, TO THE POWER INDICATED BY THE RADICAL INDEX, MULTIPLY THE RESULT BY THE QUANTITY ALREADY UNDER THE SIGN, AND WRITE THE PRODUCT UNDER THE SIGN.

Addition and Subtraction of Surds.

ART. 91. Manifestly radicals involving different quantities cannot be added or subtracted. For instance, $3\sqrt{2}$ and $2\sqrt{5}$ cannot be added or subtracted, except by indication, as $3\sqrt{2} \pm 2\sqrt{5}$, for the square root of 2 is very different from the square root of 5. It would be as possible to add $3b$ and $2a$.

But $3\sqrt{2}$, $2\sqrt{2}$, $5\sqrt{2}$, etc., can be as easily added as $3a$, $2a$, and $5a$; thus, $3\sqrt{2} + 2\sqrt{2} + 5\sqrt{2} = 10\sqrt{2}$.

Nor can surds of different orders be combined for the same reason. For instance, $3\sqrt{2}$ $\sqrt[3]{2}$, $4\sqrt[4]{2}$ cannot be added, because each is entirely distinct, like a, b, c, etc.

Multiplication and Division of Surds.

ART. 92. $3\sqrt{2}$ and $2\sqrt[3]{2}$ are as distinct entities as a and b, for the square root of 2 is plainly as different from the third root of 2 as if it were an entirely unlike quantity under the radical sign.

If it were required to multiply $(a+b)^2$ by $(a+b)$, it would be incorrect to multiply $(a+b)$ with the exponent 2, by $(a+b)$ with exponent 1, thus, $(a+b)^2 \times (a+b) = (a^2+2ab+b^2)^2$, because $(a+b)^2$ is of a quite different order from $(a+b)^1$. It is then necessary to reduce both to the same exponent, thus, $(a+b)^2 = (a^2+2ab+b^2)^1 \times (a+b)^1 = [(a^2+2ab+b^2)\ (a+b)]^1 = (a^3+3a^2b+3ab^2+b^3)^1$, or $(a+b)^4 \times (a+b)^2 = (a^2+2ab+b^2)^2 \times (a+b).^2$ Likewise, if $3\sqrt{2}$ is to be multiplied by $2\sqrt[3]{2}$, these surds must be reduced to the same order.

The least common order to which each of these surds can be reduced is clearly the 6th; thus:

$$3\sqrt{2} = 3\sqrt[6]{(2)^3} = 3\sqrt[6]{8}$$

$$2\sqrt[3]{2} = 2\sqrt[6]{(2)^2} = 2\sqrt[6]{4}.$$

Then $3\sqrt[6]{8} \times 2\sqrt[6]{4} = (3 \times 2)\sqrt[6]{4 \times 8} = 6\sqrt[6]{32}$.

Rule: REDUCE THE SURDS TO THE SAME ORDER. MULTIPLY THE COEFFICIENTS OF THE SURDS TOGETHER, FOR THE COEFFICIENT OF THE PRODUCT, AND THE QUANTITIES UNDER THE RADICAL SIGN FOR THE SURD PART OF THE PRODUCT.

Division of Surds.

ART. 93. Division likewise requires the reduction of the surds to the same order. Surds of different order can no more be multiplied or divided by one another than can bushels be multiplied or divided by feet. Hence,

Rule: REDUCE THE SURDS TO THE SAME ORDER. DIVIDE THE COEFFICIENTS AND THE QUANTITIES UNDER THE RADICAL SIGN SEPARATELY AND EXPRESS THE QUOTIENT AS THE PRODUCT OF THESE TWO PARTIAL QUOTIENTS.

EXAMPLE. Divide $3a\sqrt[3]{a^2b}$ by $2ab\sqrt[4]{ab^2}$.

$$3a\sqrt[3]{a^2b} = 3a\sqrt[12]{(a^2b)^4} = 3a\sqrt[12]{a^8b^4}$$

$$2ab\sqrt[4]{ab^2} = 2ab\sqrt[12]{(ab^2)^3} = 2ab\sqrt[12]{a^3b^6}$$

$$3a\sqrt[12]{a^8b^4} \div 2ab\sqrt[12]{a^3b^6} = \frac{3a}{2ab}\frac{\sqrt[12]{a^8b^4}}{\sqrt[12]{a^3b^6}} = \frac{3}{2b}\sqrt[12]{\frac{a^5}{b^2}}$$

or

$$\frac{3}{2}b^{-1}\sqrt[12]{a^5b^{-2}}.$$

The order to which the surds are reduced should be the L.C.M. of the indices of the original surds.

Comparison of Surds.

ART. 94. If it is necessary to compare any quantities, they must be expressed in the same unit. Likewise, if surds are to be compared, they must be reduced to the same order. Hence,

REDUCE THE EXPRESSIONS TO COMPLETE SURDS. REDUCE THESE COMPLETE SURDS TO THE SAME ORDER, AND THE SURD HAVING THE GREATEST QUANTITY UNDER THE SIGN IS THE GREATEST.

For example, compare $2\sqrt[3]{3}$, $3\sqrt{2}$, and $\frac{5}{2}\sqrt[4]{4}$.

$$2\sqrt[3]{3} = \sqrt[3]{8 \times 3} = \sqrt[3]{24}$$
$$3\sqrt{2} = \sqrt{9 \times 2} = \sqrt{18}$$
$$\frac{5}{2}\sqrt[4]{4} = \sqrt[4]{\frac{625 \times 4}{16}} = \sqrt[4]{\frac{625}{4}} = \sqrt[4]{156.25}.$$

12 is the L.C.M. of the indices.

$$\sqrt[3]{24} = \sqrt[12]{(24)^4} = \sqrt[12]{13{,}824}$$
$$\sqrt[2]{18} = \sqrt[12]{(18)^6} = \sqrt[12]{34012224}$$
$$\sqrt[4]{156.25} = \sqrt[12]{(156.25)^3} = \sqrt[12]{38{,}411.389385}.$$

Hence, $\sqrt{18}$ is the greatest.

Rationalizing Denominators Containing Surds.

ART. 95. It is usually undesirable to have surds in the denominators of fractions, owing to the difficulty of estimating values in such forms.

The principle derived from factoring, that the product of the sum and difference of two quantities equals the difference of their squares, enables us to remove these surds.

For example: To rationalize the denominator of the fraction $\frac{6}{\sqrt{7}-2}$.

Since the multiplication of both terms of a fraction by the same quantity does not change its value, we can choose any multiplier we please. If we choose the quantity $\sqrt{7} + 2$, we will have the sum of the quantities $\sqrt{7}$ and 2, whose difference is the denominator of the fraction, and since the product of this sum and difference gives the difference of their squares, the radicals will disappear. Thus:

$$\frac{6}{\sqrt{7}-2} \times \frac{\sqrt{7}+2}{\sqrt{7}+2} = \frac{6\sqrt{7}+12}{(\sqrt{7})^2-(2)^2} = \frac{6\sqrt{7}+12}{7-4} = 2\sqrt{7}+4.$$

Again, to rationalize the denominator of

$$\frac{2+\sqrt{3}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}.$$

This fraction may be written thus:

$$\frac{2+\sqrt{3}}{(\sqrt{2}-\sqrt{3})+\sqrt{5}}.$$

Multiply by $\dfrac{(\sqrt{2}-\sqrt{3})-\sqrt{5}}{(\sqrt{2}-\sqrt{3})-\sqrt{5}}$

$$\frac{2+\sqrt{3}}{(\sqrt{2}-\sqrt{3})+\sqrt{5}}\times\frac{(\sqrt{2}-\sqrt{3})-\sqrt{5}}{(\sqrt{2}-\sqrt{3})-\sqrt{5}}$$

$$=\frac{2\sqrt{2}-2\sqrt{3}-2\sqrt{5}+\sqrt{6}-\sqrt{15}-3}{(\sqrt{2}-\sqrt{3})^2-(\sqrt{5})^2}$$

$$=\frac{2\sqrt{2}-2\sqrt{3}-2\sqrt{5}-\sqrt{15}+\sqrt{6}-3}{2\sqrt{6}}$$

Multiply again by $\dfrac{\sqrt{6}}{\sqrt{6}}$;

$$\frac{2\sqrt{2}-2\sqrt{3}-2\sqrt{5}+\sqrt{6}-\sqrt{15}-3}{2\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$$

$$=\frac{2\sqrt{12}-2\sqrt{18}-2\sqrt{30}+6-\sqrt{90}-3\sqrt{6}}{12}$$

$$=\frac{4\sqrt{3}-6\sqrt{2}-2\sqrt{30}+6-3\sqrt{10}-3\sqrt{6}}{12}$$

$$=\frac{1}{3}\sqrt{3}-\frac{1}{2}\sqrt{2}-\frac{1}{6}\sqrt{30}+\frac{1}{2}-\frac{1}{4}\sqrt{10}-\frac{1}{4}\sqrt{6}.$$

This last expression may be readily computed.

ART. 96. A quadratic surd in its simplest form cannot equal the sum of a rational quantity and a surd.

For if it were possible let $\sqrt{x} = y + \sqrt{z}$.

Squaring both sides $x = y^2 + 2y\sqrt{z} + z$;

transposing, $x - y^2 - z = 2y\sqrt{z}$;

that is, a surd is equal to a rational quantity, which is manifestly impossible.

$\therefore \sqrt{x} \neq y + \sqrt{z}$. ($\neq$ means *not* equal.)

ART. 97. The sum or difference of two dissimilar quadratic surds, in their simplest form, cannot equal a rational quantity or be expressed as a single surd.

If it were possible let $\sqrt{a} \pm \sqrt{b} = c$.

Squaring, $a \pm 2\sqrt{ab} + b = c^2$;

transposing, $\pm 2\sqrt{ab} = c^2 - a - b$;

but a surd cannot equal a rational quantity,

$$\therefore \sqrt{a} \pm \sqrt{b} \neq c.$$

If $\sqrt{a} \pm \sqrt{b} = \sqrt{c}$.

Squaring, $a \pm 2\sqrt{ab} + b = c$;

transposing, $a + b - c = \pm\sqrt{2ab}$,

which is impossible, $\therefore \sqrt{a} \pm \sqrt{b} \neq \sqrt{c}$.

ART. 98. If the sum of a rational quantity and a surd equals the sum of a rational quantity and a surd, then the rational quantities must be equal and the surds equal.

That is, if $\sqrt{x} + m = \sqrt{y} + n$, $x = y$ and $m = n$.

For, transposing, $\sqrt{x} - \sqrt{y} = n - m.$

This violates the previous article unless both sides are equal to zero, that is, unless $x = y$ and $n = m$.

Roots of Quadratic Surds.

ART. 99.

EXAMPLE. Extract the square root of $9 + 4\sqrt{5}$.

Suppose $\sqrt{9 + 4\sqrt{5}} = \sqrt{x} + \sqrt{y}.$

Squaring, $9 + 4\sqrt{5} = x + y + 2\sqrt{xy}.$

By last article $x + y = 9$ . . . . (1)

and $2\sqrt{xy} = 4\sqrt{5}$ . . . . (2)

or $x + y = 9$ . . . . (1)

squaring (2) and dividing by 4

$xy = 20$ . . . . (3)

Equations (1) and (3) tell us that the sum of the two numbers x and y is 9, and their product 20.

Evidently the numbers are 4 and 5; say $x = 4$ and $y = 5$, as it makes no difference about the order of arrangement of the numbers ($y = 4$ and $x = 5$ would serve as well).

$$\therefore \quad \sqrt{9+4\sqrt{5}} = \sqrt{x} + \sqrt{y} = \sqrt{4} + \sqrt{5} = 2 + \sqrt{5}.$$

Verify, $(2 + \sqrt{5})^2 = 4 + 4\sqrt{5} + 5 = 9 + 4\sqrt{5}.$

Evidently a surd with a minus sign, as $4\sqrt{5} - 9$, would have the form $(\sqrt{x} - \sqrt{y})^2$.

ANOTHER METHOD.

ART. 100. Since the squares of the sum and of the difference of two surds differ only in the sign of the surd terms in these squares:

as, $$(\sqrt{x}+\sqrt{y})^2 = x + y + 2\sqrt{xy}$$
$$(\sqrt{x}-\sqrt{y})^2 = x + y - 2\sqrt{xy},$$

in the example just solved

if $$(\sqrt{x}+\sqrt{y})^2 = 9 + 4\sqrt{5},$$
then $$(\sqrt{x}-\sqrt{y})^2 = 9 - 4\sqrt{5},$$

or $$\sqrt{x}+\sqrt{y} = \sqrt{9+4\sqrt{5}}$$
$$\sqrt{x}-\sqrt{y} = \sqrt{9-4\sqrt{5}}.$$

Multiply, $$x - y = \sqrt{(9)^2 - (4\sqrt{5})^2} = \sqrt{1} = 1$$
as before $$x + y = 9.$$

Add, and $$2x = 10 \qquad x = 5$$
subtract, $$2y = 8 \qquad y = 4$$

$$\therefore \sqrt{9+4\sqrt{5}} = \sqrt{x}+\sqrt{y} = \sqrt{5}+\sqrt{4} = \sqrt{5}+2,$$
as before.

EXERCISE XV.

Simplify:

1. $\sqrt[4]{48x^5b^6c^8}$.
2. $\sqrt{a^3 - 2a^2b + ab^2}$.
3. $\sqrt{4a^3b - 8a^2b^2 + 4ab^3}$.
4. $\sqrt[3]{16a^5x^9}$.
5. $\sqrt[10]{(a^2 - 2ax + x^2)^3}$.

Express as complete surds:

6. $a^2b\sqrt{bc}$.
7. $5abc\sqrt{a^{-2}bc^{-1}}$.
8. $(x+y)\sqrt{\dfrac{xy}{x^2+2xy+y^2}}$
9. $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.

Perform indicated operations and simplify:

10. $\frac{2\sqrt{10}}{3\sqrt{27}} \times \frac{7\sqrt{48}}{3\sqrt{14}} \div \frac{4\sqrt{15}}{15\sqrt{21}}$.

11. $\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}}$.

12. $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}$.

13. $2\sqrt{3} + 3\sqrt{1\frac{1}{3}} - \sqrt{5\frac{1}{3}}$.

14. $\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}$.

Reduce to same order:

15. $3\sqrt[3]{2}$, $\frac{1}{2}\sqrt{6}$ and $2\sqrt[4]{\frac{1}{3}}$. 16. $5\sqrt{1.2}$, $\frac{3}{2}\sqrt[5]{2}$, $6\sqrt[3]{3}$.

17. $9\sqrt{3}$, $4\sqrt[3]{4}$, $5\sqrt[4]{3}$ (which is the largest?).

Rationalize denominators:

18. $\sqrt{1\frac{1}{16}}$, $\sqrt{3\frac{1}{8}}$, $\frac{1}{2}\sqrt[3]{\frac{a^2}{b}}$.

19. $\frac{8}{3\sqrt[3]{4}}$, $\frac{1}{2-\sqrt{3}}$, $\frac{1+\sqrt{2}}{2-\sqrt{2}}$.

20. $\frac{2-\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$, $\frac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$.

21. $\frac{8}{\sqrt{3}+\sqrt{2}}$, $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.

Extract the square root:

22. $12 - 2\sqrt{35}$.

23. $4 + 2\sqrt{3}$.

24. $8 - 4\sqrt{3}$.

25. $70 - 30\sqrt{5}$.

26. $18 + 8\sqrt{5}$.

IMAGINARIES.

ART. 101. Since the squares of both positive and negative quantities * are positive always, the square root of a negative quantity is something essentially different from the quantities heretofore considered.

Such roots of negative quantities are called imaginaries. They will arise occasionally in the solution of quadratic equations.

ART. 102. A *pure imaginary* is of the form

$$a\sqrt{-b} \text{ or } a\sqrt{b}\sqrt{-1}.$$

A *complex number* is of the form $a \pm \sqrt{-b}$. Define each.

Imaginaries are added and subtracted according to the usual rules for surds.

Multiplication of Imaginaries.

ART. 103. The application of the general laws of multiplication must be made with some care in the products of imaginaries.

For example: $(\sqrt{-a}) \times (\sqrt{-a})$ does *not* equal $\sqrt{a^2} = \pm a$ as the ordinary process would indicate, but $(\sqrt{-a}) \times (\sqrt{-a}) = -a$, which restricts the value of this product to one value instead of two ($+a$ and $-a$).

It is customary and simplest to reduce all imaginary terms to the form $\sqrt{a}\sqrt{-1}$ $(\sqrt{-a})$, wherein the factor $\sqrt{-1}$ always appears and the general rule, that the multiplying of a quadratic radical expression by itself removes the radical sign, may be applied without confusion.

* A pure number (or its representative, a letter) independent of sign is often called a scalar number or merely a scalar.

Hence:

$$(\sqrt{-a}) \times (\sqrt{-b}) = (\sqrt{a}\sqrt{-1}) \times (\sqrt{b}\sqrt{-1}) = \sqrt{ab}\,(-1) = -\sqrt{ab}.$$

$$(-\sqrt{-a}) \times (\sqrt{-b}) = (-\sqrt{a}\sqrt{-1}) \times (\sqrt{b}\sqrt{-1}) = -\sqrt{ab}\,(-1) = +\sqrt{ab}.$$

$$(+\sqrt{-a}) \times (-\sqrt{-b}) = +\sqrt{ab}.$$

$$(-\sqrt{-a}) \times (-\sqrt{ab}) = (-\sqrt{a}\sqrt{-1}) \times (-\sqrt{b}\sqrt{-1}) = (+\sqrt{ab})\,(-1) = -\sqrt{ab}.$$

ART. 104. The product or quotient of two complex quantities is in general a complex quantity. Verify this by examples.

Every complex quantity can be expressed in the form $a + b\sqrt{-1}$ evidently.

ART. 105. Two complex quantities both consisting of the same terms, but united by contrary signs, are called conjugate complex quantities; as,

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1},$$

or $$-x + y\sqrt{-1} \text{ and } -x - y\sqrt{-1}.$$

ART. 106. The product of two conjugate complex quantities is a real quantity. For example,

$$(5 + 2\sqrt{-1})\,(5 - 2\sqrt{-1}) = [5^2 - (2\sqrt{-1})^2] = 25 - (-4) = 25 + 4 = 29,$$

or $$(-m + n\sqrt{-1})\,(-m - n\sqrt{-1}) = [(-m)^2 - (n\sqrt{-1})^2] = m^2 - (-n^2) = m^2 + n^2, \text{ etc.}$$

ART. 107. Clearly the sum of two conjugate complex quantities is real and their difference is a pure imaginary.

EXAMPLE. $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) =$

$$a + b\sqrt{-1} + a - b\sqrt{-1} = 2a.$$

Also, $(a + b\sqrt{-1}) - (a - b\sqrt{-1})$

$$= a + b\sqrt{-1} - a + b\sqrt{-1} = 2b\sqrt{-1}.$$

ART. 108. By reference to the similar propositions under surds, it will be clear that:

If two complex quantities are equal, their real parts must be equal and their imaginary parts equal. Thus:

if $x + y\sqrt{-1} = m + n\sqrt{-1}, x = m$

and $y\sqrt{-1} = n\sqrt{-1}$ or $y = n$.

If a complex quantity equals zero, both real and imaginary parts are zero. Thus,

if $$a + b\sqrt{-1} = 0,$$

$$a = 0 \text{ and } b = 0.$$

EXERCISE XVI.

Multiply:

1. $4 + \sqrt{-3}$ by $4 - \sqrt{-3}$.
2. $\sqrt{3} - 2\sqrt{-2}$ by $\sqrt{3} + 2\sqrt{-2}$.
3. $5 + 2\sqrt{-8}$ by $3 - 5\sqrt{-2}$.

Divide and rationalize denominators:

4. $26 \div (3 + \sqrt{-4}.)$
5. $(3 + \sqrt{-1}) \div (4 + 3\sqrt{-1}.)$
6. $63\sqrt[3]{-16} \div \sqrt{-81}$.
7. $(a + \sqrt{-x}) \div (a - \sqrt{-x}.)$
8. $1 \div (3 - 2\sqrt{-3})$.

CHAPTER VII.

INDETERMINATE EQUATIONS.

ARTICLE 109. A system of equations containing a less number of equations than of unknown quantities, is called indeterminate. For instance, one equation containing two unknowns is an indeterminate; two equations containing three unknowns are indeterminate, etc.

ART. 110. As the name "indeterminate" signifies, such equations have no single solution.

For instance, an equation like $3x + 5y = 16$ may be satisfied by an infinite number of values of x and y, for we may give either one any value we please, and by substituting it in the equation, find the value of the other unknown that, with it, will satisfy the equation.

ART. 111. We may, however, limit the number of solutions, by confining the values that the unknowns may have, to integers, and still further limit them by specifying that the values of the unknowns must be positive integers.

The latter offers the only aspect of practical interest to these equations, because we deal in real experience only with positive quantities and largely with integers.

ART. 112. As an illustration of how these restrictions affect the number of solutions, consider the equation above, $3x + 5y = 16$. Without any restrictions $3x + 5y = 16$ may by satisfied by any of the following sets of values:

$x = 0$	$y = 3\frac{1}{5}$	$x = -1$	$y = 3\frac{4}{5}$
$x = 1$	$y = 2\frac{3}{5}$	$x = -2$	$y = 4\frac{2}{5}$
$x = 2$	$y = 2$	$x = -3$	$y = 5$
$x = 3$	$y = 1\frac{2}{5}$	$x = -4$	$y = 5\frac{3}{5}$
$x = 4$	$y = \frac{4}{5}$	$x = -5$	$y = 6\frac{1}{5}$
$x = 5$	$y = \frac{1}{5}$	$x = -6$	$y = 6\frac{4}{5}$
$x = 6$	$y = -\frac{2}{5}$		

and so on indefinitely.

If we restrict the values to positive quantities, all the negative values above will be excluded, if we restrict the values to positive integers, it removes all but one pair of values from above.

ART. 113. It is often desirable then to determine the possible solutions of an indeterminate equation in terms of positive integers:

It is plainly quite impracticable to make out a complete list of values, for they are infinite; but it is possible to arrive at the result in another way, indicated in the following solution.

EXAMPLE. Solve in positive integers $3x + 5y = 16$. Divide by the coefficient of one of the unknowns, preferably by the smaller, in this case 3.

It gives, $$x + y + \frac{2y}{3} = 5 + \frac{1}{3}.$$

Transpose, $$x + y - 5 = \frac{1}{3} - \frac{2y}{3} = \frac{1-2y}{3}.$$

Since x and y must be integers and 5 is an integer, $\frac{1-2y}{3}$ must be really an integer, although of fractional form, for one side of an equation cannot be integral and the other fractional, that is, y must have such a value that $\frac{1-2y}{3}$ will reduce to an integer.

Say, $$\frac{1-2y}{3} = m \text{ (an integer)},$$

then $$1 - 2y = 3m \text{ or } 2y = 1 - 3m$$

$$y = \frac{1-3m}{2}$$

which is still fractional in form.

To avoid this repetition of fractional form, we have recourse to a simple process, based on the truth that an integer multiplied by an integer will give an integral product.

If $\frac{1 - 2y}{3}$ is multiplied by an integer, that will make the coefficient of y one greater than some multiple of 3, the repetition of fractional form will be avoided.

Observe that this multiplication does not affect our result, for we are seeking any integral value not a particular one, for $\frac{1 - 2y}{3}$.

To be as simple as possible, the smallest number that will suffice for our purpose is chosen for a multiplier; in this case evidently 2, for $2 \times 2y = 4y$, and 4 leaves a remainder 1 when divided by 3.

Then $\left(\frac{1 - 2y}{3}\right) \times 2 = \frac{2 - 4y}{3} = -y + \frac{2 - y}{3}$. Since y is an integer we need only consider the *apparent* fraction $\frac{2 - y}{3}$, which we will equate to some integer, say n; then $\frac{2 - y}{3} = n$, $2 - y = 3n$, $y = 2 - 3n$, which is cleared of fractions.

Substituting this value of y in terms of the integer n (whose actual value we do not yet know) in the original equation,

$$3x + 5(2 - 3n) = 16$$
$$3x + 10 - 15n = 16$$
$$3x = 6 + 15n$$
$$x = 2 + 5n.$$

Now we have two condition equations to limit the value of n.

$$x = 2 + 5n \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and
$$y = 2 - 3n \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Remembering that x and y must be positive, whole quantities, we can set limits for n.

In (1), plainly n may be any positive number from 1 to ∞, but no negative number, because if $n = -1, x = -3$, which violates our condition. Clearly any other negative number will make x negative.

In (2), n can be any negative number from -1 to $-\infty$, but no positive number, for a like reason.

Therefore n, to satisfy both (1) and (2), can be neither a positive nor negative number; it can then be only 0. Hence, if $n = 0$, $x = 2$ (from (1)) and $y = 2$ (from (2)), showing that $x = 2$ and $y = 2$ is the only solution for $3x + 5y = 16$, if limited to positive integers.

Take another example.

Solve in positive integers $8x + 5y = 74$.

Divide by 5, $$x + y + \frac{3x}{5} = 14 + \frac{4}{5}.$$

Transpose, $$x + y - 14 = \frac{4}{5} - \frac{3x}{5} = \frac{4 - 3x}{5}$$

$$\left(\frac{4 - 3x}{5}\right)2 = \frac{8 - 6x}{5} = 1 - x + \frac{3 - x}{5}$$

$$\frac{3 - x}{5} = n \text{ (an integer)}, \; 3 - x = 5n, \; x = 3 - 5n.$$

Substituting $x = 3 - 5n$, in $8x + 5y = 74$.

$$8(3 - 5n) + 5y = 74$$
$$24 - 40n + 5y = 74$$
$$5y = 50 + 40n$$
$$y = 10 + 8n \quad \ldots\ldots \quad (1)$$
$$x = 3 - 5n \quad \ldots\ldots \quad (2)$$

From (1), n can be -1, 0, $+1$, $+2$, etc., anything greater than -2, that is $n > -2$.

From (2), n can be 0 or any negative number, but no positive number. Hence, from both (1) and (2) n can be -1 or 0.

If $n = -1, x = 8, y = 2.$

If $n = 0, x = 3, y = 10.$

That is, there are two solutions in this case.

ART. 114. It is sometimes desirable to know numbers with predetermined remainders when divided by given numbers. Suppose, for instance, it is required to find the least number which when divided by 3, 5, and 6 leaves respectively the remainders 1, 3, and 4.

Let x be the required number.

Then $\frac{x-1}{3}$ = some integer, because by the conditions of the problem, if 1 be subtracted from x, 3 will exactly divide it.

Also $$\frac{x-3}{5} = \text{an integer} \quad \ldots \quad (2)$$

and $$\frac{x-4}{6} = \text{an integer} \quad \ldots \quad (3)$$

say, $$\frac{x-1}{3} = m \text{ (an integer)}$$

$$x = 3m + 1 \quad \ldots \quad (1)$$

Substituting this value of x in (2), $\frac{3m+1-3}{5}$ = an integer, that is, $\frac{3m-2}{5}$ = an integer.

Multiply $3m - 2$ by 2 (to make coefficient of m one greater than a multiple of 5, as explained in last article).

$$\frac{3m-2}{5} \times 2 = \frac{6m-4}{5} = m + \frac{m-4}{5} = \text{an integer.}$$

Since m is an integer by supposition, $\frac{m-4}{5}$ must be an integer, say $\frac{m-4}{5} = n$ (an integer).

$$m - 4 = 5n.$$
$$m = 5n + 4.$$

Therefore, from (1),

$$x = 3(5n + 4) + 1 = 15n + 13 \quad . \quad . \quad (4)$$

Substituting this value of x in (3)

$$\frac{15n + 13 - 4}{6} = \frac{15n + 9}{6} = \frac{5n + 3}{2} = 2n + 1 + \frac{n + 1}{2}$$
$$= \text{an integer.}$$

Since $2n + 1$ is integral, $\frac{n+1}{2}$ must be an integer; say $\frac{n+1}{2} = s$ (an integer); $n = 2s - 1$.

From (4), $x = 15(2s - 1) + 13 = 30s - 2 \quad . \quad . \quad (5)$

From (5), the least value s can have, that will give x a positive value, is 1 (s cannot be 0, why?).

If s is 1 in (5), $x = 28$, the required number.

Verification, $\frac{28}{3} = 9 +$ remainder 1

$\frac{28}{5} = 5 +$ " 3

$\frac{28}{6} = 4 +$ " 4.

The principle applied is this; that if any number of equations (each one expressing one condition for the unknown quantity) be combined, the resulting combination equation contains all the conditions expressed by the component equations.

Thus when the value of x in (1), expressing the condition that x is divisible by 3 with a remainder 1, is substituted in (2), the resulting value of x in (4) contains also the condition expressed by (2), that is, the x in (4) is not only divisible by 3 with remainder 1, but also divisible by 5 with remainder 3.

When this value of x from (4) is substituted in (3), the resulting value of x in (5) contains all three conditions, hence, by giving s any integral value we please, a value of x will result from (5) that will fulfill the three requirements of the problem. Since we want the *least* value of x that fulfills these requirements, we choose the least value of s, which is 1.

EXERCISE XVII.

Indeterminate Equations.

Solve in positive integers:

1. $2x + 11y = 83.$

2. $\frac{3}{4}x + 5y = 92.$

3. $\frac{2}{5}x + \frac{3}{4}y = 53.$

4. $2x + 3y = 25.$

5. $12x + 13y = 175.$

6. $\begin{cases} x + 3y + 5z = 44. \\ 3x + 5y + 7z = 68. \end{cases}$

7. Divide 89 into two parts, one of which is divisible by 3 and the other by 8.

8. What is the smallest number which gives a remainder 4, when divided by 5 or 7?

9. In how many ways can 300 lbs. be weighed with only 7 and 9 lbs. weights?

10. Divide $\frac{79}{117}$ into two parts, having respectively denominators 13 and 9.

11. A wheel with 17 teeth meshes with a wheel having 13 teeth. After how many revolutions of each wheel will each tooth occupy its original position?

12. How many times each must a 7-inch rule and a 13-inch rule be applied to measure 4 feet, using both at the same time?

CHAPTER VIII.

QUADRATIC EQUATIONS.

ARTICLE 115. An equation containing the second and no higher power of a quantity, is said to be a quadratic equation in that quantity. Write three quadratic equations.

Quadratics in Single Unknown.

ART. 116. When the quantity whose square is involved is a single variable (like x or y) and no other variable enters the equation, it is a simple quadratic of one unknown quantity.

The general form is $a z^2 + b z + c = 0$, where a, b, and c are constants.

Kinds of Simple Quadratics.

ART. 117. If b is zero in $a z^2 + b z + c$, the resulting equation, $a z^2 + c = 0$, is called an incomplete or pure quadratic; the equation $a z^2 + bz + c$ is called a complete or affected quadratic.

Roots of a Quadratic.

ART. 118. If in the equation $a z^2 + b z + c = 0$, we substitute for z, the value

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac},}{2a}$$

thus

$$a\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) + c = 0,$$

we get $-c + c = 0$. Verify.

These two values of z, which when substituted for z make the two sides of an equation identical (or satisfy the equation, as it is said), are called roots of z for this equation. Formulate a general definition for the roots of an equation.

Solution of an Incomplete Quadratic.

ART. 119. The equation $ay^2 + c = 0$ can be put in the form $y^2 = -\frac{c}{a}$ by transposing c and dividing by a. What advantage for solution arises from this operation? Since the solution of the equation is the finding of the value or values of y that will satisfy it, that is, its roots, how would you complete the solution? Formulate rule.

Solution of Complete Quadratics.

ART. 120. What would above solution suggest as to first steps in the solution of the equation

$$ax^2 + bx + c = 0?$$

By Binomial Theorem $(x+m)^2 = x^2 + 2mx + m^2 = n^2$ say, which is a general form of a quadratic equation in x. Observe the relation between the third term m^2 and the coefficient of x. Suppose this expression to be put in the form of an equation, thus:

$$x^2 + 2mx = -m^2 + n^2.$$

How would you restore the form of a perfect square to left hand member without altering the truth of the equation? Complete the solution and formulate a rule.

Geometrical Illustration.

ART. 121.

$$AB = AC - BC = n - m \text{ (see Fig. 4.)}$$
$$n^2 = CEFA = BGHA + BCDG + GKFH$$
$$+ DEKG = x^2 + mx + mx + m^2$$
$$= x^2 + 2mx + m^2.$$

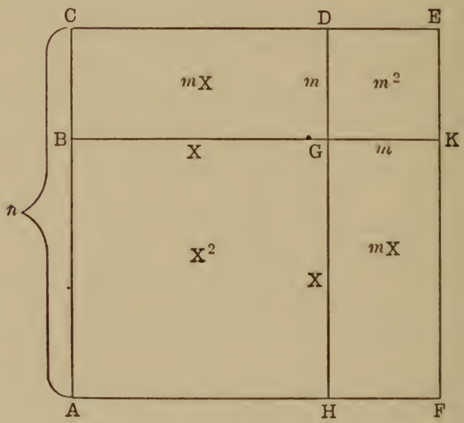

Fig. 4.

LITERAL AND NUMERICAL EQUATIONS.

ART. 122. Equations such as we have considered, involving letters as coefficients, are called literal equations; if the coefficients are numbers, they are said to be numerical equations. Write three numerical quadratic equations.

Solution.

ART. 123. What are the essential differences between the use of letters and the use of numbers in solution?

Observe carefully the following steps; by Binomial Theorem,

$$(x-5)^2 = x^2 - 10x + 25 = 49, \text{ say}, \quad . \quad (a)$$

transposing,

$$x^2 - 10x = 24 \text{ or } x^2 - 10x - 24 = 0, \quad . \quad (b)$$

compare $$ax^2 + bx + c = 0. \quad . \quad (c)$$

An examination of (a) shows that a simple extraction of the square root as in case of pure quadratic, will give the values of x, thus:

$$x - 5 = \pm 7, \quad \text{whence} \quad x = 12 \text{ or } -2.$$

Plainly then, to solve a quadratic like (b) or (c), we must put it in (a) form.

What must be added to $x^2 - 10x$ in (b) to restore form $(x-5)^2$? What relation does this added quantity bear to coefficient of x?

Again, $$(x + \tfrac{2}{3})^2 = x^2 + \tfrac{4}{3}x + \tfrac{4}{9} = 1,$$

$$x^2 + \tfrac{4}{3}x = \tfrac{5}{9}, \text{ transposing.}$$

Complete solution.

Solution by Factoring.

ART. 124. Solve $x^2 - 7x + 10 = 0$

$$x^2 - 7x = -10 \qquad \text{[transposing]}$$

$$x^2 - 7x + (\tfrac{7}{2})^2 = (\tfrac{7}{2})^2 - 10 = \tfrac{9}{4} \quad \text{[completing square]}$$

$$x - \tfrac{7}{2} = \pm \tfrac{3}{2}$$

whence $$x = \tfrac{7}{2} + \tfrac{3}{2} = 5$$

or $$x = \tfrac{7}{2} - \tfrac{3}{2} = 2.$$

But $x^2 - 7x + 10$ may be resolved into the factors $(x-2)$ and $(x-5)$; hence $(x-2)(x-5) = 0$.

By inspection, it is plain that if either 2 or 5 is substituted for x, the equation is satisfied; for $(2 - 2)(2 - 5) = (0) \times (-3) = 0$ (since any finite quantity multiplied by 0 equals 0), and $(5 - 2)(5 - 5) = (3) \times (0) = 0$.

Hence, the roots of the equation are 2 and 5, as found above. This is known as solution by factoring and is of advantage when the equation is easily factorable. Factoring may be accomplished in any case as follows: take same equation, $x^2 - 7x + 10 = 0$; to complete the square, $\frac{49}{4}$ must be added to $x^2 - 7x$; write then the equation thus, (*a*) $x^2 - 7x + \frac{49}{4} - \frac{9}{4} = 0$, which does not alter the value, merely the form of the equation.

The last equation can also be written $(x - \frac{7}{2})^2 - (\frac{3}{2})^2 = 0$, which is the difference of two squares, hence factorable into the product of the sum and difference of the square roots of these terms, *i.e.* $(x - \frac{7}{2} + \frac{3}{2})(x - \frac{7}{2} - \frac{3}{2}) = 0$, or $(x - 2)(x - 5) = 0$, as before.

To get a general result, let us take $ax^2 + bx + c = 0$.

Whence $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

To make complete square of $x^2 + \frac{b}{a}x$, we must add

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}, \text{ hence } x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$$

or $$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0,$$ a difference of two squares hence equals product of two factors.

$$\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Solution by Substitution.

ART. 125. Since in the above solution letters alone were involved, they may evidently stand for any numbers we please.

Let us compare the two equations just used,

$$x^2 - 7x + 10 = 0 \quad \text{and} \quad ax^2 + bx + c = 0;$$

we may if we choose, say that $a = 1$, $b = -7$, and $c = 10$; $ax^2 + bx + c = 0$ then becomes $x^2 - 7x + 10 = 0$.

The two values of x found above, if $a = 1$, $b = -7$, and $c = 10$, will then plainly become the values of x for $x^2 - 7x + 10 = 0$; *i.e.*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ becomes } x = \frac{7 \pm \sqrt{49 - 40}}{2}$$

$$= \frac{7 \pm 3}{2} = 5, \text{ or } 2 \text{ as before.}$$

Why is $ax^2 + bx + c = 0$ called the general form of the quadratic equation?

EQUATIONS INVOLVING RADICALS AND REDUCIBLE TO QUADRATICS.

ART. 126. The process of removing the radical expressions from an equation by squaring, not infrequently introduces extraneous roots; it is therefore necessary to verify the results carefully in each case, thus:

$$\sqrt{x+3} + \sqrt{4x+1} = \sqrt{10x+4} \quad . \quad . \quad . \quad (a)$$

whence squaring,

$$x + 3 + 2\sqrt{(x+3)(4x+1)} + 4x + 1 = 10x + 4.$$

$$2\sqrt{(x+3)(4x+1)} = 5x \qquad \text{[collecting]}$$

$$16x^2 + 52x + 12 = 25x^2 \qquad \text{[squaring again]}$$

$$x^2 - \frac{52x}{9} = \frac{12}{9} = \tfrac{4}{3} \; (b), \qquad \text{[collecting and dividing by 9]}$$

whence $x = 6$ or $-\frac{2}{9}$. Complete the solution of (b).

Substitute 6 in (a), we get $3 + 5 = 8$, hence 6 is a root of (a).

But if we substitute $-\frac{2}{9}$ in (a) we get $\frac{5}{3} + \frac{1}{3} = \frac{4}{3}$, which is false. It is plain, however, that if the second radical is negative, the value $-\frac{2}{9}$ would satisfy, for

$$\frac{5}{3} - \frac{1}{3} = \frac{4}{3}.$$

Have we any right to use negative sign with $\sqrt{4x + 1}$? Why? $-\frac{2}{9}$ is then a root of

$$\sqrt{x + 3} - \sqrt{4x + 1} = \sqrt{10x + 4} \quad . \quad . \quad (c);$$

squaring (c)

$$x + 3 - 2\sqrt{(x + 3)(4x + 1)} + 4x + 1 = 10x + 4.$$

Whence, $\qquad -2\sqrt{(x + 3)(4x + 1)} = 5x.$

Squaring, $16x^2 + 52x + 12 = 25x^2$ as before. Why? Hence both (a) and (c) lead to same quadratic whose roots are 6 and $-\frac{2}{9}$.

Emphasis then must be laid on the examination of all roots, where we are required to square terms of the equation.

EQUATIONS OF HIGHER DEGREE SOLUBLE AS QUADRATICS.

ART. 127. Recalling the definition of a quadratic equation, it will be observed that the term *quantity* means not necessarily a simple letter, nor even a monomial.

We may extend the definition thus: *Any equation involving only such expressions in the variable as may be collected into two exactly similar groups, whose exponents shall have the ratio of* 2 *to* 1, *may be solved or at least partially solved as a quadratic.*

Solution.

ART. 128.

$$x^{\frac{1}{4}} + x^{\frac{1}{2}} - 20 = 0.$$

Let $x^{\frac{1}{4}} = y$, and hence expressed in quadratic form

$y + y^2 - 20 = 0$ Complete solution.

Again, $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11$

may be written, $(x^2 - 2x + 5)^1 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16.$

Since the exponents of the similar groups $(x^2 - 2x + 5)^1$ and $(x^2 - 2x + 5)^{\frac{1}{2}}$ have ratio of 2 to 1,

let $(x^2 - 2x + 5)^{\frac{1}{2}} = y.$

Then the equation becomes,

$$y^2 + 6y = 16,$$

whence, $y = 2$ or -8; *i.e.*, $(x^2 - 2x + 5)^{\frac{1}{2}} = 2$ or -8,

whence, $x^2 - 2x + 5 = 4$ or 64;

$$x^2 - 2x = -1 \text{ or } 59,$$

$$x = 1, 1, (1 + 2\sqrt{15}) \text{ or } (1 - 2\sqrt{15}).$$

Again, $4x^4 - 12x^3 + 5x^2 + 6x - 15 = 0,$

arranged, $4x^4 - 12x^3 + 9x^2 - 4x^2 + 6x = 15,$

or $(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15,$

plainly of quadratic form. Complete solution.

The grouping and arrangement of such equations is a pure matter of judgment and ingenuity and can be subjected to no general rules.

Solution of Higher Equations by Factoring.

ART. 129. Any equation of the form $x^n + bx^{n-1} + cx^{n-2} + \ldots = 0$ whose left hand member can be resolved into factors of degree not higher than the second may be solved completely by methods already known.

ART. 130. Solution.

$$x^3 - 6x^2 + 11x - 6 = 0, \text{ may be written}$$
$$x^3 - 6x^2 + 12x - 8 - x + 2 = 0,$$
or
$$x^3 - 6x^2 + 12x - 8 - (x - 2) = 0,$$
or
$$(x - 2)^3 - (x - 2) = 0,$$

which may be factored thus:

$$(x - 2)[(x - 2)^2 - 1] = 0, \text{ or } (x - 2)[(x - 2) - 1][(x - 2) + 1] = 0, \text{ or } (x - 2)(x - 3)(x - 1) = 0.$$

Plainly, if $x = 2$ or 3 or 1, the equation is satisfied. These values arise from setting the factors successively equal to 0.

Again, $(x - 1)(x - 2)(x^2 - 4x - 5) = 0.$

Clearly any value of x that will reduce any one factor to zero will satisfy the equation, provided it does not make another factor infinite. These values will result from the solution of the three equations

$$x - 1 = 0$$
$$x - 2 = 0$$
$$x^2 - 4x - 5 = 0. \quad \text{(Why?)}$$

$x - 1 = 0$ gives $x = 1$, $x - 2 = 0$ gives $x = 2$, and $x^2 - 4x - 5 = 0$ gives $x = +5$ and $x = -1$, hence $1, 2, -1, +5$ substituted for x in the original equation identically satisfy it.

CHARACTER OF ROOTS.

ART. 131. Since the solution of an equation with literal coefficients gives general results, we may derive useful information from a study of the roots of the equation

$$ax^2 + bx + c = 0, \text{ i.e., } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If we examine these roots carefully, we see that the difference in their values arises from the addition of $\frac{\sqrt{b^2 - 4ac}}{2a}$ in one case and its subtraction in the other from the same quantity $\frac{-b}{2a}$. Put this statement into a rule, remembering what a and b are in the equation.

If, then, $\sqrt{b^2 - 4ac}$ is 0, the roots will both be the same, hence this is the condition for equal roots, which has important applications in other branches of mathematics. Put this condition into a rule.

Again, $\sqrt{b^2 - 4ac}$ determines whether the roots shall be rational, real, or imaginary.

What condition must $b^2 - 4ac$ fulfill in each case?

ILLUSTRATION.

$5x^2 + 6x = 8$ can be written $5x^2 + 6x - 8 = 0$.

Compare,
$$5x^2 + 6x - 8 = 0$$
$$ax^2 + bx + c = 0.$$

If $a = 5$, $b = 6$, $c = -8$, the results of the solution for $ax^2 + bx + c = 0$, will be those for $5x^2 + 6x - 8 = 0$. Why?

Then,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

becomes,

$$\frac{-6 + \sqrt{36 + 160}}{10} \text{ or } \frac{-6 - \sqrt{36 + 160}}{10}; \text{ i.e., } \frac{4}{5} \text{ or } -2.$$

Now $b^2 - 4ac = 196 = 14^2$ is a perfect square, hence the rational roots $\frac{4}{5}$ and -2.

ILLUSTRATION.

In $3y^2 - 8y + 10 = 0$, $a = 3, b = -8, c = 10$. Here $b^2 - 4ac = (-8)^2 - (4 \times 3 \times 10) = 64 - 120 = -56$.

Hence roots are

$$\frac{8 + 2\sqrt{-14}}{6} \text{ and } \frac{8 - 2\sqrt{-14}}{6}.$$

Which condition prevails here?

ILLUSTRATION.

$$6 + \frac{5t}{2} = 6t^2. \text{ Here } a = 6,\ b = -\frac{5}{2},\ c = -6.$$

Hence

$$b^2 - 4ac = \left(-\frac{5}{2}\right)^2 - (4 \times 6 \times -6) = \frac{25}{4} + 144 = \frac{601}{4}.$$

Write roots and state condition.

EXERCISE XVIII.

Quadratics.

Verify results in every case.

1. $2x^2 - 27 = 9x - x^2 + 3$.
2. $y^2 - 5y - 24 = 0$.
3. $5t^2 - 3 = 10t - 3t^2$.
4. $s = \frac{1}{2}gt^2$ (solve for t).
5. $\frac{g}{2l} = \frac{\pi^2}{T^2}$ (Pendulum formula).
6. $s = vt + \frac{1}{2}at^2$.
7. $5x^2 = 8x$.
8. Solve $x^2 + 6x = 0$ by general rule, and then show how this equation can be solved by shorter method.
9. Prove from a solution of the general equation $ax^2 + bx + c = 0$, that if $c = 0$ one root is 0, and hence derive a rule relative to the absolute term.
10. $2y^2 - 5y = 3y + 234$.

11. $3y^2 - 8y = 2y(y-4) + 9$.

12. What must be added to $P^2 - 5P$ to make the expression a perfect square?

13. Make $9a^2x^4 + 12ax^2$ a perfect square.

14. $3x^2 + 5x - 4 = x^2 - 2x + 3$.

15. $6 + 5t = 6t^2$.

16. $x^2 + 2 = \dfrac{(x-1)^3 - x + 24}{x+2}$.

17. $12x^2 - cx - 20c^2 = 0$.

18. $\dfrac{2x+5}{3x-2} - \dfrac{2x+7}{3x-4} - \dfrac{3}{4} = 0$.

19. $8x - x^2 - 12 = 0$.

20. $(2y-3)^2 = 6(y+1) - 5$.

21. $\dfrac{1}{2}x - \dfrac{3x^2}{4} + 2 = 0$.

22. $15y^2 - 7y - 2 = 0$.

23. $x^2 + (m+n)x + mn = 0$.

24. $\dfrac{3}{a} + 2 = \dfrac{m}{n}t - \dfrac{am}{2n}$.

25. $3u^2 - 4u - 10 = 0$.

EXERCISE XIX.

Solve following examples by the process that seems most expeditious to you.

1. $6x^2 - 7x + \frac{5}{3} = 0$.

2. $x^2 - 21x + 104 = 0$.

3. $(x-5)^2 + x^2 = 16(x+3)$.

4. $5x^2 - \dfrac{28}{3}x + 4 = 0$.

5. $x^2 - 13x = -42$.

6. $\dfrac{x+1}{17} + (x+1)(x+2) = 0$.

7. $x + \dfrac{x+6}{x-6} = 2(x-2)$.

8. $x^2 - (a+b)\,x = -ab.$ **9.** $\frac{3}{4\,x^2} - \frac{1}{6\,x^2} = \frac{7}{3}.$

10. $x^2 + bx + a = bx\,(1 - bx).$

11. $x\,(10 + x) = -21.$ **12.** $x^2 = x.$

13 $\frac{1}{x^2} - \frac{1}{x} = 6.$ **14.** $x^2 + 9\,x = 8.5.$

15. $4.05\,x^2 - 7.2\,x = 1476.$

16. $\frac{6 + 5\,x}{4\,(5 - x)} - \frac{3\,x - 4}{5\,(5 + x)} + \frac{5 - 7\,x}{25 - x^2} - \frac{89}{105} = 0.$

17. $25\,x = 6\,x^2 + 21.$ **18.** $x + 22 - 6\,x^2 = 0.$

19. $2\,x^2 - ax + 2\,bx = ab.$ **20.** $2\,x^2 - 7\,x + 3 = 0.$

21. $3x^2 + 3 = \frac{2\,x^2}{3} + 24.$ **22.** $\frac{x}{x+3} + \frac{2}{x+6} = \frac{13}{20}.$

23. $\frac{x+5}{x+2} + 1 = 3\,x.$ **24.** $\frac{1}{x} - 2 + x = \frac{2}{x}.$

25. $b\,(a - x)^2 = (b - 1)\,x^2.$

Verify results in each case.

26. The number of square inches in the area of a square exceeds the number of inches in its perimeter by 32. What is the area?

27. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way it would take 128 tiles. Find size of tile.

28. A wheel driving a drum makes 15 less revolutions than the drum in rolling up 440 feet of rope. If they were each 2 feet more in circumference, the wheel would make 10 less revolutions. Find circumference of each.

29. A lever, cut from a bar weighing 4.2 pounds per foot, balances at a point 2.3 feet from one end if 54 pounds is suspended from that end. Find length of bar.

30. When a lever *AB* is supported at its *C.G.* it is found that a weight *W* at *A* will balance 2.5 pounds at *B*; but *W* at *B* requires 19 pounds at *A* to balance it. Find *W*.

EXERCISE XX.

1. $x^3 - x^{\frac{3}{2}} = 56.$
2. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756.$
3. $x + 16 - 7\sqrt{x + 16} = 10 - 4\sqrt{x + 16}.$
4. $\sqrt{x + 12} + \sqrt[4]{x + 12} = 6.$
5. $x^6 + 7x^3 = 8.$
6. $3\sqrt{x} - 3x^{-\frac{1}{2}} = 8.$
7. $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0.$
8. $2x^2 + 3x + 1 = \dfrac{30}{2x^2 + 3x}.$
9. $3x(3 - x) = 11 - 4\sqrt{x^2 - 3x + 5}.$
10. $\left(x + \dfrac{1}{x}\right)^2 - 4\left(x + \dfrac{1}{x}\right) = 5.$
11. $\dfrac{x^2 - 3}{x} + \dfrac{3x}{x^2 - 3} = \dfrac{13}{2}.$
12. $y^4 + 2y^3 + 5y^2 + 4y = 60.$
13. $z^2 - 5z + 2\sqrt{z^2 - 5z - 2} = 10.$
14. $u^4 + 2u^3 - 3u^2 - 4u - 96 = 0.$

REMARK. It is evident that the number of equations that can be solved thus is very limited; the general solution of third and fourth degree equations cannot be considered here.

Determine all roots of the following equations:

1. $x + \sqrt{x} = 4x - 4\sqrt{x}.$
2. $\sqrt{4y + 17} + \sqrt{y + 1} - 4 = 0.$
3. $\sqrt{x + 1} + (x + 1)^{-\frac{1}{2}} = 2.$
4. $\sqrt{4x + 1} - \sqrt{x + 3} = \sqrt{x - 2}.$

5. $\sqrt{x-2} + \sqrt{3+x} - \sqrt{19+x} = 0.$

6. $(1+2x)^{\frac{1}{2}} - \sqrt{4+x} + \sqrt{3-x} = 0.$

7. $(4x-2)^{\frac{1}{2}} + 2\sqrt{2-x} - \sqrt{14-4x} = 0.$

8. $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = 2.$

9. $\sqrt{x+3} - \sqrt{x+8} = 5\sqrt{x}.$

10. $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 20 = 0$ (discuss roots thoroughly).

11. $\sqrt{z+5} + \sqrt{3z+4} = \sqrt{12z+1}.$

12. $\sqrt{2x+9} + \sqrt{49-x} = \sqrt{x+16}.$

MAXIMA AND MINIMA.

ART. 132. It is often desirable to know how large or how small an expression may be made by altering the unknown quantity involved in it within rational limits.

It might for instance be required to find the largest rectangular beam that could be cut from a cylindrical log of known diameter, or how to divide up a line so that its parts would inclose the largest area when made sides of a figure of a certain general form. Say, for example, it were required to find the largest value the expression $5 + 24x - 9x^2$ can have, if x varies within real limits.

Say the value of this expression is m, where m has a changing value, of course, as x changes.

Then, $$5 + 24x - 9x^2 = m.$$

Now since the value of x depends upon the value of m, because we want to find what x is, when m is the largest possible, it is plainly desirable to solve the above equation for x, so we can see in its simplest form just the kind of dependence that x has upon m.

Transposing, then changing the sign, and completing the square:

$$9x^2 - 24x = 5 - m$$

$$x^2 - \tfrac{8}{3}x = \frac{5 - m}{9}$$

$$x^2 - \tfrac{8}{3}x + \frac{16}{9} = \frac{21 - m}{9}$$

$$x - \tfrac{4}{3} = \pm \frac{\sqrt{21 - m}}{3}$$

whence,
$$x = \frac{4 \pm \sqrt{21 - m}}{3}.$$

By an inspection of this value of x, it can be readily seen that if m has a value greater than 21, the expression under the radical $(21 - m)$ will be a negative quantity. For example,

if $m = 22$, then $21 - m = -1$ and $x = \dfrac{4 \pm \sqrt{-1}}{3}.$

an imaginary value. As only real values of x can be considered, clearly any value of m that is greater than 21 is impossible. 21 then is the largest value m can have and is called the maximum value of m. If m is 21 (its largest value), then the radical $21 - m = 0$ and $x = \frac{4}{3}$. That is, the value $\frac{4}{3}$ for x makes the expression $5 + 24x - 9x^2$ as large as it can be. m can evidently be anything less than 21, and hence the expression $5 + 24x - 9x^2$ has no minimum value. Again, let it be required to divide any number, say a, into two such parts, that their product shall be a maximum.

Let $x =$ one part.

Then, $a - x =$ the other

and $ax - x^2$ is to be a maximum.

Let

$$ax - x^2 = m$$

$$x^2 - ax + \frac{a^2}{4} = \frac{a^2 - 4m}{4}$$

$$x - \frac{a}{2} = \pm \frac{\sqrt{a^2 - 4m}}{2},$$

$$x = \frac{a \pm \sqrt{a^2 - 4m}}{2}.$$

$4m$ cannot be greater than a^2, or x will be imaginary, that is, m cannot be greater than $\frac{a^2}{4}$, and for this maximum value of m, $x = \frac{a}{2}$, hence the product of the parts is greatest when they are equal.

EXERCISE XXI.

Find maximum or minimum values of following expressions:

1. $x^2 - 6x + 13$.

2. $3 + 12x - 9x^2$.

3. $\frac{x - 6}{x^2}$.

4. $\frac{4x}{(x + 2)^2}$.

5. $\frac{x^2 - x - 1}{x^2 - x + 1}$.

6. $\frac{1}{2 + x} - \frac{1}{2 - x}$.

7. $\frac{x^2 + 3x + 5}{x^2 + 1}$.

8. $12 + x^2 - 2ax$.

9. $x^2 - 10x + 35$.

10. Find greatest rectangle that can be inscribed in circle of radius 10 inches.

11. Divide a line 12 inches long into two parts such that their product shall be a maximum.

12. Find length of the sides of the largest rectangle having perimeter 16.

13. If you were three miles from shore in a boat, and could row four miles per hour and walk five miles per hour, and wanted to reach a point on the beach five miles down in the shortest time, where would you land?

14. Find maximum value of x for real values of r in equation $1 = xrd + x^2 - r^2$.

15. $2x^2 - 3yx - 5y^2 + 18 = 0$, find minimum value of y, if y is always positive.

EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES OF A DEGREE HIGHER THAN THE FIRST.

ART. 133. Equations involving more than one unknown quantity and of certain forms can be readily solved by special methods, usually reducible to the quadratic solution.

Such equations may be classified; first as *homogeneous* equations of the second degree involving two unknowns, *i.e.*, two simultaneous equations such that the terms containing the unknowns are all of the second degree in both equations.

Thus,
$$x^2 + 2xy - y^2 = 28 \quad . \quad . \quad . \quad . \quad (a)$$
$$3x^2 + 2xy + 2y^2 = 72 \quad . \quad . \quad . \quad . \quad (b)$$

All such equations may be solved by substituting $y = mx$ or $x = ny$.

SOLUTION. Then (a) becomes $x^2 + 2mx^2 - m^2x^2 = 28$ (sub. $y = mx$) and (b) becomes $3x^2 + 2mx^2 + 2m^2x^2 = 72$.

Whence, from (a)
$$x^2 = \frac{28}{1 + 2m - m^2}$$

and from (b)
$$x^2 = \frac{72}{3 + 2m + 2m^2}.$$

$$\therefore \frac{28}{1 + 2m - m^2} = \frac{72}{3 + 2m + 2m^2} \text{ or } m^2 - \frac{22m}{32} = -\frac{3}{32}.$$

Whence, $m = \frac{1}{2}$ or $\frac{3}{16}$

and $\therefore x^2 = \dfrac{28}{1 + 1 - \frac{1}{4}} = 16,$

$x = \pm 4 \quad y = \pm 2$, etc.

In special cases briefer methods may be employed which depend entirely upon the ingenuity of the solver. As in same equations;

add the equations; $4x^2 + 4xy + y^2 = 100,$

whence, extracting the square root, $2x + y = \pm 10,$

whence, $y = 10 - 2x$ or $-10 - 2x$.

Substitute first value in (a), and a quadratic in x results. Finish solution.

ART. 134. When one equation is linear (of first degree), the method of substitution is generally most effective, as indicated at the conclusion of last article.

SOLUTION.

$$\begin{cases} 2s + 3t = 10, \text{ whence } s = \dfrac{10 - 3t}{2} \\ t(s + t) = 25 \end{cases}$$

$\therefore t\left(\dfrac{10 - 3t}{2} + t\right) = 25$, (substituting in 2d equation)

$$-t^2 + 10t = 50$$

$$t^2 - 10t = -50.$$

Whence, $t = 5(1 \pm \sqrt{-1})$

and $s = 5\dfrac{(-1 \mp 3\sqrt{-1})}{2}$. Verify.

The exercise of a little judgment and ingenuity will often simplify the solution of problems of this kind also. It may be said as a general remark, that there is a large field in algebra for the application of legitimate artifice to shorten labor of calculation. For instance,

$$x - 3y + 9 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$xy - y^2 + 4 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Transposing and squaring in (*a*) $x^2 - 6xy + 9y^2 = 81$

Multiply (*b*) by 8 and add $8xy - 8y^2 = -32$

$$x^2 + 2xy + y^2 = 49$$

Extract square root $x + y = \pm 7 \quad . \quad . \quad (c)$

Subtract (*a*) from (*c*) $x - 3y = -9$

$$4y = 16 \text{ or } 2, \text{ etc.}$$

Lose no opportunity to apply such methods, but remember they are worth while only when quickly observed.

ART. 135. When the equations are both *symmetrical*, they may often be readily solved by substituting

$$x = u + v.$$
$$y = u - v.$$

Equations are symmetrical when the unknown quantities may be interchanged without affecting the equation, as,

$x^4 + y^4 = 706 \; (a)$ and $x^2 + 3xy + y^2 = 125.$

$x + y = 2 \; (b)$ and $x^5 + x^4y + xy^4 + y^5 = 1020.$

Making above substitution in (*a*) and (*b*),

$$(u + v)^4 + (u - v)^4 = 706$$

or $2u^4 + 12u^2v^2 + 2v^4 = 706.$

Whence, $u^4 + 6u^2v^2 + v^4 = 353 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$

and $u + v + u - v = 2.$ Substituting in (*b*)

Whence, $u = 1.$

Substituting $u = 1$ in (*a*), $1 + 6v^2 + v^4 = 353.$

Whence, $v = \pm 4$ or $\pm\sqrt{-22}$.

Whence, $x = u + v = 1 \pm 4$ or, $1 \pm \sqrt{-22}$; $y = u - v$, etc.

Again,

$$x^5 - y^5 = 211 \quad \ldots\ldots \quad (a)$$
$$x - y = 1 \quad \ldots\ldots \quad (b)$$

Divide (a) by (b); $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 211$

raise (b) to 4th power, $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 1$

subtract, $5x^3y - 5x^2y^2 + 5xy^3 = 210$

$$x^3y - x^2y^2 + xy^3 = 42$$

square (b) and multiply by xy,

$$x^3y - 2x^2y^2 + xy^3 = xy$$

subtract, $x^2y^2 + xy = 42$

complete square,

$$x^2y^2 + xy + \tfrac{1}{4} = \tfrac{169}{4}.$$

$$xy + \tfrac{1}{2} = \pm\frac{13}{2},\ xy = 6 \text{ or } -7 \quad \ldots \quad (c)$$

Multiply (c) by 4 and add to square of (b);

$$x^2 - 2xy + y^2 = 1$$
$$4xy = 24 \text{ or } -28$$
$$x^2 + 2xy + y^2 = 25 \text{ or } -27$$

$$x + y = \pm 5 \text{ or } \pm 3\sqrt{-3}$$
$$x - y = 1 \quad (b)$$

which indicates how a general solution may be varied in special cases.

ART. 136. One equation may be divisible by the other, as,

SOLUTION.

$$x^4 + x^2y^2 + y^4 = 931 \quad \ldots\ldots \quad (a)$$
$$x^2 + xy + y^2 = 49 \quad \ldots\ldots \quad (b)$$

Divide (a) by (b) to get (c);

$$x^2 - xy + y^2 = 19 \quad \ldots\ldots \quad (c)$$

Subtract (c) from (b) to get (d);

$$2\,xy = 30;\ xy = 15 \quad \ldots\ldots \quad (d)$$

Add (d) to (b) and subtract (d) from (c);

Whence $x^2 + 2\,xy + y^2 = 64$, or $x + y = \pm 8$
$x^2 - 2\,xy + y^2 = 4$, $x - y = \pm 2$, etc.

Again,
$$x^3 - y^3 = 7\,xy \quad \ldots\ldots \quad (a)$$
$$x - y = 2 \quad \ldots\ldots \quad (b)$$

Divide (a) by (b); $x^2 + xy + y^2 = \frac{7}{2}xy$.

or
$$x^2 - \frac{5\,xy}{2} + y^2 = 0 \quad \ldots\ldots \quad (c)$$

Divide (c) by (y^2).

$$\frac{x^2}{y^2} - \frac{5}{2}\left(\frac{x}{y}\right) + 1 = 0 \ \left[\text{quadratic in } \left(\frac{x}{y}\right)\right].$$

or
$$\left(\frac{x}{y}\right)^2 - \frac{5}{2}\left(\frac{x}{y}\right) = -1,$$

Whence, $\frac{x}{y} - \frac{5}{4} = \pm\frac{3}{4}$, $\frac{x}{y} = 2$ or $\frac{1}{2}$.

Whence, $x = 2\,y$ or $\frac{1}{2}\,y$. Complete.

EXERCISE XXII.

Compose examples of each type indicated and solve them.

1. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{19}{6} \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}. \end{cases}$

2. $\begin{cases} x^2 + y^2 = \dfrac{5\,xy}{2} \\ x + y = \dfrac{5}{6}. \end{cases}$

3. $\begin{cases} x^2y^2 - 16\,xy + 60 = 0 \\ x + y = 7. \end{cases}$

4. $\begin{cases} x^3 + y^3 = 1 - 3\,xy \\ x^2 + y^2 = xy + 37. \end{cases}$

5. $\begin{cases} x^2 + y^2 = axy \\ x + y = bxy. \end{cases}$

6. $\begin{cases} x^2 - 3\,xy + y^2 = 5 \\ x^4 + y^4 = 2. \end{cases}$

7. $\begin{cases} x^2 + y^2 + 5\sqrt{x^2 + y^2} = 50 \\ x^2 - y^2 = 7. \end{cases}$

8. $\begin{cases} 3\,x^{-2} - y^{-2} = 1 \\ 5\,x^{-2} - (xy)^{-1} + 2\,y^{-2} = 3. \end{cases}$

9. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3} \\ x^2 + y^2 = 45. \end{cases}$

10. $\begin{cases} x^2 + 3\,xy + 3\,(x - y) = 2 \\ x^2 + 2\,xy - 3\,y^2 = 0. \end{cases}$

11. $\begin{cases} x^2 + xy + y^2 = 63 \\ x + y = -3. \end{cases}$

12. $\begin{cases} \frac{1}{6}x^2 + \frac{1}{4}y^2 - 60 = 0 \\ \frac{1}{6}x + \frac{1}{4}y - 5 = 0. \end{cases}$

13. $\begin{cases} x^2 + 3\,xy = 54 \\ xy + 4\,y^2 = 115. \end{cases}$

14. $\begin{cases} x^3 - y^3 = 127 \\ x^2y - xy^2 = 42. \end{cases}$

15. $\begin{cases} x^2 + xy + y^2 = 84 \\ x - \sqrt{xy} + y = 6. \end{cases}$

16. $\begin{cases} \sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{10}{3} \\ x + y = 10. \end{cases}$

17. $\begin{cases} x^2 + y^2 - z^2 = 21 \\ 3\,xz + 3\,yz - 2\,xy = 18 \\ x + y - z = 5. \end{cases}$

18. $\begin{cases} x^2\,y^4 - 6\,xy^2 = -9 \\ xy - y = 2. \end{cases}$

19. $\begin{cases} x^2 + xy + 2\,y^2 = 74 \\ 2\,x^2 + 2xy + y^2 = 73. \end{cases}$

20. $\begin{cases} x^2 + xy = 15 \\ xy - y^2 = 2. \end{cases}$

21. $\begin{cases} x^2 - 4y^2 = 9 \\ xy + 2y^2 = 4. \end{cases}$

22. $\begin{cases} 4\,(x + y) = 3\,xy \\ x + y + x^2 + y^2 = 26. \end{cases}$

23. $\begin{cases} xy\,(x + y) = 30 \\ x^3 + y^3 = 35. \end{cases}$

24. $\begin{cases} x^2 + y^2 = 65 \\ xy = 28. \end{cases}$

25. $\begin{cases} .1x^2 + .5y - 2 = 0 \\ .1x - .25y - 3 = 0. \end{cases}$

EXERCISE XXIII.

Problems.

1. The product of the number 2 x 3 and 4 x 6 in the decimal system is 115 368. What is the digit, x?

2. The sum of a number and its square root is 42. Find the number.

3. The area of a rectangle is 120 square feet and its diagonal is 17 feet. Find length and breadth.

4. A square and a rectangle have together the area 220 square yards. The breadth of the rectangle is 9 yards and its length equals the side of the square. Find area of square.

5. From the vertex of a right angle two bodies move on the sides of the angle, one at rate of 1.5 feet and other 2 feet per second. After how long are they 50 feet apart?

6. If the sides of an equilateral triangle are shortened 8, 7, and 6 inches respectively, a right angled triangle is formed. Find the side of the equilateral.

7. About the point of intersection of the diagonals of a square as a center, a circle is described; the circumference passes through the mid-points of the semi-diagonals; the area between the circumference and the sides of the square is 971.68 square inches. Find the length of side of square ($\pi = 3\frac{1}{7}$).

8. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel. If the circumference of each were increased 2 feet, the fore wheel would turn only 88 times more. Find the circumferences.

9. A cistern can be filled by 2 pipes; one can fill it in 2 hours less than the other; it can be filled by both pipes running at once in $1\frac{7}{8}$ hours. Find time for each.

10. *A* and *B* are laying a cement walk. At *A's* rate of work he could finish the job himself in 18 hours; *B* lays 9 running yards per hour. *A* finishes his portion in as many hours as B lays yards per hour. Find amount laid by each.

11. Two cubical tanks have together 407 cubic feet contents. The sum of their edges (outside measure) = 11 feet 1 inch. Tanks are made of $\frac{1}{4}$ inch steel. Find amount of steel necessary for them.

12. A body starts from rest under acceleration of 18 feet per second, find the time required to pass over the first foot; the second; the third.

13. In going 173.25 yards the front wheel of a wagon makes 165 revolutions more than the rear wheel, but if the circumference of each wheel were 27 inches more, the front wheel would, in going same distance, make only 112

revolutions more than rear one. Find circumference of each.

14. Two points, *A* and *B*, start at same time from a fixed point and move about circumference of a circle in opposite directions, each at a uniform rate, and meet after 6 seconds. The point *A* passes over the entire circumference in 9 seconds less time than *B*. Find the time taken *A* and *B* to travel entire circumference.

15. A reservoir has a supply pipe, *A*, and a discharge pipe, *B*. *A* can fill the reservoir in 8 minutes less time than *B* can empty it. If both pipes are open, the reservoir is filled in 6 minutes. Required number of minutes it will take to fill, if *A* is open and *B* closed.

16. A body is projected vertically upward with a velocity of 80 feet per second. When will it reach a height of 64 feet?

17. A lawn 25 feet wide and 40 feet long has a brick walk of uniform width around it. The area of the walk is 750 square feet. Find the width.

18. The perimeter of a rectangular field is 184 rods and the field contains 12 acres. What are its dimensions?

CHAPTER IX.

LOGARITHMS.

ARTICLE 137. The logarithm of a number is the power to which a given number, called the *base*, must be raised that this power may equal the number.

For instance, take 2 as a base,

then,	$2^1 = 2$,	exponent 1.
	$2^2 = 4$,	exponent 2.
	$2^3 = 8$,	exponent 3.
	$2^4 = 16$, etc.	exponent 4, etc.

Hence,
1 is the logarithm of 2 to base 2.
2 is the logarithm of 4 to base 2.
3 is the logarithm of 8 to base 2.
4 is the logarithm of 16 to base 2.

Plainly any number, except 0 or 1, may be selected as a *base*. Why not 0 or 1?

ART. 138. It has become customary to use 10 as the base for logarithms, principally for the reason that 10 is also the base of our number systems, both integral and decimal, and hence is best adapted for the base of logarithms of these numbers.

Take then a series of powers of 10; thus:

$$10^1 = 10.$$
$$10^2 = 100.$$
$$10^3 = 1000.$$
$$10^4 = 10{,}000, \text{ etc.}$$

Then with the base, 10, the

$$\log 10 = 1.$$
$$\log 100 = 2.$$
$$\log 1000 = 3.$$
$$\log 10{,}000 = 4.$$

By inference the logarithm of any number between 10 and 100 is between 1 and 2, of numbers between 100 and 1000 is between 2 and 3, etc.

Hence, to represent all numbers, it is necessary to employ fractional powers, for instance, the logarithm of

$$29 = 1.4624 +$$

of $327 = 2.5145 +$ etc.

That is, $29 = 10^{1.4624+}$

and $327 = 10^{2.5145+}$ etc.

These fractional powers of 10, which we call the logarithms of the numbers to which they correspond, are found by computation from a series, which is of no especial interest here.

The first logarithms for general use were based upon an incommensurable decimal, 2.7182818 + (usually represented by e), and known as Naperian logarithms from their discoverer, Baron Napier.

Art. 139. Since logarithms are exponents (usually exponents of 10), they obey the laws of exponents; namely, in multiplying, exponents, and hence logarithms, are added ; in dividing, exponents, and hence logarithms, are subtracted, etc. Hence, the following rules :

The logarithm of a product equals the sum of the logarithms of the factors. The logarithm of a quotient equals the logarithm of the dividend, minus that of the divisor. State the rules for powers and roots, from analogy to exponents.

ART. 140. The use of the base 10 makes it possible not only to simplify calculation by logarithms, but also to express them in a much more compact tabulated form. For example, take the series of powers of 10 again.

$$\begin{array}{lll}
\log = 1. + & \left\{\begin{array}{l} 10^1 = 10 \\ 10^2 = 100 \end{array}\right\} & \text{two digits} \\
\log = 2. + & \left\{\begin{array}{l} 10^2 = 100 \\ 10^3 = 1000 \end{array}\right\} & \text{three digits} \\
\log = 3. + & \left\{\begin{array}{l} 10^3 = 1000 \\ 10^4 = 10{,}000 \end{array}\right\} & \text{four digits, etc.}
\end{array}$$

It is apparent that any number between 10 and 100 has a logarithm 1 + a fraction; any number between 100 and 1000 has a logarithm 2 + a fraction, etc.

But every number between 10 and 100 is composed of two digits in its integral part, for instance 23, 29.375, 57.5, etc.

Every number between 100 and 1000 is composed of three digits, as 237, 676, 253, 987, 234.2, etc.

Hence, the whole part of the logarithm of a number is always one less in absolute value than the number of digits in the integral part of the number.

This fact may be shown in tabulated form:

Between 10 and 100	log 23	= 1.3617;	whole part is 1.
	log 67.6	= 1.8299;	whole part is 1.
	log 98.2	= 1.9921;	whole part is 1.

Between 100 and 1000	log 235	= 2.3711;	whole part is 2.
	log 595.35	= 2.7769;	whole part is 2.
	log 802	= 2.9042;	whole part is 2, etc.

ART. 141. The whole part of a logarithm is called its *characteristic*, and the decimal part is called its *mantissa*.

ART. 142. Since numbers which have the same figures (digits) arranged in the same order, differ from one another only by some multiple of ten, and since the logarithms of multiples of ten are always whole numbers, it follows that the decimal part of the logarithm remains the same so long as the digits are unchanged, no matter where the decimal point be placed.

EXAMPLE. 23456, 2345.6, 234.56, 23.456, 2.3456, .23456, .023456, etc., all have the same mantissa in their logarithms. This makes it possible to find the logarithms of all numbers from any table of logarithms, as shown later.

Logarithms of Decimals.

ART. 143. A pure decimal always indicates a fraction with a denominator which is a higher power of ten than the numerator, hence, since a fraction means a division, by the law of division by logarithms, the logarithm of a fraction, decimal or otherwise, is negative.

It is customary to keep the decimal part of a logarithm always positive, and to make the characteristic bear the negative sign.

For example, $.04324 = \frac{4324}{100000}$

$$\therefore \log .04323 = \log 4324 - \log 100000$$

$$\log 4324 = 3.6359$$

$$\log 100000 = 5.0000$$

$$\log .04324 = -1.3641 = -2. + .6359.$$

This result is usually written, $\bar{2}.6359$, to indicate that the 2 alone is negative, while the decimal is positive. It is an advantage of uniformity entirely.

Again, find log .235.

$$\log .235 = \log \frac{235}{1000} = \log 235 - \log 1000$$

$$\log 235 = 2.3711$$
$$\underline{\log 1000 = 3.0000}$$
$$\log 235 = -.6289 = \bar{1}.3711.$$

By an inspection of these results a general rule for pure decimals may be stated, thus:

Find the decimal part of the logarithm from the table, ignoring the decimal point. The characteristic is equal to a number one greater than the number of zeros following the decimal point or is equal to the number representing the position of the first significant figure (*that is the first one not zero*) *after the decimal point.*

ART. 144. To find the logarithm of a number from a table of logarithms.

Find the log 23.7625.

Say the table runs to 1000 and gives the log to four decimal places. Since the decimal part of the log is independent of the decimal point, the point may be placed to best advantage; in this case between 7 and 6, because the table gives the logarithms of numbers of three digits. Mantissa of 237.625 lies, evidently, between that of 237 and 238 in the table. If the change of logarithms between 237 and 238 is uniform, the mantissa of 237.625 should be the mantissa of 237 plus .625 of the difference between the mantissas of 237 and 238. Thus:

$$\text{man } 238 = .3766$$
$$\underline{\text{man } 237 = .3747}$$

Difference for $1 = .0019.$

Difference for $.625 = .0019 \times .625 = .0012 +$

$$\therefore \text{man } 237.625 = .3747 + .0012 = .3759$$
$$\therefore \log 23.7625 = 1.3759.$$

The same thing may be represented schematically, thus:

$$\begin{array}{lll}
 & \text{man } 238 = .3766 & \\
\text{Diff. of one unit} = & \text{man } 237.625 = .3759 & = \text{Diff. of .0019 corresponding to one unit} \\
\text{Diff. of .625} = & \text{man } 237 = .3747 & = \text{Diff. of .0012 corresponding to .625}
\end{array}$$

COLOGARITHMS.

ART. 145. To avoid negative logarithms, where a smaller quantity is to be divided by a larger, the logarithm of the reciprocal of a number is employed, and is called the *cologarithm* of the number itself.

For example, find $\log \frac{239}{562}$.

$$\log \frac{239}{562} = \log 239 - \log 562,$$

which would give a negative result.

$$\frac{239}{562} \text{ also equals } 239 \times \frac{1}{562}$$

$$\therefore \log \frac{239}{562} = \log 239 + \log \frac{1}{562},$$

which may be expressed thus:

$$\log \frac{239}{562} = \log 239 + \text{colog } 562$$

$$\log \frac{1}{562} = \log 1 - \log 562$$

$$\log 1 = 0.0000$$

$$\log 562 = 2.7497.$$

$$\therefore \log \frac{1}{562} = 0 - 2.7497.$$

But 0 may be expressed as $4-4$ or $5-5$ or $10-10$, etc. For uniformity we say $0 = 10-10$.

$$\therefore \log \frac{1}{562} = (10 - 10) - 2.7497$$

$$= (10 - 2.7497) - 10 = 7.2503 - 10.$$

Hence, $$\log \frac{1}{562} = \text{colog } 562 = 7.2503 - 10$$

$$\therefore \log \frac{239}{562} = \log 239 + \text{colog } 562 =$$

$$2.3784 + 7.2503 - 10 = 9.6287 - 10 = \bar{1}.6287.$$

The result may be stated thus:

TO FIND THE COLOGARITHM OF A NUMBER, FIND ITS LOGARITHM, SUBTRACT THIS LOGARITHM FROM 10 AND WRITE − 10 AFTER THE REMAINDER.

The statement of the law for division by logarithms may be amended thus: *The logarithm of the quotient of two numbers equals the logarithm of the dividend plus the cologarithm of the divisor.*

To Find a Number From Its Logarithm.

ART. 146. The number is often called the antilogarithm. The process of finding a number from its logarithm is evidently the reverse of the process for finding the logarithm of a number.

Find the antilog of 3.8764.

In finding a number from its logarithm the characteristic is at first ignored, because only mantissas are given in the tables and the characteristics are readily found by the simple rules already enunciated.

An examination of the table shows no such mantissa as .8764. The two nearest to it are,

$$.8768 = \text{man } 753$$

and

$$.8762 = \text{man } 752.$$

The difference .0006 corresponds to difference 1 in numbers.

The mantissa .8764 being between .8768 and .8762 the number corresponding to it must be between 753 and 752, which correspond respectively to the mantissas .8768 and .8762.

If the change in the mantissa corresponds to the change in the numbers, the difference between the smaller mantissa .8762 and .8764 will have the same ratio to the difference between .8768 and .8762 as the difference between 752 and the number corresponding to .8764 has to 1, the difference between 752 and 753, *i.e.*, $.0002 : .0006 :: (x) : 1$.

$$\therefore x = \frac{.0002}{.0006} = \tfrac{1}{3} = .333 +$$

$$\therefore \text{ man } .8764 \text{ corresponds to } 752.333 +$$

$$\therefore \log 3.8764 = 7523.33 +$$

Since the characteristic of a logarithm is found by taking one less than the number of figures in the whole part of the number, the pointing off of the whole places in the number from the logarithm is the reverse. That is, there will be one more place in the whole part of the number than there are units in the characteristic. For example, the antilog of 1.2345 has two places in whole part. The antilog of 3.0642 has four places in whole part, etc.

If the characteristic is negative, the decimal point is placed so that the first significant figure in the number shall occupy a place after the decimal point of the same order as the number of units in the characteristic. For example:

$$\bar{2}.1790 = \log .0151$$
$$\bar{1}.7803 = \log .603$$
$$\bar{4}.6191 = \log .000416, \text{ etc.}$$

A solution of a general problem, by logarithms, may assist in the comprehension of the process.

Find value of

$$\sqrt[4]{\frac{.008541^2 \times 8641 \times 4.276^{\frac{1}{3}} \times .0084}{(.00854)^3 \times 182.63^{\frac{1}{4}} \times 82^{\frac{1}{3}} \times 487.27^{\frac{1}{4}}}}.$$

Log of above expression equals $\frac{1}{4}$[2 log .008541 + log 8641 + $\frac{1}{3}$ log 4.276 + log .0084 + 3 colog .00854 + $\frac{1}{4}$ colog 182.63 + $\frac{1}{3}$ colog 82 + $\frac{1}{4}$ colog 487.27].

Man .008541 = man 854.1
man 855 = .9320
man 854 = .9315
diff for 1 = .0005
diff for .1 = .00005 = .0001 (dropping the 5)
man 854.1 = .9316 [.9315 + .0001]
log .008451 = $\bar{3}$.9316; 2 log .008541 = $\bar{5}$.8632
or 2 log .008541 = 5.8632 − 10 [adding and subtracting 10, which does not change value].

2 log .008541 = 5.8632 − 10
log 8641 = 3.9366
$\frac{1}{3}$ log 4.276 = .2103
log .0084 = 7.9243 − 10
3 colog .00854 = 6.2055 [(10 − $\bar{7}$.7945) − 10]
$\frac{1}{4}$ colog 182.63 = 9.4346 − 10
$\frac{1}{3}$ colog 82 = 9.3621 − 10
$\frac{1}{4}$ colog 487.27 = 9.3281 − 10

Log of original expression = $\frac{1}{4}$ (52.2647 − 50)
= $\frac{1}{4}$ (2.2647) = .5662

∴ original expression = 3.6833 + = antilog of .5662.

It is to be observed that the colog of a decimal, since it is the log of the reciprocal of the decimal, is really the log of a whole or mixed number, since the reciprocal of a decimal must be such a number. Hence the 10 may be subtracted from the characteristic after the colog is found without giving a negative quantity as above

$$\begin{aligned} \log .00854 &= \bar{3}.9315 \\ & \underline{\qquad\quad 3} \\ 3 \log .00854 &= \bar{7}.7945 \end{aligned}$$

[the two carried over the decimal point is positive, hence $-3 \times 3 + 2 = -7$].

$$\begin{aligned} & 10.0000 \quad - 10 \\ & \underline{\bar{7}.7945} \\ \text{colog } .00854^3 = \; & 16.2055 - 10 = 6.2055. \end{aligned}$$

It must be remembered that the mantissa is *always* positive no matter what the characteristic may be. In the above example, since 1 has been borrowed from the 10 to subtract the .7 from, there are only 9 left from which to subtract -7, hence $9 - (-7) = 9 + 7 = 16$.

EXERCISE XXIV.

Logarithms.

Find the logarithms of:

1. 235.6.

2. 1.7456.

3. 1023.5.

4. .00235.

5. $(125.6)^2$.

6. $(23.67)^{\frac{2}{3}}$.

Find the antilogarithms of:

7. 1.301362.

8. $\bar{2}.441201$.

9. 3.673092.

10. 9.720387–10.

11. 2.800046.

Find the cologarithms of:

12. 216.93. **14.** .2765.

13. .01672. **15.** 9929.7.

Find the value by logarithms of:

16. $237.95 \times .0192$. **17.** $67.25 \div 3.2719$.

18. $(2.356)^{\frac{1}{2}} \times (77.777)^{\frac{2}{3}}$.

19. $$\frac{\sqrt{62.31} \times 92086}{.035671}.$$

20. $$\sqrt[6]{\frac{.03195^2 \times 62.932 \times .83678^3}{29.312 \times (.00261)^4}}.$$

21. $$\sqrt[3]{\frac{6.6251 \times \sqrt{.19672} \times .01872}{\sqrt{.51672} \times 11.137 \times .09823}}.$$

22. $$\frac{(67.025) \times (1.06)^{12}}{(1.06)^{12} - 1}.$$

23. $\frac{2}{3}\sqrt[3]{-7} \div \sqrt[4]{22}$.

24. $$\left[\frac{(\frac{5}{8})^2 \times (\frac{3}{4})^3 \times (3\frac{1}{7})^4}{6\frac{1}{3}}\right]^{\frac{3}{5}}.$$

25. $$\sqrt[3]{\frac{2.72\sqrt[5]{-4.6307}}{.317^2\sqrt{124.61}}}.$$

CHAPTER X.

INEQUALITIES.

ARTICLE 147. If $x - y$ is positive, x is said to be greater than y, written thus, $x > y$.

If $x - y$ is negative, x is less than y and this relation is represented thus, $x < y$.

ART. 148. Two inequalities with the inequality sign turned in the same direction are said to be in the same sense; as $x > y$, $5 > 4$, $a > b$, etc.

ART. 149. Just as we have equations involving unknown quantities, we have also inequalities involving unknown quantities. By the solution of equations we get values of the unknown to satisfy the equations; by the solution of inequalities we get the greatest or smallest value that the unknown may have without violating the conditions of inequality. These values are called maximum and minimum respectively.

ART. 150. As equations are subject to certain rules of transformation in order that solution may be accomplished, so inequalities obey certain laws, which must be determined before they can be handled legitimately. They are as follows:

ART. 151. If both terms of an inequality are multiplied by a positive quantity, the inequality is unchanged. If multiplied by a negative quantity, the sign is reversed.

Let $a > b$, then $ma > mb$, but $-ma < -mb$. For if $a > b$ then $a - b =$ some positive quantity, say c; that is, $a - b = c$, $\therefore ma - mb = mc$, still a positive quantity. $\therefore ma > mb$.

But $-ma - (-mb) = -mc$, a negative quantity, since $a > b$, $\therefore -ma < -mb$.

ART. 152. If $x > y$ then $x^m > y^m$.

For $x - y = c$, a positive quantity.

Or $(x^{m-1} + x^{m-2}y + x^{m-3}y^2 \ldots xy^{m-1} + y^m)(x - y) = c(x^{m-1} + x^{m-2}y + \text{etc.})$ which is plainly positive.

But $(x^{m-1} + x^{m-2}y + x^{m-3}y^2 \ldots xy^{m-1} + y^m)(x-y) = x^m - y^m$, $\therefore x^m - y^m =$ positive quantity, $\therefore x^m > y^m$.

ART. 153. If $a \neq b$ (read a is *not* equal to b) then $a^2 + b^2 > 2ab$.

For $(a - b)^2 > 0$ (because the square of either a positive or negative quantity is positive, hence greater than 0) that is, $a^2 - 2ab + b^2 > 0$. Add $2ab$ to both sides.

$$a^2 + b^2 > 2ab.$$

That is, the sum of the squares of two unequal quantities is always greater than twice their product.

EXAMPLE. Find minimum value of x if

$$2x^2 - 8x + 21 > x^2 - 2x + 37 \qquad \text{(collect)}$$
$$x^2 - 6x > 16 \qquad \text{(add 9 to both sides)}$$
$$x^2 - 6x + 9 > 25 \qquad \text{(extract square root)}$$
$$x - 3 > 5 \quad \therefore \quad x > 8.$$

Hence, x cannot be as small as 8.

EXAMPLE. Find the area of the largest rectangle having the perimeter 20 inches.

Let $x =$ one side, then since the perimeter is 20, $10 - x =$ other side

$$\therefore 10x - x^2 = \text{area}.$$

Say, $10x - x^2 = y$ (the area)

then, $x^2 - 10x + 25 = 25 - y$

$$x - 5 = \pm\sqrt{25 - y}$$
$$x = 5 \pm \sqrt{25 - y}.$$

If y is greater than 25, $(25 - y)$ will be negative and $\sqrt{25 - y}$ will be imaginary.

Therefore, $y \not> 25$ (y is *not* greater than 25).

Then 25 is the maximum value of y.

When, $$10x - x^2 = 25 \quad (= y)$$

$$x^2 - 10x + 25 = 0$$

$$x = 5$$ the maximum value for a side. $$10 - 5 = 5,$$

hence, at the maximum, the rectangle becomes a square, 5 inches on the side.

RATIO, VARIATION, AND PROPORTION.

ART. 154. The ratio of one quantity to another is the fraction whose numerator is the first quantity and whose denominator the second as $\frac{a}{b}$ = the ratio of a to b. This is often written $a : b$.*

In such a ratio, a is called the *antecedent* and b the *consequent*.

If a and b both change values, but maintain always the same ratio, a is said to vary as b, written $a \propto b$. Calling m the constant ratio between a and b, this may be written $a = mb$. Clearly one variable quantity may vary as several others together; for instance, a may vary as b, c, d, etc. This is expressed thus:

$$a \propto b.c.d.$$

Or a may increase as b decreases (or vice versa); a is then said to vary inversely as b, written

$$a \propto \frac{1}{b}.$$

* The line between the numerator and denominator of a fraction is probably an evolution from the ratio sign, : .

ART. 155. There are certain laws governing ratio which may be stated as follows:

If the antecedent is the greater, the ratio is said to be of greater inequality.

A ratio of greater inequality is diminished, and a ratio of less inequality is increased by adding any positive quantity to both terms.

Take $a > b$ in the ratio $\frac{a}{b}$ and let m be any positive quantity.

Add m to both terms, $\frac{a+m}{b+m}$

$$\frac{a+m}{b+m} > \, = \text{ or } < \frac{a}{b}$$

According as

$ab + bm > \, = \text{ or } < ab + am$ [clearing of fractions]

$bm > \, = \text{ or } < am$ [subtracting ab from both sides]

$b > \, = \text{ or } < a$ [dividing by m]

but $a > b$ by hypothesis,

hence, $\frac{a}{b} > \frac{a+m}{b+m}$. That is, $\frac{a}{b}$ was diminished.

If $a < b$, then $\frac{a}{b} < \frac{a+m}{b+m}$, that is, $\frac{a}{b}$ was increased.

EXAMPLES. By Boyle's law of physics if P is the pressure on a volume V, of gas, then $P \propto \frac{1}{V}$. A certain gas has a volume of 1200 c.c. under a pressure of 1033 g. to 1 sq. cm. What is the volume when the pressure is 1250 g.?

Let m be the ratio in the variation $P \propto \frac{1}{V}$.

Then $P = \frac{m}{V}$ or $PV = m$. . . . . (a)

By first condition $P = 1033$ when $V = 1200$
then $m = 1033 \times 1200$

$$\therefore \; P = \frac{1033 \times 1200}{V} \text{ (substituting } m \text{ in } (a))$$

In second condition $P = 1250$; $1250 = \dfrac{1033 \times 1200}{V}$

$$\therefore V = \frac{1033 \times \overset{24}{\cancel{1200}}}{\underset{25}{\cancel{1250}}} = \frac{1033 \times 24}{25} = 991.68 \text{ c.c.}$$

EXERCISE XXV.

Variation.

1. If $y \propto x$ and $y = 5$ when $x = 3$, find x when $y = 9$.

2. If $I \propto \dfrac{E}{R}$ and $I = .45$ when $E = 110$ and $R = 244$, find E when $I = .48$ and $R = 254$.

3. If the rate of discharge of water from an orifice varies as the square root of the depth, and 11 gallons per minute are discharged when the height is 49 feet, what is the discharge when the height is 77.44 feet?

4. The distance a body falls, due to gravity, varies as the square of the time of fall. If a body falls 257.6 feet in 4 seconds, how long will it be in falling 788.9 feet?

5. The square of the time of revolution of a planet about the sun varies as the cube of its distance from it. If the distance of the earth is 93,000,000 miles and of Saturn 886,000,000 miles, what is Saturn's period about the sun?

6. A shell 1 foot in diameter weighs $\frac{91}{216}$ as much as it would if solid. Find the thickness of the shell, remembering that the volumes of spheres vary as the cubes of their radii or diameters.

7. The penetration of a bullet varies as its momentum, or if the mass remains the same, it varies as the velocity. If a bullet having a velocity of 187 meters per second will penetrate 8.92 cm. into a target, what velocity is necessary to penetrate 14.8 cm.?

8. The light received upon a surface varies inversely as the square of its distance from a source of light. If a screen is 25 feet from an incandescent lamp, to what distance must it be removed to receive $\frac{1}{3}$ as much light?

9. The weight of a body on the surface of a material sphere varies directly as the mass of the sphere and inversely as the square of its radius. If a body on the earth's surface weighs 24 lbs., taking the earth's radius as 3963 miles, what would it weigh on the surface of the moon, whose radius is 1081.5 miles and whose mass is $\frac{1}{80}$th of the earth's mass?

10. If the electric resistance of a wire varies directly as its length and inversely as the square of its diameter, and if a wire 137 cm. long and .038 mm. diameter has a resistance of 19.3 ohms, what will be the resistance of a wire of the same material 235 cm. long and 1.2 mm. diameter?

11. Three metal spheres whose radii are 3, 4, and 5 inches respectively, are melted and cast into one sphere. What is the radius of this sphere, the volumes of spheres being known to vary as the cube of their radii?

PROPORTION.

ART. 156. A statement of equality between two ratios is called a proportion; thus,

$$a : b :: c : d \quad \text{or} \quad a : b = c : d \quad \text{or} \quad \frac{a}{b} = \frac{c}{d}.$$

The first and fourth terms of a ratio are called its extremes, the second and third terms are called its means. Each ratio is called a *couplet*.

When $a : b :: b : c$, b is said to be a *mean proportional* to a and c, and c or a are third proportionals to the other two.

A *continued* proportion is a series of equal ratios, as,

$$a : b :: c : d :: m : n :: x : y, \text{ etc.}$$

Laws of Proportion.

ART. 157. Every proportion admits of certain transformations as follows:

(*a*) If four quantities are in proportion, they are also in proportion, by alternation, that is the first is to the third as the second is to the fourth. To prove,

If $a : b :: c : d$ then $a : c :: b : d$.

Proof,

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{\not b} \times \frac{\not b}{c} = \frac{\not c}{d} \times \frac{b}{\not c}$$

(Multiplying both sides by $\frac{b}{c}$) or $\frac{a}{c} = \frac{b}{d}$

$$\therefore a : c :: b : d.$$

(*b*) Also the product of the extremes equals the product of the means.

For $\frac{a}{b} = \frac{c}{d} \therefore ad = bc$ (clearing of fractions).

(*c*) They are also in proportion by inversion, that is, the second is to the first as the fourth is to the third.

For $$\frac{a}{b} = \frac{c}{d}$$

then $$\frac{b}{a} = \frac{d}{c}.$$

[If two quantities are equal their reciprocals are equal.]

$$\therefore b : a :: d : c.$$

(*d*) They are also in proportion by composition, that is, the sum of the first and second is to either the first or the second as the sum of the third and fourth is to either the third or fourth.

That is, $a + b : a$ or $b :: c + d : c$ or d.

For $$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad \frac{b}{a} = \frac{d}{c}$$

then $$\frac{a}{b} + 1 = \frac{c}{d} + 1 \quad \text{or} \quad \frac{b}{a} + 1 = \frac{d}{c} + 1$$

$$\frac{a+b}{b} = \frac{c+d}{d} \quad \text{or} \quad \frac{a+b}{a} = \frac{c+d}{c} \qquad \text{[adding]}$$

$$\therefore a + b : b :: c + d : d \text{ or } a + b : a :: c + d : c.$$

(*e*) Prove that they are also in proportion by division, that is, $a - b : a$ or $b :: c - d : c$ or d.

(*f*) If two proportions have a couplet in each equal, the remaining couplets are in proportion.

If $a : b :: c : d$ and $m : n :: c : d$ then $a : b :: m : n$.

For $$\frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{m}{n} = \frac{c}{d}$$

$$\therefore \frac{a}{b} = \frac{m}{n}$$

$$a : b :: m : n.$$

If couplets from each proportion form a proportion the remaining couplets are in proportion.

If $a : b :: c : d$ and $m : n :: p : q$ and $c : d :: p : q$ then $a : b :: m : n$

For $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{m}{n} = \frac{p}{q} \text{ and } \frac{c}{d} = \frac{p}{q}$$

$$\therefore \frac{a}{b} = \frac{m}{n}.$$

(*g*) In a continued proportion the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent.

That is, if $a : b :: c : d :: m : n :: x : y$, etc.
$a + c + m + x : b + d + n + y :: a : b :: c : d$, etc.

For let the common ratio be represented by r then

$$\frac{a}{b} = r \text{ or } a = br$$

$$\frac{c}{d} = r \text{ or } c = dr$$

$$\frac{m}{n} = r \text{ or } m = nr$$

$$\frac{x}{y} = r \text{ or } x = yr$$

Add; $$a + c + m + x = (b + d + n + y)\,r$$

or $$\frac{a + c + m + x}{b + d + n + y} = r = \frac{a}{b} = \frac{c}{d}, \text{ etc.}$$

It is readily proved that, if four quantities are in proportion, any one power (whole or fractional) of these quantities forms a proportion.

Also that if the product of two quantities equals the product of two other quantities, two of them may form the extremes and two the means of a proportion.

For if $ad = bc$

then $\frac{ad}{bd} = \frac{bc}{bd}$ [dividing through by bd]

or $\frac{a}{b} = \frac{c}{d}$

$$\therefore a : b :: c : d.$$

It is to be observed that a, b, c, d, etc. stand for any quantities whatever in these proportions. They are by no means restricted to monomials.

EXERCISE XXVI.

Proportion.

If $a : b :: c : d$:

1. Show that $2a + b : b :: 2c + d : d$.

2. That $5a+3b : 5a-3b :: 5c+3d : 5c-3d$.

3. Find the mean proportional to 5 and $13\frac{2}{3}$.

4. Find the third proportional to $3\frac{3}{10}$ and 5.

5. What quantity must be added to each of the quantities a, b, c, and d to make them proportional?

Find the value of unknown in the following proportions:

6. $11\frac{1}{4} : 4\frac{1}{2} :: 3\frac{3}{4} : x$.

7. $\frac{4m}{5n} : \frac{16m}{7R} :: \frac{14R}{15n} : x$.

8. $x : x^2 - 1 :: 15 - 7x : 8 - 8x$.

9. $\begin{cases} x^3 + y^3 : x + y :: 7 : 1 \\ x^2 - y^2 : x - y :: 5 : 1. \end{cases}$

10. $\frac{y + \sqrt{3 - y}}{y - \sqrt{3 - y}} = 3.$

11. Two cars running in opposite directions pass each other in 2 seconds; running in the same direction, the faster passes the slower in 30 seconds. What is the ratio of their rates?

12. What number must be added to each of the numbers 3, 7, and 13, that the second may become a mean proportional to the other two.

13. Show that there is no number that, added to each of three consecutive numbers, will make the second a mean to the other two.

14. Spheres are to each other as the cubes of like dimensions. What will be the diameter of a ball formed from two balls whose diameters are respectively 7 inches and 9 inches?

15. If a bar is supported at two points, and a weight is suspended between these points, the parts of the weight borne by the supports are inversely proportional to the distances of the weight from the supports. If a bar 18 feet long, supported at its ends, carries a weight of 234 lbs. 4 feet from the end, A, find weight sustained at each end.

16. The cubes of the planets' distances from the sun are to each other as the squares of their periods of revolution. Calling the earth's period 1 year, and that of Jupiter 12 years, what is Jupiter's distance from the sun if the earth's distance is 93,000,000 miles?

CHAPTER XI.

PROGRESSIONS.

ARTICLE 158. A *series* is a number of successive quantities, each one derived from its predecessor by some fixed law. The successive quantities are called *terms*. If the series ends, it is called finite; if it extends indefinitely, it is called infinite.

ART. 159. There are plainly an unlimited number of forms of series. *Arithmetical*, *geometrical*, and *harmonical* are the only kinds that possess any general importance.

Arithmetical Series or Progression.

ART. 160. If each term of a series is derived from the preceding term by adding (algebraically) a constant quantity, it is known as an *arithmetical series* or *arithmetical progression*. Such as 1, 3, 5, 7, etc. (adding 2); $x + y$, x, $x - y$, $x - 2y$, etc. (adding $-y$).

The general form of this series is, a, $a + d$, $a + 2d$ $a + 3d$, etc., d being the *common difference*.

ART. 161. To find an expression for any term of an A. P. (arithmetical progression) in terms of the first term, the common difference, and the number of terms, it is only necessary to inspect the general form indicated above, marking the number of each term.

(Number of term)

1st	2d	3d	4th	5th	
a,	$a + d$,	$a + 2d$,	$a + 3d$,	$a + 4d$,	etc.

It will be observed that any term is equal to a (first term) plus d (difference) taken as many times less one as the number of this term in the series; thus the fourth term $= a + (4 - 1)\,d = a + 3d$.

Calling any desired term l, its number n, first term a, and difference d, clearly;

$$l = a + (n - 1)\,d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (x)$$

ART. 162. To find the sum of any number of terms: With the same notation, in addition calling the sum s, and remembering that if, starting with the *last* term, the common difference be *subtracted* successively, the series is the same (but in reverse order) as if we started with the first term and *added* the difference, then;

$$\begin{aligned} s &= a + (a + d) + (a + 2d) \ldots \quad (l - d) + l, \\ s &= l + (l - d) + (l - 2d) \ldots \quad (a + d) + a \end{aligned}$$

Add, $2s = (a + l) + (a + l) + (a + l) + (a + l) \ldots$
$(a + l) + (a + l)$.

If there are n terms in the series, evidently there will be $n\ (a + l)$'s, that is, $2s = (a + l) + (a + l) + (a + l)$ + repeated n times; or, $2s = n\,(a + l)$

$$s = \frac{n}{2}\,(a + l) \quad . \quad . \quad . \quad . \quad . \quad . \quad (y)$$

(x) and (y) are the fundamental relations between parts of an arithmetical progression. From them and their combinations, if any three of the quantities a, d, n, s, and l are given, the other two are readily found. For example:

In an arithmetical progression, given

$$d = 7,\ n = 12,\ s = 594.$$

Substituting in (x), $\quad l = a + (12 - 1)\,7 = a + 77$

or $\quad l - a = 77 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

Substituting in (y), $594 = \frac{12}{2}(a + l) = 6(a + l)$

or $$a + l = 99 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2), $2l = 176$, $l = 88$

Subtracting (1) from (2)

$$2a = 22, a = 11.$$

Arithmetical Mean.

ART. 163. The *arithmetical mean* between two quantities has a wide field of application in practical mathematics. It is the quantity which, placed between two other quantities, forms with them an arithmetical progression.

If a and l are any two quantities, and m stands for their arithmetical mean; by definition, a, m, l is an arithmetical progression;

hence $m - a = l - m$ (since there is a constant difference)

or $$2m = a + l$$

$$m = \frac{a + l}{2}.$$

ART. 164. It is sometimes necessary to insert several arithmetical means between two quantities, and here formulae (x) and (y) assist. For example:

Insert 4 arithmetical means between 3 and 18.

If m_1, m_2, m_3, m_4, stand for the means, the series will be, 3, m_1, m_2, m_3, m_4, 18, hence $a = 3$, $n = 6$, $l = 18$, to find d.

Substituting in (x), $18 = 3 + (6 - 1)d$

whence, $$5d = 15; \; d = 3$$

$$\therefore 3, \overbrace{6, 9, 12, 15,}^{\text{means}} 18 \text{ is the series.}$$

ART. 165. It is convenient to represent unknown quantities, when they are in arithmetical progression, by the following series, the first one when the number of unknowns is odd; the second when it is even; etc.

$$x - 2y,\ x - y,\ x,\ x + y,\ x + 2y,\ \text{etc.}$$
$$x - 3y,\ x - y,\ x + y,\ x + 3y,\ \text{etc.}$$

As an illustration:

The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. Find the numbers.

Let $x - y$, x, and $x + y$ represent the numbers.

then $$(x - y) + x + (x + y) = 33 \quad . \quad . \quad . \quad (1)$$
$$(x - y)^2 + x^2 + (x + y)^2 = 461 \quad . \quad . \quad . \quad (2)$$

From (1) the wisdom of the above notation is evident, for it reduces to, $3x = 33$, $x = 11$

hence, from (2), $(11 - y)^2 + (11)^2 + (11 + y)^2 = 461$

$$y = 7.$$

EXERCISE XXVII.

Arithmetical Series.

1. Find the 8th term of the series 3, 8, 13. . . .

2. Find the 10th term of $2\frac{1}{2}$, $1\frac{5}{6}$, $1\frac{1}{6}$. . . .

3. Find the 9th term of 3, $2\frac{1}{3}$, $1\frac{2}{3}$. . . .

Find the sum of:

4. $1 + 3 + 5 + 7$ . . . to 15 terms.

5. $-3 + 1 + 5$ . . . to 10 terms.

6. $1\frac{1}{4} + 1 + \frac{3}{4}$. . . . to 20 terms.

7. $x + (3x - 2y) + (5x - 4y)$ . . . to 8 terms.

8. $\dfrac{x - 1}{x} + \dfrac{x - 3}{x} +$ . . . to 12 terms.

9. Insert 6 means between 9 and 177.

10. Given $a = 3\frac{1}{4}$, $l = 64$, $n = 82$. Find d and s.

11. Given $l = 105$, $n = 16$, $s = 840$. Find a and d.

Find parts not given in the following:

12. $d = 5$, $l = 77$, $s = 623$.

13. $s = 143\frac{1}{3}$, $a = \frac{2}{3}$, $n = 20$.

14. $n = 20$, $a = 5$, $d = 2\frac{2}{3}$.

15. $a = 200$, $l = 88$, $s = 2160$.

16. $d = 4$, $n = 14$, $s = 812$.

17. How many terms of the series $-5, -2, +1 \ldots$ must be taken to sum 63?

18. The first term of an arithmetical progression is 5, the third term is 17. Find the sum of 8 terms.

19. How many terms of the series 2, 5, 8 . . . must be taken, that the sum of the first half may be to the sum of the second half as 8:23?

20. Starting from a mark, 30 stones are placed at intervals of two feet. If, starting at the mark, the stones are collected one by one and carried back each time to the mark, how far will the collector walk?

21. The three sides of a right angled triangle, whose area is 54 square rods, are in arithmetical progression. Find the sides.

22. If a falling body descends 16.1 feet the first second, 48.3 feet the second second, 80.5 feet the third second, how far will it fall in one minute?

23. A man was given his choice of wages, either $1.00 per day, or 3 cents the first day, 6 cents the second, 9 cents the third, his wage increasing 3 cents each day. He chose the former. Did he win or lose (in 30 days) and how much?

24. What value of a will make the arithmetical mean between $a^{\frac{1}{2}}$ and $a^{\frac{1}{4}}$ equal to 6?

Geometrical Progression.

ART. 166. A *geometrical progression* is a series of quantities so related to one another that each bears a constant ratio to the preceding. Thus: 2, 4, 8, 16, etc., or in general, $a, ar, ar^2, ar^3, \ldots$ etc.

Value of Any Term.

ART. 167. Let a represent the first term, l, the last term, r, the ratio, n, the number of terms, and s, the sum of the series.

Then a G.P. (geometrical progression) is represented in general by

(Number of term) 1st 2d 3d 4th 5th

$$a, \quad ar, \quad ar^2, ar^3, \quad ar^4, \text{ etc.},$$

the numbers indicating the number of the term.

It will be observed that any term is the product of the first term, a, by r, raised to a power, which is one less than the number of the term in the series. Hence, if n represent the number of any term, l, in the series, this term will be

$$l = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is one of the fundamental relation equations for geometrical progressions.

Sum of Any Number of Terms.

ART. 168. According to the notation above, evidently,

$$s = a + ar + ar^2 + ar^3 + ar^4 + \ . \ . \ . \ ar^{n-2} + ar^{n-1} \text{ (or } l)$$

multiplying by r

$$rs = ar + ar^2 + ar^3 + ar^4 \ . \ . \ . \ ar^{n-1} + ar^n \text{ or } (lr)$$

Subtracting, $s - rs = a - ar^n$, (or $a - rl$)

or $s(r-1) = a(r^n - 1)$, (or $rl - a$)

$$s = \frac{a(r^n - 1)}{r - 1} \text{ or } \frac{rl - a}{r - 1} \quad . \quad . \quad (2)$$

ART. 169. By these formulæ or by combinations of them, as in the case of arithmetical progressions, any two of the quantities, a, d, n, l, and s, may be found when the other three are given.

EXAMPLE. Given $a = 5$, $n = 3$, $s = 285$.

From (1) $l = 5r^2$

From (2) $285 = \frac{5(r^3 - 1)}{r - 1}$

or $57 = \frac{r^3 - 1}{r - 1} = r^2 + r + 1$

or
$$r^2 + r = 56$$
$$r^2 + r + \tfrac{1}{4} = \tfrac{225}{4}$$
$$r + \tfrac{1}{2} = \pm \tfrac{15}{2}$$
$$r = 7 \text{ or } -8$$

whence, from (1), $l = 5(7)^2$ or $5(-8)^2 = 245$ or 320.

The series is, then, either 5, 35, 245 or 5, −40, 320.

Geometrical Mean.

ART. 170. A *geometrical mean* between two quantities is a quantity which bears to the first quantity the same ratio that the second quantity bears to it; that is, it forms with the two quantities a geometrical progression.

If x stand for the geometrical mean between a and b, then by definition:

$$\frac{x}{a} = \frac{b}{x} \quad \text{or} \quad x^2 = ab \qquad x = \sqrt{ab}.$$

That is, a geometrical mean between two quantities is equal to the square root of their product.

For example, the geometrical mean between 16 and 25 is $\sqrt{16 \times 25} = 20$; between $a + 2b + \frac{b^2}{a}$ and a is

$$\sqrt{a\left(a + 2b + \frac{b^2}{a}\right)} = \sqrt{a^2 + 2ab + b^2} = a + b, \text{ etc.}$$

ART. 171. If several geometrical means are inserted between two quantities, each of these means is a mean between the two means on either side of it. Hence, to insert any number of means between two quantities, it is necessary to construct the series, hence to find r.

EXAMPLE. Insert 5 geometrical means between 2 and 1458, calling the means G_1, G_2, G_3, G_4, and G_5, the series is 2, G_1, G_2, G_3, G_4, G_5, 1458 and $n = 7$.

By (1) $$1458 = 2r^6$$

or $$r^6 = 729$$

$$r = \sqrt[6]{729} = 3.$$

Hence, the series, 2, 6, 18, 54, 161, 486, 1458.

Rule: *Taking the two quantities as* a *and* l *and* n *equal to two more than the number of means, use formula* (1).

Infinite Series.

ART. 172. If the number of terms is unlimited, the geometrical progression is called an infinite series, otherwise it is finite.

Formula (2), $s = \frac{a(r^n - 1)}{r - 1}$, may be put in the form,

$$s = \frac{ar^n}{r-1} - \frac{a}{r-1}, \quad = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

If r is a fraction and n is large enough, the value of r^n may become insignificant, since each increasing power of fraction is less than the preceding one, for by definition a power of a quantity is the product of a quantity by itself a certain number of times, and if the quantity is a fraction, the product is each time multiplied by a fraction, which must decrease its value.

If then n is infinite $r^n = 0$,

$$\therefore \frac{ar^n}{1-r} = \frac{0}{1-r} = 0 \qquad \text{hence, } s = \frac{a}{1-r} \quad . \quad (3)$$

which is the formula for the sum of an infinite geometrical progression.

EXAMPLE. Find the sum of the infinite geometrical progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$

Here, $a = 1,\ r = \frac{1}{2},\ n = \infty$

$$\therefore s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

ART. 173. A common application of the formula for infinite geometrical progression, is the expression of a recurring decimal in terms of a simple fraction.

EXAMPLE. Evaluate .124545. . . .

This is equivalent to the series

$$.12 + .0045 + .000045 + \ldots$$

or $\frac{12}{100} + \frac{45}{10000} + \frac{45}{1000000} + \ldots$ to infinity, as the figures 45 are repeated indefinitely in succeeding decimal orders.

Starting with $\frac{45}{10000}$ the rest of the expression is plainly an infinite geometrical progression in which $r = \frac{1}{100}$ and $a = \frac{45}{10000}$.

$$\therefore s = \frac{a}{1-r} = \frac{\frac{45}{10000}}{1-\frac{1}{100}} = \frac{\frac{45}{10000}}{\frac{99}{100}} = \frac{45}{9900}.$$

Then $.124545 \ldots = \frac{12}{100} + \frac{45}{9900} = \frac{1233}{9900} = \frac{137}{1100}$.

The recurring digits in such a decimal are usually indicated by a dot placed over them.

Thus, in the example above, $.124545 \ldots = .12\dot{4}\dot{5}$.

Harmonical Progression.

ART. 174. A *harmonical progression* (H.P.) is a series of numbers whose reciprocals, in the same order, form an arithmetical progression.

Thus, $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ is an harmonical progression, since 3, 5, 7, 9 is an arithmetical progression.

Again, $\frac{3}{4}, 4, -\frac{6}{5}, -\frac{12}{23}$ is an harmonical progression, since $\frac{4}{3}, \frac{1}{4}, -\frac{5}{6}, -\frac{23}{24}$ is an arithmetical progression.

Hence, to solve an harmonical progression invert its terms and apply the formulas for an arithmetical progression, then reinvert.

EXERCISE XXVIII.

Geometrical Progression.

1. Find the 10th term of 3, 6, 12. . . .
2. Find the 9th term of $6\frac{1}{4}, 2\frac{1}{12}, \frac{25}{36}$. . . .
3. Find the 7th term of 32, -16, 8. . . .
4. Find the 6th term of $1\frac{3}{5}, 2\frac{2}{3}, \frac{40}{9}$. . . .

Find the sum of:

5. $5 + (-3) + 1\frac{4}{5}$ . . . to 9 terms.
6. $1 + (-\frac{2}{5}) + \frac{4}{25}$ . . . to 10 terms.
7. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9}$ . . . to infinity.
8. $\frac{1}{64} + \frac{1}{16} + \frac{1}{4} +$ . . . to 8 terms.

Find parts not given in following:

9. $a = 36, l = 2\frac{1}{4}, n = 5$.
10. $l = 1296, r = 6, s = 1555$.
11. $r = 2, n = 7, s = 635$.
12. $a = -\frac{2}{3}, n = 7, r = -\frac{1}{2}$.
13. $a = 1, l = 81, r = 3$.

14. Insert 3 geometrical means between 17 and 4352.

15. Insert 6 geometrical means between 5 and − 640.

16. The fifth term of a geometrical progression is 48 and $r = 2$. Find first term.

17. Four numbers are in geometrical progression. The sum of the first and fourth is 195, and the sum of the second and third is 60. Find the numbers.

18. The sum of the first 8 terms of a geometrical progression is 17 times the sum of the first 4 terms. Find the series.

19. Find the value of the recurring decimal

$$3.17272\ldots$$

20. Find the value of the recurring decimal

$$.153153\ldots$$

21. A blacksmith proposes to shoe a horse for $1.60 or to take for his work, 1 cent for the first 4 nails, 2 cents for the next four, 4 cents for the next four, and so on. If he used 8 nails to each of the four shoes, which proposition was the better for him?

22. A "letter chain" is started thus, for a memorial fund: three letters are sent out with a request for 10 cents, and each receiver is asked to send out three letters containing the same requests. This process is repeated 30 times. How much will be realized for the fund?

23. If $100 be placed in a savings bank, where the amount increases 4 per cent each year, how much will be to the depositor's credit at the end of 20 years if no money is withdrawn?

CHAPTER XII.

INTEREST AND ANNUITIES.

Interest.

ARTICLE 175.

DEFINITION. *Interest* is the earnings of money when loaned or invested.

DEFINITION. The *principal* is the sum thus put to use.

DEFINITION. The ratio of the earnings for one year to the principal is called the *rate of interest*, or simply the *rate*.

DEFINITION. The *amount* is the sum of principal and interest.

DEFINITION. When the interest itself earns interest at stated intervals, it is said to be compounded. Such interest is, hence, called *compound interest*.

Simple Interest.

ART. 176. Let P = principal; r = rate; n = time; A = amount; I = interest. Then by arithmetic, $I = Prn$ and $A = P + I = P + Prn = P(1 + rn)$.

Compound Interest.

ART. 177. By definition, if the interest is payable annually,

$A = P(1 + r) = PR$ [letting $1 + r = R$]; end of 1st year.
$A_2 = PRr + PR = PR(1 + r) = PR^2$; end of 2d year.
$A_3 = PR^2r + PR^2 = PR^2(1 + r) = PR^3$; end of 3d year.

A comparison of the exponent of R with the number of years will enable us to express the amount for any number of years, say n, thus;

$$A_n = PR^n.$$

The subscripts for A indicate the number of years for which the amount (A) stands.

ART. 178. Frequently the interest is compounded semi-annually or quarterly, as in savings accounts. Then again by definition, if the interest is semi-annual,

$$A_{\frac{1}{2}} = \tfrac{1}{2}Pr + P = P\left(1 + \frac{r}{2}\right), \text{ for 1st half year.}$$

$$A_1 = \tfrac{1}{2}P\left(1 + \frac{r}{2}\right)r + P\left(1 + \frac{r}{2}\right) = P\left(1 + \frac{r}{2}\right)^2, \text{ for 2d}$$

half year.

$$A_{\frac{3}{2}} = \tfrac{1}{2}P\left(1 + \frac{r}{2}\right)^2 r + P\left(1 + \frac{r}{2}\right)^2 = P\left(1 + \frac{r}{2}\right)^3; \text{ for}$$

3d half year.

Then by analogy for n years or $2n$ half years

$$A_n = P\left(1 + \frac{r}{2}\right)^{2n}.$$

If the interest is quarterly, exactly similar process gives the formula

$$A_n = P\left(1 + \frac{r}{4}\right)^{4n}.$$

Annuities.

ART. 179. An *annuity* is a fixed amount of money to be paid or set aside annually or at stated regular intervals.

If these amounts are allowed to accumulate at compound interest, the annuities constitute a *sinking fund*, which is usually a provision for eventually liquidating an indebtedness of some institution.

ART. 180. Let S be the annuity; R, the amount of one dollar for one year at the rate, r; n, the number of years, and A, the final amount of the sinking fund, at any number of years. Then by definition,

$A_1 = S$, at the end of 1 year
$A_2 = S + SR$, at the end of 2 years.
$A_3 = S + SR + SR^2$, at the end of 3 years.
$A_4 = S + SR + SR^2 + SR^3$, at the end of 4 years.

And by easy analogy,

$$A_n = S + SR + SR^2 + SR^3 + \ldots SR^{n-1} \quad (1)$$
$$\therefore A_nR = SR + SR^2 + SR^3 + \ldots SR^{n-1} + SR^n \quad (2)$$

(multiplying (1) by R)

Subtract (1) from (2), $A_nR - A_n = SR^n - S$

or
$$A_n(R - 1) = S(R^n - 1)$$
$$\therefore A_n = \frac{S(R^n - 1)}{R - 1}.$$

ART. 181. It is usually necessary in the establishment of a sinking fund to estimate the amount required as annuity, or the number of years with a given annuity, to meet the obligation assumed.

Let P = the amount of debt.
R = the amount of \$1 at rate r, for 1 year.
n = time.

Then if the debt and its accumulated interest are to be balanced by the annuity and its accumulations,

PR^n (the debt and compound interest for n years) $= \frac{S(R^n - 1)}{R - 1}$ (the annuity S, and its accumulation at the same rate). Solving this equation for S,

$$S = \frac{PR^n(R - 1)}{R^n - 1} \quad \ldots\ldots (a)$$

By the use of logarithms n may be found from (a) if S is given or vice versa.

Example. What annuity will satisfy a debt of \$4600 in ten years, money being worth 5 per cent?

$$(a) \text{ becomes, } S = \frac{4600\,(1.05)^{10}\,(.05)}{(1.05)^n - 1} = \frac{230\,(1.05)^{10}}{(1.05)^{10} - 1}$$

$$\log\,(1.05)^{10} = .021189 \times 10 = .211890$$
$$\log\ 230 = 2.361728$$
$$\log 230\,(1.05)^{10} = 2.573618$$
$$(1.05)^{10} = \text{anti log } .211890 = 1.6289$$
$$(1.05)^{10} - 1 = 1.6289 - 1 = .6289$$
$$\log S = \log 230(1.05)^{10} + \text{colog } .6289 = \begin{cases} 2.573618 \\ .201418 \end{cases}$$
$$2.775036$$
$$\therefore\ S = \$595.71.$$

EXERCISE XXIX.

Interest and Annuities.

1. Find what \$1 would amount to at 6 per cent, compounded annually in 20 years.

2. What sum will in 8 years at 5 per cent compounded annually, amount to \$1327.67?

3. A certain principal will in 7 years at a certain rate, simple interest, amount to \$1136, and in 10 years to \$1280. Find principal and rate.

4. In what time will \$960 at 6 per cent, annually compounded, amount to \$1190.48?

5. Find the difference between the amount of \$1200 when compounded annually at 6 per cent and when compounded quarterly at same rate, in ten years.

6. Find the present worth of \$7500, due in 6 years, if money is worth 5 per cent compounded annually. That is, find the principal that will amount to \$7500 in 6 years, at 5 per cent compound interest.

7. A church borrows \$1000, and renews its note every six months at an increase of 10 per cent. How long will it take the note to reach \$4594.97?

8. How long will it take a sum of money to double itself at $5\frac{1}{2}$ per cent compound interest, compounded annually?

9. An institution borrows \$10,000. What amount must it set aside yearly to pay the debt in 15 years, money being worth 5 per cent?

10. What annual premium must an insurance company charge that it may pay a policy holder \$1000 at the end of 15 years, and still make \$200, if money is worth 5 per cent?

PLANE TRIGONOMETRY.

PLANE TRIGONOMETRY.

PART I.

THE RELATION BETWEEN ANGLE AND LINE.

ARTICLE 1. Trigonometry is a branch of mathematics which is concerned with the estimation of lengths, areas, and volumes, by using the relation between angle and line as well as that between line and line.

Geometry affords no *general* relation between angles and lines; it offers no method of comparison between an angle expressed in degrees and a line expressed in feet or inches; trigonometry, however, enables us to make such comparisons.

Suppose, for instance, we wished to measure the distance from a point A to an inaccessible object P; further, imagine a wood W so situated that B would be the first point to the right of A from which an unobstructed view of P could be obtained. Now, we might lay off a known length AB (called a base line) and with a transit, measure the angles BAP and ABP; we could then calculate the

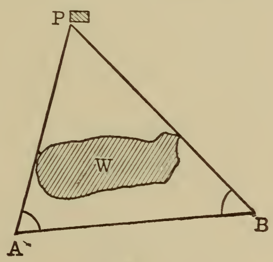

Fig. 1.

angle APB. Now by geometry we could obtain no information from these data regarding the length of AP, but trigonometry enables us to involve the known angles and side AB in calculation, so that AP can be determined in linear units.

ART. 2. The necessity of being able to combine in calculations, angles expressed in degrees, minutes, and seconds, with lines expressed in linear units, led to the invention of six new expressions called *functions* of angles. In general, *one quantity is said to be a function of another when it depends upon it for its value;* now the six trigonometrical functions are usually expressed as ratios between the sides of a right triangle, in which the angle concerned occurs, and although they are entirely linear, depend directly upon the angle for their values.

Being linear, they readily combine with the sides, and depending as they do upon the angle for their value, the latter becomes a useful element in calculation.

Trigonometric Ratios.

ART. 3. We will now take an angle KMH (see Fig. 2)

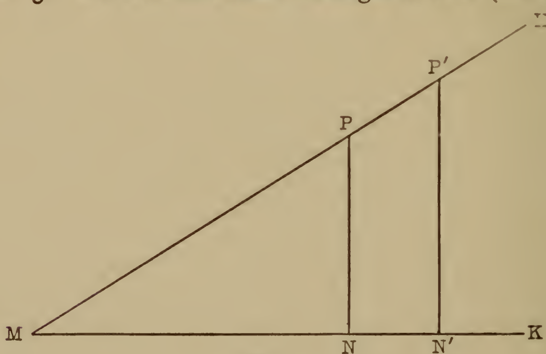

Fig. 2.

having the special size $32° \ 15'$; upon one of the sides MH take a point P and draw PN perpendicular to MK. We thus obtain a right-angled triangle PNM, with the right angle at N. To find the ratio of NP to MP we may proceed as follows: taking any unit of length, say $\frac{1}{60}$ inch, we find by careful measurement, $NP = 56.5$, $MP = 106$; hence the ratio $\frac{NP}{MP} = \frac{56.5}{106} = 0.5330$.

Constructing in like manner any other perpendicular $N'P'$ and taking the millimeter, suppose, as our linear unit, we find $N'P' = 30.75$ and $MP' = 57.7$; hence $\frac{N'P'}{MP'} = 0.5326$.

We know from geometry that $\frac{NP}{MP} = \frac{N'P'}{MP'}$; now our actual measurements agree very approximately, and the mean of the results gives 0.5328. It is possible mathematically to calculate the ratio of the side opposite to the hypotenuse for a given angle to any degree of accuracy; for the special case of $32° \ 15'$ we find

$$\frac{\text{length of opposite side}}{\text{length of hypotenuse}} = 0.5336,$$

a quantity to which our rough measurements approximate, and which is independent of the unit of linear measurement. *This quantity is called the sine* of $32° \ 15'$.

Now taking the angle SMK which is $21° \ 50'$ and draw-

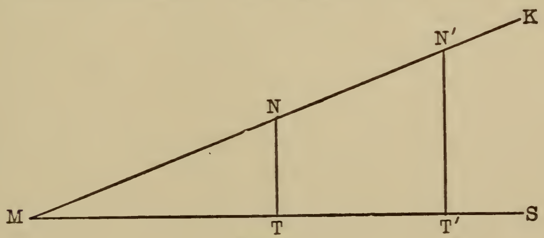

Fig. 3.

ing any two perpendiculars NT and $N'T'$, we obtain in terms of $\frac{1}{60}$ inches, $NT = 27.5$, and $MN = 74.0$, giving

$\frac{NT}{MN} = \frac{27.5}{74.0} = 0.3716$; again in millimeters, $N'T' = 19.75$, and $MN' = 53.25$, giving $\frac{N'T'}{MN'} = \frac{19.75}{53.25} = 0.3710$. The mean of these results is 0.3713, and the accurate value calculated to five places is $\frac{\text{side opposite}}{\text{hypotenuse}} = 0.37191$. Hence we say the sine of $21° 50' = 0.37191$. It is easy to see that the sine, although constant in value for any given angle, is dependent directly upon its size; hence we say *the sine of an angle is a function of it.*

ART. 4. The six trigonometric functions are as follows: — a right triangle MNR being constructed with $\angle M$ as one acute angle.

The *sine* of M is the ratio of the opposite side to the hypotenuse, or $\frac{RN}{MN}$.

The *cosine* of M is the ratio of the adjacent side to the hypotenuse, or $\frac{MR}{MN}$.

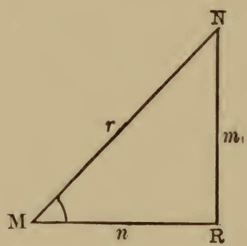

Fig. 4.

The *tangent* of N is the ratio of the opposite side to the adjacent side, or $\frac{RN}{MR}$.

The *cotangent* of M is the ratio of the adjacent side to the opposite side, or $\frac{MR}{RN}$.

The *secant* of M is the ratio of the hypotenuse to the adjacent side, or $\frac{MN}{MR}$.

The *cosecant* of M is the ratio of the hypotenuse to the opposite side, or $\frac{MN}{RN}$.

In the right triangle MNR the capital letters represent angles and the corresponding small letters represent the opposite sides, R being the right angle.

Write	sine	of M . . .	$\sin M$.
	cosine	of M . . .	$\cos M$.
	tangent	of M . . .	$\tan M$.
	cotangent	of M . . .	$\cot M$.
	secant	of M . . .	$\sec M$.
	cosecant	of M . . .	$\csc M$.

Then

$$\sin M = \frac{m}{r} \quad (a). \qquad \cot M = \frac{n}{m} \quad (d).$$

$$\cos M = \frac{n}{r} \quad (b). \qquad \sec M = \frac{r}{n} \quad (e).$$

$$\tan M = \frac{m}{n} \quad (c). \qquad \csc M = \frac{r}{m} \quad (f).$$

ART. 5. It will be observed by comparing (a) with (f), (b) with (e), and (c) with (d), that $\csc M = \frac{1}{\sin M}$, $\sec M = \frac{1}{\cos M}$, and $\cot N = \frac{1}{\tan M}$; hence if the three functions, sine, cosine, and tangent, are known, the others may be easily found by taking the reciprocals of these.

The functions most commonly used are the sine, cosine, tangent, and cotangent.

ART. 6. By producing the two sides including the angle M, and drawing perpendiculars from various points of NM produced, to NR produced, it is evident, by similar triangles, that so long as M remains the same, the ratios do not change, although both terms of any ratio

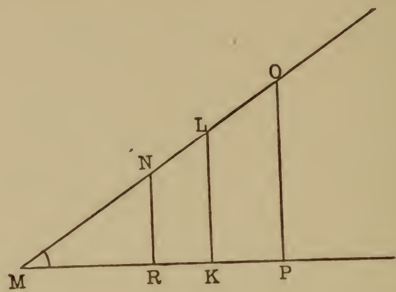

Fig. 5.

alter in value; thus, Produce MN to O and MR to P; draw perpendiculars to MP from MO. Then sin $M = \frac{RN}{MN}$ in triangle MNR; sine $M = \frac{KL}{ML}$ in MKL, and sine $M = \frac{OP}{MO}$ in MOP, but $\frac{NR}{MN} = \frac{KL}{ML} = \frac{OP}{MO}$ by similar triangles.

On the other hand, if M changes, the ratios change. For, let M increase, and suppose the hypotenuse to remain the same, then the side opposite increases, and hence the ratio of opposite side to hypotenuse increases, etc.

When the angle M is zero, since the side M decreases with the angle, and ultimately vanishes, the sine of zero degrees takes the form $\frac{m}{r} = \frac{0}{r} = 0$. In the case of the tangent we have $\frac{0}{n} = 0$. Thus we find sin $0° = 0$, tan $0° = 0$. We further note that, with a decreasing angle, the hypotenuse r becomes more and more nearly equal to n.

and when the angle M is finally zero, $r = n$, and hence the cosine and secant each become $\frac{n}{n} = 1$. In the case of the cotangent and cosecant, since it is the side m that now decreases as the angle becomes less, when M vanishes we have for these functions the ratios $\frac{n}{0}$ and $\frac{r}{0}$, each of which equals infinity. We thus obtain the values $\sin 0° = 0$, $\cos 0° = 1$, $\tan 0° = 0$, $\cot 0° = \infty$, $\sec 0° = 1$, $\operatorname{cosec} 0° = \infty$. By a similar process of reasoning we find $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90° = \infty$, $\cot 90° = 0$, $\sec 90° = \infty$, $\operatorname{cosec} 90° = 1$.

A comparison of the results, which are of considerable importance and should be carefully noted, shows that the values of the sine and cosine of an angle from zero to 90° cannot be greater than unity, while the values of the other functions vary between zero to infinity.

ART. 7. Since 60° and 45° are angles of equilateral or isosceles triangles, geometry enables us to find their functions very easily, and also those of 30°, which is half the angle of an equilateral triangle.

To Find the Functions of 45°.

ART. 8. Let xyz be any isosceles right triangle, y being the right angle. Then $\angle x = \angle z = 45°$, and as the sides xy and yz can be of any length, we will put

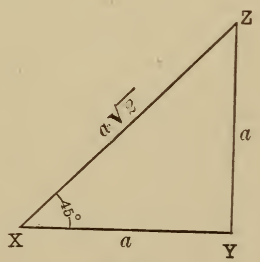

Fig. 6

$$xy = yz = a.$$

Now $$\overline{xz}^2 = \overline{xy}^2 + \overline{yz}^2.$$

$$\therefore \overline{xz}^2 = a^2 + a^2 = 2a^2$$

and $$xz = a\sqrt{2}.$$

Thus we get,

$$\sin 45^\circ = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{a}{a} = \frac{1}{1} = 1$$

$$\cot 45^\circ = \frac{a}{a} = \frac{1}{1} = 1$$

$$\sec 45^\circ = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{cosec}\, 45^\circ = \frac{a\sqrt{2}}{a} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

It will be noticed that,

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2} = 0.7071 +$$

$$\tan 45^\circ = \cot 45^\circ = 1$$

$$\sec 45^\circ = \cos 45^\circ = \sqrt{2} = 1.4142 +$$

To Find the Functions of 60° and 30°.

Art. 9. Let ABC be an equilateral triangle. Draw

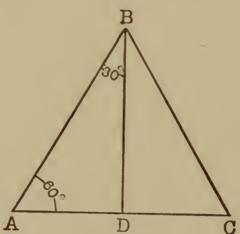

Fig. 7.

the perpendicular BD from B to AC; then $\angle BAD = 60°$ and $\angle ABD = 30°$, also $\angle ADB = 90°$.

Now $AD = \frac{1}{2} AB$; let x be the length of AB, then $AD = \frac{x}{2}$;

and since $$\overline{DB}^2 + \overline{AD}^2 = \overline{AB}^2$$

we get $$\overline{DB}^2 + \left(\frac{x}{2}\right)^2 = x^2.$$

$$\therefore DB^2 = x^2 - \frac{x^2}{4} = \frac{3}{4}x^2$$

and $$BD = \frac{\sqrt{3}}{2}x.$$

Hence $$\sin 60° = \frac{\frac{\sqrt{3}}{2}x}{x} = \frac{\sqrt{3}}{2} \qquad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos 60° = \frac{\frac{1}{2}x}{x} = \frac{1}{2} \qquad \sec 60° = 2$$

$$\tan 60° = \frac{\frac{\sqrt{3}}{2}x}{\frac{1}{2}x} = \sqrt{3} \qquad \cos 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

If we now note that DB is the side adjacent to the angle $ABD = 30°$ and AD the side opposite to it, we further obtain

$$\sin 30° = \frac{\frac{1}{2}x}{x} = \frac{1}{2} \qquad \cot 30° = \sqrt{3}$$

$$\cos 30° = \frac{\frac{\sqrt{3}}{2}x}{x} = \frac{\sqrt{3}}{2} \qquad \sec 30° = \frac{2\sqrt{3}}{3}$$

$$\tan 30° = \frac{\frac{1}{2}x}{\frac{\sqrt{3}}{2}x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cos 30° = 2.$$

Comparing the above two sets of values, we find,

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2} = 0.8660$$

$$\cos 60^\circ = \sin 30^\circ = \frac{1}{2} = 0.5000$$

$$\tan 60^\circ = \cot 30^\circ = \sqrt{3} = 1.7321$$

$$\cot 60^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3} = 0.5774$$

$$\sec 60^\circ = \cos 30^\circ = 2 = 2.0000$$

$$\cos 60^\circ = \sec 30^\circ = \frac{2\sqrt{3}}{3} = 1.1547$$

A simple reference to the figure has shown that the functions of 60° are the co-functions of 30°, and conversely; this would clearly be true for any right triangle and any acute angles.

Since the latter are complementary, the significance of the prefix "Co" is revealed. It is simply an abbreviation for *complementary*. That is, *a function of any angle is the corresponding co-function of its complementary angle, and conversely.*

Since angles of 45°, 30°, and 60° are of frequent occur-

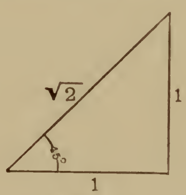

Fig. 6a.

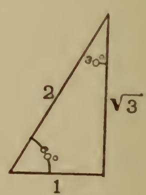

Fig. 7a.

rence, the functions of these angles are important. It will be found easy to recall the numerical results of Articles 8 and 9 if we assign special values to the sides of the triangles in Figs. 6 and 7; this we are at liberty to do, as the values of the functions are independent of the unit of length chosen. In Article 8, putting $a = 1$ we have $xz = \sqrt{2}$; again, in Article 9, if $x = 2$, $AD = 1$, and $BD = \sqrt{3}$. We thus get Figs. 6*a* and 7*a*, from which it is easy to obtain the numerical value of any function of 45°, 30°, or 60° when required; for instance, a glance at Fig. 7*a* shows that $\sin 60 = \frac{\sqrt{3}}{2}$, $\sin 30 = \frac{1}{2}$, etc.

The functions of angles in general cannot be found thus simply by geometry, but are estimated from series, into which the six functions have been developed by methods of higher mathematical analysis. These values and their logarithms are set down in tables, which record them to single minutes, and sometimes to smaller parts of a degree. By a method called interpolation, explained in connection with these tables, the functions of any angle, or the logarithms of these functions, may be found.

ART. 10. As already explained, the six functions can be grouped into three pairs of reciprocals (see § 5); thus in the triangle ABC,

$$\left.\begin{array}{l} \csc A \sin A = 1 \\ \sec A \cos A = 1 \\ \tan A \cot A = 1 \end{array}\right\} (1)$$

The question now arises, How can we find further relationships among the functions? We have a right-angled triangle to work upon, and have already used the complementary property of the acute angles to discover that the functions of the one are the co-functions of the other. Now what further property of the right triangle remains to be investigated? Can we obtain new relations among the functions from the equation $a^2 + b^2 = c^2$? Clearly we can by division; that is, if we take each term separately and divide by it we shall obtain three new equations involving the squares of the functions.

Taking

$a^2 + b^2 = c^2$	$a^2 + b^2 = c^2$	$a^2 + b^2 = c^2$
Dividing by c^2	Dividing by b^2	Dividing by a^2
$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$	$\frac{a^2}{b^2} + 1 = \frac{c^2}{b^2}$	$1 + \frac{b^2}{a^2} = \frac{c^2}{a^2}$
$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$	$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$	$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$
$\sin^2 A + \cos^2 A = 1$	$\tan^2 A + 1 = \sec^2 A$	$1 + \cot^2 A = \csc^2 A$

We thus obtain

(2) $\sin^2 A + \cos^2 A = 1$; $\sin A = \sqrt{1 - \cos^2 A}$; $\cos A = \sqrt{1 - \sin^2 A}$

(3) $\sec^2 A = 1 + \tan^2 A$; $\sec A = \sqrt{1 + \tan^2 A}$; $\tan A = \sqrt{\sec^2 A - 1}$

(4) $\text{cosec}^2 A = 1 + \cot^2 A$; $\text{cosec}\, A = \sqrt{1 + \cot^2 A}$; $\cot A = \sqrt{\csc^2 A - 1}$

It will be noticed that since $\sin A = \frac{a}{c}$ and $\cos A = \frac{b}{c}$ $\therefore$

$$\frac{\sin A}{\cos A} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} = \tan A.$$

We thus find two important relations:

$$\tan A = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A}.$$

EXERCISE I.

Functions as Ratios.

With the usual notation, large letters representing the angles of a right triangle, and the corresponding small letters the sides opposite, find the functions of B, A being the right angle, when,

1. $a = 17, b = 8, c = 15.$

2. $a = 15, b = 9, c = 12.$

3. $a = 12, b = 10, c = 7\frac{1}{2}.$

4. $a = 35.7,\ b = 31.5,\ c = 16.8.$
5. $a = 5.55,\ b = 4.44,\ c = 3.33.$
6. $a = 17.85,\ b = 8.4,\ c = 15.75.$
7. $a = 25.8,\ b = 15.48.$
8. $b = 158.1,\ c = 74.4.$
9. $a = 61,\ b = 11.$
10. $a = 1.\quad c = \frac{3}{5}.$
11. $c = 2b.$
12. $c + b = 1\frac{2}{5}a.$
13. $a - c = \frac{1}{2}b.$

Find the other sides of the right triangle, if,

14. $\sin B = \frac{4}{11}$ and $a = 22.$
15. $\cos B = \frac{3}{5}$ and $c = 6.$
16. $\tan B = \frac{11}{60}$ and $a = 61.$
17. $\sec C = \frac{5}{3}$ and $b = 9.6.$

18. Two straight roads make an angle of 40°; a man walks down one road $1\frac{1}{2}$ miles, and then crosses over to the second road in a straight line, meeting it at right angles. How far from the starting point will he be in a direct line, if $\sin 40° = .6428$?

19. The grade of a railroad track is about 10%, that is, it makes an angle of 6° with the horizontal. What weight can a locomotive pull up the grade, if it can haul 500 tons on the level? $\sin 6° = .105.$

Find the other functions, if,

20. $\sin x = \frac{3}{5}.$
21. $\sin 50° = .766.$
22. $\tan 45° = 1.$
23. $\sin 90° = 1.$
24. $\cot 60° = \frac{1}{3}\sqrt{3}.$
25. $\sec 30° = \dfrac{2}{\sqrt{3}}.$

Find the angle A, given;

26. $\sin A = \cos 2A$.

Solution. To compare the two sides of this equation, it is necessary to express both in the same function, and since $\sin A = \cos (90 - A)$ [or $\cos 2A = \sin (90 - 2A)$], $\cos (90 - A) = \cos 2A$ [or $\sin A = \sin (90 - 2A)$].

If the same function of two angles are equal, the angles themselves must be equal, supposing them both in the same quadrant.

$$\therefore 90 - A = 2A, \text{ whence } A = 30^\circ$$

27. $\tan B = \cot (45 - \frac{1}{2} B)$.

28. $\sin 3x = \cos (2x - 270)$.

29. $\sec (2x - 30^\circ) = \csc (180 - x)$.

30. $\cos \frac{A}{4} = \sin \left(60 + \frac{A}{4}\right)$.

Identities

Art. 11. It will be remembered that an identity differs from an equation in that the two terms are equal for *all* values of the unknown quantity, hence the two terms are exactly the same in value, but differ merely in form. In other words, the same relation is expressed in two different ways. For example, the words "air" and "atmosphere" are two different ways of expressing the same idea. Likewise, $\tan x = \frac{\sin x}{\cos x}$.

It is often required for convenience or for simplicity to change the form of a trigonometric expression, and the fundamental relations already found, namely, $\sin^2 x + \cos^2 x = 1$, $\tan x = \frac{\sin x}{\cos x}$; $\tan x \cdot \cot x = 1$, $\sin x \cdot \csc x = 1$, $\cos x \cdot \sec x = 1$, make it readily possible to do this.

EXAMPLE. Prove that $\csc^2 A\ (1 - \sin^2 A) = \cot^2 A$.

Since $1 - \sin^2 A = \cos^2 A$ and $\csc A = \dfrac{1}{\sin A}$

$$\therefore\ \csc^2 A\ (1 - \sin^2 A) = \frac{\cos^2 A}{\sin^2 A} = \cot^2 A.$$

As a general rule, it is advisable to reduce all the terms of an identity to their simplest terms and to perform the indicated operations in order to show the equality.

EXERCISE II.

Identities.

Prove the following identities:

1. $\tan x \sin x + \cos x = \sec x$.
2. $\sin A \csc A \tan A = \tan A$.
3. $\sec^2 m \csc^2 m \sin^4 m + 1 = \sec^2 m$.
4. $3 \sin^2 x - 2 + 3 \cot x \cos x \sin x = 1$.
5. $\dfrac{1 - 2 \sin x \cos x}{\cos^2 x \sin^2 x} = (\sec x - \csc x)^2$
6. $\sin^4 x - \cos^4 x + 2 \cos^2 x = 1$.
7. $\dfrac{\cos^2 y}{1 - \sin y} = 1 + \sin y$.
8. $\dfrac{\sin A}{\cos A \tan^2 A} = \cot A$.
9. $\cos^2 x (\sec x + 1) + \dfrac{\sin^2 x}{\sec x + 1} = 2 \cos x$.
10. $\dfrac{\cos n \cot n - \sin n \tan n}{\csc n - \sec n} = 1 + \sin n \cos n$

Trigonometric Equations.

ART. 12.

EXAMPLE. Sec $x + \tan x = \sqrt{3}$. (1) Find x.

In order to solve this equation, since it involves the two unknowns, sec x and tan x, it is clearly necessary to have another independent equation involving the same unknowns. We can always find at least one equation between any two functions among the relations already established between the functions. In this case we have,

$$\tan^2 x + 1 = \sec^2 x,$$

or $$\sec^2 x - \tan^2 x = 1 \quad (2)$$

Dividing (2) by (1); $\sec x - \tan x = \dfrac{1}{\sqrt{3}}$ (3)

Adding (1) to (3);

$$\sec x + \tan x = \sqrt{3}$$

$$2 \sec x = \frac{1}{\sqrt{3}} + \sqrt{3} = \tfrac{1}{3}\sqrt{3} + \sqrt{3} = \tfrac{4}{3}\sqrt{3}$$

$$\therefore \sec x = \tfrac{2}{3}\sqrt{3}$$

whence $$x = 30^\circ.$$

Another method which can be frequently used, is to express all the trigonometric quantities involved in the equation in terms of the *same* function of the unknown angle; this method often results in a quadratic equation which can be solved in the usual manner.

Suppose we have $2\sqrt{1 - \sin^2 A} + \sec A = 3$

then $$2 \cos A + \frac{1}{\cos A} = 3$$

$$2 \cos^2 A + 1 = 3 \cos A$$

$$2 \cos^2 A - 3 \cos A + 1 = 0$$

$$(2 \cos A - 1)(\cos A - 1) = 0$$

$$2 \cos A - 1 = 0. \quad \text{also, } \cos A - 1 = 0$$

$$\therefore \cos A = \tfrac{1}{2} \quad \text{and } \cos A = 1$$

The angle whose cosine is $\frac{1}{2}$ is 60°; the latter result where the cosine is 1 gives an angle 0°. Hence we have two answers, namely, 60° and 0°.

EXERCISE III.

Trigonometric Equations.

Find the angle in the following equations:

1. $\csc x = \frac{2}{3} \tan x$.
2. $\tan A + \cot A = 2$.
3. $\sec^2 x + \csc^2 x = 4$.
4. $\sin y + \csc y = -\frac{5}{2}$.
5. $3 \sin A = 2 \cos^2 A$.
6. $\sin^2 x = 2 - 3 \cos^2 x$.
7. $\cot x + \sin x = \dfrac{5}{4 \sin x}$.
8. $\sin^2 x + \sin^2 x \tan^2 x = 1$.

PART II.

SOLUTION OF TRIANGLES.

The Right-Angled Triangle.

Art. 13. The process by which the unknown numerical values of the parts of a triangle are computed from the known parts is called solving the triangle.

By the use of the six trigonometric functions, any right triangle may be completely solved when two of its parts, one of which is a side, are known.

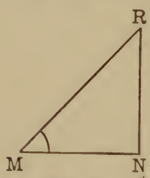

Fig. 8.

Given a right triangle MNR, right-angled at N. Also $\angle M = 35^\circ\ 36'\ 20''$ and $n = 21.674$ feet. Required $\angle R$ and the sides r and m.

I. To find R.

$\because \angle M + \angle R = 90^\circ. \quad \therefore \angle R = 90^\circ - \angle M = 54^\circ\ 23'\ 40''.$

II. To find r, having given M and n.

Here it is necessary to use a formula which includes M, n, and r; since n is the hypotenuse and r the side adjacent to M, the cosine of M is suggested.

Now $\cos M = \dfrac{r}{n}$, whence $r = \cos M \times n$.

$\therefore \log r = \log n + \log \cos M = \log 21.674 + \log \cos 35^\circ\ 35'\ 20''$

$$\begin{aligned} \log 21.674 &= 1.33594 \\ \log \cos 35^\circ\ 36'\ 20'' &= 9.91011 - 10 \\ \hline \log r &= 11.24605 - 10 \\ &= 1.24605 \\ r &= 17.622 \text{ feet.} \end{aligned}$$

III. To find m.

It is clear that M, n, and m must occur in the formula selected; since n is the hypotenuse and m is the side opposite M, the sine of M is suggested.

$$\sin M = \frac{m}{n}, \text{ whence } m = n \sin M.$$

$\therefore \log m = \log n + \log \sin M = \log 21.674 + \log \sin 35° 36' 20''$

$$\begin{aligned} \log 21.674 &= 1.33594 \\ \log \sin 35° 36' 20'' &= 9.76507 - 10 \\ \hline \text{Add, } \quad \log m &= 1.10101 \\ \therefore m &= 12.619 \text{ feet.} \end{aligned}$$

Given a right triangle ABC, right-angled at B. Also $\angle A = 63° 12' 25''$ and $a = 112.34$ feet. Required $\angle C$ and the sides b and c.

I. To find $\angle C$.

$\angle C = 90 - (63° 12' 25'') = 26° 47' 35''$.

II. To find b.

It is necessary to choose a function containing the given parts, A and a, and the required part, b. Since b is the hypotenuse and a is the side opposite to A, the sine is suggested; hence,

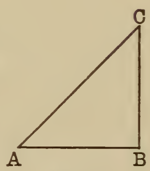

Fig. 9.

$$\sin A = \frac{a}{b}, \text{ whence } b = \frac{a}{\sin A} \text{ and}$$

$\log b = \log a + \text{colog} \sin A = \log 112.34 + \text{colog.} \sin 63° 12' 25''$

$$\begin{aligned} \log 112.34 &= 2.05053 \\ \text{colog} \sin 63° - 12' - 25'' &= 0.04932 \\ \hline \log b &= 2.09985 \\ b &= 125.85 \text{ feet.} \end{aligned}$$

III. To find c.

Here A and a are concerned, which suggests the tangent or cotangent. We may use either $\tan A = \frac{a}{c}$ or $\cot A = \frac{c}{a}$.

Since c is required, the latter is preferable;

whence, $$c = a \cot A.$$

$$\log c = \log a + \log \cot A = \log 112.34 + \log \cot 63^\circ\ 12'\ 25'',$$

$$\begin{aligned} \log 112.34 &= 2.05053 \\ \log \cot 63^\circ\ 12'\ 25'' &= 9.70328 - 10 \\ \hline \log c &= 1.75381 \\ c &= 56.730 \text{ feet.} \end{aligned}$$

The Isosceles Triangle.

ART. 14. Since a perpendicular from the vertex of an isosceles triangle bisects the base, the solution of an isosceles triangle easily resolves itself into that of a right triangle. Let MNP be an isosceles triangle in which $MN = PN$. Drop the perpendicular NR; then in the right triangle MNR, $\angle MNR = \frac{1}{2} \angle N$, and $MR = \frac{1}{2} MP$; hence if any two of the unequal parts, one of which is a side, be given, the unknown parts may be found.

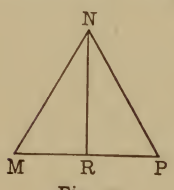

Fig. 10.

For example, suppose we have $MN = x = NP = 62.231''$ and $\angle N = 102^\circ\ 34'\ 12''$, to find $MP = n$ and $\angle M = \angle P$.

Since $\angle MNR = \frac{1}{2} \angle M$, $\angle MNR = 51^\circ\ 17'\ 06''$.

I. To find $\angle M$.

$$M = 90 - (51^\circ - 17' - 06'') = 38^\circ - 42' - 54''.$$

II. To find MR.

$MR = \frac{1}{2} MP = \frac{1}{2} n$; here x, $\frac{1}{2} N$ and $\frac{1}{2} n$ are involved,

hence $\sin \frac{1}{2} N = \frac{\frac{1}{2} n}{x}$ or $\frac{1}{2} n = x \sin \frac{N}{2}$.

$$\therefore \log\ (\tfrac{1}{2}\ n) = \log x + \log \sin \tfrac{1}{2} N$$

$$\log 62.231 = 1.79401$$

$$\log \sin 51^\circ\ 17'\ 06 = 9.89224 - 10$$

$$\log\ (\tfrac{1}{2}\ n) = 1.68625$$

$$\tfrac{1}{2} n = 48.557$$

$$n = 97.114.$$

Solution of Regular Polygons.

ART. 15. Since a regular polygon may be divided into isosceles triangles by lines from its center to its several vertices, its solution depends directly upon that of the isosceles triangle, and, therefore, ultimately upon the solution of a right triangle. For example:

Let $ABCDE$ be a regular pentagon; from its center O draw OA, OB, OC, OD, and OE, to the vertices, thus dividing it into five isosceles triangles. A solution of one of these triangles will lead to a solution of all, and hence to a solution of the polygon.

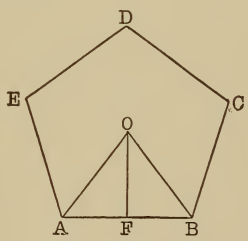

Fig. 11.

By geometry the sum of the angles at the center of the pentagon = 360°, and since the five angles there formed are all equal, each one, say AOB, is equal to $\frac{1}{5}$ of 360° = 72°. In general, if the number of sides of a polygon be n, then each central angle will be $\frac{360^\circ}{n}$, or one half of each central angle will be $\frac{180^\circ}{n}$.

Drop a perpendicular OF from O to AB, then

$$\angle AOF = \frac{180}{n} = \frac{180}{5} = 36°.$$

In general, calling the apothem of a polygon h, and one side of the polygon as AB, x; then if the number of sides is represented by n, and the radius OA by r, we have

$$\sin\frac{180°}{n} = \frac{\frac{1}{2}x}{r}, \text{ whence } r = \frac{\frac{1}{2}x}{\sin\frac{180°}{n}}$$

and
$$\frac{1}{2}x = r\sin\frac{180°}{n};$$

Again,
$$\tan\frac{180°}{n} = \frac{\frac{1}{2}x}{h}, \text{ whence } h = \frac{\frac{1}{2}x}{\tan\frac{180°}{n}},$$

$$\therefore h = \frac{1}{2}x\cot\frac{180°}{n} \text{ and } \frac{1}{2}x = h\tan\frac{180°}{n}.$$

In the pentagon $ABCDE$ let AB = 9.7232 inches = x. Then $\angle\frac{1}{2} AOB = \angle AOF = \frac{180°}{5} = 36°$.

I. To find r.

Here 36°, $\frac{1}{2}x$ and r, are involved in the right triangle AOF. Since r is the hypotenuse and $\frac{1}{2}x$ the side opposite the angle 36°, we have,

$$\sin 36° = \frac{\frac{1}{2}x}{r}, \text{ whence } r = \frac{\frac{1}{2}x}{\sin 36}.$$

$$\therefore \log r = \log\tfrac{1}{2}x + \text{colog}\sin 36° = \log 4.8616 + \text{colog}\sin 36°$$

$$\log 4.8616 = 0.68678$$
$$\text{colog}\sin 36° = 0.23078$$
$$\log r = 0.91756$$
$$\therefore r = 8.271 \text{ inches.}$$

To find the perimeter:

We had $\sin 36° = \dfrac{\frac{1}{2}x}{r}$

$$\therefore \ \tfrac{1}{2}x = \sin 36° \, r$$

$$\begin{aligned} \log \sin 36° &= 9.76922 \\ \log r &= \ \ .91756 \\ \hline \log \tfrac{1}{2}x &= 0.68678 \\ \log 10 &= \underline{1.00000} \\ & \quad\ 1.68678 \end{aligned}$$

$$\therefore \text{ perimeter} = 48.618 \text{ inches.}$$

Since the perimeter $= 5x = 10(\frac{1}{2}x) = p$ (suppose) $\therefore \log p = \log 10 + \log \frac{1}{2}x$.

To find h:

$36°$, $\frac{1}{2}x$ and h are involved; now $\tan 36 = \dfrac{\frac{1}{2}x}{h}$ or $\cot 36° = \dfrac{h}{\frac{1}{2}x}$, whence $h = \frac{1}{2}x \cot 36°$; $\therefore \log h = \log \frac{1}{2}x + \log \cot 36° = \log 4.8616 + \log \cot 36°$.

$$\begin{aligned} \log 4.8616 &= 0.68678 \\ \log \cot 36° &= 0.13874 \\ \hline \log h &= \ \ .82552 \\ h &= \ \ 6.6915. \end{aligned}$$

Areas.

ART. 16. To find the area of a triangle.

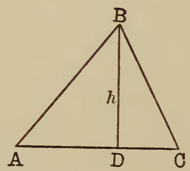

Fig. 12.

In the triangle ABC, call the base b, and the altitude h; then by geometry,

$$\text{area } ABC = \tfrac{1}{2} b \times h. \quad (1)$$

This formula applies when base and altitude are given.

Again, in the right triangle ABD, $\sin \angle A = \dfrac{h}{c}$ (small letters represent the sides opposite the angles indicated by large letters), whence $h = c \sin A$. Substituting this value of h in (1), we get

$$\text{area } ABC = \tfrac{1}{2} bc \sin A. \quad (2a)$$

This formula applies when two sides and the included angle are given. It is evident, by drawing perpendiculars successively from the other vertices A and C, we obtain,

$$\text{area } ABC = \tfrac{1}{2} ac \sin B \quad (2b)$$
$$\text{and area } ABC = \tfrac{1}{2} ab \sin C. \quad (2c)$$

Again, in ABD, $\cot A = \dfrac{AD}{h}$, and in BDC, $\cot C = \dfrac{DC}{h}$

$$\therefore \cot A + \cot C = \frac{AD + DC}{h} = \frac{b}{h}$$

whence $h = \dfrac{b}{\cot A + \cot C}$. Substituting this value of h in (1), we get,

$$\text{area } ABC = \frac{\tfrac{1}{2} b^2}{\cot A + \cot C}. \quad (3a)$$

By drawing the altitude from A and C successively, it easily follows that,

$$\text{area } ABC = \frac{\tfrac{1}{2} a^2}{\cot B + \cot C} \quad (3b)$$

$$\text{and area } ABC = \frac{\tfrac{1}{2} c^2}{\cot A + \cot B}. \quad (3c)$$

This formula applies when two angles and the included side are given.

The above formulæ (3 a), (3 b) and (3 c) will be found later to reduce to the forms:

$$\text{area } ABC = \frac{b^2 \sin A \sin C}{2 \sin (A + C)} = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}$$
$$= \frac{c^2 \sin A \sin B}{2 \sin (A + B)}.$$

EXAMPLE. Find the area of the triangle ABC, if $\angle B = 32° 16'$; $\angle C = 25° 39'$ and $a = 23.27$ inches.

Here
$$\text{area } ABC = \frac{\frac{1}{2} a^2}{\cot B + \cot C}$$
$$= \frac{\frac{1}{2} (23.27)^2}{\cot (32° 16') + \cot (25° 39')}$$

$$\cot 32° 16' = 1.5839$$
$$\cot 25° 39' = 2.0825$$
$$\cot B + \cot C = 3.6664$$

$$\log \text{area } ABC = \text{colog } 2 + 2 \log 23.27 + \text{colog } 3.6664$$

$$\text{colog } 2 = 9.69897 - 10$$
$$2 \log 23.27 = 2.73360$$
$$\text{colog } 3.6664 = 9.43576 - 10$$
$$\log \text{area} = 21.86833 - 20$$
$$= 1.86833$$
$$\text{area} = 73.847 \text{ sq. in.}$$

It is evident, provided we have the necessary known parts, that the area of an isosceles triangle may always be found from formula 1, since the altitude and base may be calculated, if not given (see Art. 16).

Area of the Regular Polygon.

ART. 17. Since the regular polygon may always be divided into as many isosceles triangles as it has sides, by drawing radii to its vertices, its area is readily found from previous formulæ. Calling the perimeter, p; one side, c; and its apothem, h; then by geometry, area of polygon $= \frac{1}{2}ph$, and $p = nc$, where n is the number of sides; also the angle of each isosceles triangle at the center equals $\frac{360^\circ}{n}$ or the half angle $= \frac{180^\circ}{n}$.

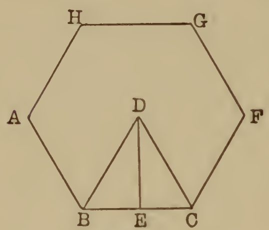

Fig. 13.

With these data p and h can be calculated from the given parts.

EXAMPLE. To find the area of a regular hexagon, $ABCFGH$, given $p = 74.116''$. Draw the perpendicular DE in the triangle BDC (D being the center).

$$\angle BDE = \frac{180^\circ}{6} = 30^\circ, \text{ and } c = \frac{74.116}{6} = 12.353.$$

In the triangle BDE, $\cot BDE = \frac{DE}{BE}$; $\therefore DE = BE \cot BDE$. Now $DE = h$ and $BE = \frac{1}{2}c$. $\therefore h = \frac{1}{2}c \cot BDE = \frac{1}{2}c \cot 30^\circ$. $\log h = \log \frac{1}{2}c + \log \cot 30^\circ = \log 6.176 + \log \cot 30^\circ$.

$$\begin{aligned} \log 6.176 &= 0.79071 \\ \log \cot 30^\circ &= 0.23856 \\ \hline \log h &= 1.02927 \\ h &= 10.697 \end{aligned}$$

Area $= \frac{1}{2}hp$, whence log area $=$ colog 2 $+ \log 10.697 + \log 74.116$,

$$\begin{aligned} \text{colog } 2 &= 9.69897 - 10 \\ \log 10.697 &= 1.02927 \\ \log 74.116 &= 1.86992 \\ \hline \log \text{area} &= 2.59816 \\ \text{area} &= 396.43 \text{ sq. in.} \end{aligned}$$

It is also possible to find the parts of a regular polygon having a given area, since the central angles can always be obtained if the number of sides is known.

Example. Find the perimeter of a regular decagon whose area is 336.72 sq. ft.

Since area $= \frac{1}{2} hp$ and $p = 10\,c$

$$\tfrac{1}{2} hp = 336.72 \text{ or } 5\,hc = 336.72 \qquad (1)$$

Calling the central angles each C, then $\tan \frac{1}{2} C = \dfrac{\frac{1}{2} c}{h}$.

$\therefore\ h = \dfrac{\frac{1}{2} c}{\tan \frac{1}{2} C}$. Substituting this value of h in (1),

area $= A = 5\,c\,\dfrac{\frac{1}{2} c}{\tan \frac{1}{2} C}$. $\therefore\ \frac{5}{2} c^2 = 336.72 \tan \frac{1}{2} C$, hence

$c^2 = 134.688 \tan \frac{1}{2} C = 134.688 \tan 18° \left(\text{since } \frac{1}{2} C = \dfrac{180°}{10} = 18°\right)$; whence $2 \log c = \log 134.688 + \log \tan 18°$.

$$\begin{aligned} \log 134.688 &= 2.12933 \\ \log \tan 18° &= 9.51178 - 10 \\ \hline 2 \log c &= 1.64111 \\ \log c &= .82056 \\ c &= 6.6154 \end{aligned}$$

To find h.

$$h = \tfrac{1}{2} c \cot \tfrac{1}{2} C = \tfrac{1}{2}(6.6154) \cot 18°$$
$$\log h = \text{colog } 2 + \log 6.6154 + \log \cot 18°$$

$$\begin{aligned} \text{colog } 2 &= 9.69897 - 10 \\ \log 6.6154 &= 0.82056 \\ \log \cot 18° &= 0.48822 \\ \hline \log h &= 11.00775 - 10 \\ \log h &= 1.00775 \\ h &= 10.18 \end{aligned}$$

EXERCISE IV.

Right Triangle.

Solve the right triangle (right-angled at C), given:

1. $a = 2.3756$, $b = 6.1023$.
2. $A = 29° 13' 23''$, $b = 27.132$.
3. $B = 57° 19' 32''$, $c = 112.67$.
4. $b = .02567$, $a = .06211$.
5. $a = 3.6378$, $A = 69° 23' 45''$.

6. The shadow of a steeple 102 ft. high is 116 ft. long. Find the elevation of the sun above the horizon.

7. The guy-ropes of a derrick are 76 ft. long, and make an angle of $43° 25'$ with the ground. What is the height of the derrick, and how far from its foot are the guy-ropes anchored?

8. The elevation of a tower is $18° 12' 16''$ at a distance of 500 ft. What is the height?

9. From a point A, immediately opposite a stake B, on the opposite bank of a river, a distance of 83.25 yards is measured to C at right angles to AB, and the angle ACB is found to be $62° 19' 8''$. What is the breadth of the river?

10. From the top of a lighthouse 98 ft. high, how far is it to the most remote visible point at sea, regarding the earth as a sphere 7918 miles in diameter?

11. What is the angle of an inclined plane which rises 1 ft. in 55 ft., measured horizontally?

12. What must be the slope of a roof for a garret 42 ft. wide, that the ridge may be 16 ft. above the garret floor?

13. In a circle of 6.275 in. radius, what angle at the center will be subtended by a chord 10 inches long?

14. The angle between two lines is $44° 32' 10''$. At what distance from the point of intersection will lie the center of a circle of 6235.2 ft. in diameter, tangent to both lines?

15. The diameters of two wheels, one on a shaft, the other on a machine, are 28 inches and 21 inches respectively, and their centers are 20 ft. apart. What length of belt is necessary for them?

16. Find the perimeter of an equilateral triangle circumscribed about a circle whose radius is 13 inches.

17. In the 15th example, what change would be necessary in gear wheel and belt to double the speed of the machine?

18. What is the length of 1° on the circle of latitude through Pittsburg, latitude $40° 27'$, if the radius of the earth (regarded as a sphere) is 3959 miles?

19. From the top of a hill the angles of depression of two stakes, set in straight line with the hill, 1200 yards apart, are observed to be 18° and 8° respectively. What is the height of the hill?

20. Two roads are non-parallel. From a certain point on one of them the angles between the perpendicular to the other road and lines joining the point with two successive milestones on the other, are respectively $6° 30'$ and $12° 15'$. What is the distance between the roads from the point, on a perpendicular, to the second road?

21. Calling S the area of a right triangle,

$A = 31° 20' 27''$, $c = 211.89$. Find S.

22. $c = 12.117$, $a = 9.208$. Find S.

23. $S = 134.263$, $B = 33° 12'$. Find other parts of triangle.

24. $S = 32.73$, $c = 35.86$. Find other parts of triangle.

25. $a = \sqrt{5}$, $b = \sqrt{3}$. Find other parts of triangle.

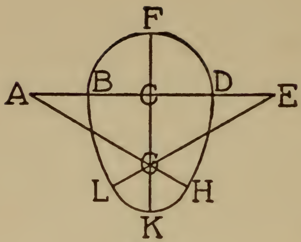

Fig. 13a.

26. How many square feet in the section of a sewer (see figure) whose dimensions are as follows:

$AB = BC = FC = CD = DE = 6'; EL = AH = 3\,AB; GL = GK = GH = 3'$?

27. A railway is 10′ from the curb on two streets intersecting at an angle of 120°; its curve at the corner is 4′ from that corner at the nearest point. The radii at the ends of the curve are ⊥ to the curb. What is the radius of the curve?

EXERCISE V.

Regular Polygons.

Call the perimeter, p; apothem, h; side, c; radius, r; number of sides, n. Solve completely the regular polygons following:

1. $n = 8$, $c = 2.7284$.

2. $n = 11$, $h = 9.2706$.

3. $c = 16.208$, $h = 24.941$. Find n.

4. $S = 224.92$, $h = 8.562$. Find p.

5. $S = 196.22$, $h = 6.768$. Find r.

6. The areas of a hexagon and an octagon are both 302.64 sq. ft. Find the difference between their perimeters.

7. How many hexagonal tiles, 3 in. on a side, will it take to pave a hallway containing 225 sq. ft.?

8. The corners of a board 2 ft. square are cut away, leaving a regular octagon. What is the area of the octagon?

9. A pentagonal fort is to have a diagonal of 500 ft. What will be the length of wall necessary to inclose it?

10. How many cu. ft. in the walls of a chimney in the form of an octagonal prism, if the apothem of a section is 1′, the thickness of the wall 4″, and the height 40′?

PART III.

FURTHER RELATIONS BETWEEN ANGLE AND LINE.

ARTICLE 18. In Part I we have already discussed certain relations between the trigonometric functions. We will now extend our investigations in this direction. It is customary to give the name goniometry to this branch of trigonometric analysis.

In Trigonometry the same rules govern the direction of lines, which are already familiar to the student, through the graphical representation of equations in Algebra. There are, however, some further conventions which we will now explain.

Definition: A directed line is one having a definite direction by which it is distinguished.

A line AB is understood as directed, and therefore measured from A towards B, while by BA we mean a line taken in the contrary direction, hence $AB = -BA$.

We refer all points in the plane to two lines at right angles: the first, CA, is horizontal and is called the Abscissa or X-axis; the second, DB, is vertical and called the ordinate or Y-axis; these cut at a point O, known as the Origin.

Lines	measured	in the	direction	OA	are	positive.
"	"	"	"	OB	"	positive.
"	"	"	"	OC	"	negative.
"	"	"	"	OD	"	negative.

An angle is conceived as generated by a line revolving from its initial position OA, which coincides with the X-

axis (Fig. 14), and extends along it towards the right. The revolution may take place either counter-clockwise or clockwise: in the former case the angle (AOA', Fig. 14) is said to be positive, in the latter (AOA'', Fig. 14), negative.

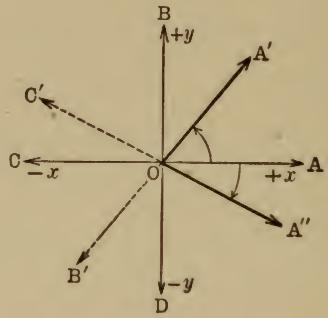

Fig. 14.

The revolving line when fixed, bounds or terminates the angle, and therefore is often alluded to as the terminal line.

The size of an angle is estimated from the horizontal diameter to the right of O, either counter-clockwise or clockwise to the terminal line, and can therefore be of any number of degrees up to plus or minus 360°, in one revolution, or may be made to contain as many positive or negative degrees as desired by repeated rotation of the revolving line.

When a line such as OA' occupies any position *between* the X and Y axes we consider it as directed from O towards A', and that lengths measured from the origin O toward the extremity A' are positive, while those taken in the opposite direction, as from O towards B', *relatively to* OA', are negative. This is true for all positions of a line between the X and Y axes, and hence we note in Fig. 14 that while OA'' is positive, OC', being in the opposite relative direction, is negative.

Heretofore only the functions of angles less than 90° have been considered. The question now arises whether the idea of the trigonometric functions can receive general extension.

We have seen that by means of a revolving line an

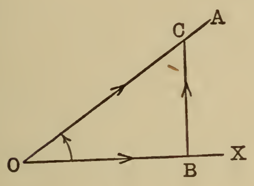

Fig. 15.

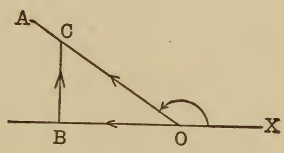

Fig. 16.

angle of any size can be obtained. Let us consider the functions of the angle *XOA* in each of the figures 15, 16, 17, and 18. In every case the angle *XOA* is traced by

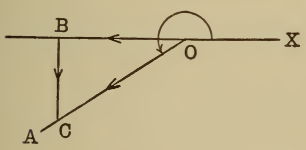

Fig. 17.

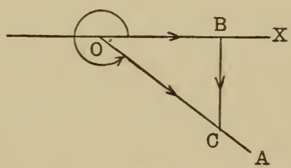

Fig. 18.

the revolving line from the initial position, as shown by the curved arrow, to a terminal position *OA*. We notice that for angles greater than 90° we no longer, as heretofore, have a side opposite; instead, however, we have a perpendicular drawn *from the X-axis to the terminal line*, so that in each case considered we have a right-angled triangle formed by three directed lines *BC*, *OB*, and *OC*. Taking careful note of the directions of these lines, we obtain the following:

Angle XOA between	Fig. 15. $0^\circ - 90^\circ$	Fig. 16. $90^\circ - 180^\circ$
$\sin XOA =$	$\frac{BC\,(+)}{OC\,(+)} = +\frac{BC}{OC}$	$\frac{BC\,(+)}{OC\,(+)} = +\frac{BC}{OC}$
$\cos XOA =$	$\frac{OB\,(+)}{OC\,(+)} = +\frac{OB}{OC}$	$\frac{OB\,(-)}{OC\,(+)} = -\frac{OB}{OC}$
$\tan XOA =$	$\frac{BC\,(+)}{OB\,(+)} = +\frac{BC}{OB}$	$\frac{BC\,(+)}{OB\,(-)} = -\frac{BC}{OB}$

Angle XOA between	Fig. 17. $180^\circ - 270^\circ$	Fig. 18. $270^\circ - 360^\circ$
$\sin XOA =$	$\frac{BC\,(-)}{OC\,(+)} = -\frac{BC}{OC}$	$\frac{BC\,(-)}{OC\,(+)} = -\frac{BC}{OC}$
$\cos XOA =$	$\frac{OB\,(-)}{OC\,(+)} = -\frac{OB}{OC}$	$\frac{OB\,(+)}{OC\,(+)} = +\frac{OB}{OC}$
$\tan XOA =$	$\frac{BC\,(-)}{OB\,(-)} = +\frac{BC}{OB}$	$\frac{BC\,(-)}{OB\,(+)} = -\frac{BC}{OB}$

Since the reciprocal of a trigonometric function has the same sign as the function, we can easily obtain the cotangents, secants, and cosecants, with their proper signs prefixed, from an inspection of the above table.

It should be noted that to each positive angle XOA corresponds one negative angle of size $360^\circ - XOA$, the trigonometric functions of which are exactly the same as those of the positive angle.

Art. 19. Now as the function values do not depend upon the unit chosen, we might select some convenient length OC, on the revolving line, as our unit.

We thus get, $\sin XOA = \dfrac{BC}{OC} = \dfrac{BC}{1} = BC$ (Fig. 15).

Again, $\cos XOA = \dfrac{OB}{OC} = \dfrac{OB}{1} = OB$ (Fig. 15).

In the remaining positions of OC we will always have $\pm BC$ and $\pm OB$ representing the sines and cosines.

This suggests that we should draw a circle of *unit* *radius*, with O as center, and endeavor, by suitable geometrical construction, to express the remaining functions as lengths, and not, as previously, by a ratio; such

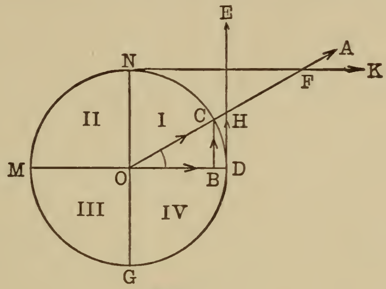

Fig. 19.

a procedure would tend toward simplicity. Extending this conception, we will define the functions as certain lines determined by the angle involved in a circle of unit radius, known as a unit circle.

Let $DNMG$ be such a circle. With O as center, draw a horizontal and a vertical diameter. These two diameters divide the circle into four parts called quadrants, numbered I, II, III, IV.

Suppose the radius stops in the position OC, then, since OD was the initial position, and OC is now the terminal line, the angle described is DOC. From C drop a perpendicular upon the horizontal diameter OD, call it BC, *and consider it as being directed from B towards C*, or in a positive direction.

At D draw a tangent DE; let it cut the unit radius OC produced in H. Also draw a tangent NK at N, and let OC produced cut it at F. Then in the triangle OBC, representing $\angle BOC$ by x,

$$\sin x = \frac{BC\ (+)}{OC\ (+)} = + \frac{BC}{1} = + BC$$

$$\cos x = \frac{OB\ (+)}{OC\ (+)} = + \frac{OB}{1} = + OB.$$

Again, in the right triangle OHD,

$$\tan x = \frac{DH\ (+)}{OD\ (+)} = +\frac{DH}{1} = +DH$$

$$\sec x = \frac{OH\ (+)}{OD\ (+)} = +\frac{OH}{1} = +OH.$$

Now the two remaining functions, the cotangent and cosecant, will both be positive, since they are the reciprocals of the tangent and sine respectively. In the triangle NOF, which is right-angled at N, we have $\angle NFO = \angle x$,

hence, $$\cot x = \frac{NF}{NO} = \frac{NF}{1} = NF$$

and $$\operatorname{cosec} x = \frac{FO}{NO} = \frac{FO}{1} = FO.$$

We notice that all the signs of the functions in the first quadrant are positive, and further, that the three directed lines BC, OB, DH, give us by their directions the signs of the sine, cosine, and tangent, from which those of the remaining three functions, being the reciprocals of these, can be determined.

If the angle is in the second, third, or fourth quadrant, a similar construction to the above will in each case enable us to express a function as a line of definite length and direction. Before, however, we endeavor to obtain these results, it will be necessary to define in general terms, and for any angle, the trigonometric functions referred to a unit circle.

Definitions.

ART. 20. The sine of an angle in a unit circle is the perpendicular to the horizontal diameter extending from it to the extremity of the moving radius.

The cosine of an angle in a unit circle is the distance from the center of the circle to the foot of the sine along the horizontal diameter.

The tangent of an angle in a unit circle is that part of the tangent to the circle *at the right-hand extremity* of the horizontal diameter, between the point of tangency and the point where the tangent intersects the moving radius produced forward or backward to meet it.

The cotangent of an angle in a unit circle is that part of the tangent drawn to the circle *at the upper end* of the vertical diameter included between the point of tangency and the point where the tangent meets the moving radius produced forward or backward to meet it.

The secant of an angle in a unit circle is the distance measured along the moving radius from the center to its intersection with the tangent.

The cosecant of an angle in a unit circle is the distance measured along the moving radius from the center to its intersection with the cotangent.

ART. 21. Keeping the above definitions carefully in view, the student will have no difficulty in obtaining the lines representing the functions of an angle in any quadrant, together with their signs, as given in the table below. (See Fig. 20.)

QUAD. I.	QUAD. II.
$\sin DOC = +BC$	$\sin DOC' = +B'C'$
$\cos DOC = +OB$	$\cos DOC' = -OB'$
$\tan DOC = +DH$	$\tan DOC' = -DH'$
$\cot DOC = +NF$	$\cot DOC' = -NF'$
$\sec DOC = +OH$	$\sec DOC' = -OH'$
$\operatorname{cosec} DOC = +OF$	$\operatorname{cosec} DOC' = +OF'$

QUAD. III.	QUAD. IV.
$\sin DOC'' = -B''C''$	$\sin DOC''' = -B'''C'''$
$\cos DOC'' = -OB''$	$\cos DOC''' = +OB'''$
$\tan DOC'' = +DH$	$\tan DOC''' = -DH'$
$\cot DOC'' = +NF$	$\cot DOC''' = -NF'$
$\sec DOC'' = -OH$	$\sec DOC''' = +OH'$
$\operatorname{cosec} DOC'' = -OF$	$\operatorname{cosec} DOC''' = -OF'$

Since the sine has the same sign in the quadrants that are side by side or on a horizontal line; the cosine has

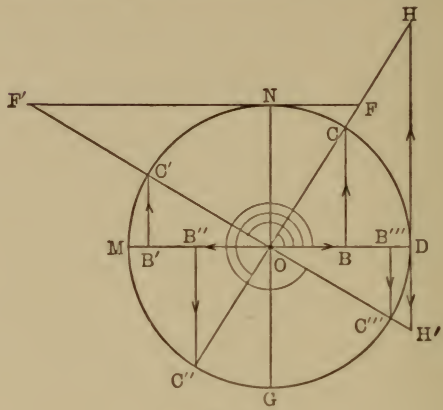

Fig. 20.

the same sign in the quadrants lying on a vertical line, and the tangents have the same sign in the quadrants lying on diagonal lines, the result may be plotted thus:

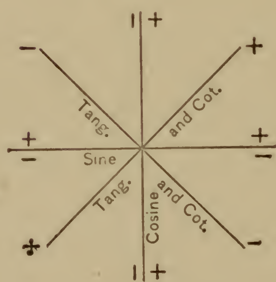

Fig. 21.

Then if the sign of the cosine of any angle, for example, is in question, it is only necessary to observe whether the angle is in a quadrant on a vertical line with the first quadrant or not, etc.

It should be noted that these results are in accord as to sign with those in Art. 18; had we drawn our tangent lines at M or G this would not have been the case; hence our reason for selecting D and N as points of tangency. The student should carefully note this fact.

An analysis of the quadrants indicates that the variations tabulated below take place among the functions while the angle is increasing from 0° to 360°.

Angle Increases from	0 — 90°	90° — 180°	180° — 270°	270° — 360°
Sine	increases from 0 to + 1	decreases from + 1 to 0	decreases from 0 to — 1	increases from — 1 to 0
Cosine	decreases from + 1 to 0	decreases from 0 to — 1	increases from — 1 to 0	increases from 0 to + 1
Tangent	increases from 0 to + ∞	increases from — ∞ to 0	increases from 0 to + ∞	increases from — ∞ to 0
Cotangent	decreases from + ∞ to 0	decreases from 0 to — ∞	decreases from + ∞ to 0	decreases from 0 to — ∞
Secant	increases from + 1 to + ∞	increases from — ∞ to — 1	decreases from — 1 to — ∞	decreases from + ∞ to + 1
Cosecant	decreases from + ∞ to + 1	increases from + 1 to + ∞	increases from — ∞ to — 1	decreases from — 1 to — ∞

To Express the Functions of any Angle in Terms of the Functions of an Angle in the First Quadrant.

ART. 22. The values of the trigonometric functions are compiled in tables, which tables will be found to contain only angles in the first quadrant, or those between 0° and 90°; the reason for this lies in the fact that it has been found easy to reduce the functions of an angle in any quadrant, to those of an acute angle; this is of much practical importance, and will now claim our attention.

To express the function of any angle in terms of the functions of an angle in the first quadrant.

Let $AOC = x$ be an angle in the first quadrant (Fig.

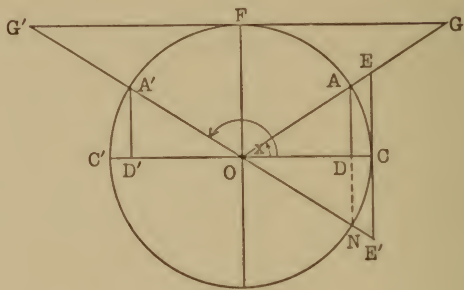

Fig. 22.

22), and COA' an angle in the second quadrant, such that its supplement $A'OC' = x$, then $180 - x = COA'$.

Assuming a unit circle, and making the construction shown in Fig. 22, then,

$$\sin x = DA \qquad \cot x = FG$$
$$\cos x = OD \qquad \sec x = OE$$
$$\tan x = CE \qquad \operatorname{cosec} x = OG$$

Noting that $\angle COA' = 180° - x$, we have,

$$\sin(180 - x) = D'A' \qquad \cot(180 - x) = FG'$$
$$\cos(180 - x) = OD' \qquad \sec(180 - x) = OE'$$
$$\tan(180 - x) = CE' \qquad \operatorname{cosec}(180 - x) = OG'$$

Now in the right triangles DOA and $D'OA'$ we have $OA = OA'$ (being radii), and the angle DOA is equal to the angle $D'OA'$ (by construction), therefore these triangles are equal; and taking notice of their directed lines, we see that $D'A' = DA$, and $OD' = -OD$ (since their directions are opposite). Again, in the equal triangles COE and COE' we have $CE' = -CE$, and $OE' = OE$; lastly, in the triangles FOG and FOG' which are also equal, $FG' = -FG$, and $OG' = OG$.

Now by reference to the above values of the functions, we obtain:

$$\sin(180 - x) = +\sin x \qquad \cot(180 - x) = -\cot x$$
$$\cos(180 - x) = -\cos x \qquad \sec(180 - x) = -\sec x$$
$$\tan(180 - x) = -\tan x \qquad \operatorname{cosec}(180 - x) = +\operatorname{cosec} x$$

The functions of $(360° - x)$ can be easily obtained from Fig. 22, by drawing DN and considering ON as the terminal line of a reflex angle* CON in the fourth quadrant. The angle $CON = x$ and the reflex angle $CON = 360° - x$. We thus have, $\sin(360 - x) = DN = -DA$, or $\sin(360 - x) = -\sin x$, and in like manner for the other functions, see Art. 23. The remaining case where the angle is $180° + x$ can be obtained in a similar manner to the above, and is left to the ingenuity of the student. The results, however, are given in Art. 23.

ART. 23. We will next consider an angle of $(90 + x)$ in the second quadrant. Let $COB = x$ be an angle in the first quadrant, and the angle $COF' = 90 + x$, the angle GOF' being equal to x.

Making the construction shown in Fig. 23, we have,

$$\sin x = DB \qquad \cot x = GF$$
$$\cos x = OD \qquad \sec x = OE$$
$$\tan x = CE \qquad \operatorname{cosec} x = OF$$

Noting that $\angle COF' = 90° + x$, we get,

$$\sin(90 + x) = D'B' \qquad \cot(90 + x) = GF'$$
$$\cos(90 + x) = OD' \qquad \sec(90 + x) = OE'$$
$$\tan(90 + x) = CE' \qquad \operatorname{cosec}(90 + x) = OF'$$

* A reflex angle is one greater than 180° and less than 360°, and therefore in the III or IV quadrant.

In the right triangles DOB and $D'OB'$, we have, $OB = OB'$ and $\angle D'B'O = \angle DOB$, $\therefore \triangle DOB = \triangle D'OB'$. Further, $D'B' = OD$, and $OD' = -DB$. Again, in the

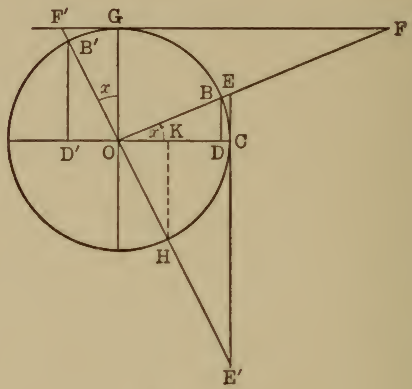

Fig. 23.

equal triangles EOC and $F'OG$ we have $OF' = OE$, and $GF' = -CE$; while in the triangles GOF and COE', which are also equal, $OE' = -OF$, and $CE' = -GF$. Referring to the above values of the functions, we obtain,

$$\sin(90 + x) = \cos x \qquad \cot(90 + x) = -\tan x$$
$$\cos(90 + x) = -\sin x \qquad \sec(90 + x) = -\operatorname{cosec} x$$
$$\tan(90 + x) = -\cot x \qquad \operatorname{cosec}(90 + x) = \sec x$$

By drawing KH and regarding the reflex angle COH as $270 + x$, we may obtain in like manner the relations between the functions of $(270° + x)$ and those of x. The case where the angle considered is $(270° - x)$ involves a similar construction to the above, and should be investigated by the student.

We will now present in tabulated form the results of the previous article.

II QUAD.	III QUAD.
$\sin(180° - x) = + \sin x$	$\sin(180° + x) = - \sin x$
$\cos(180° - x) = - \cos x$	$\cos(180° + x) = - \cos x$
$\tan(180° - x) = - \tan x$	$\tan(180° + x) = + \tan x$
$\cot(180° - x) = - \cot x$	$\cot(180° + x) = + \cot x$
$\sec(180° - x) = - \sec x$	$\sec(180° + x) = - \sec x$
$\operatorname{cosec}(180° - x) = + \operatorname{cosec} x$	$\operatorname{cosec}(180° + x) = - \operatorname{cosec} x$

IV QUAD.	II QUAD.
$\sin(360° - x) = - \sin x$	$\sin(90° + y) = + \cos y$
$\cos(360° - x) = + \cos x$	$\cos(90° + y) = - \sin y$
$\tan(360° - x) = - \tan x$	$\tan(90° + y) = - \cot y$
$\cot(360° - x) = - \cot x$	$\cot(90° + y) = - \tan y$
$\sec(360° - x) = + \sec x$	$\sec(90° + y) = - \operatorname{cosec} y$
$\operatorname{cosec}(360° - x) = - \operatorname{cosec} x$	$\operatorname{cosec}(90° + y) = + \sec y$

III QUAD.	IV QUAD.
$\cos(270° - y) = - \sin y$	$\cos(270° + y) = + \sin y$
$\sin(270° - y) = - \cos y$	$\sin(270° + y) = - \cos y$
$\tan(270° - y) = + \cot y$	$\tan(270° + y) = - \cot y$
$\cot(270° - y) = + \tan y$	$\cot(270° + y) = - \tan y$
$\sec(270° - y) = - \operatorname{cosec} y$	$\sec(270° + y) = + \operatorname{cosec} y$
$\operatorname{cosec}(270° - y) = - \sec y$	$\operatorname{cosec}(270° + y) = - \sec y$

These results may be epitomized in the following rules:

1. Any function of an angle which is equal to 180° or 360° plus or minus an acute angle, is equal to the same function of the acute angle, and will be positive or negative according as the original function was positive or negative.

For example, $\cos(180° - x) = - \cos x$.

2. Any function of an angle which is equal to 90° or 270° plus or minus an acute angle, is equal to the co-

named function of the acute angle; or, if the original function is a co-function, it will be equal to the plain function of the acute angle. The sign of the last function will agree with the sign of the original function.

For example, $\sin(270° - x) = -\cos x$.

In both the above rules the sign of the final function is determined by the quadrant in which the original angle occurred.

It will be noticed that any function of an angle greater than 90° can be reduced to the function of an angle less than 45°, or, if desired, to one of an angle between 45° and 90°.

ART. 24. From the definition of negative angles it is evident that the moving radius, having described a negative angle, will arrive at the same point as if it had described the positive angle, represented by 360° minus the number of degrees in the negative angle; hence the functions of the negative angle and of this corresponding positive angle will be exactly the same.

For example, the functions of $-75°$ are exactly the same as the functions of $+(360° - 75°) = +285$.

Therefore, to find the functions of a negative angle, subtract this angle (as if it were positive) from 360°, and find the functions of the positive angle resulting.

For example, $\tan -125° = \tan(360° - 125°) = \tan 235° = \tan(270° - 35°) = \cot 35°$.

ART. 25. In Art. 18 we drew attention to the important part played by the directed lines of a right triangle in determining the trigonometric functions. Up to this point we have considered the initial line as horizontal, but this is often not the case, as a triangle may obviously occupy any position in the plane. To facilitate the recognition of the functions of an angle in varying positions, we suggest the following rules:

I. The perpendicular is always at right angles to the initial line, or the initial line prolonged backwards; it is directed *from the initial to the terminal line*, and is *positive* for acute and obtuse angles, and *negative* for reflex angles.

II. The hypotenuse is invariably directed *from the vertex to the perpendicular*, and is positive in all positions.

III. The remaining side, the adjacent side, if the angle be acute, is directed *from the vertex to the foot of the perpendicular.* It is positive for acute angles and reflex angles between 270° and 360°, and negative for angles between 90° and 270°.

In giving the ratio expressing any trigonometric function of an angle, care should be taken to give the correct directions of lines to which reference is made.

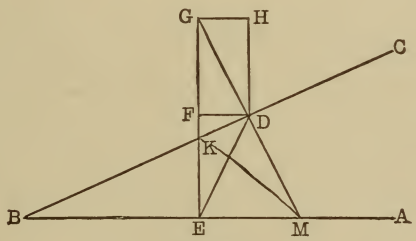

Fig. 24.

In Fig. 24, keeping the above rules in mind, we have: $\cos EBK = \frac{BE}{BK}$, $\cos FED = \frac{EF}{ED}$, $\cos FGD = \frac{GF}{GD}$, $\cos\ GDH = \frac{DH}{DG}$. Note that in each case the directed lines are measured from the vertex to the perpendicular, according to Rules II and III. Again, $\tan\ EBK = \frac{EK}{BE}$. $\tan FED = \frac{FD}{EF}$, $\tan\ FGD = \frac{FD}{GF}$, $\tan\ GDH = \frac{HG}{DH}$

Further, in triangle DBM, tan $DBM = \frac{DM}{BD}$, while cot DKG in the triangle $DKG = \frac{KD}{DG}$. Taking the obtuse angles BKG and BKM, we have tan $BKG = \frac{DG}{KD}$, and tan $BKM = \frac{DM}{KD}$. By Rule III the latter functions are negative.

Art. 26. In articles 22 and 23, we found certain relations existing between the functions of an angle formed by increasing or diminishing 90°, 180°, etc., by an acute angle x, and the functions of x; we obtained, for example, $\sin(x + 90) = \cos x$. It would seem natural to inquire at this point, whether it would not be possible to increase or diminish x by *any other angle y*, and find relations between the functions of the new angle $(x \pm y)$ and those of x and y. A geometrical investigation of such a problem clearly includes several cases, according as the terminal lines of x and y lie in the several quadrants; it will be found convenient at first to confine our attention simultaneously to the two cases where $(x + y) < 90$ and $(x + y) > 90$, in the latter of which the angle x is acute and y extends into the second quadrant.

To express sin $(x + y)$ and cos $(x + y)$ in Terms and Functions of x and y.

Art. 27. In each of the Figs. 25 and 26, let the angle $DBC = x$ and the angle $CBN = y$; in Fig. 25, the sum of these angles, x and y, is less than 90°, while in Fig. 26, it is greater than 90°, but the individual angles x and y are each less than a right angle.

In both cases take a point A, *upon the line bounding the* *angle* y, and let fall two perpendiculars, one AG, *upon the initial line of* x (or the initial line produced backward,

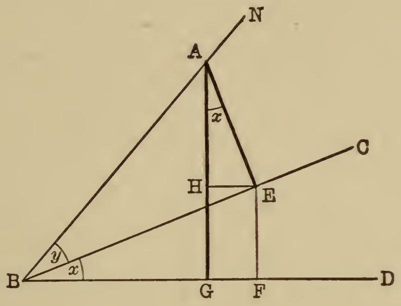

Fig. 25.

as in Fig. 26); the other AE, *upon the terminal line of* x. Note that the angle included between these perpendiculars to the sides of the angle x is $GAE = x$ (by Geometry).

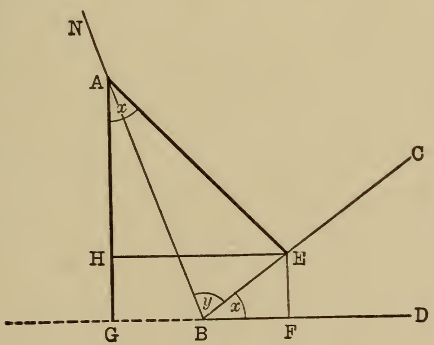

Fig. 26.

Now in each case we have a right-angled triangle GBA formed by the line AG and $\sin DBA = \sin(x+y) = \frac{GA}{BA}$.

Again, the perpendicular AE completes a right triangle AEB which contains the angle $EBA = y$.

We have, however, as yet, *no right triangle with x as an angle.* Now at least one such triangle must be constructed, if it is our purpose to establish relations between the sides of the triangle ABG, whose ratios represent the functions of $(x + y)$, and line ratios representing functions of the individual angles x and y. This suggests our drawing $EF \perp BD$ and $EH \perp GA$. We thus obtain two new triangles BFE and AHE, the former right-angled at F, the latter at H, *each of which contains an angle x.*

It now remains to investigate geometrically the relations existing between the sides of the triangle ABG, whose ratios represent the functions of $(x + y)$, and those of the triangles ABE, BFE, and AHE, giving careful attention to the directed lines.

We have, in each figure, $\sin(x + y) = \dfrac{GA}{BA}$.

Now, $$\frac{GA}{BA} = \frac{GH + HA}{BA} = \frac{FE + HA}{BA} = \frac{FE}{BA} + \frac{HA}{BA}.$$

Note that in these ratios each numerator and denominator is the side of a triangle containing an angle equal to x or y; it was, in fact, to obtain this result that we broke up GA. Again, FE is a side of a right triangle FBE, and BA of a right triangle ABE; *both these have a common side* BE; hence if BE is introduced into the ratio $\dfrac{FE}{BA}$, by breaking it up into two ratios without altering its value, thus, $\dfrac{FE}{BA} \cdot \dfrac{BE}{BE} = \dfrac{FE}{BE} \cdot \dfrac{BE}{BA}$, the former will be a function of x, namely, $\sin x$, and the latter a function of y, or, as we see, $\cos y$.

Again, taking the ratio $\frac{HA}{BA}$, we note AH is a side of the triangle AHE, and BA of BAE; these have a common side AE; introducing this as above, and observing the directed lines, we get,

$$\frac{AH}{BA} = \frac{AH}{BA} \cdot \frac{EA}{AE} = \frac{AH}{AE} \cdot \frac{EA}{BA} = \cos x \sin y.$$

Collecting our results, we finally obtain,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

We have discussed this proof at length because of its great importance. The steps are few and simple, but it is desirable that the reasons for them should be clearly understood by the student before proceeding further, as other similar proofs follow which form a basis for many of the most important formulæ in trigonometry.

For the sake of clearness we will now give concisely the necessary steps and construction discussed above.

ART. 28. I. *To show that sin* $(x + y) =$ *sin x cos y* $+$ *cos x sin y*. Let

$$\angle DBC = x \text{ and } \angle CBN = y; \text{ then } \angle DBN = x + y.$$

Take a point A on BN, *the bounding line of y*, and draw $AG \perp BD$, $AE \perp BC$, also $EH \perp AG$ and $EF \perp BD$. The angle $GAE = x$, $\because$ its sides are $\perp$ to those of x

$$\sin (x + y) = \frac{GA}{BA} = \frac{GH + HA}{BA} = \frac{FE + HA}{BA} = \frac{FE}{BA} + \frac{HA}{BA}$$

$$= \frac{FE}{BE} \cdot \frac{BE}{BA} + \frac{AH}{AE} \cdot \frac{EA}{BA},$$

$$\therefore \sin (x + y) = \sin x \cos y + \cos x \sin y. \quad . \quad . \quad (1)$$

II. *To show that* $\cos(x+y) = \cos x \cos y - \sin x \sin y$, making the same construction as above, we get,

Fig. 27.

$$\cos(x+y) = \frac{BG}{BA} = \frac{BF - GF}{BA}$$

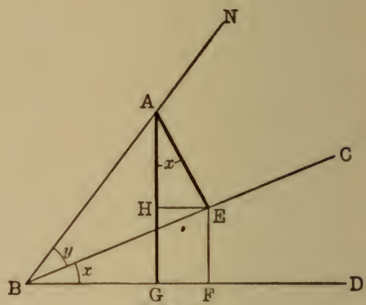

Fig. 27.

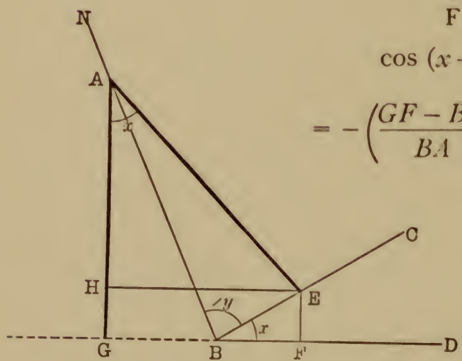

Fig. 28.

$$\cos(x+y) = -\frac{BG}{BA} = -\left(\frac{GF - BF}{BA}\right) = \frac{BF - GF}{BA}$$

Fig. 28

Hence in both cases,

$$\cos(x+y) = \frac{BF - GF}{BA} = \frac{BF}{BA} - \frac{HE}{BA} = \frac{BF}{BE} \cdot \frac{BE}{BA} - \frac{HE}{AE} \cdot \frac{EA}{BA},$$

(Notice again the use of the common side of the triangles.)

$$\therefore \cos(x+y) = \cos x \cos y - \sin x \sin y. \quad . \quad . \quad (2)$$

III. *To show that sin* $(x-y) = $ *sin* x *cos* $y -$ *cos* x *sin* y. Given $\angle x < 90$. Let $\angle DBC = x$ and $\angle CBN = y$, then $DBN = x - y$.

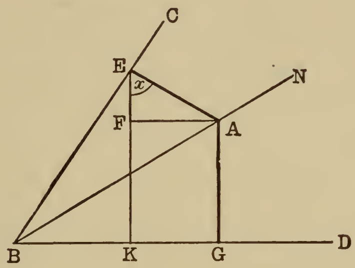

Fig. 29.

Making the construction indicated in Fig. 29, and observing that the point A is again taken upon the line bounding the angle y; also that the $\angle KEA = x$, we have,

$$\sin(x-y) = \frac{GA}{BA} = \frac{KF}{BA} = \frac{KE - FE}{BA}$$

$$= \frac{KE}{BA} - \frac{FE}{BA}$$

$$= \frac{KE}{BE} \cdot \frac{BE}{BA} - \frac{EF}{EA} \cdot \frac{EA}{BA}$$

$$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y. \quad . \quad . \quad (3)$$

IV. *To show that cos* $(x-y) = $ *cos* x *cos* $y +$ *sin* x *sin* y. Referring to Fig. 29, we see that

$$\cos(x-y) = \frac{BG}{BA} = \frac{BK + KG}{BA} = \frac{BK}{BA} + \frac{FA}{BA}$$

$$= \frac{BK}{BE} \cdot \frac{BE}{BA} + \frac{FA}{EA} \cdot \frac{EA}{BA}$$

$$\therefore \cos(x-y) = \cos x \cos y + \sin x \sin y. \quad . \quad . \quad (4)$$

ART. 29. It might now occur to the student to inquire whether the proofs of the four formulæ just given could not be simplified by making use of the unit circle, and, should we attempt such a method, which line would lend itself best to selection as a unit radius? A glance at the four cases just investigated shows that the line extending from the vertex B to the point A, *selected upon the line bounding the angle y*, always appears as a denominator, hence our choice obviously lies with this line. Making

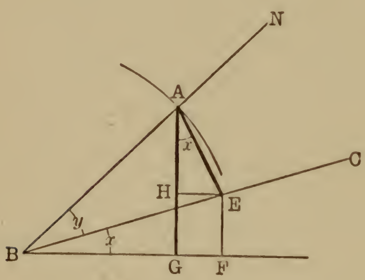

Fig. 30.

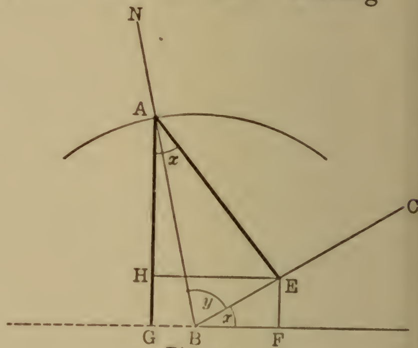

Fig. 31.

the same construction as before for case I, and in addition describing an arc through A with B as center, further, putting $BA = 1$, we obtain

$$\sin(x + y) = GA = GH + HA = FE + HA.$$

Now $\quad \sin x = \dfrac{FE}{BE} \quad \therefore \quad FE = \sin x \, BE$

and $\quad \cos x = \dfrac{AH}{AE} \quad \therefore \quad AH = \cos x \, AE,$

but $\quad BA = 1$

$\therefore \quad BE = \cos y$ and $AE = \sin y$.

Substituting these values, we get

$$FE = \sin x \cos y \text{ and } AH = \cos x \sin y.$$

Hence $\quad \sin(x + y) = \sin x \cos y + \cos x \sin y.$

A similar method may be adopted in each of the other cases.

ART. 30. We might now ask, Are the formulæ that we have just derived, general truths? Do they hold when the dimensions of the angles x and y are unrestricted? Also when either or both are negative?

The student will readily see that it would be tedious to attempt to investigate all possible cases geometrically. A more simple method is to show analytically that the formulæ still hold good if the angles x and y be increased by a right angle, or if equal negative angles be substituted, and hence establish the general truth.

Thus, $\sin\{(x+90)+y\} \equiv \sin\{90+(x+y)\}$.

By Art. 18, $\sin(90+A) = \cos A$.

If $A \equiv (x+y)$ we get

$$\sin\{90+(x+y)\} = \cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Now $\cos x = \sin(90+x)$ and $\sin x = -\cos(90+x)$.

Substituting, we get

$$\sin\{(90+x)+y\} = \sin(90+x)\cos y + \cos(90+x)\sin y.$$

From the above equation it follows that if $\sin(x+y) = \sin x \cos y + \cos x \sin y$ be true for any special quadrant, it also holds when x is in the following quadrant. But this equality has been proved for the first quadrant, hence it is true when x is in the second, and thus the limit may be indefinitely increased. A similar method of procedure shows that if either or both the angles be increased at will, or negative angles substituted, the truths expressed by the four formulæ of the previous article remain — and hence they are universal.

ART. 31. From the relation $\tan A = \frac{\sin A}{\cos A}$, it is easily possible to find an expression for $\tan(x+y)$ as follows:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Divide by $\cos x \cos y$;

$$\tan(x+y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Likewise,

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cot x \cot y - 1}{\cot y + \cot x} \quad . \quad . \quad . \quad (6)$$

Tan $(x - y)$ and cot $(x - y)$ are found in identically the same manner as tan $(x + y)$ and cot $(x + y)$.

ART. 32. Suppose in the formulæ:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

and $$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$$

y be made equal to x, then these formulæ become,
$\sin 2x = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$. (7)

$$\cos 2x = \cos^2 x - \sin^2 x \quad \ldots \ldots \quad (8)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \quad \ldots \ldots \quad (9)$$

and $$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x} \quad \ldots \ldots \quad (10)$$

If $2x = A$ in these formulæ, then $x = \frac{1}{2} A$, and substituting we have,

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A \quad \ldots \quad (7)$$

$$\cos A = \cos^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A \quad \ldots \quad (8)$$

$$\tan A = \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A} \quad \ldots \ldots \quad (9)$$

and $$\cot A = \frac{\cot^2 \frac{1}{2} A - 1}{2 \cot \frac{1}{2} A} \quad \ldots \ldots \quad (10)$$

That is, the sine of any angle equals twice the product of sine and cosine of half the angle. The cosine of any angle equals the square of the cosine of half the angle minus the square of the sine of half the angle. Write the corresponding rules for tangent and cotangent. These formulæ give the functions of an angle, when the functions of its half are known.

Functions of Half an Angle.

ART. 33. In the formula $\cos A = \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$

since $$\cos^2 \tfrac{1}{2} A = 1 - \sin^2 \tfrac{1}{2} A$$

$$\cos A = 1 - \sin^2 \tfrac{1}{2} A - \sin^2 \tfrac{1}{2} A = 1 - 2 \sin^2 \tfrac{1}{2} A$$

whence $$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A$$

or, $$\sin^2 \tfrac{1}{2} A = \frac{1 - \cos A}{2} \quad \ldots \ldots \quad (11)$$

Making the contrary substitution, $\sin^2 \frac{1}{2} A = 1 - \cos^2 \frac{1}{2} A$; $\cos A = \cos^2 \frac{1}{2} A - (1 - \cos^2 \frac{1}{2} A) = \cos^2 \frac{1}{2} A - 1 + \cos^2 \frac{1}{2} A = 2 \cos^2 \frac{1}{2} A - 1$,

whence, $2 \cos^2 \frac{1}{2} A = 1 + \cos A$,

or,
$$\cos^2 \tfrac{1}{2} A = \frac{1 + \cos A}{2} \quad \ldots \ldots \quad (12)$$

Since
$$\tan \tfrac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}$$

$$\tan^2 \tfrac{1}{2} A = \frac{\sin^2 \frac{1}{2} A}{\cos^2 \frac{1}{2} A} = \frac{\frac{1 - \cos A}{2}}{\frac{1 + \cos A}{2}} = \frac{1 - \cos A}{1 + \cos A} \quad . \quad (13)$$

and
$$\cot^2 \tfrac{1}{2} A = \frac{1 + \cos A}{1 - \cos A} \quad \ldots \ldots \quad (14)$$

These formulæ make it readily possible to find the functions of half an angle when any function of the whole angle is given.

For example, to find the functions of 30° having $\tan 60° = \sqrt{3}$ given.

From the formulæ, $\tan^2 60° + 1 = \sec^2 60°$

it follows that, $3 + 1 = \sec^2 60°$

$\sec 60° = 2$

hence, $\cos 60° = \frac{1}{2}$.

From (11) and (12) are derived respectively,

$$\sin^2 30° = \frac{1 - \cos 60°}{2} = \frac{1 - \frac{1}{2}}{2} = \tfrac{1}{4}$$

whence, $\sin 30° = \frac{1}{2}$

$$\cos^2 30° = \frac{1 + \cos 60°}{2} = \frac{1 + \frac{1}{2}}{2} = \tfrac{3}{4}$$

$$\therefore \cos 30 = \tfrac{1}{2}\sqrt{3}$$

$$\tan^2 30 = \frac{1 - \cos 60}{1 + \cos 60} = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

$$\therefore \tan 30 = \sqrt{\tfrac{1}{3}} = \tfrac{1}{3}\sqrt{3}, \text{ etc.}$$

It will be observed that these results tally with those obtained in Art. 9, page 10.

Sum and Difference of Functions.

ART. 34. Returning to the formulæ,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad . \quad (1)$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y \quad . \quad (3)$$

If $(x + y)$ is replaced by a single angle, say P, and $(x - y)$ is replaced by another angle, say Q, then the addition of the two formulæ above will give the sum of the sines of P and Q, thus:

$$x + y = P$$
$$x - y = Q$$

add, $2x = P + Q$

$$x = \tfrac{1}{2}(P + Q)$$

subtract, $2y = P - Q$

$$y = \tfrac{1}{2}(P - Q).$$

Adding (1) and (3) above,

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y.$$

Substituting values of x and y, $x + y$, and $x - y$, assumed above,

$$\sin P + \sin Q = 2 \sin \tfrac{1}{2}(P + Q) \cos \tfrac{1}{2}(P - Q). \quad (15)$$

Subtracting (3) from (1), and substituting,

$$\sin P - \sin Q = 2 \cos \tfrac{1}{2}(P + Q) \sin \tfrac{1}{2}(P - Q). \quad (16)$$

In the same way, taking (2) and (4),

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \quad . \quad (2)$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y \quad . \quad (4)$$

Adding and substituting,

$$\cos P + \cos Q = 2 \cos \tfrac{1}{2}(P + Q) \cos \tfrac{1}{2}(P - Q) \quad . \ (17)$$

Subtracting (4) from (2), and substituting,

$$\cos P - \cos Q = -2 \sin \tfrac{1}{2}(P + Q) \sin \tfrac{1}{2}(P - Q) \quad (18)$$

EXERCISE. State formulæ (15), (16), (17), and (18) as rules.

EXERCISE VI.

Goniometry.

Express the following functions in terms of the functions of angles less than 45°.

1. $\left\{\begin{array}{l} \sin 113°;\ \cos 216°;\ \tan 97°;\ \tan 315°;\ \cot 263°; \\ \sec 190°;\ \csc 181°;\ \cos 302°;\ \sin 220°;\ \tan 175°; \\ \cot 316°;\ \cos 156°;\ \cot 142°. \end{array}\right.$

2. $\sin -72°$; $\cos -118°$; $\tan -217°$; $\cot -105°$.

By application of formulæ find the simplest value of the following function:

3. $\sin (180 - x)$; $\cos (180 + x)$; $\tan (90 + x)$.

4. $\sin (270 + y)$; $\cos (270 - y)$; $\sin (360 - x)$; $\tan (360 - y)$.

Prove following relations:

5. $\tan (45 - x) = \dfrac{1 - \tan x}{1 + \tan x}.$

6. $\cos 3x = 4 \cos^3 x - 3 \cos x.$

7. $\sin 3x = 3 \sin x - 4 \sin^3 x$

8. $\sec A \csc A = 2 \csc 2A.$

9. $\cot \frac{1}{2} y + \tan \frac{1}{2} y = 2 \csc y.$

10. $\sin x = \frac{9}{41}$; $\cos x = \frac{40}{41}$; $\sin y = \frac{5}{13}$; $\cos y = \frac{12}{13}$: Find $\sin (x + y)$ and $\cos (x - y)$.

11. $\cos 26° = .9$. Find $\sin 13°$ and $\cos 13°$.

12. $\sin 53° 8' = .8$. Find sine, cosine, and tangent of $106° 16'$.

By formulæ (15), (16), (17), and (18) show that,

13. $\sin (45 + x) + \sin (45 - x) = 2 \sin 45 \cos x$
$= \sqrt{2} \cos x$.

14. $\sin (150 + x) - \sin (90 - x) = 2 \cos 120 \sin (30 + x) = - \sin (30 + x)$.

15. If x, y, and z are the three angles of a triangle, prove,
$\sin x + \sin y + \sin z = 4 \cos \frac{1}{2} x \cos \frac{1}{2} y \cos \frac{1}{2} z$.

Transform the following into expressions suitable for use of logarithms:

16. $\tan x + \tan y$.

17. $\cot x + \tan x$.

18. $\tan x \tan y + 1$.

19. $\cot x \cot y - 1$.

20. $\sin x - \cos^2 x \sin x$

21. $\dfrac{\sin x}{\sqrt{1 - \cos x}}$.

22. $\dfrac{1 - \cos 2A}{1 + \cos 2A}$.

23. $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

24. $\dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

Inverse Trigonometrical Functions.

ART. 35. In an algebraic equation involving more than one unknown quantity the equation may be solved for any one of the unknowns in terms of the others, thus: if $3xy = 4$, solving for x, $x = \frac{4}{3} y^{-1}$; or for y, $y = \frac{4}{3} x^{-1}$.
Likewise in Trigonometry the expression, $y = \sin x$ may be solved for x, by adopting a notation like this:

$x = \sin^{-1} y$, read x is the angle whose sine is y, or
$x =$ anti-sine of y.

The symbol -1 *above the sine symbol must not be mistaken for an exponent*, although it is adopted from the analogy of the process to exponential division: $\sin^{-1}$ is a single symbol and inseparable, having a definite meaning, distinct from *sin*.

Likewise, we have $\cos^{-1}A$, $\tan^{-1}A$, $\cot^{-1}A$, etc.; read respectively anti-cosine, anti-tangent, etc.

ART. 36. These inverse functions may be readily converted into direct functions by setting them equal to another quantity, representing their value.

For example, $\sin^{-1}\frac{1}{2} = 30°$

whence $\frac{1}{2} = \sin 30°$

or, in general, $\sin^{-1}A = B$, whence $A = \sin B$.

It is to be observed that the expression $\sin^{-1}A$ represents the angle and not its function, namely, the angle whose sine is A, which above is called B.

EXAMPLE. Prove $\cot^{-1}a + \cot^{-1}b$

$$= \cot^{-1}\frac{ab-1}{a+b}$$

Let $\cot^{-1}a = x$, hence $a = \cot x$

and $\cot^{-1}b = y$, hence $b = \cot y$.

Substituting in formulæ $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

$$\cot(\cot^{-1}a + \cot^{-1}b) = \frac{ab-1}{a+b}$$

or $$\cot^{-1}a + \cot^{-1}b = \cot^{-1}\frac{ab-1}{a+b}.$$

PART IV.

SOLUTION OF OBLIQUE TRIANGLES.

ARTICLE 37. In the right triangle the right angle is always known, and it is always possible to find remaining parts, when two are given, provided the two given parts are not two angles. In the oblique triangle there

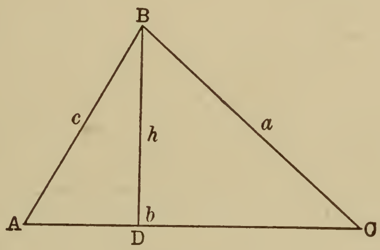

Fig. 32.

are in general six variable parts, three of which must be known, in order that the triangle may be completely solved.

It is plainly necessary to divide the oblique triangle into two right triangles, that the relations of its parts may be found, through the medium of angle functions, since these latter are defined as ratios in right triangles. In the triangle ABC, then, draw the perpendicular BD, and call the sides a, b, and c, using small letters for the sides opposite the angle denoted by the corresponding large letters.

In the right triangle ABD,

$$\sin A = \frac{h}{c} \quad (\text{where } BD = h) \qquad (a)$$

In triangle BDC,

$$\sin C = \frac{h}{a} \quad \ldots\ldots \quad (b)$$

Divide (a) by (b)

$$\frac{\sin A}{\sin C} = \frac{a}{c} \quad \ldots\ldots \quad (19)$$

By drawing perpendiculars from the other vertices, successively, in the same manner may be shown,

$$\frac{\sin A}{\sin B} = \frac{a}{b} \quad \ldots\ldots \quad (20)$$

$$\frac{\sin B}{\sin C} = \frac{b}{c} \quad \ldots\ldots \quad (21)$$

With (19), (20), and (21), if two angles and any side are given, the remaining parts may be found.

ART. 38. A slight transformation produces a formula which makes it possible to find the unknown parts when two sides and the included angle are given.

Taking (19) by division and composition, according to the theory of proportion,

$$\frac{\sin A - \sin C}{\sin A + \sin C} = \frac{a - c}{a + c} \quad \ldots \quad (m)$$

Dividing (16) by (15),

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)}{2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)}$$

$$= \frac{\cos \frac{1}{2}(P + Q)}{\sin \frac{1}{2}(P + Q)} \times \frac{\sin \frac{1}{2}(P - Q)}{\cos \frac{1}{2}(P - Q)}$$

$$= \cot \tfrac{1}{2}(P + Q) \tan \tfrac{1}{2}(P - Q).$$

Replacing P and Q by A and C respectively in this formula,

$$\frac{\sin A - \sin C}{\sin A + \sin C} = \cot \tfrac{1}{2}(A + C) \times \tan \tfrac{1}{2}(A - C).$$

Substituting this value of $\frac{\sin A - \sin C}{\sin A + \sin C}$ in (m),

$$\cot \tfrac{1}{2}(A + C) \tan \tfrac{1}{2}(A - C) = \frac{a - c}{a + c}$$

or

$$\tan \tfrac{1}{2}(A - C) = \frac{a - c}{a + c} \tan \tfrac{1}{2}(A + C) \quad . \quad (22)$$

Since $A + C = 180 - B,$

this formula makes it possible to find the remaining parts when the two sides a and c and their included angle B are given.

By an exactly analogous process, using (20) and (21), may be derived,

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B) \quad . \quad . \quad (23)$$

and

$$\tan \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \tan \tfrac{1}{2}(B + C) \quad . \quad . \quad (24)$$

which meet all requirements, when any two sides and their included angle are given.

ART. 39. In the above case the third side may be found directly, without finding the two unknown angles, by employing the geometrical theorem relative to the square of a side opposite an acute angle.

If ABC is any triangle, BD being a perpendicular from B upon AC, then by geometry,

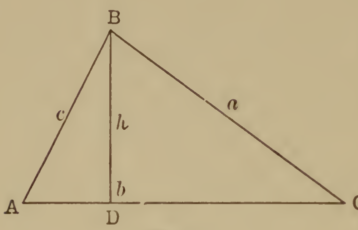

Fig. 33.

$$c^2 = a^2 + b^2 - 2b \times DC,$$

but, in the right triangle BDC, $\cos C = \frac{DC}{a}$ or $DC = a \cos C$.

Substituting this value of DC above,

$$c^2 = a^2 + b^2 - 2ab \cos C \quad . \quad (25)$$

By drawing perpendiculars from the other vertices and applying the same theorem, are obtained the following:

$$a^2 = b^2 + c^2 - 2\,bc \cos A \quad . \quad . \quad . \quad . \quad (26)$$
$$b^2 = a^2 + c^2 - 2\,ac \cos B \quad . \quad . \quad . \quad . \quad (27)$$

ART. 40. There is plainly a third case that arises in the solution of oblique triangles, namely, when the three sides are given.

Solving (25) for $\cos C$, $\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$.*

Subtracting each side from 1 and then adding each side to 1, are obtained,

$$1 - \cos C = 1 - \frac{a^2 + b^2 - c^2}{2\,ab} = \frac{2\,ab - a^2 - b^2 + c^2}{2\,ab}$$
$$= \frac{c^2 - (a - b)^2}{2\,ab} \quad . \quad . \quad (R)$$

* Formulæ (25), (26) and (27), each of which involves the three sides and one angle, enable us to find these angles, as C, in the equation above, but the result is in very inconvenient form; adding to and subtracting from 1, is to bring the $2ab$ from the denominator into numerator to combine with $a^2 + b^2$, forming a perfect square.

$$1 + \cos C = 1 + \frac{a^2 + b^2 - c^2}{2\,ab} = \frac{2\,ab + a^2 + b^2 - c^2}{2\,ab}$$

$$= \frac{(a + b)^2 - c^2}{2\,ab} \quad . \quad . \qquad (S)$$

Factoring (R) and (S), observing that the numerator of the right-hand members of both equations are each the difference of two squares,

$$1 - \cos C = \frac{[c - (a - b)]\,[c + a - b]}{2\,ab} = \frac{(c - a + b)\,(c + a - b)}{2\,ab}$$

$$1 + \cos C = \frac{(a + b - c)\;(a + b + c)}{2\,ab}.$$

From (11) and (12),

$$1 - \cos C = 2\sin^2 \tfrac{1}{2}\,C = \frac{(c - a + b)\;(c + a - b)}{2\,ab} \qquad (u)$$

$$1 + \cos C = 2\cos^2 \tfrac{1}{2}\,C = \frac{(a + b - c)\;(a + b + c)}{2\,ab}. \qquad (v)$$

Putting $s = \frac{1}{2}\,(a + b + c)$ or $2\,s = a + b + c$,
then,

$$2\,(s - a) = c - a + b$$
$$2\,(s - b) = c + a - b$$
$$2\,(s - c) = a + b - c$$

Substituting these values in (u) and (v),

$$2\sin^2 \tfrac{1}{2}\,C = \frac{4\,(s - a)\;(s - b)}{2\,ab} = \frac{2\,(s - a)\;(s - b)}{ab}$$

$$2\cos^2 \tfrac{1}{2}\,C = \frac{4\,s\,(s - c)}{2\,ab} = \frac{2\,s\,(s - c)}{ab}$$

or,

$$\sin \tfrac{1}{2}\,C = \sqrt{\frac{(s - a)\,(s - b)}{ab}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

$$\cos \tfrac{1}{2}\,C = \sqrt{\frac{s\,(s - c)}{ab}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

Divide (28) by (29),

$$\tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

By an exactly analogous process, corresponding expressions for $\frac{1}{2} A$ and $\frac{1}{2} B$ are found as follows:

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-c)(s-b)}{s(s-a)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

A comparison of (30), (33), and (36) will show that,

$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ is a common multiple.

If this expression be represented by r,

then (30) may be written, $\tan \frac{1}{2} C = \frac{r}{s-c}$

(33) may be written, $\tan \frac{1}{2} A = \frac{r}{s-a}$

and (36) may be written, $\tan \frac{1}{2} B = \frac{r}{s-b}$

which reduces the calculation of the three angles to the determination of the value of one radical expression, r.

EXERCISE VII.

Oblique Triangles.

Calling the angles A, B, and C, the sides respectively opposite a, b, and c, and the area S, solve the following triangle:

1. $A = 69^\circ\, 21'\, 30''$, $C = 23^\circ\, 11'\, 17''$, $a = 123.23$.
2. $B = 101^\circ\, 42'\, 21''$, $A = 47^\circ\, 12'\, 19''$, $b = 10.029$.
3. $B = 99^\circ\, 12'\, 10''$, $C = 35^\circ\, 0'\, 40''$, $a = 1027.2$.
4. $A = 11^\circ\, 17'\, 33''$, $B = 77^\circ\, 15'$, $c = 3.4576$.
5. $A = 82^\circ\, 12'\, 36''$, $b = 62.117$, $c = 90.741$.
6. $B = 109^\circ\, 49'\, 38''$, $a = 22.222$, $c = 19.34$.
7. $C = 67^\circ\, 58'\, 58''$, $a = 393.611$, $c = 208.47$.
8. $B = 23^\circ\, 27'\, 50''$, $b = .08679$, $a = .07241$.
9. $a = 111$, $b = 425$, $c = 238$.
10. $a = 1023.75$, $b = 978.36$, $c = 1321.13$.
11. $a = 18.705$, $b = 23.202$, $c = 9.667$.
12. Find S in each of the above examples:

13. A line AB, 225 yds. long, is measured off on level ground. The angles formed with it by imaginary lines to C, a point in the same plane, are respectively $98^\circ\, 12'\, 23''$ and $78^\circ\, 9'\, 21''$. Find the distance from A to C.

14. In running a line from B to C, two points in a survey, an impenetrable swamp is encountered. A third point D is chosen, from which B and C are both visible and accessible. The distances DB, DC, and the angle CDB are then measured and found to be, $DB = 429.58$ ft., $DC = 319.26$ ft., and $\angle CDB = 18^\circ\, 21'\, 36''$. Find length and direction of BC.

15. Two forces of 116.5 and 200 pounds per sq. in. respectively make an angle of $110^\circ\, 25'$ with each other. Find the intensity and direction of their resultant.

16. Three forces of 95.265, 68.21, and 105.2 lbs. respectively are in equilibrium. Find the angle between the first two.

NOTE. Additional problems on the oblique triangle on page 252.

17. To find the height of a steeple, a line *mn* 100 ft. long is measured on the ground, and the horizontal angles at *m* and *n* made by *mn* with imaginary lines drawn to the point directly below the top of the steeple on the ground, are found to be $80° 9' 25''$ and $72° 31' 13''$ respectively. Also the elevation of the top of the steeple from *m* is $14° 2' 30''$. What is the height of the steeple?

18. From a point in a 25% slope, the angle subtended by a tower higher up the slope is $29° 16' 25''$. From a point 75 feet higher up it subtends an angle of $42° 12' 17''$. Find height of tower.

To Express Angles in Radians.

Art. 41. In addition to the unit of angular measure, the degree, used in Geometry, Trigonometry employs a unit called a *radian*.

A *radian* is the central angle, in any circle, whose arc is equal in length to the radius.

Hence the number of radians in a given angle is the number of times its arc contains the radius of the circle at whose center its vertex is placed.

Since the total of the angles at the center of any circle is 360 degrees and the circumference is $2\pi r$, where $\pi = 3.1416$ and $r =$ radius, 360 degrees $= 2\pi r$ (central angles are measured by their arcs).

$$\therefore \frac{360°}{2\pi} = \frac{180°}{\pi} = 57.3° = r.$$

That is, an arc which equals r, subtends an angle of $57.3°$, or, more accurately, $206,265''$.

Since the total circumference is 2π times r, and r represents a radian, the circumference contains 2π radians; also the arc subtending an angle has the same ratio to the entire circumference that the angle has to 360 degrees. Hence the angle will contain the same part of 2π radians that it does of 360 degrees, or, what is the same thing, it will contain the same part of π radians that it does of 180 degrees.

Find the value of 30°, 45°, 65°, 90°, 225° in radians.

$$30^\circ = \frac{30}{180} \text{ or } \frac{1}{6} \text{ of } 180^\circ$$

$$\therefore\ 30^\circ = \frac{30}{180}\pi \text{ radians } = \frac{\pi}{6} \text{ radians.}$$

$$45^\circ = \frac{45}{180} \text{ or } \frac{1}{4} \text{ of } 180^\circ$$

$$\therefore\ 45^\circ = \frac{45}{180}\pi \text{ radians } = \frac{\pi}{4} \text{ radians.}$$

$$65^\circ = \frac{65}{180} \text{ or } \frac{13}{36} \text{ of } 180^\circ$$

$$\therefore\ 65^\circ = \frac{13}{36}\pi \text{ radians.}$$

$$90^\circ = \frac{90}{180} \text{ or } \tfrac{1}{2} \text{ of } 180^\circ$$

$$\therefore\ 90^\circ = \frac{\pi}{2} \text{ radians.}$$

$$225^\circ = \frac{225}{180} \text{ or } \frac{5}{4} \text{ of } 180^\circ$$

$$\therefore\ 225^\circ = \frac{5\pi}{4} \text{ radians, etc.}$$

Express, $22\frac{1}{2}^\circ$, 40°, 135°, 300°, 270° in radians.

An angle is plainly the same part of 180° that it is of π radians, hence the process of expressing radians in degrees is the exact reverse of the above.

For example, $\dfrac{\pi}{3}$ radians $= \dfrac{1}{3}$ of $180^\circ = 60^\circ$

$$\frac{2\pi}{5} \text{ radians } = \frac{2}{5} \text{ of } 180^\circ = 72^\circ, \text{ etc.}$$

1. Express in radians, 130°; 90°; 75°; 225°; $67\frac{1}{2}$ 11° 15′; 312°; 720°; 32° 12′ 20″.

2. Express in degrees, $\frac{2}{3}\pi$ rad.; $\frac{\pi}{6}$ rad.; .23 π rad.; $2\frac{1}{2}\pi$ rad.; .25 π rad.; $\frac{4}{3}\pi$ rad.

3. If a circular object subtends an angle of 1° at a distance of 114.6′, what is its diameter?

4. If a wheel makes 20 revolutions per second, what is its angular velocity in radians?

5. What is the radius of a circle if an arc of 2100 miles subtends an angle of 57.3 minutes at the center?

6. If the difference in latitude between two places on the earth (regarded as a sphere) is 7° 12′, and their distance apart is 495.8 miles, what is the diameter of the earth?

7. At 3 o'clock what is the angle expressed in radians between the hands of a watch?

8. The moon is 239,000 miles from the earth (approx.), and its diameter is 2162 miles. What angles does it subtend to us?

PART V.

SPHERICAL TRIGONOMETRY.

ARTICLE 1. A spherical polygon is a portion of the surface of a sphere inclosed by intersecting arcs of great circles.

Hence the sides are measured in degrees, minutes, and seconds, instead of linear units.

Knowing the radius of the sphere of whose surface the polygon is a part, the length of its sides can be also easily expressed in linear units, for any side will be the same part of a circumference (found from the formula, $2\pi r$) as its number of degrees is of 360°.

It is to be remembered that an arc of a great circle bears the same relation to a spherical surface that a straight line does to a plane surface.

ART. 2. By Solid Geometry, the sum of the sides of a spherical triangle is always less than 360°; and the sum of its angles is greater than 180° and less than 540°.

Also the essential theorems relating to plane triangles apply equally to spherical triangles.

Right Spherical Triangles.

ART. 3. As in Plane Trigonometry, the right triangle furnishes the simplest relations between its parts, and hence it provides the natural starting-point. Let, then, *ABC* (see Fig. 34) (notation being the same as before) be a spherical right triangle, with sides a, b, and c; C being the right angle. To avail ourselves of the known relations of Geometry, let O be the center of the sphere, of whose surface ABC is part. Join O with A, B, and C. Through the vertex A pass a plane $\perp$ to OB, intersecting the face OAB in AD, the face OAC in AE, and the face OBC in DE. Then since OB is $\perp$ to ADE, its plane

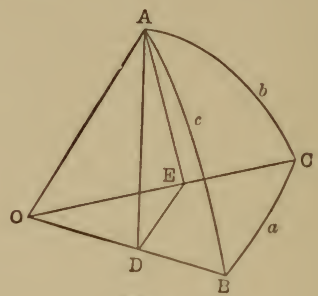

Fig. 34.

OBC is $\perp$ to ADE; and hence AE (a line in ADE drawn through a point of the intersection of these two perpendicular planes) is $\perp$ to OBC, and hence is $\perp$ to OC and DE, lines of the plane OBC.

That is, AED, AEO, ADO, and ODE are right angles, and ADE is the plane angle of the dihedral whose edge is OB, or $\angle ADE = \angle B$ (by Geometry).

The radius of this sphere may clearly be taken as unity for simplicity's sake, without in any way affecting results.

Remembering that the central angles AOC, AOB, and BOC are measured by their arcs, respectively b, c, and a, the plane right triangles, AOE, AOD, DOE, and ADE will clearly furnish relations between a, b, c, and B. For example:

By Goniometry (since radius = 1), $\cos AOB = \cos c = OD$, but in $\triangle ODE$, $\cos DOE = \cos a = \dfrac{OD}{OE}$ or $OD = OE \cos a = \cos b \cos a$, since $OE = \cos AOC = \cos b$.

$$\therefore \cos c = \cos b \cos a \quad . \quad . \quad . \quad . \quad (1)$$

Formula (1) may be stated thus: In a right spherical triangle the cosine of the hypotenuse equals the product of the cosines of the two legs.

It bears the same relation to the spherical right triangle that the Pythagorean theorem does to the plane right triangle. Again,

$$\sin c = AD = \frac{AE}{\sin ADE} = \frac{\sin b}{\sin B} \text{ or } \sin b = \sin c \sin B \quad (2^a).$$

Put this formula into a rule.

By changing the construction of the figure (drawing the perpendicular plane through B), it can be similarly proved that, $\sin a = \sin c \sin A$ (2^b). This formula could be inferred by analogy.

Again,

$$\tan a = \frac{DE}{OD} = \frac{AD \cos B}{OD} = \frac{\sin c \cos B}{\cos c} = \frac{\sin c}{\cos c} \times \cos B$$

$$= \tan c \cos B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3^a)$$

By analogy, $\tan b = \tan c \cos A$ . . . . (3^b)

Again,

$$\cos B = \frac{DE}{AD} = \frac{OE \sin a}{AD} = \frac{\cos b \sin a}{\sin c}; \text{ but from } (2^b)$$

$$\frac{\sin a}{\sin c} = \sin A. \quad \therefore \cos B = \cos b \times \frac{\sin a}{\sin c}$$

$$= \cos b \sin A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^a)$$

By analogy, $\cos A = \cos a \sin B$ . . . . (4^b)

ART. 4. Thus a variety of combinations may be made and each relation proved geometrically.

By grouping and comparing these various formulæ, Baron Napier, a famous Scotch mathematician, discovered a very simple device for reproducing them. Understand, his rules are purely empirical, that is, found by trial, and are not proofs in any sense; but since these formulæ *can* be proved rigidly, Napier's rules make their reproduction easy.

Ignoring the right angle, and taking the other five parts in a circle just in the order they occur, but using the complements of the two angles and of the hypotenuse, the rules are as follows (see Fig. 35):

(1) The sine of any part is equal to the product of the tangents of the two parts adjacent to it.

(2) The sine of any part is equal to the product of the cosines of the two parts opposite (not adjacent) to it.

For example:

$$\sin(\text{Co. } A) = \tan(\text{Co. } c) \tan b.$$

Since, $\sin(\text{Co. } A) = \cos A$ and $\tan(\text{Co. } c) = \cot c$,

$$\cos A = \cot c \tan b,$$

which is formula (3^b) already found.

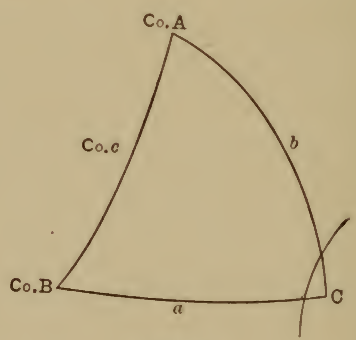

Fig. 35.

Again, $\sin(\text{Co. } A) = \cos(\text{Co. } B) \cos a,$

or, $\cos A = \sin B \cos a$, which is formula (4^b).

Again, $\sin(\text{Co. } c) = \cos a \cos b,$

or, $\cos c = \cos a \cos b$, which is (1), etc.

ART. 5. It is to be observed that if three certain parts are to be combined in an equation, one must be chosen, to which the others are either *both* adjacent or *both* opposite, in order to use Napier's rules.

Suppose, for example, A and c are given in a right spherical triangle and the other parts are required. The case stands thus: $\left.\begin{matrix} A \\ c \end{matrix}\right\}$ given $\left.\begin{matrix} B \\ a \\ b \end{matrix}\right\}$ required.

First, to find B: since A and c are the known parts and B is to be found, an equation between A, c, and B is necessary. Of the three, A, c, and B, A will not answer for the middle part, for c is adjacent and B opposite; c, however, has both A and B adjacent to it, hence by rule 1:

$$\sin (\text{Co. } c) = \tan (\text{Co. } A) \tan (\text{Co. } B)$$

or,

$$\cos c = \cot A \cot B$$

whence, $\cot B = \dfrac{\cos c}{\cot A} = \cos c \tan A$ (whence B is found).

Second, to find a: of the three, A, c, and a, A cannot be middle, nor can c, for in neither case do the other two occupy the same position relative to it; but a has both A and c opposite to it. Hence, by rule 2,

$$\sin a = \cos (\text{Co. } A) \cos (\text{Co. } c)$$

or,

$$\sin a = \sin A \sin c \text{ (whence } a \text{ is found).}$$

By a like procedure, find the formula for b.

ART. 6. A quadrantal triangle is one having at least one side a quadrant (90°) in length. Its solution can be easily reduced to that of a right spherical triangle by using its polar.

EXAMPLE. Solve the triangle in which $a = 90°$, $B = 65°$, $c = 80°$. Constructing the polar triangle and calling corresponding sides and angles a', b', c', A', B', C', we have by Geometry, $a' = 180 - A$, $b' = 180 - B$, $c' = 180 - C$, $A' = 180 - a$, $B' = 180 - b$, $C' = 180 - c$; hence $A' = 180° - 90° = 90°$ and $A'B'C'$ is a right spherical triangle, a' being the hypotenuse.

To find b.

$$\therefore \sin(\text{Co. } B') = \cos(\text{Co. } C') \cos b' \text{ or } \cos B' = \sin C' \cos b'$$

whence, $\cos(180 - b) = \sin(180 - c) \cos(180 - B)$

or, $-\cos b = (\sin c)(-\cos B) = -\sin c \cos B$

that is, $\cos b = \sin c \cos B$, etc.

EXERCISE I.

Right Spherical Triangle (C = 90°).

1.	$a = 39° 27' 32''$,	$b = 69° 21' 13''$
2.	$B = 112° 10' 11''$,	$A = 88° 14' 17''$
3.	$b = 56° 25' 42''$,	$c = 61° 23' 27''$
4.	$A = 76° 30' 52''$,	$a = 110° 17' 24''$
5.	$A = 67° 29' 39''$,	$b = 79° 19' 19''$
6.	$B = 42° 47' 58''$,	$a = 25° 32' 47''$
7.	$B = 98° 45' 46''$,	$b = 58° 8''$
8.	$c = 90°$, $C = 104° 10' 15''$,	$B = 70° 16' 26''$
9.	$c = 163° 14' 12''$,	$b = 112° 38' 10''$
10.	$c = 102° 27' 6''$,	$a = 99° 11' 33''$
11.	$A = 53° 49' 36''$,	$B = 78° 29' 14''$
12.	$B = 83° 44' 22''$,	$c = 10° 19' 25''$
13.	$b = 32° 47' 18''$,	$A = 80° 30' 20''$
14.	$a = 29° 18' 18''$,	$B = 142° 39' 27''$
15.	$A = 152° 21' 21''$,	$B = 149° 7' 9''$
16.	$a = 90°$, $c = 94° 20' 37''$,	$b = 75° 15' 28''$

Isosceles Triangle.

ART. 7. The spherical isosceles triangle depends upon the right spherical triangle for solution in exactly the same way that the plane isosceles triangle depends upon the plane right triangle. An arc of a great circle drawn through the vertex perpendicular to the base, divides the isosceles triangle into two equal right triangles, which are readily solved.

ART. 8. The analogy in process between plane and spherical trigonometry is maintained in the solution of an oblique spherical triangle, which is made to depend upon the solution of right spherical triangles, by drawing an arc of a great circle through one vertex perpendicular to the opposite side, thus forming two right triangles, which, however, are not equal unless the original triangle is isosceles.

ART. 9. These two right triangles having a common side (the arc), enable us through it to find the relation between the opposed parts of the oblique triangle, since

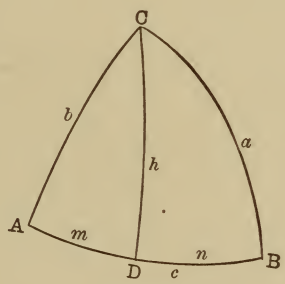

Fig. 36.

they are on both sides related to this common part in a way we have learned to know.

Let ABC (Fig. 36) be a spherical triangle, with sides a, b, c. Through C draw the great circle arc $CD \perp$ to AB at D. Call CD, h; AD, m; and DB, n.

In the right triangle ACD (b being the hypotenuse), taking h as a middle part, with A and b; $\sin h = \cos (\text{Co. } A) \cos (\text{Co. } b) = \sin A \sin b$.

Likewise, in CDB; $\sin h = \sin B \sin a$.

$$\left.\begin{array}{ll} \therefore & \sin A \sin b = \sin B \sin a, \\ \text{or,} & \sin A : \sin B :: \sin a : \sin b \ (1^m). \\ \text{Similarly,} & \sin A : \sin C :: \sin a : \sin c \ (1^n) \\ & \sin B : \sin C :: \sin b : \sin c \ (1^o). \end{array}\right\} \begin{array}{l} \text{Two angles and} \\ \text{opposite side, or} \\ \text{two sides and} \\ \text{opposite angle.} \end{array}$$

Put these formulæ into the form of a rule.

ART. 10. Again, in the right triangle CBD, $\cos a = \cos h \cos n = \cos h \cos (c - m)$, (since $n = c - m$) $= \cos h \cos c \cos m + \cos h \sin c \sin m$ (1) $[\cos (c - m) = \cos c \cos m + \sin c \sin m]$, but $\cos h \cos m = \cos b$ (in the right triangle ACD) and,

$$\cos h \sin m = \frac{\cos b}{\cos m} \times \sin m$$

$$\left[\text{for } \cos b = \cos h \cos m, \quad \therefore \cos h = \frac{\cos b}{\cos m}\right]$$

$$= \cos b \times \frac{\sin m}{\cos m} = \cos b \tan m = \cos b \frac{\cos A}{\cot b}$$

$$= \sin b \cos A \left[\text{for } \cos A = \cot b \tan m,\right.$$

$$\left.\therefore \tan m = \frac{\cos A}{\cot b}, \text{ also } \frac{\cos b}{\cot b} = \sin b\right].$$

Substituting these values for $\cos h \cos m$ and $\cos h \sin m$ in (1),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad . \quad . \quad . \quad (2^m)$$

By similar process or by analogy,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \quad . \quad . \quad . \quad (2^n)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad . \quad . \quad . \quad (2^o)$$

Napier's rules are applied in every case above.

ART. 11. From (2^m), solving the equation for cos A,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (x)$$

which gives A, but in inconvenient form.

Hence,

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$ (subtracting both sides from 1)

$$= \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c} = \frac{\cos (b-c) - \cos a}{\sin b \sin c}$$

$$\frac{\cos (b-c) - \cos a}{\sin b \sin c} = \frac{-2 \sin \frac{1}{2}(b-c+a) \sin \frac{1}{2}(b-c-a)}{\sin b \sin c}.$$

[By formula 18, Plane Trigonometry, calling $P = (b-c)$ and $Q = a$,]

$$= \frac{2 \sin \frac{1}{2}(b-c+a) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}$$

[for $-\sin x = \sin(-x)$, hence, $-\sin \frac{1}{2}(b-c-a) = \sin \frac{1}{2}[-(b-c-a)] = \sin \frac{1}{2}(a-b+c)$.]

Let $s = \frac{1}{2}(a+b+c)$

then, $s - a = \frac{1}{2}(b-a+c)$

$s - b = \frac{1}{2}(a-b+c)$

$s - c = \frac{1}{2}(a+b-c)$.

Substituting these values above,

$$1 - \cos A = \frac{2 \sin \frac{1}{2}(b-c+a) \sin \frac{1}{2}(a-b+c)}{\sin b \sin c}$$

$$= \frac{2 \sin (s-c) \sin (s-b)}{\sin b \sin c},$$

but, $1 - \cos A = 2 \sin^2 \frac{1}{2} A$ (by Goniometry),

$$\therefore 2 \sin^2 \frac{1}{2} A = \frac{2 \sin (s-c) \sin (s-b)}{\sin b \sin c}$$

or, $$\sin^2 \frac{1}{2} A = \frac{\sin (s-c) \sin (s-b)}{\sin b \sin c} \quad . \qquad (3^m)$$

By a similar process with (2^n) and (2^o) or by analogy,

$$\sin^2 \tfrac{1}{2} B = \frac{\sin (s - a) \sin (s - c)}{\sin a \sin c} \quad . \quad . \quad . \quad (3^n)$$

$$\sin^2 \tfrac{1}{2} C = \frac{\sin (s - a) \sin (s - b)}{\sin a \sin b} \quad . \quad . \quad . \quad (3^o)$$

By adding 1 to each side of equation (x) the value of $1 + \cos A = \cos^2 \tfrac{1}{2} A$ can be easily found to be:

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c} \quad . \quad . \quad . \quad . \quad (4^m)$$

Likewise, $$\cos^2 \tfrac{1}{2} B = \frac{\sin s \sin (s - b)}{\sin a \sin c} \quad . \quad . \quad . \quad . \quad (4^n)$$

and $$\cos^2 \tfrac{1}{2} C = \frac{\sin s \sin (s - c)}{\sin b \sin a} \quad . \quad . \quad . \quad . \quad (4^o)$$

Dividing (3^m) by (4^m); (3^n) by (4^n); (3^o) by (4^o),

$$\frac{\sin^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} A} = \tan^2 \tfrac{1}{2} A = \frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)} \quad (5^m)$$

$$\tan^2 \tfrac{1}{2} B = \frac{\sin (s - a) \sin (s - c)}{\sin s \sin (s - b)} \quad . \quad . \quad (5^n)$$

$$\tan^2 \tfrac{1}{2} C = \frac{\sin (s - a) \sin (s - b)}{\sin s \sin (s - c)} \quad . \quad (5^o)$$

ART. 12. (5^m), (5^n), (5^o) have the least common multiple,

$$\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s} = r^2, \text{ say};$$

$$\text{or, } r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}.$$

If r^2 be divided successively by (5^m), (5^n), and (5^o), and the roots of the quotients extracted, the results are :

$$\left.\begin{aligned}\tan \tfrac{1}{2} A &= \frac{r}{\sin\ (s-a)} \quad . \quad . \quad . \quad (6^m)\\ \tan \tfrac{1}{2} B &= \frac{r}{\sin\ (s-b)} \quad . \quad . \quad . \quad (6^n)\\ \tan \tfrac{1}{2} C &= \frac{r}{\sin\ (s-c)} \quad . \quad . \quad . \quad (6^o)\end{aligned}\right\} \text{Three sides.}$$

Attention is called to the analogy between these results and the corresponding formulæ under Plane Trigonometry. They will be found to have exactly similar application, and the use of r is as before a great simplification of the labor in solution of triangles.

ART. 13. Reverting to formula (2^m),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

and substituting the values of a, b, c, and A in terms of the sides and angles of the polar triangle, (x) becomes

$$\cos (180 - A') = \cos (180 - B') \cos (180 - C')$$
$$+ \sin (180 - B') \sin (180 - C') \cos (180 - a')$$

or, $-\cos A' = (-\cos B')\ (-\cos C') + (\sin B')\ (\sin C')$
$(-\cos a')$

or, $\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a'.$

It is clear that the accents have no significance except to distinguish the parts of one triangle from the corresponding parts of its polar. Since a relation has been found between the parts of this single triangle among themselves, and since this triangle, although it happens to be polar to a certain other triangle, is not in any sense a special kind of triangle, the above result is perfectly general, and the accents may be dropped; hence,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \quad (7^m)$$

Likewise,

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \quad . \quad (7^n)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad . \quad (7^o)$$

ART. 14. By treating (7^m), (7^n), and (7^o) exactly as we did (2^m), the following formulæ arise:

$$\sin^2 \tfrac{1}{2} a = \frac{-\cos S \cos (S - A)}{\sin B \sin C} \quad (8^m) \quad \left[\text{where } S = \tfrac{1}{2}(A + B + C)\right]$$

$$\sin^2 \tfrac{1}{2} b = \frac{-\cos S \cos (S - B)}{\sin A \sin C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8^n)$$

$$\sin^2 \tfrac{1}{2} c = \frac{-\cos S \cos (S - C)}{\sin A \sin B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8^o)$$

$$\cos^2 \tfrac{1}{2} a = \frac{\cos (S - B) \cos (S - C)}{\sin B \sin C} \quad . \quad . \quad . \quad . \quad . \quad (9^m)$$

$$\cos^2 \tfrac{1}{2} b = \frac{\cos (S - A) \cos (S - C)}{\sin A \sin C} \quad . \quad . \quad . \quad . \quad . \quad (9^n)$$

$$\cos^2 \tfrac{1}{2} c = \frac{\cos (S - A) \cos (S - B)}{\sin A \sin B} \quad . \quad . \quad . \quad . \quad . \quad (9^o)$$

$$\tan^2 \tfrac{1}{2} a = \frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)} \quad . \quad . \quad . \quad . \quad . \quad (10^m)$$

$$\tan^2 \tfrac{1}{2} b = \frac{-\cos S \cos (S - B)}{\cos (S - A) \cos (S - C)} \quad . \quad . \quad . \quad . \quad . \quad (10^n)$$

$$\tan^2 \tfrac{1}{2} c = \frac{-\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)} \quad . \quad . \quad . \quad . \quad . \quad (10^o)$$

The G. C. D. of (10^m), (10^n), and (10^o) is found to be

$$\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)} = R^2, \text{ say.}$$

$$\left.\begin{aligned} \therefore\ \tan \tfrac{1}{2} a &= R \cos (S - A) \quad . \quad . \quad (11^m) \\ \tan \tfrac{1}{2} b &= R \cos (S - B) \quad . \quad . \quad (11^n) \\ \tan \tfrac{1}{2} c &= R \cos (S - C) \quad . \quad . \quad (11^o) \end{aligned}\right\} \text{Three angles}$$

NOTE. — It is to be observed that $A + B + C = 2S$ is always greater than 180° and less than 540, by Geometry, and hence S is always greater than 90° and less than 270°; and hence $\cos S$ is always negative, by Goniometry. Therefore, $-\cos S$ must be always positive; so that the values of the radicals in this last article are never imaginary in a real triangle.

ART. 15. Dividing (5^m) by (5^n) we get

$$\frac{\tan^2 \frac{1}{2} A}{\tan^2 \frac{1}{2} B} = \frac{\sin^2 (s-b)}{\sin^2 (s-a)}$$

or,

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \frac{\sin (s-b)}{\sin (s-a)}$$

or,

$$\tan \tfrac{1}{2} A : \tan \tfrac{1}{2} B :: \sin (s-b) : \sin (s-a).$$

By composition and division,

$$\tan \tfrac{1}{2} A + \tan \tfrac{1}{2} B : \tan \tfrac{1}{2} A - \tan \tfrac{1}{2} B :: \sin (s-b) + \sin (s-a) : \sin (s-b) - \sin (s-a),$$

whence,

$$\frac{\tan \frac{1}{2} A - \tan \frac{1}{2} B}{\tan \frac{1}{2} A + \tan \frac{1}{2} B} = \frac{\sin (s-b) - \sin (s-a)}{\sin (s-b) + \sin (s-a)} \quad . \quad (P)$$

But

$$\frac{\tan \frac{1}{2} A - \tan \frac{1}{2} B}{\tan \frac{1}{2} A + \tan \frac{1}{2} B} = \frac{\dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} - \dfrac{\sin \frac{1}{2} B}{\cos \frac{1}{2} B}}{\dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} + \dfrac{\sin \frac{1}{2} B}{\cos \frac{1}{2} B}}$$

$$= \frac{\sin \frac{1}{2} A \cos \frac{1}{2} B - \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B}$$

$$= \frac{\sin (\frac{1}{2} A - \frac{1}{2} B)}{\sin (\frac{1}{2} A + \frac{1}{2} B)} = \frac{\sin \frac{1}{2} (A - B)}{\sin \frac{1}{2} (A + B)} \quad (P_1)$$

Again,

$$\frac{\sin(s-b)-\sin(s-a)}{\sin(s-b)+\sin(s-a)} = \frac{2\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)}{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}$$

[Let $P = (s-b)$ and $Q = (s-a)$ in (15) and (16)]

$$= \frac{\cos\frac{1}{2}c}{\sin\frac{1}{2}c} \times \frac{\sin\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a-b)} = \cot\tfrac{1}{2}c\tan\tfrac{1}{2}(a-b). \quad (P_2)$$

[Since $2s = a+b+c$, ∴ $2s-a-b = c$]

∴ substituting (P_1) and (P_2) in (P)

$$\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} = \cot\tfrac{1}{2}c\tan\tfrac{1}{2}(a-b);$$

or

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c. \quad . (12^m)$$

[Two angles and included side (12^m) pairs with (12^x), page 239.]

By the same process, using (5^n) with (5^o) and (5^m) with (5^o), we get,

$$\tan\tfrac{1}{2}(b-c) = \frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}\tan\tfrac{1}{2}a \quad (12^n)$$

$$\tan\tfrac{1}{2}(a-c) = \frac{\sin\frac{1}{2}(A-C)}{\sin\frac{1}{2}(A+C)}\tan\tfrac{1}{2}b \quad (12^o)$$

Two angles and included side (12^n) and (12^o) pair respectively with (12^y) and (12^z), page 239.

ART. 16. Using the polar triangle, and substituting in (12^m), (12^n), and (12^o) the values of their parts in terms of the supplementary parts of the polar as was done in Art. 13, arise the corresponding formulæ:

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \quad (13^m)$$

$$\tan\tfrac{1}{2}(B-C) = \frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A \quad (13^n)$$

$$\tan\tfrac{1}{2}(A-C) = \frac{\sin\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a+c)}\cot\tfrac{1}{2}B \quad (13^o)$$

Two sides and included angle (13^m), (13^n), (13^o) pair respectively with (13^x), (13^y), $(13)^z$, page 240.

ART. 17. Multiplying together (5^m) and (5^n),

$$\tan^2 \tfrac{1}{2} A \tan^2 \tfrac{1}{2} B = \frac{\sin^2 (s-c)}{\sin^2 s}$$

or,
$$\frac{\tan \frac{1}{2} A \tan \frac{1}{2} B}{1} = \frac{\sin (s-c)}{\sin s}.$$

As before, taking this proportion by composition and division:

$$\frac{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} \quad . \quad . \quad (y)$$

But
$$\frac{1 + \tan \frac{1}{2} A \tan \frac{1}{2} B}{1 - \tan \frac{1}{2} A \tan \frac{1}{2} B} = \frac{1 + \dfrac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B}}{1 - \dfrac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B}}$$

$$= \frac{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\cos \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A+B)}.$$

Also,

$$\frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} = \frac{2 \sin \frac{1}{2} (2s-c) \cos \frac{1}{2} c}{2 \cos \frac{1}{2} (2s-c) \sin \frac{1}{2} c}$$

[substituting s for P, and $(s-c)$ for Q in (15) and (16)]

$= \tan \frac{1}{2} (a+b) \cot \frac{1}{2} c$ [since $2s = a+b+c$; $2s - c = a+b$]

∴ (y) becomes, substituting these values for its members,

$$\frac{\cos \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A+B)} = \tan \tfrac{1}{2} (a+b) \cot \tfrac{1}{2} c,$$

or,

$$\tan \tfrac{1}{2} (a+b) = \frac{\cos \frac{1}{2} (A-B)}{\cos \frac{1}{2} (A+B)} \tan \tfrac{1}{2} c \quad (12^x)$$

Likewise,

$$\tan \tfrac{1}{2} (b+c) = \frac{\cos \frac{1}{2} (B-C)}{\cos \frac{1}{2} (B+C)} \tan \tfrac{1}{2} a \quad (12^y)$$

and

$$\tan \tfrac{1}{2} (a+c) = \frac{\cos \frac{1}{2} (A-C)}{\cos \frac{1}{2} (A+C)} \tan \tfrac{1}{2} b \quad (12^z)$$

} Two angles and included side.

By using polar triangle in application to (12^x), (12^y), (12^z), we get,

$$\left.\begin{aligned}\tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \quad &(13^x)\\ \tan\tfrac{1}{2}(B+C) &= \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A \quad &(13^y)\\ \tan\tfrac{1}{2}(A+C) &= \frac{\cos\frac{1}{2}(a-c)}{\cos\frac{1}{2}(a+c)}\cot\tfrac{1}{2}B \quad &(13^z)\end{aligned}\right\}\text{Two sides and included angle}$$

ART. 18. By using formulæ (1^m), (1^n), (1^o), when 2 sides and an opposite angle, or 2 angles and an opposite side are given; formulæ (6^m), (6^n), (6^o), when 3 sides are given; formulæ (11^m), (11^n), (11^o), when 3 angles are given; formulæ $(12^m, 12^x)$, $(12^n, 12^y)$, $(12^o, 12^z)$, when two angles and the included sides are given; formulæ $(13^m, 13^x)$, $(13^n, 13^y)$, $(13^o, 13^z)$, when two sides and the included angle are given, any spherical triangle may be completely solved.*

ART. 19. EXAMPLE. Given $A = 135° 21' 21''$; $a = 117° 10' 18''$; $b = 78° 23' 40''$.

To find B. $\sin A : \sin B : : \sin a : \sin b$

$$\sin B = \frac{\sin A \sin b}{\sin a}.$$

$$\log \sin B = \log \sin A + \log \sin b + \text{colog} \sin a$$

$$\begin{aligned}\log \sin A &= \log \sin (180 - A)\\ &= \log \sin 44° 38' 39'' = 9.846771 - 10\\ & \log \sin 78° 23' 40'' = 9.991029 - 10\end{aligned}$$

$$\begin{aligned}\text{colog} \sin a &= \text{colog} \sin (180 - a)\\ &= \text{colog} \sin 62° - 49 - 42'' = 0.050785\end{aligned}$$

$$\begin{aligned}\log \sin B = \quad & 19.888585 - 20\\ &= 9.888585 - 10\end{aligned}$$

$$B = 50° 41' 21''.$$

* Solutions of examples involving formulæ (12) and (13) will be found on page 251.

To find c. Formula (12^m) contains the known parts a, b, A, B and the unknown c, with no others; hence by solving it for $\tan \frac{1}{2} c$, the value of c may be found.

In (12^m) then, $\tan \frac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \frac{1}{2}(a-b)$

$$\begin{array}{ll} A = 135^\circ\ 21'\ 21'' & a = 117^\circ\ 10'\ 18'' \\ B = 50^\circ\ 41'\ 21'' & b = 78^\circ\ 23'\ 40'' \\ \hline A+B = 186^\circ\ 2'\ 42'' & a-b = 38^\circ\ 46'\ 38'' \\ A-B = 84^\circ\ 40'\ 00'' & \frac{1}{2}(a-b) = 19^\circ\ 23'\ 19'' \\ \frac{1}{2}(A+B) = 93^\circ\ 1'\ 21'' & \\ \frac{1}{2}(A-B) = 42^\circ\ 20' & \end{array}$$

$\log \tan \frac{1}{2} c = \log \sin \frac{1}{2}(A+B) + \text{colog} \sin \frac{1}{2}(A-B)$
$+ \log \tan \frac{1}{2}(a-b)$

$\log \sin \frac{1}{2}(A+B) = \log \sin [180 - \frac{1}{2}(A+B)] = \log \sin 86^\circ\ 58'\ 39'' = 9.999396 - 10$

$\text{colog} \sin \frac{1}{2}(A-B) = \text{colog} \sin 42^\circ\ 20'$
$= 0.171699$

$\log \tan \frac{1}{2}(a-b)$
$= \log \tan 19^\circ\ 23'\ 19'' = 9.546459 - 10$

$$\log \tan \tfrac{1}{2} c = 9.717554 - 10$$
$$\tfrac{1}{2} c = 27^\circ\ 33'\ 30''$$
$$c = 55^\circ\ 7'$$

To find C, use formula (13^m) in the same way.

Example. Given $A = 110^\circ\ 36'\ 24''$; $B = 122^\circ\ 8'\ 42''$; $C = 140^\circ\ 20'\ 18''$.
Here the three angles are given to find three sides, hence formulæ (11^m), (11^n), (11^o) apply.

$2 \log R = \log(-\cos S) + \text{colog} \cos (S - A)$
$+ \text{colog} \cos (S - B) + \text{colog} \cos (S - C)$.

$$A = 110^\circ\ 36'\ 24''$$
$$B = 122^\circ\ 8'\ 42''$$
$$C = 140^\circ\ 20'\ 18''$$

$$2S = 373 - 5 - 24$$
$$S = 186 - 32 - 42$$
$$S - A = 75 - 56 - 18$$
$$S - B = 64 - 24 - 00$$
$$S - C = 46 - 12 - 24$$

$$\log(-\cos S) = \log - [-\cos(180 - S)]$$
$$= \log\cos 6^\circ\ 32'\ 42'' = 9.997180 \quad - 10$$
$$\text{colog}\ \cos(S - A)$$
$$= \text{colog}\cos 75^\circ\ 56'\ 18'' = 0.614152$$
$$\text{colog}\ \cos(S - B) = \text{colog}\cos 64^\circ\ 24' = 0.364430$$
$$\text{colog}\ \cos(S - C)$$
$$= \text{colog}\cos 46^\circ\ 12'\ 24'' = 0.159751$$
$$2\log R = 1.135513$$
$$\log R = .5677565$$

$$\log\tan\tfrac{1}{2}a = \log R + \log\cos(S - A)$$
$$\log\tan\tfrac{1}{2}b = \log R + \log\cos(S - B)$$
$$\log\tan\tfrac{1}{2}c = \log R + \log\cos(S - C)$$

$$\log R = .5677565$$
$$\log\cos(S - A) = 9.385848 \quad - 10$$
$$\log\tan\tfrac{1}{2}a = 9.9536045 \quad - 10$$
$$\tfrac{1}{2}a = 41^\circ\ 56'\ 43''$$
$$a = 83^\circ\ 53'\ 26''$$

$$\log R = .5677565$$
$$\log\cos(S - B) = 9.635570 \quad - 10$$
$$\log\tan\tfrac{1}{2}b = 10.2033265 \quad - 10$$
$$\tfrac{1}{2}b = 57^\circ\ 56'\ 51''$$
$$b = 115^\circ\ 53'\ 42''$$

$$\log R = .5677565$$
$$\log\cos(S - C) = 9.840349 \quad - 10$$
$$\log\tan\tfrac{1}{2}c = 10.4081055 \quad - 10$$
$$\tfrac{1}{2}c = 68^\circ\ 39'\ 26''$$
$$c = 137^\circ\ 19'\ 52''$$

EXERCISE II.

Find unknown parts of following triangles:

1.	*a* = 57° 56′ 42″	*b* = 137° 22′ 18″	*C* = 94° 47′ 12″
2.	*B* = 131° 17′ 24″	*C* = 94° 48′ 24″	*a* = 57° 56′ 36″
3.	*A* = 68° 34′	*B* = 130° 48′ 24″	*C* = 94° 1′ 36″
4.	*a* = 149° 24′ 24″	*b* = 129° 48′ 24″	*c* = 67° 19′ 12″
5.	*c* = 88° 12′ 20″	*b* = 124° 8′ 17″	*C* = 50° 2′ 1″
6.	*A* = 76° 13′ 42″	*b* = 96° 49′ 6″	*c* = 83° 18′ 25″
7.	*a* = 48° 48′ 48″	*B* = 139° 20′ 30″	*c* = 84° 39′ 29″
8.	*A* = 65° 41′ 16″	*B* = 109° 33′ 22″	*c* = 78° 42′ 36″
9.	*B* = 111° 44′ 46″	*b* = 102° 37′ 14″	*C* = 89° 27′ 15″
10.	*a* = 83° 40′ 40″	*B* = 68° 18′ 17″	*C* = 49° 11′ 10″
11.	*b* = 26° 56′ 48″	*B* = 39° 10′ 45″	*C* = 145° 35′ 36″
12.	*a* = 100° 47′ 9″	*B* = 99° 36′ 13″	*c* = 87° 49′ 27″
13.	*A* = 127° 32′ 25″	*B* = 112° 57′ 42″	*C* = 75° 55′ 45″
14.	*a* = 68° 38′ 48″	*b* = 73° 42′ 37″	*c* = 58° 17′ 16½″
15.	*C* = 113° 10′ 7″	*B* = 98° 43′ 14″	*c* = 71° 21′ 8″
16.	*a* = 39° 7′ 7″	*b* = 77° 33′ 11″	*C* = 82° 23′ 52″
17.	*B* = 109° 22′ 11″	*b* = 119° 12′ 43″	*C* = 102° 37′ 19″
18.	*A* = 133° 6′ 4″	*B* = 91° 48′ 24″	*C* = 78° 43′ 58″
19.	*b* = 44° 33′ 20″	*B* = 86° 25′ 18″	*c* = 22° 16′ 40″
20.	*A* = 97° 27′ 32.4″	*b* = 62° 14′ 17.3″	*c* = 59° 52′ 4″
21.	*a* = 49° 57′ 57″	*b* = 51° 42′ 37″	*c* = 82° 42′ 18″
22.	*A* = 112° 46′ 33″	*B* = 109° 27′ 23″	*C* = 98° 7′ 36″
23.	*A* = 56° 56′ 56″	*b* = 79° 28′ 43″	*c* = 62° 30′ 21″
24.	*B* = 121° 19′ 39″	*a* = 81° 57′ 16″	*C* = 85° 47′ 32″

PART VI.

APPLICATION OF SPHERICAL TRIGONOMETRY.

ARTICLE 20. Most frequent application of Spherical Trigonometry is made in Practical Astronomy and Navigation.

In these sciences, the Spherical Triangle takes a specific form, known as the *Astronomical Triangle*, the earth's surface being regarded as spherical, and hence the meridians as great circles.

In the accompanying figure, 37, ABC is the horizon;

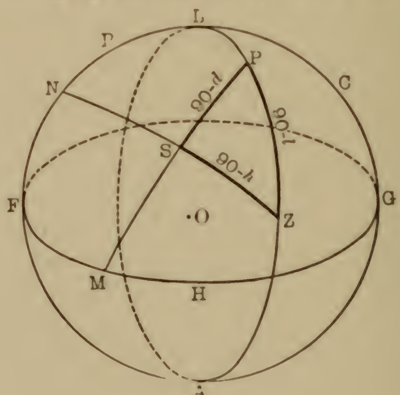

Fig. 37.

FGH is the equator projected on the sky, called the equatorial; PZA is the meridian.

The observer is supposed to be at the center, O, the point Z being the zenith, and P the north pole. S, being any celestial object, SM is called its declination, (d); SN its altitude, (h); PL the latitude (l) of the observer at O.

Hence in the Spherical Triangle ZSP, $ZS = 90 - h$; $SP = 90 - d$; $ZP = 90 - l$; also the angle AZN between the great circle (vertical circle) through S and Z, and PZA the meridian, reckoned from the south point through the west point, is called the azimuth, (a) of S.

The angle ZPS, between the great circle (hour circle) PSM through S and P, and the meridian, PZA, is called the hour angle, (t), of S.

Hence in the triangle $PZS = 180 - AZN$ (in the figure) and AZN is found by subtracting the azimuth from 360, when it exceeds 180°.

ART. 21. EXAMPLE. What time does the sun set in St. Petersburg, lat. 59° 56′, on the longest day of the year?

On the longest day of the year the sun is farthest north, and its declination on that day is always 23° 27′, the angle its apparent path makes with the equinoctial. Also at setting it is on the western horizon, hence its latitude is zero. ∴ in the triangle ZSP, $ZP = 90° - l = 90° - (59° 56') = 30° 4'$; $SP = 90 - d = 90° - (23° 27') = 66° 33'$; and $SZ = 90° - h = 90°$.

We have then a triangle with three sides given, which is

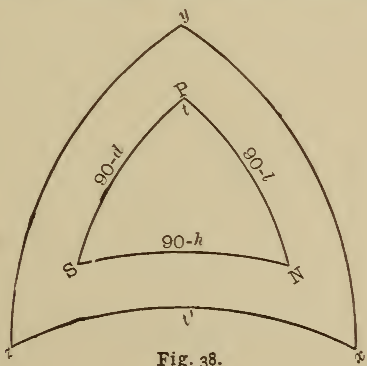

Fig. 38.

solvable by the method explained previously, or since it is also a quadrantal triangle, we can use its polar which will be a right triangle.

Then in the figure (38), drawing the polar $x\ y\ z$, since the hour angle t is required, we must find side t'; $t' = 180° - t$.

$y = 180° - 90° = 90°$; $z = 180° - (90° - l) = 90° + l$
$x = 180° - (90° - d) = 90° + d$, and t' is hypotenuse.

By Napier's rules, t' being a middle part to x and z the other known parts, sin (co. t') = tan (co. x) tan (co. z) or $\cos t' = \cot x \cot z$. Substituting values above, $\cos(180 - t) = \cot(90 + d) \cot(90 + l)$,

whence, $-\cos t = (-\tan d)(-\tan l)$, or $\cos t$
$= -\tan d \tan l$.

That is, $\cos t = -\tan(23° \; 27') \tan(59° \; 56')$

$$\begin{aligned} \log \tan 23° \; 27' &= 9.63726 - 10 \\ \log \tan 59° \; 56' &= 10.23739 - 10 \\ \hline \log(-\cos t) &= 9.87465 - 10 \\ t = 180° - (48° \; 31' \; 44'') &= 131° \; 28' \; 16'' \end{aligned}$$

or in time, $t = (131° \; 28' \; 16'') \div 15$
$= 8$ hr. $- 45$ min. $- 53$ sec.

That is, the sun sets about 8.46 o'clock P.M.

Again: On a given day the sun's declination is 18° 35′ *N*. At 3 o'clock P.M. its altitude is 48° 22′. What is the latitude of the place?

In the triangle ZPS, we have here, ZPS (t) = 45° [3 × 15;] $SP = 90° - d = 90° - [18° \; 35'] = 71° \; 25'$; and $ZS = 90° - h = 90° - [48° \; 22'] = 41° \; 38'$, to find $ZP = 90 - l$. That is, we have two sides and one angle given, from which the third side ZP is readily found.

Art. 22. Since the longitude of a place is the same as the difference between its local time and Greenwich time, if Greenwich time is known at any observation, the hour angle as calculated above will give local time, and hence the longitude is easily found.

Every ship carries chronometers with Greenwich time, and therefore this method gives its longitude readily.

ART. 23. There is another class of problems whose solution is much simplified by the use of Spherical Trigonometry. For example, let it be required to find the angle between the lateral faces of a regular octagonal pyramid, whose edges meet at an angle of 18° at the vertex.

In the pyramid *ABCDEFGH* – *K* to find the angle between the faces, say between *ABK* and *CBK*. Take *B* as the center of a sphere of any convenient radius; the surface of this sphere will intersect the three faces *ABK*, *CBK*, and *ABCDEFGH* in the sides of a spherical triangle, which will be isosceles, because the pyramid is regular.

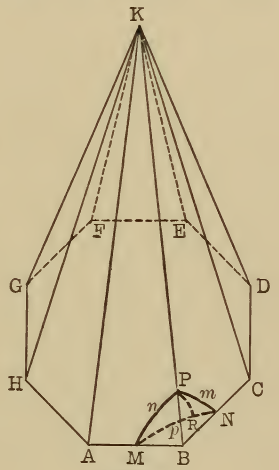

Fig. 39.

Call this triangle *MNP* (as represented in Fig. 39), the sides being *m*, *n* and *p*, according to our usual designation.

By geometry, $\angle ABC$ = arc $p = \frac{1}{8}$ [2 right angles × (8 – 2)] = $\frac{3}{2}$ right angles = 135°. Also, since the pyramid is regular, *KBC* (or *KBA*) is isosceles. Hence, since *BKC* (or *BKA*) = 18°, *KBC* (or *KBA*) = $\frac{1}{2}$ (180° – 18°) = 81°. That is, arcs *m* and *n* = 81°.

By dropping a perpendicular arc from *P* to *MN*, say at *R*, the isosceles triangle is divided into two equal right triangles, wherein *m* (or *n*) = 81° and *RN* (or *RM*) = $\frac{1}{2}$ (135°) = 67° 30′. Whence $\frac{1}{2} \angle P$ (the required angle) is easily found.

Again: Through the foot of a rod making an angle *m* with a plane, a straight line is drawn making an angle *n*, with the projection of the rod on the plane.

What angle does the rod make with this line? Let *MN* be the plane, and *OA* the rod, *OC* its projection, and *OB* the line in the plane. With *O* as a center describe a sphere with any convenient radius. It will intersect the planes of the three lines in the right spherical tri-

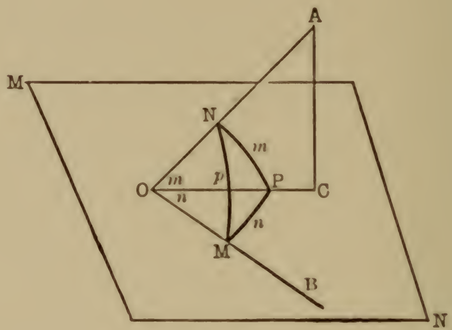

Fig. 40.

angle *MNP*, whose sides will be *m*, *n*, and *p*, of which *m* and *n* are known. With *m* and *n* known, it will be easy to find *p*, which equals the required angle *AOB*. Why is the spherical triangle a right one? It will be observed that while these problems can be solved by Plane Trigonometry methods, the solutions are greatly simplified by the application of the spherical.

Area of Spherical Triangle.

ART. 24. By Solid Geometry the area of a spherical triangle is given by the formula: $\frac{ER^2\pi}{180}$, *E* being the spherical excess $[(A + B + C) - 180]$ in the triangle, expressed in spherical degrees, and *R* is the radius of the sphere.

To use this formula it is clearly necessary to know the three angles.

However, the value of E can be found from the three sides, by the formula: $\tan^2 \frac{1}{4} E = \tan \frac{1}{2} S \tan \frac{1}{2} (S-a) \tan \frac{1}{2} (S-b) \tan \frac{1}{2} (S-c)$; wherein a, b, and c are the three sides and $S = \frac{1}{2}(a+b+c)$.

EXAMPLE. Find the area of the spherical triangle, whose sides are: $a = 69° 15' 6''$, $b = 120° 42' 47''$, $c = 159° 18' 33''$, on a sphere whose radius is 7918 miles.

$$\begin{array}{rl} a = & 69° \; 15' \; 6'' \\ b = & 120° \; 42' \; 47'' \\ c = & 159° \; 18' \; 33'' \\ \hline a + b + c = 2S = & 349° \; 16' \; 26'' \end{array}$$

$$\begin{array}{rlrl} S = & 174° \; 38' \; 13'' & \frac{1}{2} S = & 87° \; 19' \; 6\frac{1}{2}'' \\ S - a = & 105° \; 23' \; 7'' & \frac{1}{2}(S-a) = & 52° \; 41' \; 33\frac{1}{2}'' \\ S - b = & 53° \; 55' \; 26'' & \frac{1}{2}(S-b) = & 26° \; 57' \; 43'' \\ S - c = & 15° \; 19' \; 40'' & \frac{1}{2}(S-c) = & 7° \; 39' \; 50'' \end{array}$$

$$\begin{array}{rl} \log \tan \frac{1}{2} S = & 11.32942 - 10 \\ \log \tan \frac{1}{2}(S-a) = & 10.11805 - 10 \\ \log \tan \frac{1}{2}(S-b) = & 9.70644 - 10 \\ \log \tan \frac{1}{2}(S-c) = & 9.12893 - 10 \\ \hline \log \tan^2 \frac{1}{4} E = & 20.28284 - 20 \\ \log \tan \frac{1}{4} E = & 10.14142 - 10 \\ \frac{1}{4} E = & 54° \; 10' \; 5'' \\ E = & 216° \; 40' \; 20''. \end{array}$$

$\frac{\pi R^2 E}{180}$ may be expressed $\frac{\pi R^2 E''}{648000}$, if E be reduced to seconds (since 180 must also be multiplied by 3600).

$$\begin{array}{rl} \log \frac{\pi}{648000} = & 4.68557 - 10 \\ \log E'' = & 5.89210 \\ \log 7918^2 = & 7.79724 \\ \hline \log \text{area} = & 8.37491 \\ \text{area} = & 237088889 \text{ sq. miles.} \end{array}$$

EXERCISE III.

Applications of Spherical Trigonometry.

1. A ship's captain observes the sun's altitude to be $14° 18'$ at 6 o'clock A.M. The almanac gives its declination as $18° 36'$ N. What is the ship's latitude?

2. If a ship in latitude $50° 13'$ finds the sun's altitude to be $16° 20'$ at 9 o'clock A.M., Greenwich time, the sun's declination being $21° 6'$, what is its longitude?

3. At what time will the sun rise at Melbourne, lat. $37° 49'$ S. on the longest day in the southern hemisphere, sun's declination being $23° 27'$ S?

4. What angle does the shadow on a sun-dial plate make with the gnomon at 3 P.M. in latitude $40° 37'$?

5. Find the latitude of the place at which the sun sets at 9.30 P.M. on the longest day.

6. In what latitude will the sun rise exactly in the northeast point on the longest day?

7. The moon's path makes an angle of $5° 8'$ with the ecliptic, in which the axis of the earth's shadow lies. A section of this shadow is circular in form with its center on the ecliptic. If the radius of the moon is $15° 45''$, how far must the moon be from the intersection of its path with the ecliptic, that it may just touch the shadow, that is, begin an eclipse?

EXAMPLE 1. Two sides and included angle.

Given,	$a = 92° 37' 40''$;		$C = 108° 48' 16''$	
	$b = 44\ \ 52\ \ 12$;		$\frac{1}{2} C = 54\ \ 24\ \ 8$	
	$a + b = 137\ \ 29\ \ 52$;		$\frac{1}{2}(a+b) = 68\ \ 44\ \ 56$	
	$a - b = 47\ \ 45\ \ 28$;		$\frac{1}{2}(a-b) = 23\ \ 52\ \ 44$	

The pair of formulæ applying here is (13^m) and (13^x),

$$\begin{cases} \tan \frac{1}{2}(A - B) = \dfrac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \frac{1}{2} C & . \quad (13^m) \\ \tan \frac{1}{2}(A + B) = \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \frac{1}{2} C & . \quad (13^x) \end{cases}$$

$$\begin{array}{rl} \log \sin \frac{1}{2}(a-b) = & 9.607245 - 10 \\ \text{colog} \sin \frac{1}{2}(a+b) = & .030584 \\ \log \cot \frac{1}{2} C = & 9.854906 - 10 \\ \hline \log \tan \frac{1}{2}(A-B) = & 9.492735 - 10 \end{array}$$

$$\begin{array}{lll} \frac{1}{2}(A-B) = 17^\circ\ 16'\ 29'' & & \\ \frac{1}{2}(A+B) = 61\quad 1\quad 49 & & \log \cos \frac{1}{2}(a-b) = 9.961138 - 10 \\ \hline \text{Add; } A = 78\quad 18\quad 18 & & \text{colog} \cos \frac{1}{2}(a+b) = 0.440745 \\ \text{Sub.; } B = 43\quad 45\quad 20 & & \log \cot \frac{1}{2} C = 9.854906 - 10 \\ & & \hline & & \log \tan \frac{1}{2}(A+B) = 10.256789 - 10 \end{array}$$

EXAMPLE 2. Two angles and included side.

$$\begin{array}{lrl} \text{Given,} & B = 128^\circ\ 50'\ 18'' & a = 69^\circ\ 8'\ 38'' \\ & C = 54\quad 46\quad 10 & \frac{1}{2} a = 34\quad 34\quad 19 \\ \hline & B+C = 183\quad 36\quad 28 & \frac{1}{2}(B+C) = 91\quad 48\quad 14 \\ & B-C = 74\quad 4\quad 8 & \frac{1}{2}(B-C) = 37\quad 2\quad 4 \end{array}$$

The pair of formulæ applying is (12^n) and (12^y);

$$\begin{cases} \tan \frac{1}{2}(b - c) = \dfrac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \tan \frac{1}{2} a & . \quad . \quad (12^n) \\ \tan \frac{1}{2}(b + c) = \dfrac{\cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)} \tan \frac{1}{2} a & . \quad . \quad (12^y) \end{cases}$$

It is to be observed that $\frac{1}{2}(B+C)$ being greater than 90°, its cosine is negative. Its sine and cosine are found in table by observing the rule, $\sin x = \sin (180 - x)$, and $\cos x = - \cos (180 - x)$.

$$\log \sin \tfrac{1}{2}(B-C) = 9.779809 - 10$$
$$\text{colog} \sin \tfrac{1}{2}(B+C) = 0.000221$$
$$\log \tan \tfrac{1}{2} a = 9.838302 - 10$$
$$\log \tan \tfrac{1}{2}(b-c) = 9.618332 - 10$$

$\frac{1}{2}(b-c) = 22° \; 33' \; 6''$
$\frac{1}{2}(b+c) = 86 \; 43 \; 30$
$b = 109 \; 16 \; 36$
$c = 64 \; 10 \; 24''$

$$\log \cos \tfrac{1}{2}(B-C) = 9.902152 - 10$$
$$\text{colog} \cos \tfrac{1}{2}(B+C) = 1.501988$$
$$\log \tan \tfrac{1}{2} a = 9.838302 - 10$$
$$\log \tan \tfrac{1}{2}(b+c) = 11.242442 - 10$$

The finding of c in Example 1 and A in Example 2 has been explained in Article 19.

Additional Problems on Oblique Triangle.

1. Two observers notice a rocket explode in the air at elevations of 60° 30′ and 45° 15′ 25″ respectively. If the interval between the flash and the report of the explosion is $2\frac{1}{4}$ sec. for the first observer, what is the interval for the second?

2. A ball is thrown south with a velocity of 36.8′ per sec. from a train running 78.6′ per sec. south 30° 15′ east. What is the velocity of the ball with respect to the earth?

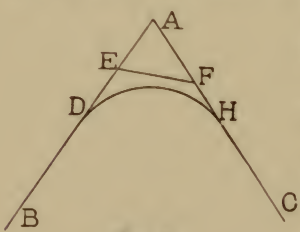

Fig. A.

3. In laying out a railway curve (see Fig. A) to connect two pieces of straight track DB and HC, two points E and F are taken in the prolongation of DB and HC (meeting at A). By measurement $\angle AEF = 43° \; 12'$, $\angle AFE = 27° \; 44'$, and $EF = 413'$. Find $\angle A$ and the distances DE and FH, radius of curve being 2000′.

4. In bevel gear wheels (Fig. *B*), $AD = 2r$, $AB = 2r'$, $\angle DCO = m$, $\angle ACP = n$, $y = \angle OCP$ between shafts. Show that $\frac{r}{r'} = \frac{\sin m}{\sin (y - m)}$. If the rates of revolution (ω and ω') are inversely as the radii, show that $\tan m = \frac{\omega' \sin y}{\omega + \omega' \sin y}$.

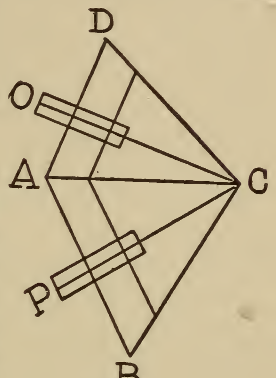

Fig. B.

5. *AB* is the crank of an engine (Fig. *C*), *BE* the connecting rod, *D* and *C* the extreme positions of the end *E*, so that *DC* is the length of the stroke. If in any position *AD* makes angles θ and ϕ with *AB* and *BE* as shown, prove that $AE = AB \cos\phi + BE \cos\theta$; and since $AD = AB + BE$, find *DE*, distance traversed by the piston while the crank moves through any given angle.

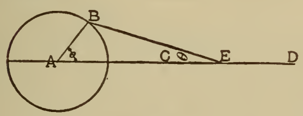

Fig. C.

6. The distance from a point on the ground 24′ from the foot of an abutment to a point 49′ up its inclined face is 58′. What is the inclination of the abutment?

7. A section of a tunnel is the shape of a rectangle capped by a segment of a circle. The height of the rectangle is 12′, its width is 18′, and the height of the center of the arch is 17′. If the tunnel is ½ mile long, find the amount of excavation and the radius of the arch.

8. The distance between conning towers on a battleship is 250′; the range-finders stationed in these towers show respectively angles of 87° 15′ and 88° 25′ when focused on a distant ship. What is the range?

9. In making a survey the line *AB* runs directly through a barn. A line *BC* 698.27′ long is then run so that *A* is visible from *C*, whence the angle *ACB* is found to be

59° 18′ 30″, and the distance CA measures 964.12′. Find AB in length and direction.

10. To find the height of a distant flagstaff AB, a line xy was measured on the level ground at place of observation, 520′ long. Also the horizontal angles xyB (to the base of the staff) and yxB were found to be 139° 25′ 20″ and 26° 28′ 30″ respectively. What was the height if the elevation of the top of the staff at y was 29° 24′?

11. In running a line for a survey a swamp was encountered. A point M was then taken from which both ends of the line AB through the swamp were visible. The line MA was found to be 679.26′, MB, 859.74′, and angle AMB, 70° 23′ 40″. Find length and direction of AB if AM ran S. 28° 15′ E.

PART VII.

VECTORS.

ART. 1. Geometrically a line has been regarded as possessing merely extent, without regard to its direction except as an incidental matter. It is, however, useful to attach a special importance to its direction as well.

With this additional quality a line is known as a *vector*. Hence a *vector* may be defined as a *directed line*.

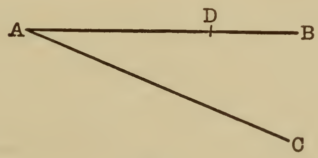

Fig. 1.

For example, the line AB (Fig. 1) is carefully distinguished from BA.

As these lines are coincident and equal in extent, the method of distinguishing them by signs suggests itself. That is, if AB is positive, BA is negative, and it may be said that

$$BA = -AB.$$

Again, if from A a line extends in any other direction than AB, it is clearly different from AB, although it may be of the same length, as AC (Fig. 1), for direction is now an essential quality of a line.

As vectors have extent as well as direction, AD (Fig. 1) would be distinguished from AB although it is part of it.

A pure number, as for example the length of AB, AC or AD in ordinary linear units, is called a *scalar*. As illustrations of scalars might be mentioned magnitude, weight, time, etc. As illustrations of vectors we have forces, accelerations, trigonometric functions, etc.

A vector consists then of both a scalar and a vector part.

It may be said, for example, that $AD = \frac{2}{3} AB$, where $\frac{2}{3}$ is the scalar and AB is the vector part of AD.

The idea of direction is the essential quality of the vector part; magnitude is the essential quality of the scalar part. A vector is said to equal zero, or to be a *null vector* as it is called, when its magnitude is zero. Evidently it may be regarded as a geometric point.

Art. 2. As a line may be regarded as a path of a moving point, it is sometimes useful to consider vectors from this standpoint, especially in vector addition.

Since vectors are determined both by magnitude and direction, parallel vectors, having the same extent and taken in the same direction, are equal; and hence also, vectors not parallel *cannot* under any circumstances be equal.

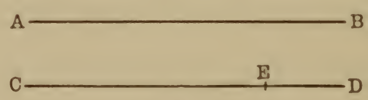

Fig. 2.

That is, if two vectors AB and CD (Fig. 2), are equal, they must be parallel.

Also if

$$CE = \frac{3}{4} CD$$

then
$$CE = \frac{3}{4} AB \text{ (since } AB = CD).$$

So a vector may always be expressed in terms of a parallel vector. How would EC be expressed?

ART. 3. *Addition and Subtraction of Vectors.*

To meet the new conditions it is plainly necessary to agree upon laws for the fundamental operations of addition, subtraction, etc.

These laws are arbitrary, but are recommended by experience.

Recurring to the idea of a moving point, it may be said that *the sum of two vectors is the straight line from the starting-point to the final position of the point that successively and continuously traces them.*

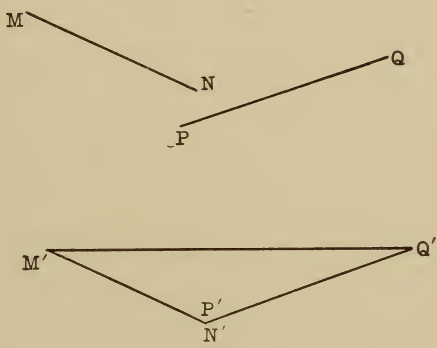

Fig. 3.

For example, to add MN and PQ, starting with the point M' let a point trace $M'N'$ equal in length and parallel to MN (hence vector $M'N'$ = vector MN), then $P'Q' = PQ$. Clearly the point would arrive at Q' as well by traversing the vector $M'Q'$, hence it is said $M'Q' = M'N' + P'Q'$ or

$$M'Q' = MN + PQ.$$

It follows, of course, from the fact that parallel vectors of equal extent are equal, that it does not affect a vector in any way to transport it parallel to its first position.

This same definition of addition is easily extended to any number of vectors.

Let it be required to add the five vectors A, B, C, D and E, say (Fig. 4). Let the point trace successively the vectors

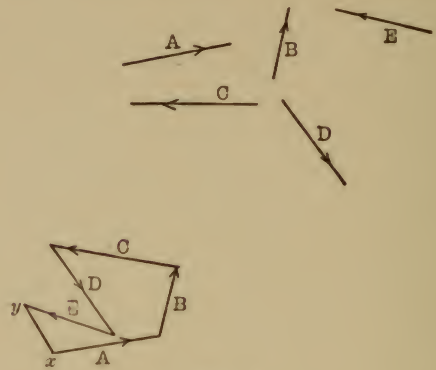

Fig. 4.

beginning with A, the directions being indicated by the arrowheads; it will end at y. It would have clearly reached the same point by traversing the vector xy, hence,

$$xy = A + B + C + D + E.$$

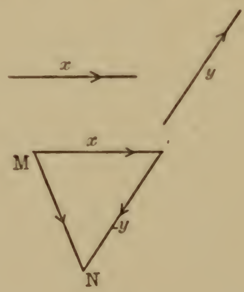

Fig. 5.

As algebraic subtraction is merely a form of addition, so is vector subtraction a form of addition. Bearing in mind that reversal of direction reverses the sign of a vector, so that $BA = -AB$ for example, it follows that $AB - CD$ may be written

$$AB + (-CD) = AB + DC, \text{ since } DC = -CD.$$

Let it be required to find the difference between two vectors x and y. Let $x - y$ be required. Tracing x and then $-y$, directions being indicated by arrowheads, the result is equivalent to the vector MN. (Fig. 5.)

Find $A + B - C + D - E$ (Fig. 4).

Art. 4. It is evident from the explanation of vector addition that the order in which the component vectors are added is indifferent.

If A, B, C, D, etc., are a number of vectors whose sum is required, the final position of the traversing point will clearly be the same whatever be the order in which it describes the vectors; that is,

$A + B + C + D = B + C + A + D = D + A + C + B$ etc.

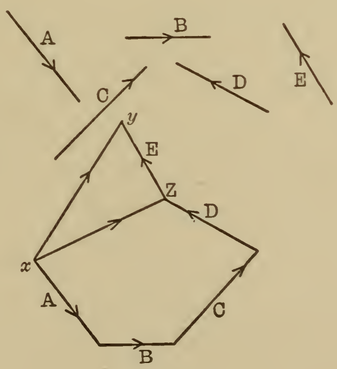

Fig. 6.

Since subtraction is merely a phase of addition it makes no difference if some of the vectors are negative. Hence the ordinary commutative law of algebraic addition applies here. Again, if the sum of two or more vectors is added to another vector, the result is the same as if the vectors had all been added successively in one sum. For example, let A, B, C, D and E be five vectors (Fig. 6), then $A + B + C + D + E = xy$, also $A + B + C + D = xz$ and $xz + E = xy$,

$$(A + B + C + D) + E = A + B + C + D + E.$$

Likewise,

$$(A + B + C) + D + E = (A + B + C) + D + E$$
$$= A + B + C + D + E.$$

Hence the *associative law* of addition also holds true with vectors.

ART. 5. It will be observed that the sum of two vectors is the diagonal of the parallelogram of which the two vectors are adjacent sides, the diagonal being the one joining their extremities, when both vectors extend in the same direction and hence can be described by a point moving always forward from the origin of the first to the extremity of the second.

If both vectors radiate from one point, the diagonal drawn from their common point represents their sum, as

Fig. 7.

(Fig. 7) vectors NM and NP have NQ as sum, for from P (starting with N) draw the vector $PQ = NM$, then by our rule $NP + PQ = NQ$, or $NP + NM = NQ$ (since $PQ = NM$).

ART. 6. From the definitions and illustrations of vector addition it is plain that the sum of any number of vectors forms one side of a polygon, whose other sides are the component vectors, and hence if the terminus of the last vector in a sum coincides with the origin of the first vector, thus forming a closed polygon, the sum is zero. This has its illustration in physics where any number of forces are in equilibrium.

The sum of two vectors radiating from the same point, represented by the diagonal of the parallelogram constructed upon them, suggests immediately the physical law of composition of two forces.

As might be inferred from the similarities cited, vector processes have a wide and very effective application to physical problems.

It is to be observed that the rules laid down above for addition and subtraction, as well as those for multiplication and division to be hereafter established, are purely arbitrary. The sum of two vectors might have been given an entirely different meaning if desired, but the method of addition already explained has been found to give best results. Once defined and adhered to, they are entirely effective.

ART. 7. A simple reference to the definition makes it plain that the algebraic rule for signs in addition and subtraction holds with vectors. For example, it is evident that $AD + (-BC) = AD - BC$, for this latter expression is the same as $AD + CB$ (where BC is reversed) by our understanding of vector addition, and $+ CB = - BC$.

Hence with vectors as with scalars, $+ - = -$; $- - = +$, etc. Again, by a simple application of the principles of similar polygons, it follows that $x(A + B) = xA + xB$, where x is a scalar, and A and B, vectors. Thus: in Fig. 8a let $MN = xA$ and $PQ = xB$. Then adding MN and PQ, (Fig. 8b), where $NQ = PQ$, $MQ = MN + PQ$. Also, let $ab = A$; $bc = B$ (Fig. 8c), then $ac = A + B$.

But $\triangle$ MNQ and abc are similar since xA must be // to A (x being a scalar coefficient only) and xB is // to B, hence $\angle N = \angle b$ and $MN : NQ :: ab : bc$ (having the same ratio, x); $\therefore$ MQ has the same ratio to ac; that is,

$$MQ = xac \quad . \; . \; . \; . \; . \; . \quad (1)$$

But $\qquad MQ = MN + PQ = xA + xB$

and $\qquad ac = A + B$

$$xA + xB = x(A + B) \text{ (by (1))}.$$

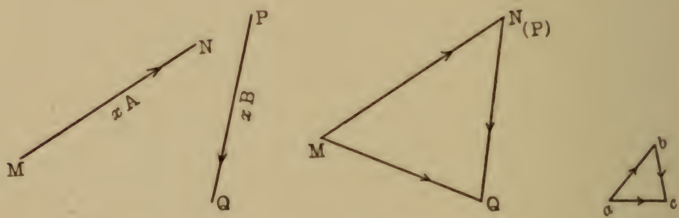

Fig. 8a. Fig. 8b. Fig. 8c.

EXERCISE.

1. Prove $x(A - B) = xA - xB$.
2. Prove $(x + y) A = xA + yA$ (where x and y are scalars).
3. Prove $-(A + B) = -A - B$.
4. Prove $x(A + B + C) = xA + xB + xC$.
5. Prove $x(A - B + C - D) = xA - xB + xC - xD$.

Multiplication.

ART. 8. Since a vector has magnitude as well as direction, and magnitude is a scalar quantity, we can represent any vector as made up of a unit vector, indicating direction, and a scalar coefficient, indicating its magnitude. It is to be understood that multiplying a vector by a scalar alters its magnitude only and does not at all affect its direction. Representing unit vectors by the small letters, corresponding to the capitals representing the entire vectors, and representing the scalar coefficient by a capital S with a subscript or by the unknown quantity letters x, y, z, etc., we may say, for example,

$$\text{Vector } A = S_a a.$$

ART. 9. Since scalars are ordinary algebraic or arithmetical quantities, the usual associative and commutative laws of algebra apply to them; for example,

$$xyzA = x(yz)A = (xz)yA = (yz)xA, \text{ etc.}$$

It may further be assumed that if

$$A = S_a a$$

then

$$a = \frac{A}{S_a}, \text{ etc.}$$

ART. 10. Since the ordinary rules of algebra apply to scalars, vector equations may be treated, through their scalar coefficients, as are algebraic equations.

For it is the coefficients in any equation that determine the relation of its parts or its relation to other equations. For example, in the algebraic equations,

$$ax + by = c,$$
$$dx + ey = f,$$

the coefficients a, b, c, etc., determine the relations of x and y

So in the vector equation,

$$xA + yB + zC = 0,$$

it may be said that

$$xA = -(yB + zC),$$
$$A = -\frac{(yB + zC)}{x}, \text{ etc.}$$

Also two or more vector equations may be combined as simultaneous, as, for example,

$$A + 2\,B = C,$$

$$3\,A + 4\,B = D,$$

give as results in the usual way,

$$2\,B = 3\,C - D$$

and

$$A = D - 2\,C.$$

Definitions.

ART. 11. Vectors parallel to the same straight line are said to be *collinear;* parallel to the same plane are *coplanar.*

If no straight line can be drawn parallel to two or more vectors, they are said to be *non-collinear*; if no plane can be drawn parallel to *three* or more vectors they are *non-coplanar.*

Evidently, from geometry, a plane can always be drawn parallel to any *two* vectors, but two vectors that are not parallel may be regarded as non-coplanar with respect to all but one plane.

Since a vector may be transported parallel to itself, if three or more vectors are coplanar they can be moved until all are in the same plane and they would there intersect (or their prolongations would intersect) unless parallel, and hence they could be expressed in terms of each other (by addition or subtraction). If any two or more were parallel any one of these parallels could be expressed in terms of the others by using scalar coefficients.

It is clear that if there are but two vectors neither could be expressed in terms of the other unless they were parallel, for they could only intersect in one point, and hence two vectors may be always treated as if non-coplanar, since it is the ability or non-ability to express vectors in terms of each other that is important.

ART. 12. As a result of these definitions the following rule may be established:

If two equal vectors are expressed in terms of other non-coplanar vectors, the coefficients of like vectors in the two expressions are equal.

That is, if

$$M = S_1 A + S_2 B + S_3 C \quad . \quad . \quad . \quad . \quad (1)$$

and

$$N = S_4 A + S_5 B + S_6 C \quad . \quad . \quad . \quad . \quad (2)$$

and

$$M = N$$

then

$$S_1 = S_4,\ S_2 = S_5,\ S_3 = S_6.$$

For, subtracting (1) from (2),

$$M - N = (S_1 - S_4)\,A + (S_2 - S_5)\,B + (S_3 - S_6)\,C. \quad (3)$$

But

$$M - N = 0,$$

and since A, B and C are non-coplanar they can have no relation to each other, hence the only way that

$$0 = (S_1 - S_4)\,A + (S_2 - S_5)\,B + (S_3 - S_6)\,C$$

is

$$S_1 - S_4 = 0, \text{ that is } S_1 = S_4;$$

$$S_2 - S_5 = 0, \text{ that is } S_2 = S_5;$$

$$S_3 - S_6 = 0, \text{ that is } S_3 = S_6.$$

From what was said in the previous article this relation is always true when equal vectors are expressed in only two other vectors, unless these latter are parallel.

Applications.

ART. 13. To prove that the diagonals of a parallelogram bisect each other.

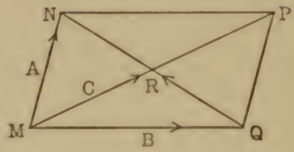

Fig. 9.

Let $MNPQ$ be a ▱ (Fig. 9), the sides MN and MQ being the vectors A and B respectively. Let the diagonals MP and QN intersect at R; to show, say,

$$MR = \tfrac{1}{2} MP.$$

$$MP = A + B \text{ (by definition of addition)}$$

$$C = MR = xMP = x(A + B) \quad . \quad . \quad (1)$$

(where x = ratio of MR to MP, to be shown $= \frac{1}{2}$).

Also $\quad QN = A - B$ (Art. 3).

Then $\quad QR = yQN = y(A - B)$

(y being ratio of QR to QN, unknown).

In triangle MQR

$$C = MR = MQ + QR = B + y(A - B),$$

or

$$C = yA + (1 - y)B \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2),

$$x = y \text{ and } x = 1 - y \text{ (by Art. 12),}$$

whence $\quad x = \frac{1}{2}, y = \frac{1}{2}.$

That is, $\quad MR = \frac{1}{2} MP$ and $QR = \frac{1}{2} QN$.

Observe that the object has been to get two independent expressions for the line involved, MR, since there were two unknowns (x and y), just as would have been done in an algebraic problem. The same result would have been achieved by expressing MR in terms of MN and NR (or RN), and also in terms of MQ and RQ (or QR); thus,

$$MR = C = MN + NR = A + x\,(B - A)$$

and

$$MR = C = MQ + QR = MQ - RQ$$
$$= B - y\,(B - A);$$

$$x = \text{ratio of } NR \text{ to } NQ;\ y = \text{ratio of } RQ \text{ to } NQ,$$

whence $\quad C = (1 - x)\,A + xB$

and $\quad C = (1 - y)\,B + yA.$

$$\therefore x = y.$$

Hence $\quad NR = RQ.$

To Divide a Line in a Given Ratio.

ART. 14. Let it be required to divide the line MN in the ratio $p : q$ (Fig. 10). Take any convenient point as O, out-

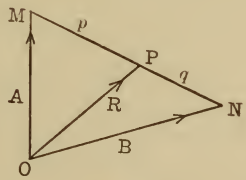

Fig. 10.

side, for reference, and draw vectors $OM = A$; $ON = B$; and $OP = R$, P being the point that divides the line in the given ratio. Then $R = A + MP = A + \dfrac{p}{p+q}\,(MN)$ (by addition), also $MN = B - A$ (by principle of addition).

$$\therefore R = A + \frac{p}{p+q}(B - A) = \frac{qA + pB}{p+q}.$$

This last equation completely determines the length and position of R.

Clearly the position of O is a matter of indifference, so that we can fix it for greatest convenience.

Center of Gravity.

ART. 15. The principles of vector relations already enunciated furnish simple solutions of the problems involving center of gravity. The two following laws of physics are readily applied to that end through vectors; namely,

1. *The center of gravity of two masses (regarded as points), lies on the line connecting them, and divides this line into two segments which are inversely proportional to the masses.*

2. *For the purpose of locating the C. G. of two systems of bodies, each system may be replaced by a simple mass equaling the aggregate mass of all the bodies composing the system, and situated at the C. G. of the system.*

Multiplication of Vectors.

ART. 16. As the vector has a twofold quality, namely scalar and vector, two kinds of multiplication are suggested, scalar and vector multiplication.

The scalar product is usually referred to as the *direct product;* the *vector* product is known as the *skew product.*

The scalar product of two vectors is arbitrarily defined as the algebraic product of their magnitudes and the cosine of the angle between them.

This product which is evidently a scalar quantity (being the product of scalars, hence the name) may be represented thus, letting A and B be vectors,

$$A \dot{\vee} B = S_a\, S_b \cos (A, B).$$

From this method of indicating the multiplication by a dot in a V it is sometimes called dot multiplication.

It follows immediately from this definition that if two vectors are parallel their product is equal to the numerical product of their lengths, since the angle between them is zero if they extend in the same direction; 180°, if they extend in opposite directions. In the first case the product is positive, since cos 0 = 1; in the second case it is negative, since cos 180° = − 1. Hence also the product of a vector by itself, represented in the usual way, as its square, is

$$A \dot{\vee} A = S_a^2.$$

Again, if two vectors are perpendicular the angle between them being 90°, and cos 90° = 0, their product is zero.

If the dot product of two vectors is 0, then, either one of them is a null vector or they are perpendicular. *Hence, if the dot product of two vectors neither of which is null, equals zero, they are perpendicular.*

Likewise, if the dot product of two vectors is equal numerically to the product of their lengths, they are parallel.

The scalar or dot product of two vectors follows the ordinary rules of Algebra, as to the associative and distributive principles.

ART. 17. The scalar product of two vectors may be given a very simple geometric interpretation as follows:

The projection of one line upon another is always equal to the length of the projected line multiplied by the cosine

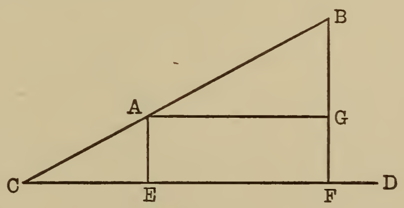

Fig. 11.

of the angle between them; as (Fig. 11), *EF* (= *AG*) is the projection of *AB* on *CD*. *EF* = *AG* = *AB* cos *BAG* = *AB* cos *BCD*.

The definition of the dot product of two vectors is then the product of the magnitude of one by the projection of the other upon it.

It is easy to prove from the laws of projections the statement made above — that scalar multiplication obeys the distributive law.

Plane Areas as Vectors.

ART. 18. Since planes have direction, as well as straight lines, there is no inconsistency in regarding plane areas as vector quantity. Certain conventions are then necessary to so represent them. A plane area bounded by a closed curve (which does not cut itself) is regarded as positive, if, with reference to a pencil describing it, it lies always to the left looking down upon it; negative if it lies to the right.

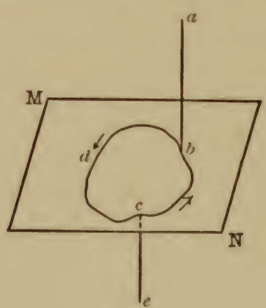

Fig. 12.

Clearly the same area would be positive on one side of the plane and negative on the other (Fig. 12). The plane area *cbd* would be positive when described by the pencil *ab* above the plane; negative if described by *ce* below the plane, always looking toward the plane. The definition may be modified thus: a plane area is positive with respect to a point, when its boundary is described in a counter-clockwise direction, looking from the point toward the plane; in the clockwise direction, negative.

It is agreed, further, to represent the area (since it is a vector quantity) by a linear vector whose magnitude equals numerically the area, and whose direction is that of a perpendicular to its plane on the positive side; extending

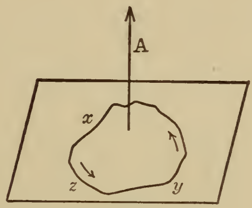

Fig. 13.

away from the plane; thus (Fig. 13) A is the vector representing the area yxz, if the length of A equals in linear units the area of yxz in square units. This conception of a closed area has a special application to electric currents flowing in closed circuits, as the lines of force pass from the negative to the positive side of the plane.

ART. 19. As a geometrical application of the product of two vectors, one of which is an area bounded by a closed

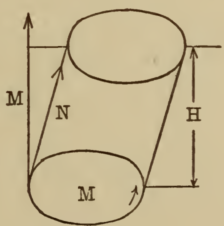

Fig. 14.

curve, the following will serve. Let M (Fig. 14) be such an area represented by the vector M, and N be any other vector. The volume of a cylinder of height H and base M is $M \times H$

[M merely representing area of base], but H is evidently the projection of N on M, that is,

$$H = S_2 \cos (M, N) \text{ (where } M = S_1 m; \ N = S_2 n).$$

$\therefore$ volume $= M \times H = S_1 S_2 \cos (M, N)$ [S_1 and S_2 being the magnitudes of M and N].

That is, volume $= M \dot{\vee} N$ [M being here the vector representing area M].

The dot product of two vectors, one of which is a plane area bounded by a closed curve, is the volume of a cylinder with the area as base and the other vector as element.

Applications of Dot Multiplication.

ART. 20. To prove that in any triangle the sum of the squares of two of the sides equals twice the square of half

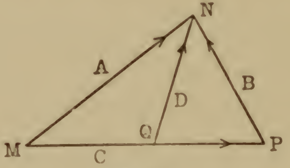

Fig. 15.

the third side plus twice the square of the median. Let (Fig. 15) A, B and C be the vector sides of the triangle MNP, and D be the median.

Then $A = \frac{1}{2} C + D$ (Q being middle of MP).

By multiplication law,

$$A \dot{\vee} A = A^2 = \tfrac{1}{4} C^2 + D^2 + C \dot{\vee} D \quad . \quad . \quad . \quad (1)$$

Also, $\qquad B = D - \frac{1}{2} C.$

Whence $\quad B \dot{\vee} B = B^2 = D^2 + \frac{1}{4} C^2 - C \dot{\vee} D \quad . \quad . \quad (2)$

Adding (1) and (2):

$$A^2 = \tfrac{1}{4} C^2 + D^2 + C \cdot D$$
$$B^2 = \tfrac{1}{4} C^2 + D^2 - C \cdot D$$
$$A^2 + B^2 = 2 (\tfrac{1}{2} C)^2 + 2 D^2 \quad . \quad . \quad . \quad . \quad (3)$$

Say the scalar lengths of A, B, C and D are respectively S_a, S_b, S_c, S_d.

Then by principle of scalar multiplication (3) becomes,

$S_a^2 + S_b^2 = 2 (\tfrac{1}{2} S_c)^2 + 2 (S_d)^2$, or in the figure geometrically, $\overline{MN}^2 + \overline{PN}^2 = 2 \overline{MQ}^2 + 2 \overline{QN}^2$, as was required.

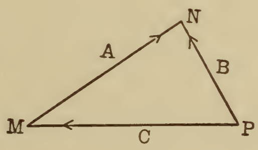

Fig. 16.

Again: to derive the trigonometric formula for one side of a triangle in terms of the other two sides and the included angle.

Let A, B and C be the vector sides of the triangle (Fig. 16),

then $\qquad C = A - B,$

whence $\qquad C \vee C = C^2 = A^2 + B^2 - 2 A \vee B.$

If S_a, S_b and S_c are the scalar lengths of A, B and C, respectively,

$$S_c^2 = S_a^2 + S_b^2 - 2 S_a S_b \cos (A, B).$$

That is, geometrically,

$$PM^2 = MN^2 + PN^2 - 2 MN \cdot PN \cos MNP.$$

EXERCISE.

1. Prove that the sum of the squares of a parallelogram's diagonals equals twice the sum of the squares of two of its sides.

2. Show that the square of the hypotenuse of a right triangle equals the sum of the squares of its legs.

3. Prove that if the perpendicular from the vertex of a triangle upon its base, bisects its base, the triangle is isosceles.

4. In a right triangle, show that the square of either leg equals the product of the hypotenuse and the projection of that leg upon it.

ART. 21. It is a natural inference that since vectors consist of both scalar and vector parts, these should both manifest themselves in a combination of vectors. The product already considered is a *pure* scalar, hence the designation, *scalar product.*

There is also a *vector product*, as is to be expected from the twofold nature of vectors.

The vector product of two vectors is a vector, normal (perpendicular) to the positive side of their plane and extending *from* it, whose magnitude is the product of the magnitudes of the two vectors and the sine of the angle between them, estimated from the first vector (in the product) to the second.

This product is represented by the cross multiplication sign in a V; thus,

$$A = B \overset{\times}{\vee} C,$$

and is known as the *cross product* or *vector product.*

Let the unit vectors be represented by the small letters, then in the equation above,

$$B \overset{\times}{\vee} C = S_b S_c \sin (B, C)\, a = A.$$

Since sin 0 = sin 180 = 0, and sin 90° = 1, if two vectors are parallel their vector product (cross product) is zero.

Hence if *neither of two vectors is a null vector and their vector product is zero, they are parallel* (*or coincident*).

Hence, also, $A \overset{\times}{\vee} A = 0$.

If B and C are adjacent sides of a parallelogram, their product, $B \overset{\times}{\vee} C = S_b S_c \sin (B\ C)\ a$, say, is the area of the parallelogram.

This is often taken as the definition of the vector product of two vectors.

Art. 22. This vector product has two obvious and simple applications to mechanics that may be cited here.

For example, if F and $-F$ are two forces forming a couple, then if G is a vector drawn from any point of F to any point of $-F$, the product,

$$F \overset{\times}{\vee} G,$$

represents the moment of the couple. Again, the velocity of a particle rotating about an axis (whether it is an isolated particle or belongs to a rotating body) is the product of its angular velocity and the radius of the circle it describes. For instance, a particle on the earth's surface, 23° 27′ N. latitude, describes the Tropic of Cancer; its velocity is the product of its angular velocity and the radius of this tropic.

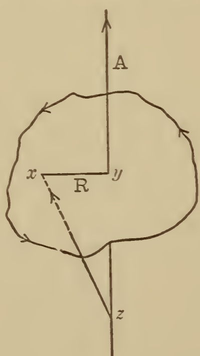

Fig. 17.

Let A (Fig. 17) be a vector along the axis, in magnitude (S_a) representing the angular velocity. Let x be a particle and B a vector from any point, z, of the axis to x. Then $A \overset{\times}{\vee} B = S_a S_b \sin (A, B)\ c$ where c is a unit vector $\perp$ to the plane of A and B (by definition of vector product). But $S_b \sin (A, B) = xy = R$, and $S_a =$ angular velocity; $\therefore$ velocity of $x = S_a R = A \overset{\times}{\vee} B$.

That is, the product of two vectors, one of which represents in direction the axis of rotation of a particle and in magnitude its angular velocity, while the other is drawn from any point of the first to the rotating particle, represents the velocity of the particle.

ART. 23. If, in the product $A \overset{\times}{\vee} B$, rotation from A to B is positive, then rotation from B to A is clearly negative; that is, $A \overset{\times}{\vee} B$ would be a normal from the positive; $B \overset{\times}{\vee} A$, from the negative side of the plane. Hence, the factors in a vector product cannot be reversed without changing the sign of the product.

ART. 24. The distributive law applies to vector multiplication, provided that the order of the factors is carefully observed. That is,

$$(A \overset{\times}{\vee} B) \overset{\times}{\vee} C = A \overset{\times}{\vee} C + B \overset{\times}{\vee} C,$$

or $$A \overset{\times}{\vee} (B \overset{\times}{\vee} C) = A \overset{\times}{\vee} B + A \overset{\times}{\vee} C,$$

but $(A \overset{\times}{\vee} B) \overset{\times}{\vee} C$ is not equal to

$$A \overset{\times}{\vee} C + C \overset{\times}{\vee} B \text{ nor to } C \overset{\times}{\vee} A + B \overset{\times}{\vee} C.$$

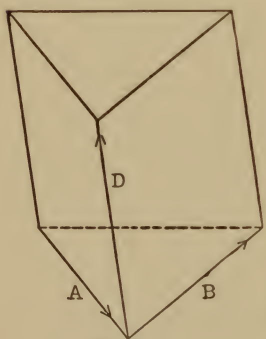

Fig. 18.

This may be proved by the aid of Geometry as follows:

Let A and B be two sides of a triangle taken successively (Fig. 18), then the third side is $-A-B$ or $-(A+B)$. Let D be another vector in a different plane. Complete a prism with the triangle as base and D as edge. The areas of the parallelogram faces are,

$$A \overset{\times}{\vee} D,\ B \overset{\times}{\vee} D \text{ and } -(A+B) \overset{\times}{\vee} D.$$

If the area of the original triangle (lower base) is $\frac{1}{2}(A \overset{\times}{\vee} B)$, then the area of the other base is $-\frac{1}{2}(A \overset{\times}{\vee} B)$, since they are viewed in exactly opposite directions (looking from outside). As the prism is a closed figure the sum of these vector faces is zero; hence,

$$A \overset{\times}{\vee} D + B \overset{\times}{\vee} D + -(A+B) \overset{\times}{\vee} D + \tfrac{1}{2}(A+B)$$

$$-\tfrac{1}{2}(A+B) = 0,$$

whence $$A \overset{\times}{\vee} D + B \overset{\times}{\vee} D = (A+B) \overset{\times}{\vee} D.$$

If the third vector D is in the plane of A and B, a fourth vector may be chosen outside this plane, and the result is the same.

Applications.

ART. 25. The addition formulæ for trigonometric functions are easily derived from the dot and cross products.

Let V and V' be two unit vectors; m and n two unit vectors ⊥ to each other in the plane of V and V'.

If x is the angle made by V with m, and y is the angle made by V' with m, then by theory of projections,

$$V = \cos x \cdot m + \sin x \cdot n$$

and $$V' = \cos y \cdot m + \sin y \cdot n.*$$

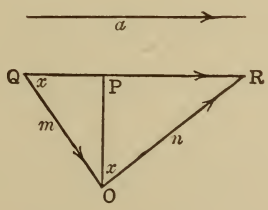

Fig. 19.

* This will be plain from accompanying figure. a may be moved // to itself to QR; draw OP ⊥ to QR from O. Then $QP = m \cos x$, $PR = n \sin x$, and $QP + PR = a$.

By law of dot products,

$V \dot{\vee} V' = \cos(V, V') = \cos(y - x)$ (since V and V' are unit vectors),

and $V \dot{\vee} V' = \cos x \cos y + \sin x \sin y$ (since $m \dot{\vee} m = 1$

and $m \dot{\vee} n = 0$, m and n being $\perp$; $\cos x$, $\cos y$, $\sin x$, $\sin y$ being scalars).

$\therefore \cos(y - x) = \cos y \cos x + \sin y \sin x.$

Again, $V \check{\vee} V' = \sin(V, V')\, c = \sin(y - x)\, c$

and $V \check{\vee} V' = (-\cos y \sin x + \sin y \cos x)c$ [since $m \check{\vee} m = 0$, $m \check{\vee} n = -1$, and $n \check{\vee} m = +1$].

$\therefore \sin(y - x) = \sin y \cos x - \cos y \sin x.$

EXERCISE.

1. Prove $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

2. Prove $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

3. If A, B and C are the vector sides of a triangle, prove that area $= \frac{1}{2} S_a S_b \sin(A, B) = \frac{1}{2} S_b S_c \sin(B, C)$, etc.

Suggestion: A triangle is half the parallelogram formed on two of its sides.

4. Show that in the triangle MNP with sides m, n, p, $m \sin P = p \sin M$, etc.

Triple Products.

ART. 26. The product of two vectors only has been considered, but the products of three vectors, known as triple products, are of equal importance at least. The products of any number of vectors can be readily reduced to triple products.

The product $(A \dot{\vee} B)\, C$ is easily interpreted, since $A \dot{\vee} B$ is a pure scalar, so that the product above is merely a vector with a scalar coefficient, and is readily understood. The parenthesis is usually omitted, as $B\ C$ could have no meaning as yet.

The product $A \dot{\vee} (B \overset{\times}{\vee} C)$, however, requires interpretation, since it is the product of two vectors, one of which is itself a vector product.

Since the dot product of two vectors is always a scalar, the above triple product is a scalar. Also, since $(B \overset{\times}{\vee} C)$ is a parallelogram whose sides are B and C (Art. 21), its product with A immediately suggests the volume of a

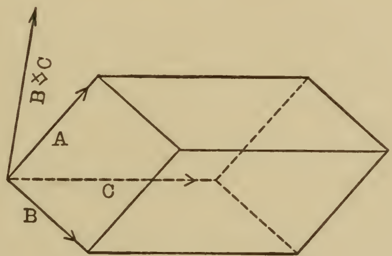

Fig. 20.

parallelopiped as in Fig. 20. Evidently this volume may be considered positive if the normal representing $B \overset{\times}{\vee} C$, and A both lie on the same side of the plane of B and C. Also, clearly, if A, B and C lie in one plane the product is zero. Hence, *if the product* $A \dot{\vee} (B \overset{\times}{\vee} C)$ *or* $(A \overset{\times}{\vee} B) \dot{\vee} C$ *of three vectors is zero they must lie in the same plane.* Hence, also, if two of the vectors are equal or collinear the product is zero. A reference to the figure will show that $(A \overset{\times}{\vee} B) \dot{\vee} C = A \dot{\vee} (B \overset{\times}{\vee} C)$ since they equal the volume of the same parallelopiped, the sign being the same. As a rule, then, *in a scalar triple product the cross and the dot may change places without affecting the product, as long as the order of the vectors is not changed.*

The parenthesis used above is unnecessary, as $A \overset{\times}{\vee} (B \dot{\vee} C)$ would mean nothing since $B \vee C$ is a scalar and A a vector, and there can be no vector product between scalar and vector.

Art. 27. A third type of triple product is still possible, viz., $A \overset{\times}{\vee} (B \overset{\times}{\vee} C)$. Since $(B \overset{\times}{\vee} C)$ is a vector $\perp$ to the plane of B and C, and the cross product of A with this vector, that is, $A \overset{\times}{\vee} (B \overset{\times}{\vee} C)$, is another vector $\perp$ to their plane, this last vector must lie *in* the plane of B and C; that is, $A \overset{\times}{\vee} (B \overset{\times}{\vee} C)$ is a vector in the (B, C) plane. Hence it can be expressed in terms of B and C (Art. 11). That is,

$$A \overset{\times}{\vee} (B \overset{\times}{\vee} C) = S_1 B + S_2 C, \text{ say.}$$

Likewise, $(A \overset{\times}{\vee} B) \overset{\times}{\vee} C$ will lie in the (A, B) plane; hence,

$$(A \overset{\times}{\vee} B) \overset{\times}{\vee} C = S_3 A + S_4 B, \text{ say.}$$

These two values are manifestly different, and therefore,

$A \overset{\times}{\vee} (B \overset{\times}{\vee} C)$ is *not* equal to $(A \overset{\times}{\vee} B) \overset{\times}{\vee} C$.

Expressed thus,

$$A \overset{\times}{\vee} (B \overset{\times}{\vee} C) \neq (A \overset{\times}{\vee} B) \overset{\times}{\vee} C.$$

The parenthesis is therefore important, and the associative law does not apply. It is likewise apparent that the order of factors cannot be changed. It is readily shown that the product of more than three vectors reduces to one of the triple products, but the exposition of this process is beyond the range of this book.*

Some General Applications to Mechanics.

Art. 28. Referring again to the principle of moments, viz., that the moment of a force about a point is the product of the force and the perpendicular upon its line of action

* See Gibb's Vector Analysis by Edwin B. Wilson, Ph.D.

from the point, it is evident from Fig. 21 that the moment of the force F about O is $F \times OM$ (ordinary multiplication). But a force may be regarded as a vector although it differs in general from a vector, in that, if the point of application of a force is changed, although the force remains parallel

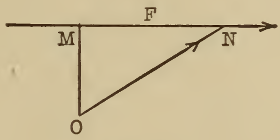

Fig. 21.

to the same line and does not change magnitude, its effect changes. However, the laws of composition are exactly like vector addition, and in general it can be treated in composition just like a vector. Hence, in the case above, the moment of F (regarded as a vector) about O is given by the equation (letting m = moment),

$$m = F \overset{\times}{\vee} G \text{ (where } G \text{ is any line from } O \text{ to } F).$$

For,

$$OM = ON \sin MNO.$$

$$\therefore OM \times F = F \times ON \sin MNO = F \overset{\times}{\vee} G.$$

The magnitude of this product will be the numerical value of the moment, and its direction indicates the direction of the impulse.

Again, it is evident from the laws of vector addition that the resultant of two forces, regarded as vectors, acting on a point is their vector sum, and that if several forces acting on a point are in equilibrium their vector sum is zero; in other words, they, or their equal vectors, will form a closed polygon. Hence, conversely, if the vector sum of any number of forces acting on a point is zero, there is equilibrium.

Since the scalar product of two vectors involves the projection of one on the other (Art. 17), the resolution of a force into components in definite directions is automatically accomplished in scalar products. For example, by mechanical law, work equals force multiplied by distance, say,

$$W = F \times D.$$

If F acts at an angle to D, the scalar product,

$$F \dot{\vee} D,$$

gives W just the same, for

$$F \dot{\vee} D = S_f S_d \cos(F, D), \text{ and}$$

$S_f \cos(F, D)$ is the component of F acting along $D = NM$ (Fig. 22).

$$\therefore F \dot{\vee} D = W.$$

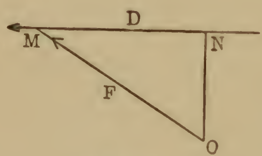

Fig. 22.

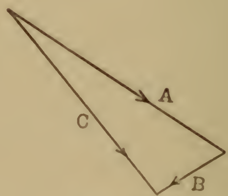

Fig. 23.

Again: Velocity is evidently a vector quantity (as is also acceleration). Hence, vector addition gives a simple graphic representation of change in velocity. For example, suppose water enters a turbine wheel at the rate of 50 feet per second, and leaves the wheel at 2 feet per second. The entry angle is 12° and the emission angle is 60°.

Let A (Fig. 23) represent the velocity at entry, and B that at issue, then C represents the fall in velocity, since

$$C = A - B, \text{ etc.}$$