FRESHMAN CALCULUS

A PRESENTATION OF

FUNDAMENTAL CONCEPTIONS AND METHODS

FOR

STUDENTS OF SCIENCE AND ENGINEERING

BY

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Preface.

The purpose of Freshman Calculus is to provide the student of science or engineering very early in his course with a familiarity with the fundamental conceptions and methods of the calculus in as far as they are of use in the elementary study of the physical sciences. Only the chief topics of the conventional calculus course are touched upon: rates and sums, limits, maxima and minima, expansions, the interpretation of slopes and areas in case of the graphs of physical functions, and differentiation and intergration of the simplest and most usual forms.

An abundance of concrete problems, not too far fetched, have been provided in which reasonable data are given and specific numerical results are required. No problems have been introduced which are to be solved by substituting into a formula. Great emphasis is laid upon the careful use of concrete numbers, the proper units being never omitted.

In the theory presented, simplicity and directness have been sought, never, (the author believes) at the expense of accuracy. The discussion is limited to functions whose graphs are the familiar smooth curves of elementary physical science: a direct appeal is made to intuition at many points where a rigorous analytical demonstration would be not convincing, nor even intelligible to the immature student.

The author wishes to acknowledge the inspiration he has received from Professors W. F. Osgood, of Harvard University, Irving Fisher, of Yale University, and D. F. Campbell, of Armour Institute of Technology, whose recent treatises on the Calculus have encouraged him to undertake the task of preparing a simple but thorough text for the use of first year students; and to express his thanks to Mr. A. Dillingham, of Tufts College, for his very material assistance with the problems.

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CONTENTS

FRESHMAN CALCULUS

In the study of arithmetic, algebra and trigonometry, the student has been concerned chiefly with the finding of the numerical values of unknown quantities. In analytical geometry one begins to think of variables and to recognize that the manner in which a quantity varies may be of as great importance as particular values assumed under particular conditions. In scientific work one is concerned very largely with quantities which vary, and whose manner of variation is a consideration of primary importance. For the treatment of such problems a knowledge of the CALCULUS is requisite.

The CALCULUS is that branch of Mathematics which treats of problems in which an essential element is the variation of quantities involved.

VARIABLES AND FUNCTIONS

Two quantities are called VARIABLES if the value of one depends upon the value of the other. One is called the INDEPENDENT variable, or ARGUMENT; the DEPENDENT variable is called a FUNCTION of the other. The statement that y is a function of x is thus abbreviated:

$$y \equiv F(x) \qquad \text{or} \qquad y \equiv \varphi(x) \quad \text{etc.}$$

A special meaning may be given to

$$F(\ ) \qquad \text{or} \qquad \varphi(\ ) \quad \text{etc.}$$

so that either shall denote what operations are to be performed upon the argument to produce the number y.

Thus if $\quad f(x) \equiv 2x^2 + \log x$

the symbol $f(\ )$ is made to mean: "Square the argument, double it, and add its logarithm." So, for various other arguments, we have:

$$f(z) = 2z^2 + \log z$$

$$f(1) = 2 \times 1^2 + \log 1 = 2$$

$$f(\sqrt{a+b}) = 2(a+b) + \tfrac{1}{2}\log(a+b)$$

Functions may have several arguments. If $\varphi(x,y,z) \equiv \frac{a}{x} - yz^2$, then $\varphi(a,z,1) = 1 - z$, etc.

A-1 Give three concrete cases of related variables. Say in each case which you naturally think of as depending on which.

There are three principal ways of expressing relation between function and argument:

1. By a FORMULA (an EQUATION, or a RULE)
2. By a GRAPH
3. By a TABLE

NOMENCLATURE

General	INDEP. VAR.	DEP. VAR.
	ARGUMENT	FUNCTION
Tables	MARGIN of Table	BODY of Table
Graphical	HORIZONTAL Dist., Run, Abscissa	VERTICAL Dist., Rise, Ordinate
Formulas	Literal quantity whose value is	
	assumed	calculated
	most involved	solved out
Symbols	$x, t, \theta, \ldots$, etc.	$y, f(x), \phi(t), \ldots$

A-2 If $F(\;) \equiv$ "square the argument, triple the result, add 5, and then multiply by the argument," what is $F(x)$? $F(0)$? $F(1)$? $F(x^2)$?

A-3 If $f(z) \equiv \sqrt{z(z-2)}$, plot the curve $y = f(x)$.

A-4 If $\phi(x) \equiv \sin(\frac{x}{10}\text{ rad.})$, tabulate $\phi(x)$ for x-values from 0 to 1, intervals of $\frac{1}{10}$.

A-5 If $F(x) \equiv x^3 - 2x^2 + 3x$, find $F(x+h)$, and find $\frac{1}{h}[F(x+h) - F(x)]$.

LIMITS

This symbol $\underset{x \doteq a}{\mathcal{L}} f(x)$

means the LIMIT approached by f( ) when its argument approaches the value a as limit. If the graph of

$$y = f(x)$$

is continuous, that is unbroken, at $x = a$, this LIMIT will be equal to

$$f(a)$$

and may be found by direct substitution.

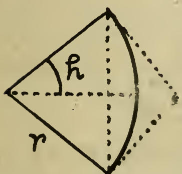
The only sort of discontinuity occurring in elementary calculus is that which involves a vanishing denominator. A function in fractional form may approach a limit when its argument approaches a certain number, and yet this limit not be ascertainable by substitution in case this results in a zero denominator. This is a very common and important case in the calculus. In such cases some transformation in the FORM of the function must be made before passing to the limit.

A-6 $\underset{h \doteq 1}{\mathcal{L}} \left[\frac{2h^2 - 5h + 3}{h - 1}\right] = ?$ A-7 $\underset{h \doteq 0}{\mathcal{L}} \frac{\sin^2 h}{\tan h} = ?$

Two limits of great importance in the Calculus, but of special difficulty, are

$$\mathop{\mathcal{L}}_{h \doteq 0}\left[\frac{\sin h}{h}\right] \quad \text{and} \quad \mathop{\mathcal{L}}_{h \doteq 0}\left[(1+h)^{\frac{1}{h}}\right]$$

The value of the first of these depends on the UNITS in which h is measured. Take $n \equiv$ the number of units in the PERIGON (360°). Then take a small arc of a circle, draw its chord and two tangents. Put the angle $\equiv h$ units:



half chord < half-arc < one tangent

or: $r \sin h < h \frac{2\pi r}{n} < r \tan h$

or: $1 < (\pi \div n) \div (\sin h \div h) < \sec h$

As $h \doteq 0$, the outer limit tends toward 1, so the limit of the INCLUDED variable must be ONE. Then the limit of its divisor must be the constant, $(2\pi \div n)$

$$\therefore \mathop{\mathcal{L}}_{h \doteq 0}\left[\frac{\sin h}{h}\right] = \frac{2\pi}{n}$$

When h is measured in DEGREES, $n \equiv 360$

$$2\pi \div n = 6.2832\ldots \div 360 = .01745+$$

When RADIANS are used this limit is ONE.

$\mathop{\mathcal{L}}_{h \doteq 0}\left[(1+h)^{\frac{1}{h}}\right] = 2.71828+$, called $\equiv e$. A proof of this is too difficult for an elementary text.

A-8 Express in the three ways the functional relation between the number of seconds a body has been falling and the number of feet it has fallen.

A-9 In the expression $L = log(N)$ is the symbol $log(\ )$ comparable with a symbol like $f(\ )$? What operation does $F(\ )$ direct one to perform if $F(x) \equiv log_{10} x$?

A-10 At what values of x is the function $\varphi(x) \equiv \frac{2x}{1-x^2}$ discontinuous?

A-11 Express, by a verbal rule, the meaning of $F(\ )$, if $F(x) \equiv (x^3 - x + 1)^{\frac{1}{2}}$.

A-12 Is there any discontinuity in the graph of $f(x) \equiv (x^2-1) \div (x-1)$? Does this function give all the values of $f(x)$ just as they are given by this: $f(x) \equiv x+1$?

A-13 $\mathcal{L}_{x \doteq 2} \left[\frac{(x-2)^2}{x-1} + \frac{x^2-4}{x-2} - 3x \right] = ?$ Ans. -2.

A-14 If $F(x) \equiv \left(\frac{x+1}{x}\right)^x$, tabulate $F(x)$ for

$x = 1, 2, 3, 10, 100$; Does it tend toward e?

A-15 Given $\varphi(x) \equiv \frac{e^x + e^{-x}}{2}$; Compare the values of $\varphi(2x)$ and $2[\varphi(x)]^2 - 1$.

A-16 Given $F(x) \equiv \frac{f(x+h) - f(x)}{h}$, find $F(x)$ when $f(x) \equiv x^3$. Ans. $3x^2 + 3xh + h^2$

A-17 If the relation between distance, d, moved in time, t, is given by the formula $d = f(t)$, what concrete quantity is denoted by $[f(t_0 + t_1) - f(t_0)]$?

A-18 $\mathcal{L}_{x \doteq 0}\left[\frac{\sin x}{\tan x} - \frac{1 - \cos x}{\sin \frac{x}{2}}\right] = ?$ Ans. 1

A-19 If $f(x) \equiv 2x^2 + 1$, find $\frac{1}{h}[f(x+h) - f(x)]$

A-20 Given $\varphi(h) \equiv (1+h)^{1/h}$ Plot $y = \varphi(h)$ for the values $h = 2, 1, .5, .1, .0001$ Does the graph tend toward $(h = 0, y = e)$?

A-21 If $y = F(x)$ is the equation of a certain graph, show by diagram that a chord between points where x is a and $a + \varepsilon$ has $[F(a+\varepsilon) - F(a)] \div \varepsilon$ for its slope.

A-22 $\mathcal{L}_{\varepsilon \doteq 0}\left[\{(x+\varepsilon)^n - x^n\} \div h\right] = ?$ Ans. nx^{n-1}

INCREMENTS

The primary question about variables is: at what RATES do they INCREASE? The ACTUAL INCREASE in a quantity is denoted by writing a Δ (delta) before the symbol for the quantity. When DELTAS are used before related variables they mean CORRESPONDING INCREMENTS. Thus if h° is the temperature at t o'-clock, Δt means a lapse of time (in hours) and Δh means the rise in temperature (in $^\circ$'s) during that interval. Δh may come out a negative number and thus represent a decrease.

The increase produced in $F(x)$ by giving x an increment, Δx, will be

$$\Delta F(x) \equiv F(x+\Delta x) - F(x)$$

ΔQ is a single number, like the symbols $\sin\theta$, $\log N$, $f(x)$, etc., NOT a product.

Example: If $Q \equiv (1-t^2)$ find ΔQ.

Solution: $Q+\Delta Q = (1-[t+\Delta t]^2)$

$$= 1 - t^2 - 2\Delta t(1-t) + (\Delta t)^2$$

Subtracting: $\Delta Q = -2\Delta t(1-t) + (\Delta t)^2$

B-1 If $F(x) \equiv ax^3$ find $F(x+\Delta x) - F(x)$

B-2 If $f(x) \equiv \sqrt{x}$ show that $\Delta f(x) = \frac{\Delta x}{\sqrt{x} + \sqrt{x+\Delta x}}$

B-3 If $y = \sin x$, find Δy when $x = 30°$ and $\Delta x = 1°$.

B-4 If $y = \varphi(x)$ be plotted, show by diagram that the slope of a certain chord is $\frac{\Delta y}{\Delta x}$.

B-5 What does $\frac{\Delta m}{\Delta h}$ give, if m miles is my distance from town at h min. past 5?

B-6 Find $\underset{\Delta x \doteq 0}{\mathcal{L}} \left[\frac{(x+\Delta x)^2 - x^2}{\Delta x} \right]$ Ans. $2x$.

B-7 If $Q = \frac{1}{x+1}$ show that $\Delta Q = \frac{-\Delta x}{(x+1)(x+\Delta x+1)}$

B-8 If $Q = \frac{1}{x+1}$ work out $\underset{\Delta x \doteq 0}{\mathcal{L}} \left[\frac{\Delta Q}{\Delta x} \right]$

B-9 If $Q = \frac{1}{x+1}$, $x = 1$, $\Delta x = .01$, $\frac{\Delta Q}{\Delta x} = ?$

B-10 T tons of earth are dug by the end of W weeks. Meaning of $T \div W$? $\Delta T \div \Delta W$?

B-11 If $y = \log_{10} \sin(a° x')$, show that Δy is a tabular difference if $\Delta x = 1$.

B-12 $R \equiv \sqrt{a^2 - x^2}$; Prove that $\frac{\Delta R}{\Delta x} = -\frac{2x + \Delta x}{2R + \Delta R}$

DIFFERENCE QUOTIENTS

Many important relations between phyical quantities occur in the form $R = S \div T$ as

Speed = Distance ÷ Time
Density = Mass ÷ Volume
Acceleration = Speed ÷ Time
Slope = Rise ÷ Run
Mileage = Fare ÷ Distanee

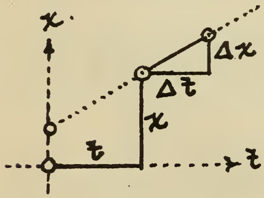
In these and similar cases the last two must be CORRESPONDING amounts, so the INCREMENT NOTATION is convenient.

Speed = Δ(distance) ÷ Δ(time) etc.

Thus if distance ≡ x mi. and time ≡ t hr.

$\frac{\Delta x}{\Delta t} \equiv$ { Speed, in mi. per hr., during the time-interval denoted by Δt.

This need not be equal to $x \div t$, for consider a case in which the journey



did not begin until t had reached the value 1 hr. Moreover the above formula holds true only if the speed is CONSTANT.

If the speed varies in the interval Δt

$$\Delta x \div \Delta t$$

is the AVERAGE, or MEAN, Speed during Δt.

Similarly, the AVG. or MEAN density of a piece of material whose volume is ΔV cu. cm. and whose mass is Δm grams is

$$[\Delta m \div \Delta g] \text{ grams per cu. cm.}$$

The ratio of two corresponding increments is called a DIFFERENCE QUOTIENT. Every average rate, every quantity measured in units named with a "PER," may be expressed by a DIF. QUO.

If two quantities are connected by a formula

$$y = F(x)$$

$$\frac{\Delta y}{\Delta x} \equiv \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

Example: Find MEAN SPEED for any interval, of ball falling $16t^2$ ft. in t seconds.

Solution: Put distance $\equiv x = 16t^2$

$$\Delta x = 16(t + \Delta t)^2 - 16t^2 = 32\,\Delta t + 16\,\Delta t^2$$

$\frac{\Delta x}{\Delta t} = (32 + 16\,\Delta t)$ ft. per sec., mean during Δt.

B-13 Find the DIF. QUO. for $y = 3x^3 - 2x + 1$.

B-14 Find the mean acceleration for any interval if the speed is $\frac{t+1}{t^2}$ ft. p. sec. at the end of t sec.

B-15 For a graph, $\frac{\Delta y}{\Delta x}$ is an AVG. WHAT?

DERIVATIVES

Since Δy is the actual increase in y produced by an increase, Δx, in x, the DIF. QUO., $\Delta y \div \Delta x$, denotes the AVERAGE, or MEAN, rate of increase in y as compared with x in the interval denoted by the Δ. It is measured in units of y per unit of x. Thus if y is in Tons, x in miles, $\Delta y / \Delta x$ is in Tons per mi.

When a very small interval is considered there is correspondingly small variation in the rate of increase of the function. The TRUE (or INSTANTANEOUS) RATE OF INCREASE of a function, $f(x)$, is defined as

$$\underset{\Delta x \doteq 0}{\mathcal{L}} \frac{\Delta f(x)}{\Delta x}$$

This whole series of operations is denoted by a simpler symbol, to wit, if $y \equiv f(x)$

$$\frac{d}{dx} f(x) \quad \text{or} \quad \frac{dy}{dx}$$

and the result is called the DERIVATIVE OF $f(x)$, (or of y), WITH RESPECT TO x.

The following are equivalent notations:

$$\text{True Rate} \equiv \underset{\text{Interval} \doteq 0}{\text{Limit}} [\text{AVG. Rate}]$$

$$\equiv \underset{\Delta x \doteq 0}{\mathcal{L}} \left[\frac{\Delta y}{\Delta x}\right] \equiv \frac{dy}{dx} \equiv \underset{\Delta x \doteq 0}{\mathcal{L}} \left[\frac{\Delta f(x)}{\Delta x}\right]$$

$$\equiv \underset{\Delta x \doteq 0}{\mathcal{L}} \frac{F(x+\Delta x) - F(x)}{\Delta x} \equiv \frac{d}{dx} f(x)$$

Example: Find the derivative of $\frac{1}{x+1}$ with respect to x. Put $Q \equiv \frac{1}{x+1}$. Then $\Delta Q = \frac{1}{x+\Delta x+1} - \frac{1}{x+1}$. Reduce to common denominator $\Delta Q = \frac{-\Delta x}{(x+\Delta x+1)(x+1)}$ and the DIF. Quo. $\frac{\Delta Q}{\Delta x} = \frac{-1}{(x+\Delta x+1)(x+1)}$. Take the limit as $\Delta x \doteq 0$ and the derivative is found:

$$\frac{dQ}{dx} = \frac{-1}{(x+1)^2}$$

Example: Find the true speed of a body which moves so that it traverses a distance of $[x^2 \div (1+x)]$ miles in the first x hours

Solution: Put $s \equiv x^2 \div (1+x)$. Then the Avg. speed $\equiv \frac{\Delta s}{\Delta x} = \frac{1}{\Delta x}\left[\frac{(x+\Delta x)^2}{1+x+\Delta x} - \frac{x^2}{1+x}\right]$ and reducing,

$$\doteq \frac{x(2+x)+\Delta x(1+x)}{(1+x+\Delta x)(1+x)}$$

Taking the limit as the interval $\Delta x \doteq 0$

$$\text{True speed} \equiv \frac{ds}{dx} = \frac{x(2+x)}{(1+x)^2} \text{ mi. per hr.}$$

B-16 If $y = 7x^2 - 6x + 1$, work out $\frac{dy}{dx} = 14x - 6$

B-17 If $Q = \frac{1}{\sqrt{s}}$, work out $\frac{dQ}{ds} = -\frac{1}{2s\sqrt{s}}$

B-18 If $z = \frac{1-2x}{x+3}$ find $\frac{dz}{dx}$ in terms of x and $\frac{dx}{dz}$ in terms of z

B-19 If $\left(\begin{smallmatrix}\text{distance}\\ \text{in feet}\end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix}\text{time in}\\ \text{seconds}\end{smallmatrix}\right)^3$, True speed = ?

B-20 If the density of a fluid in a cylindrical jar varies from the bottom up so that the mass in the lower x cm. of the jar is $\frac{10x}{x+2}$ grams, show that the density in a layer x cm. from the bottom is $[20 \div k(x+2)^2]$ gms. p. cu. cm., k sq. cm being the cross-section of the jar.

B-21 If $y = q + r$, and $q = 2t^2$ and $r = 1 \div t$ find the formula for dy/dt.

B-22. If the mass-volume formula is $M = 2V^2 - 1$, what is the formula for density?

B-23 Represent the space-time formula for a falling body, $s = \frac{1}{2}gt^2$, by a graph, and show that the slope of the tangent drawn to the curve at any point is equal to the value of $\frac{ds}{dt}$ for the value of t corresponding to the point taken.

B-24 The speed-time formula is $v = \frac{8}{t+1}$, what is the acceleration-time formula?

B-25 If $s = 7h^2 - h$, and h depends on the independent variable t (formula not given), show that $\frac{ds}{dt} = (14h - 1)\frac{dh}{dt}$.

SEVERAL DEPENDENT VARIABLES

If several variables depend upon a single independent variable we may compare their Rates of change and get a derivative of any one with respect to any other. When the increment of the INDEP. VAR. *vanishes, all the corresponding increments will vanish. Hence if u, v, w, x,.... are related it does not matter which increment is specified as approaching zero.*

Therefore these expressions:

$$\mathcal{L}_{\Delta x \doteq 0} \frac{\Delta u}{\Delta v}, \quad \mathcal{L}_{\Delta x \doteq 0} \frac{\Delta x}{\Delta w}, \quad \text{etc.}$$

are equivalent to the expressions:

$$\mathcal{L}_{\Delta v \doteq 0} \frac{\Delta u}{\Delta v}, \quad \mathcal{L}_{\Delta w \doteq 0} \frac{\Delta x}{\Delta w}, \quad \text{etc.}$$

which, by DEFINITION *(page 10), are*

$$\frac{du}{dv}, \quad \frac{dx}{dw}, \quad \text{etc.}$$

Since it is not necessary to specify which increment approaches 0, the symbol

$$\mathcal{L}$$

will be given this wider meaning:

TAKE THE LIMIT APPROACHED WHEN THE VARIOUS INCREMENTS APPROACH ZERO

Such a limit of any difference quotient must be a derivative

$$\mathcal{L} \frac{\Delta Q}{\Delta P} \equiv \frac{dQ}{dP}$$

LIMITS OF PRODUCTS AND QUOTIENTS

If we have $\mathcal{L}\, u = h$, *and* $\mathcal{L}\, v = k$, *and*
$u - h = \varepsilon$, *and* $v - k = i$,
then $\varepsilon \doteq 0$ *and* $i \doteq 0$ *in passing to limits.*

The PRODUCT
$$uv = (h+\varepsilon)(k+i)$$
$$= hk + (\varepsilon k + ih + \varepsilon i)$$
Hence $\mathcal{L}\, uv = hk$
or $\mathcal{L}\,[u \times v] = [\mathcal{L}\, u] \times [\mathcal{L}\, v]$

LIMIT OF A PRODUCT = PRODUCT OF LIMITS

The QUOTIENT
$$\frac{u}{v} = \frac{h+\varepsilon}{k+i}$$
$$= \frac{h}{k} + \left(\frac{h+\varepsilon}{k+i} - \frac{h}{k}\right)$$
$$= \frac{h}{k} + \frac{k\varepsilon - hi}{(k+i)\,k}$$
Hence $\mathcal{L}\, \frac{u}{v} = \frac{h}{k}$ { *provided only that* $k \neq 0$

or $\mathcal{L}\left[\frac{u}{v}\right] = \frac{[\mathcal{L}\, u]}{[\mathcal{L}\, v]}$ { *provided such a quotient exists*

LIMIT OF A QUOTIENT = QUOTIENT OF LIMITS

These two Theorems lead to these important relations between derivatives:

$$\frac{du}{dv} \times \frac{dv}{dw} = \mathcal{L}\frac{\Delta u}{\Delta v} \times \mathcal{L}\frac{\Delta v}{\Delta w} = \mathcal{L}\, \frac{\Delta u}{\Delta v} \times \frac{\Delta v}{\Delta w} = \mathcal{L}\frac{\Delta u}{\Delta w} = \frac{du}{dw}$$

$$\frac{dQ}{dR} \div \frac{dS}{dR} = \mathcal{L}\frac{\Delta Q}{\Delta R} \div \mathcal{L}\frac{\Delta S}{\Delta R} = \mathcal{L}\, \frac{\Delta Q}{\Delta R} \div \frac{\Delta S}{\Delta R} = \mathcal{L}\frac{\Delta Q}{\Delta S} = \frac{dQ}{dS}$$

DIFFERENTIALS

The DERIVATIVE $\frac{dy}{dx}$ is an abbreviation for

$$\mathop{\mathcal{L}}_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x} \right]$$

and $\frac{dy}{dx}$ is a single number and not a fraction. — But any single number may be regarded as a fraction: $12 = \frac{12}{1}$ or $\frac{24}{2}$ or $\frac{6}{.5}$ There are many advantages in regarding $\frac{dy}{dx}$ as the quotient of two numbers, dy and dx, which are called DIFFERENTIALS. If v is any variable connected with y and x (it may denote x itself, or y, in some cases)

$$\frac{dy}{dx} = \frac{dy}{dv} \div \frac{dx}{dv} \quad \text{and} \quad \frac{dx}{dv} \times \frac{dy}{dx} = \frac{dy}{dv}$$

In the DIFFERENTIAL NOTATION these become

$$\frac{dy}{dx} = dy \div dx \quad \text{and} \quad dx \times \frac{dy}{dx} = dy$$

where each DIFFERENTIAL signifies the DERIVATIVE with respect to the same variable (not indicated) in each case.

The differential notation makes it needless to discriminate in advance which is to be the INDEP. VAR; after it is chosen PASS TO DERIVATIVES by writing in its differential under each differential appearing.

DIFFERENTIAL FORMULAS

To get a formula for the differential of a function follow these SIX STEPS:

1. *Suppose the* INDEP. VAR. *given an increment.*
2. *Find the augmented value of the function.*
3. *Subtract to get the increment of the function.*
4. *Divide by the increment of the Indep. Var.*
5. *Pass to the limit as increments* $\doteq$ *zero.*
6. *Multiply by differential of Indep. Var.*

Thus, to find the DIFFERENTIAL *of* $y = v^2$

1. *Call the Ind. Var.* t, *and let it become* $t + \Delta t$
2. *Then* y *and* v *become* $y + \Delta y$ *and* $v + \Delta v$:

$$y + \Delta y = (v + \Delta v)^2$$

3. $\Delta y = (v + \Delta v)^2 - v^2 = 2v\,\Delta v + \Delta v^2$

4. $\frac{\Delta y}{\Delta t} = 2v\frac{\Delta v}{\Delta t} + \frac{\Delta v}{\Delta t} \cdot \Delta v$

5. $\frac{dy}{dt} = 2v\frac{dv}{dt} + \frac{dv}{dt} \cdot 0$

$$= 2v\frac{dv}{dt}$$

6. $dy = 2v\,dv$

Five easily remembered formulas enable one to differentiate any algebraic expression without taking the six steps save once for all in deducing the formulas.

In these formulas { c and n represent any CONSTANTS; u and v represent any VARIABLES }

FORMULAS

I $\quad d(c) = 0$

II $\quad d(u+v) = du + dv$

III $\quad d(cu) = c\,du$

IV $\quad d(vu) = v\,du + u\,dv$

V $\quad d(u^n) = nu^{n-1}du$

PROOFS

I Let $y \equiv c$. Then (steps 1 and 2) no matter what independent variable may change, y does not change at all, and (step 3) $\Delta y = 0$; (4) $\frac{\Delta y}{\Delta x} = 0$; (5) $\frac{dy}{dx} = 0$; and (6) dy, that is $d(c) = 0$.

II Let $y \equiv u + v$. Then (1,2,3) $\Delta y = \Delta u + \Delta v$; (4) $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$; (5) $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$; (6) dy, that is $d(u+v) = du + dv$.

III Let $y \equiv cu$. Then (1,2,3) $\Delta y = c\,\Delta u$; (4) $\Delta y/\Delta x = c\,\Delta u/\Delta x$; (5) $dy/dx = c\,du/dx$; (6) dy, that is $d(cu) = c\,du$

IV Let $y \equiv vu$. Then, (steps 1 and 2)

$$y + \Delta y = (v + \Delta v)(u + \Delta u)$$
$$= vu + v \cdot \Delta u + u \cdot \Delta v + \Delta v \cdot \Delta u.$$

(step 3) $\Delta y = v \cdot \Delta u + u\,\Delta v + \Delta v \cdot \Delta u$

(4) $\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \frac{\Delta v}{\Delta x}\cdot \Delta u$

(5) $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + \frac{dv}{dx}\cdot 0$

$= v\frac{du}{dx} + u\frac{dv}{dx}$

(6) dy, that is $d(u\,v) = v\,du + u\,dv$

V Let $y \equiv u^n$. THREE CASES. First case is when n is a positive whole number.

(1,2,3) $\Delta y = (u+\Delta u)^n - u^n$ (Expand by the Binom. Theorem and cancel u^n)

$= n\,u^{n-1}\Delta u + \frac{n(n-1)}{1\cdot 2}u^{n-2}\Delta u^2 \ldots$

(4) $\frac{\Delta y}{\Delta x} = n\,u^{n-1}\frac{\Delta u}{\Delta x} + \frac{n(n-1)}{1\cdot 2}u^{n-2}\frac{\Delta u}{\Delta x}\Delta u \ldots$

(5) $\frac{dy}{dx} = n u^{n-1}\frac{du}{dx} + \frac{n(n-1)}{1\cdot 2}u^{n-2}\frac{du}{dx}0 \ldots\ldots$

$= n\,u^{n-1}\frac{du}{dx}$

(6) dy, that is $d(u^n) = n\,u^{n-1}\,du$.

Second case is when n is a fraction, $\equiv \frac{P}{Q}$.

Let $y \equiv u^{P/Q}$

Then $y^Q = u^P$ (Treat as a case 1)

$Q y^{Q-1} dy = P u^{P-1} du$

$dy = \frac{P}{Q}u^{P-1}y^{1-Q}du = \frac{P}{Q}u^{P-1}u^{\frac{P}{Q}(1-Q)}du$

$= \frac{P}{Q}u^{\frac{P}{Q}-1}du$

Third case is when n is negative, $\equiv -m$.
Let $y \equiv u^{-m}$
Then $y \times u^{m} = 1$ (Treat the product by IV)

$$dy \times u^{m} + y \times m u^{m-1} du = d(1) = 0$$
$$dy = -y m u^{m-1} du \div u^{m}$$
$$= -u^{-m} m u^{m-1} du \div u^{m}$$
$$= -m u^{-m-1} du$$

Comparing the second and third cases with the first it is seen that whether n is plus or minus, whole or fractional,

$$d(u^{n}) = n u^{n-1} du$$

Formulas for $d(\frac{1}{u})$, $d(u+c)$, $d(u \times v \times w)$, $d(\frac{u}{v})$, $d(\sqrt{u})$ are sometimes added; but they are not needed as we may regard

$\frac{1}{u}$ as u^{-1} and use V
$u+c$ as $u+v$ and use II and I
uvw as $(uv)w$ and use IV twice
u/v as uv^{-1} and use IV and V
$\sqrt{u}$ as $u^{1/2}$ and use V.

Fractions in which either the numerator or the denominator is CONSTANT should be treated thus: regard

v/a as $\frac{1}{a}v$ and use III
a/v as av^{-1} and use III and V

DERIVATIVES BY DIFFERENTIAL FORMULAS

Every algebraic expression is primarily either a SUM, *a* PRODUCT, *or a* POWER, *of constants or variables, and may be brought under some one of the formulas: I to V. First write down* DIFFERENTIALS *by means of these formulas, then pass to the required* DERIVATIVES *by supplying the same differential as a denominator to each differential. The latter step may be regarded as* DIVIDING *by the differential of the* INDEPENDENT VARIABLE.

The derivative of a variable with respect to itself is unity: for

$$\frac{d}{dx}x \equiv \mathcal{L}\frac{\Delta x}{\Delta x} = \mathcal{L}\,1 = 1$$

So if the differential of the INDEPENDENT VAR. *occurs in an equation it divides out when we pass to derivatives and leaves the factor 1 in its place.*

Example: $y = (2x^2-1)^2$, *find* $\frac{dy}{dt}$

by V $\quad dy = 2(2x^2-1)^1 \cdot d(2x^2-1)$

by II, I $\quad = 2(2x^2-1) \cdot d(2x^2)$

by III $\quad = 2(2x^2-1) \cdot 2\,d(x^2)$

by V $\quad = 2(2x^2-1) \cdot 2 \cdot 2x\,dx$

pass to derivatives

$$\frac{dy}{dt} = 8(2x^2-1)x\frac{dx}{dt}$$

Example: $y = \frac{a}{\sqrt{x}}$, find $\frac{dy}{dx}$

Change to the form: $y = a x^{-1/2}$

by III $\quad dy = a\, d(x^{-1/2})$

by V $\quad = a(-\frac{1}{2} x^{-3/2} dx)$

pass to derivatives

$$\frac{dy}{dx} = -\frac{a}{2} x^{-3/2} \frac{dx}{dx} = -\frac{a}{2x\sqrt{x}}$$

In working out such examples, the learner must be conscious at every step of the particular formula he is using. It is not necessary, however, except at first, to indicate the steps so fully as above.

Example: $y = \frac{(2-x)^2}{x^2+1}$, $\frac{dy}{dz} = ?$

Change to the form: $y = (2-x)^2 (x^2+1)^{-1}$

by IV, V, II, I $\quad dy = 2(2-x)(-dx)\cdot(x^2+1)^{-1} + (2-x)^2 \cdot (-1)(x^2+1)^{-2}(2x\,dx)$

factoring: $\quad = 2(2-x)(x^2+1)^{-2}(-1)[(x^2+1) + x(2-x)]\,dx$

$$= -2(2-x)(x^2+1)^{-2}[1+2x]\,dx$$

dividing by dz: $\frac{dy}{dz} = \frac{2(x-2)(1+2x)}{(x^2+1)^2}\frac{dx}{dz}$

After differentiating a PRODUCT, the simplest way to make the algebraic reductions is to FACTOR out the newly introduced power of each factor, as above.

DRILL ON DIFFERENTIATION

In the examples below, a, b, c, n, represent CONSTANTS. *All the other letters represent* VARIABLES. *By differentiation by means of* I, II, III, IV, V, *verify each answer given.*

C-1 $y=(3x^2-1)(1-x)$, $\frac{dy}{dz}=(1+6x-9x^2)\frac{dx}{dz}$

C-2 $y=3x^{10}-2x^6+x^3-5$, $\frac{dy}{dx}=3(10x^7-4x^3+1)x^2$

C-3 $g=ax^2+2bx+c$, $\frac{dg}{dt}=2(ax+b)\frac{dx}{dt}$

C-4 $x=(2y^2-1)y(1+y)$, $\frac{dx}{dy}=2y(4y-1)(y+1)-1$

C-5 $r=\frac{1}{x^2+y^2}$, $\frac{dr}{dx}=\frac{-2(x+y\frac{dy}{dx})}{(x^2+y^2)^2}$

C-6 $y=\sqrt{1+x^2}$, $\frac{dy}{dx}=\frac{x}{\sqrt{1+x^2}}$

C-7 $y=x\sqrt{1+x^2}$, $\frac{dy}{dx}=\frac{2x^2+1}{\sqrt{1+x^2}}$

C-8 $y=(ax^3+b)^n x^2$, $\frac{dy}{dr}=?$

C-9 $y=\frac{4-7x}{6+5x}$. *Find* dy *in terms of* x *and* dx; *also* dx *in terms of* y *and* dy.

C-10 $A=\sqrt{s(s-a)(s-b)(s-c)}$, $\frac{dA}{ds}=?$

C-11 $y=\sqrt{s^2-x^2}$, $\frac{dy}{dx}=(s\frac{ds}{dx}-x)\div\sqrt{s^2-x^2}$

C-12 $y=x(x^3+5)^{4/3}$, $dy=5(x^3+1)(x^3+5)^{1/3}dx$

C-13 $y=\frac{1-x}{\sqrt{1+x^2}}$, $\frac{dy}{dx}=-\frac{1+x}{\sqrt{(1+x^2)^3}}$

C-14 $y=\sqrt{\frac{1+x}{1-x}}$, $dy=\frac{dx}{(1-x)\sqrt{1-x^2}}$

C-15 $y=\sqrt{ax^2+bx+c}$, $dy=\frac{2ax+b}{2\sqrt{ax^2+bx+c}}dx$

C-16 $y=\sqrt[n]{x}$, $dy=\frac{\sqrt[n]{x}}{nx}dx$

C-17 $Q=(a+x)^2(a-x)^2$, $\frac{dQ}{dy}=-4(a^2-x^2)x\frac{dx}{dy}$

C-18 $x^2+2axy+y^2=b$, $\frac{dy}{dx}=-\frac{x+ay}{ax+y}$

C-19 $r^2=x^2+y^2$, $d(\frac{1}{r})=-\frac{xdx+ydy}{r^3}$

C-20 $y=\frac{1}{x+\sqrt{1+x^2}}$, $dy=\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}dx$

C-21 $z=\frac{b}{(a^2+v^2)^3}$, $dz=\frac{-6bv\,dv}{(a^2+v^2)^4}$

C-22 $y=\left(\frac{2x^2+3x}{1-x^2}\right)^2$, $dy=?$

C-23 $y=\sqrt{(a^2-x^2)^3}$, $dy=-3x\sqrt{a^2-x^2}\,dx$

C-24 $x^{2/3}+y^{2/3}=a^{2/3}$, $1+(\frac{dy}{dx})^2=(\frac{a}{x})^{2/3}$

SUCCESSIVE DIFFERENTIATION

A derivative is itself a function that may be differentiated with respect to any related variable. All differentials must be changed to the DERIVATIVE notation before they are themselves differentiated.

Example: $y = 3x^4 + 3z^3 + 4$

$$dy = 12x^3 dx + 9z^2 dz + 0$$

$$dy/dx = 12x^3 + 9z^2\, dz/dx$$

Then $d(dy/dx) = 36x^2 dx + 9\left[2z\,dz \cdot \frac{dz}{dx} + z^2 \cdot d\left(\frac{dz}{dx}\right)\right]$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 36x^2 + 18\, z\left(\frac{dz}{dx}\right)^2 + 9z^2 \frac{d}{dx}\left(\frac{dz}{dx}\right)$$

The symbol $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written more briefly thus

$$\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2}{dx^2}y$$

In a similar way $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \equiv \frac{d^3y}{dx^3}$, etc.

Example: Given, as above,

$$\frac{d^2y}{dx^2} = 36x^2 + 18z\left(\frac{dz}{dx}\right)^2 + 9z^2\frac{d^2z}{dx^2}$$

$$\frac{d^3y}{dx^3} = 72x + 18\left[\frac{dz}{dx}\cdot\left(\frac{dz}{dx}\right)^2 + z\cdot 2\cdot\frac{dz}{dx}\cdot\frac{d^2z}{dx^2}\right]$$

$$+ 9\left[2z\frac{dz}{dx}\cdot\frac{d^2z}{dx^2} + z^2\frac{d^3z}{dx^3}\right]$$

$$= 72x + 18\left(\frac{dz}{dx}\right)^3 + 54z\frac{dz}{dx}\cdot\frac{d^2z}{dx^2} + 9z^2\frac{d^3z}{dx^3}.$$

C-25 $Q = 8r^2 + \frac{1}{2}r^3 + r$, $\frac{d^2Q}{dr^2} = 16 + 3r$

C-25 $s = x^2 + 1$, $\frac{d^2s}{dt^2} = 2\left[\left(\frac{dx}{dt}\right)^2 + x \cdot \frac{d^2x}{dt^2}\right]$

C-27 $y = \frac{s}{x}$, $\frac{d^2y}{ds^2} = 2\left[\frac{s}{x^3}\left(\frac{dy}{ds}\right)^2 - \frac{1}{x^2}\left(\frac{dx}{ds} + \frac{s}{2} \cdot \frac{d^2x}{ds^2}\right)\right]$

C-28 $Q = v^n$; $\frac{d^3Q}{dx^3} = n(n-1)v^{n-3}\left[(n-2)\left(\frac{dv}{dx}\right)^3 + 3v\frac{dv}{dx} \cdot \frac{d^2v}{dx^2} + \frac{v^2}{n-1} \cdot \frac{d^3v}{dx^3}\right]$

C-29 $y = \frac{x^3}{1-x}$, $\frac{d^4y}{dx^4} = \frac{24}{(1-x)^5}$

C-30 $y^2 = 2ax$, $\frac{d^2y}{dx^2} = -\frac{a^2}{y^3}$

C-31 $x^2 + y^2 = a^2$, $\frac{d^2y}{dx^2} = -\frac{a^2}{y^3}$

C-32 $y = x^{12}$; $\frac{d^4y}{dx^4} = \frac{\underline{|12}}{\underline{|5}} x^5$, $\frac{d^{12}y}{dx^{12}} = \underline{|12}$

and if $n > 12$, $\frac{d^ny}{dx^n} = 0$.

C-33 $x^2 + y^2 = a^2$,
Show that: $\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \pm\frac{1}{a}$

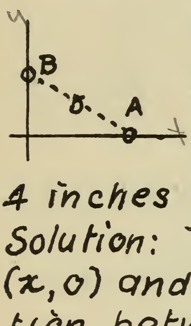
C-34 $y = sx$ $\frac{d^3y}{dt^3} =$
$s\frac{d^3x}{dt^3} + 3\frac{ds}{dt}\frac{d^2x}{dt^2} + 3\frac{d^2s}{dt^2} \cdot \frac{dx}{dt} + \frac{d^3s}{dx^3} \cdot x$

C-35 $y = x^2 \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{1-2x}{x^2} \cdot \frac{dy}{dx}$

SUBSTITUTION INTO DERIVATIVES

Calculus deals with the relations between variables as they change, and not with special values of the variables. Accordingly if a calculus problem gives or requires special values, GENERAL FORMULAS must be got first, and the special values substituted for the variables as the LAST step.

Example: The bead, B, moves up the y-axis at the rate of 3 in. per min. and drags the bead, A, along the x-axis by means of a cord 5 in. long. How fast is A moving when B is 4 inches from the origin?



Solution: The coordinates of A and B are $(x, 0)$ and $(0, y)$ and the general relation between the variables, x and y, is:

$$x^2 + y^2 = 5^2$$

The 3 in. per min. is a special value of B's speed, and the general speed $\equiv dy \div dt$ if t is the time elapsed in minutes. A's required speed is $dx \div dt$ when $y = 4$.

Differentiating: $2x\,dx + 2y\,dy = 0$

∴ In general: $\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{y}{\sqrt{25-y^2}} \frac{dy}{dt}$

When $y=4$, $\frac{dx}{dt} = -\frac{4}{3} \cdot 3$, or: Down 4 in. per min.

An abbreviation frequently used for the derivative of a function, $f(\ )$, is $f'(\ )$, in which the PRIME indicates that the function has been differentiated with respect to its argument. In short

$$f'(x) \equiv \frac{d}{dx} f(x).$$

This notation is extended to higher derivatives:

$$f^{iv}(x) \equiv \frac{d^4}{dx^4} f(x) \qquad \text{etc.}$$

When the argument in a derivative is to be replaced (say by a constant) this notation

$$f'(c)$$

means $\begin{cases} 1^{st} \text{ get } f'(x) \text{ by working out } \frac{d}{dx} f(x) \\ 2^{nd} \text{ substitute } c \text{ for the argument, } x. \end{cases}$

Example: Given $F(y) \equiv 3y^3 - y^2$. Find $F''(\frac{1}{3})$.

Solution: $F'(y) \equiv \frac{d}{dy} F(y) = 9y^2 - 2y$

$F''(y) \equiv \frac{d}{dy} F'(y) = 18y - 2$, $\therefore F''(\frac{1}{3}) = 6 - 2 = 4$

C-36 $f(x) \equiv \dfrac{\sqrt{x}+1}{2x^2}$, $f'(2) = ?$ Ans. $-\dfrac{3\sqrt{2}+4}{32}$

C-37 $\varphi(x) \equiv 2x^3 - 3x^2 + 4$, Show that $\varphi'(1) = \varphi''(\frac{1}{2}) = \varphi'(0)$

C-38 $\varphi(x) \equiv 2x^3 - 3x^2 + 4$, Show that $\varphi(1) = \varphi(0) + \varphi'(0) + \frac{1}{2}\varphi''(0) + \frac{1}{6}\varphi'''(0) + \frac{1}{24}\varphi^{iv}(0)$

C-39 $\varphi(x^2) \equiv \dfrac{x^4}{x^2+1}$, $\quad \varphi'(y^3) = \dfrac{y^6 + 2y^3}{(1+y^3)^2}$

SHAPES OF CURVES

Plotting a function, f(x), by points is very much shortened by using f'(x) and f''(x) to get the SLOPE *and the* SHAPE *(or Curvature) at a few points. Given* $y = f(x)$

AVERAGE SLOPE $\equiv$ rise $\div$ run $= \Delta y \div \Delta x$

$\therefore$ SLOPE, $\$ \equiv \lim \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx} = f'(x)$

Note that where the curvature is

DOME-SHAPE   BOWL-SHAPE

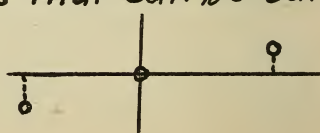
$\$$ is decreasing and $\frac{d}{dx}\$ \equiv f''(x)$ is NEG.

$\$$ is increasing and $\frac{d}{dx}\$ \equiv f''(x)$ is POS.

If f''(x) comes out zero, take points a little to each side. There are four cases:

The two middle cases are called INFLECTIONS.

Example: Plot $y = \frac{x^3}{3} - \frac{x}{4}$ *from a* FEW *points.*
Solution: Put $\frac{x^3}{3} - \frac{x}{4} \equiv f(x)$
FIRST STEP: *Plot points that can be calculated by inspection; in this case, these:* $(0,0)$, $(1, \frac{1}{12})$, $(-1, -\frac{1}{12})$.



SECOND STEP: Get general formula for slope and show graphically

$\mathcal{S} = f'(x) = x^2 - \frac{1}{4}$

$f'(-1) = \frac{3}{4}$; $f'(0) = -\frac{1}{4}$; $f'(1) = \frac{3}{4}$

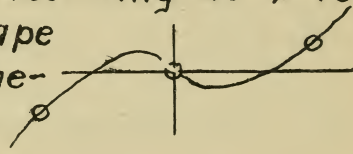
THIRD STEP: Get $f''(x)$ and plot shapes.

$\frac{d}{dx}\mathcal{S} = f''(x) = 2x$

$f''(-1) = -2$; $f''(0) = 0$; $f''(1) = 2$

Dome; Inflec.; Bowl.

FOURTH STEP: Inspect the formulas for general facts about the signs of $f(\ )$, $f'(\ )$, $f''(\ )$ in this case $f''(\ )$ is $\pm$ according as x is $\pm$, and gives bowl-shape to the right and dome-shape to the left.



Plot the graphs of these equations, calculaling a few points only:

C-40 $\quad y = x^3 - 4x - \frac{1}{2}(x^2 - 1)$

C-41 $\quad y = x^4 - 32x$

C-42 $\quad 3x = (y-1)(y-2) + xy$

C-43 $\quad y^2 = \dfrac{x}{1-x}$

EXACT RATES OF INCREASE

The rate at which any quantity Q (which may represent a distance, a volume, a speed, etc.,) increases as compared with another variable, v, (which may represent a lapse of time, a changing radius, etc., or any cause or concomitant of the change in Q) is represented by the DERIVATIVE

$$dQ/dv$$

for this is DEFINED as the result of finding the ratio of the increase in Q to the increase in v and then finding the value toward which this (AVERAGE RATE) tends as the interval for which it is calculated is made smaller and approaches zero as a limit. As on pages 10, 11, so in general:

$$\left.\begin{array}{l}\frac{\Delta Q}{\Delta v} = \text{Average Rate} \\ \frac{dQ}{dv} = \text{True Rate}\end{array}\right\} \text{of increase of } Q \text{ in units of } Q \text{ per unit of } v.$$

In studying calculus, bring to mind continually various important and familiar physical rate-quantities, of the same nature as derivatives.

These are well known definitions:

Avg. Speed = Length of motion ÷ Time of mo.
Avg. Accel. = Gain in speed ÷ Lapse of time
Avg. Slope = Rise ÷ Run
Avg. Current = Amt. of flow ÷ Time of flow
Avg. Density = Mass of piece ÷ Vol. of piece
Avg. Power = Work done ÷ Time taken
Avg. Cross Sec. = Vol. of piece ÷ its Length
Avg. Pressure = Force on Area ÷ the Area

Avg. Rate or Dif. Quo. = {Increment of DEPEND. VAR.} ÷ {Incr. of an INDEP. VAR.}

All such definitions lead to exact formulas giving EXACT RATES as DERIVATIVES.

Speed $\equiv v = \frac{dx}{dt}$ cm. per sec.	$x \equiv$ dist. cm.	$t \equiv$ time sec.
Slope $\equiv \mathcal{S} = \frac{dy}{dx}$ cm. per cm.	$y \equiv$ rise cm.	$x \equiv$ run cm.
Acceleration $\equiv \alpha = \frac{dv}{dt}$ cm. p. sec. p. sec.	$v \equiv$ speed cm. p. sec.	$t \equiv$ time sec.
Cross-Section $\equiv S = \frac{dV}{dx}$ sq. cm.	$V \equiv$ Volume Cu. cm.	$x \equiv$ length cm.
Pressure $\equiv P = \frac{dF}{dA}$ dynes p. sq. cm.	$F \equiv$ Force dynes	$A \equiv$ Area sq. cm.

DERIVATIVE RATE PROBLEMS

Attack all these problems in this general way: 1st, note what variables are involved; 2nd, express the relations between them by means of equations; 3rd; note what rates are involved; 4th, obtain the general formulas for such rates; and last, substitute given values of the Variables.

D-1 A pebble dropped into still water creates a circular disturbance whose radius lengthens 12 cm. p sec. At what rate is the disturbed area increasing when the radius is one meter?

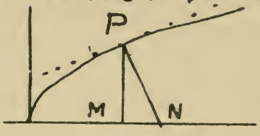
Ans. About 75 sq. meters p. sec.

D-2 The diagonals of a rectangular plate are increasing at the rate of 2 in. per sec., and the plate is lengthening at the rate of $2\frac{1}{2}$ in. per sec. When its dimensions are 5 in. x 12 in. how fast is the plate narrowing or widening?

Ans. Narrowing $\frac{4}{5}$ in. per sec.

D-3 If PN is normal to the curve

$y^2 = 2cx$ at the point P whose x-coordinate is a, and PM is normal to the x-axis, calculate the distance MN.



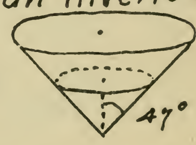
Ans. MN = c.

D-4 A body starts from rest and in t seconds gains a speed of

$$t(6 + 8\sqrt{t \div 5} + 3t/5) \text{ ft. p. sec.}$$

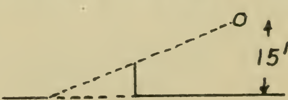
Plot the time-acceleration curve for the first five seconds.

D-5 A vessel in the form of an inverted circular cone of semi-vertical angle 47° is being filled with water at the uniform rate of 3 cu. cm. per sec. At what rate is the surface rising when the water has reached the depth of x cm? 5 cm? 10 cm?



Ans. $.83 \div x^2$; .033; .0083 cm. p. sec.

D-6 A man, 6 ft. high, walking at the rate of $3\frac{1}{2}$ mi. p. hr., passes under a light 15 ft. above his path, which is straight and level. Get general formula in terms of t, the number of



minutes since he was under the light for the length of his shadow and the rate at which it is lengthening.

D-7 In the same case as D-6 find the speed with which the end of his shadow moves along the ground.

D-8 A gasoline tank of irregular shape is being filled. The number of gallons in it when the gasoline stands at a depth of x ft. is $\frac{1}{4}(5x^3+x^2+7x)$. There are about $7\frac{1}{2}$ gallons in a cu. ft. Find the area of the bottom of the tank and of cross-sections at depths 1 ft., 2 ft., 3 ft.

Ans. .23, .80, 2.36, 4.93 sq. ft.

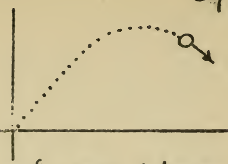
D-9 The curves $y=\frac{2}{3}x^3+x^2$ and $y=\frac{1}{3}x^4+x+2$ intersect where $x=2$. Which curve is steeper at this point?

Ans. Their slopes are 12 and $11\frac{2}{3}$.

D-10 The side of an Equilat. Triangle increases at a rate of 10 ft. per min., the area at a rate of 10 sq. ft. per sec. How large is the triangle?

Ans. One side = 69.3 ft.

D-11 The equations for a projectile are $x = 400t$ and $y = 340t - 16t^2$, in feet and sec. Find its direction of motion when $t = 5$ sec., 10 sec., 15 sec.

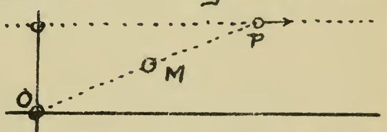


Ans. 24° up, 3° up, 19° down from level.

D-12 The cost of digging a pit, is $\$\frac{3}{4}$ multiplied by the cross-sec. in sq. yds. times the sum of the depth in yds. plus one-tenth its square. At what rate does one pay for excavation at the bottom of a 40 ft. pit of uniform cross-section?

Ans. $10\frac{1}{2}$ cents per cu. ft.

D-13 The point P moves along the line $y = 8$ at the rate of 7 units per sec. How fast is the middle point of OP receding from the point (0,8) at the instant when OP = 10 units?



Ans. 2.1 units per sec.

D-14 Which increases more rapidly as x passes through the value $x = 4$:

$(2x)^{3/2}$ or $4x\sqrt[3]{3x}$?

Ans. Their rates are as 1 : 1.44.

D-15 A ship sails due north 10 m. per hr. A steamer, 7 mi. south, 42 mi. west, steams due east at twice that speed. At what rate is the dist. between them decreasing? How far does the steamer go before their dist. begins to increase?

Ans. 18.1 mi. per hr.; 30.8 miles.

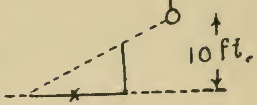
D-16 A wave crest approaches a light house at a rate of 48 ft. per sec. The light is 36 ft. above the sea. How far from the foot of the tower is the wave crest when it is approaching the light at a rate of 44 ft. p. sec.? 82.6 ft.

D-17 The time-distance formula for a moving point is $x = 2t(1-t)^2$. Work out the time-speed, and time-acceleration equations and plot all three, $t = 0$ to $t = 1$.

D-18 The total force on the upper y ft of a rectang. dam 30 ft. long is $900y^2$ lbs. What is the pressure 8 ft. from top of the dam? 480 lbs. p. sq. ft.

D-19 A man, 6 ft., tall, walks directly

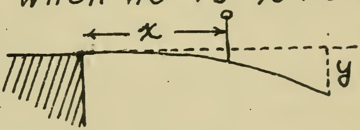
away from a lamp-post, 10 ft. high, at the rate of 4 m. per hr. How fast is the middle of his shadow moving along the pavement?



Ans. 7 mi. per hr.

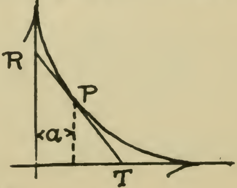
D-20 If $Q = 3x \div (1+S)$ and $S = \sqrt{1-x^2}$, find the rate at which Q increases when x is $\frac{1}{2}$ and is increasing at the rate of $\frac{1}{3}$ units per sec. Ans. .608 units p.sec.

D-21 As a man walks out along a spring-board, one end sinks to a distance of $y = \frac{1}{15}x^2(x+2)$ inches when he is x feet from the wharf end. If he moves along at the rate of 2 ft. per sec., how fast is the end sinking when he starts? When he has gone 10 ft.?



Ans. 0 and 3.77 ft. per sec.

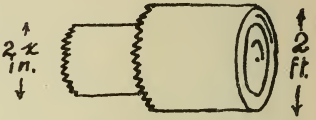
D-22 The equation of the curve SPQ is $x^{2/3} + y^{2/3} = 4$. The tangent at P whose x-coordinate is a, is the line RT. Find the length of RT.



Ans. RT = 8 units

D-23 When a ball is thrown straight up, it reaches a height of $(6 - 16t^2 + 140t)$ feet in t sec. When does its speed change from up to down? Ans. $t = 4.38$.

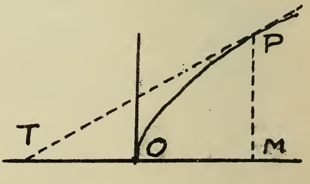
D-24 A log in the form of a cylinder, 12 ft. long, 2 ft. in diameter, consists of year-rings whose density increases toward the center so that the mass of a core whose radius is x in. is $[330x^2 \div (20+x)]$ lbs. Find the density at the center and at the bark.



Ans. 63 and 32 lbs. per cu. ft.

NOTE: In some problems data are given which would be difficult or impossible to obtain in an actual case, while the quantities asked for could easily be found by direct measurement. It is nevertheless worth while to solve such problems for the sake of the light they throw on the reversed problem.

D-25 PT is any tangent to the curve $y^2 = 2ax$. Prove that OM = OT.



D-26 The speed of a meteor before it reaches the denser part of the earth's atmosphere is given in ft. per sec. as

$$v = \frac{\sqrt{r \times 10^{21} - r^2}}{200r}$$

where r is its distance from the center of the earth in feet. Find its acceleration at a height of 15000 mi. above the surface of the earth.

Ans. $1\frac{1}{4}$ ft. per sec. per sec.

D-27 If f is a number of ft., i, a number of inches, m, a number of min., and s a number of sec., write down equations of relation and from them deduce

$$\frac{df}{dm} = \cdot 5 \, \frac{di}{ds}$$

D-28 An automatic record shows that the work done by a certain engine in h hours, beginning at 8 AM, is

$$18\,h\left(10 + 5h - \tfrac{2}{3}h^2\right)10^6 \text{ ft. lbs.}$$

Find the power being used at 9 AM and at 10.30 AM in horse-powers, one horse-power being equal to 550 ft. lbs. per sec.

Ans. 164 HP. and 205 HP.

D-29 From Regnault's experiments it

appeared that the mumber of heat units, q, required to raise the temperature of b gms. of water from 0° to T° Cg. is given by the equa.

$$q = [T + 2T^2 10^{-5} + 3T^3 10^{-7}]b$$

If heat is being supplied to 20 grams of water at the rate of 10 heat units per sec., find the rate at which the temperature rises when T = 50°. Ans. .497° per sec.

D-30 A rectangular plate is expanding at the rate of 10 sq. in. per sec. in such a way that its diagonals remain the same length. At what rate are the two sides changing when the dimensions of the plate are 4 in. x 20 in.?

Ans. Incr. .522, and Decr. .104 in. per sec.

D-31 The point P is on the curve $y^2 = x^3$, at the point where $x = 4$. A tangent is drawn from P to the x-axis. Find its length. Ans. 8.4328 units.

D-32 Assume that the work done in compressing a vapor from volume V_0 to volume V_1 is $15\sqrt{V_0 \div V_1}$ inch lbs.; 36 cu. in. of vapor are introduced into an empty cylinder, 9 sq. inches cross-section, and by means of a piston this vapor is compressed to 1 cu. in. Find the force being used at the end of compression.

Ans. 405. lbs.

D-33 The altitude of the sun is 30° when a ball is thrown vertically up to a height of 64 feet. How fast is its shadow travelling along the ground just before the ball strikes the earth?

Ans. 111. ft. per sec.

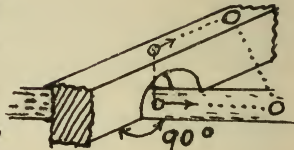
D-34 A circular metal plate is expanding so that its radius increases 1½ m.m. per min. At what rate does its area increase when its diameter is 10 cm?

Ans. 4.7 sq. cm. per min.

D-35 · Using the term "length of shadow" to denote the distance between a ball and its shadow on the ground, find a formula for the rate at which the shadow lengthens when a ball is thrown up so that its height is $(80t - 16t^2)$ ft. at the end of t sec., the shadow making an angle of 45° with the ground.

Ans. $\frac{ds}{dt} = 16\sqrt{2}[5-2t]$ f.p.s.

D-36 A man is walking over a bridge at the rate of 4 m.p.h. A boat passes under the bridge just below him. It is towed 8 m.p.h. and the canal is 20 ft. below the roadway. How fast are the man and the boat separating 3 minutes later?



Ans. $\frac{1}{5}$ mile per hour.

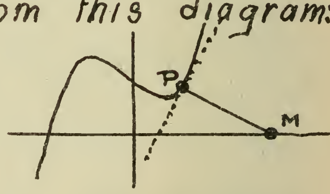
D-37 A soap bubble remains spherical and it's diameter increases at the rate of 2 cm. per sec. At what rate is its volume increasing at the instant it becomes 15 cu.cm.?

Ans. 29.4 cu. cm. per sec.

D-38 Taking data from this diagram: the equation of the curve being

$$y = 6x^3 - 2x + \frac{1}{2}$$

calculate the length of PM, which is drawn perpendicular to the tangent at P, where $x = \frac{1}{2}$. .673

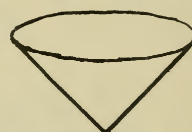


D-39 In D-38 calculate the distance from M to the origin. 1.125

D-40 The cost per mile of running a steam-boat varies as the cube of the speed. Show that the cost per hour varies as the fourth power of the speed.

D-41 A balloon rises to a height of $[10m \div \sqrt{4000+m^2}]$ miles in m min. At what rate is it rising at the end of the first half-hour? Ans. About 7 m.p.h.

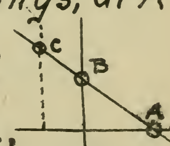
D-42 Water is poured into a conical cup at the rate of 14 cu.cm. per sec. and fills the cup in 11 sec. If its depth then is 7 cm., how fast is the water rising just before it overflows? Ans. .212 cm. p. sec.



D-43 At what angles does the curve $y = 2x^3 - 6x$ cut the x-axis?

Ans. $+85°15'$ and $-80°35'$

D-44. A rod slides through rings, at A and B, which slide on the axes, and at C which slides on the line $x+1=0$. If $AO = (9-t^2)$ cm., $OB = 6t$ cm., find speed of C when $t = 2$ seconds.



Ans. 9.12 units per sec. down.

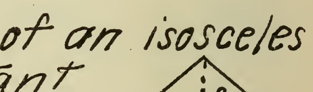
D-45 H is a number of hours; M, its value in minutes; S, its value in seconds: set down the equations of relation and from them deduce that

$$dQ/dH = 60\ dQ/dM = 3600\ dQ/dS$$

D-46 A meteor is falling to the earth. Its distance from the center of the earth is $[7000 - \sqrt{10\,000t - t^2}]$ miles, the t being a number of seconds. Show that it strikes the earth when $t = 1000$. with a speed of 80 miles per minute.

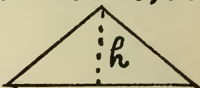
D-47 Locate the points on the curve

$$y = 5x^3 + 8x^2 - 7x$$



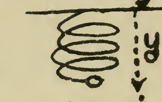
at which the tangents are horizontal. Ans. $(\frac{1}{3}, -1.26)$, $(-\frac{7}{5}, +19.6)$.

D-48 The perimeter, P, of an isosceles triangle remains constant while the altitude increases at a rate of S cm. per sec. At what rate is the area increasing?



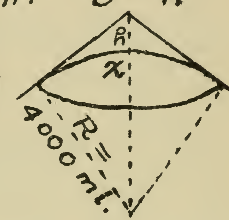
Ans. $(P^2 - 12h^2)S \div 4P$ sq. cm. per sec.

D-49 A weight hangs from a spring and rises and falls so that its speed is $2\sqrt{8y-y^2}$ cm. per sec., y cm. being its distance from the upper end of the spring. Work out a formula for the acceleration, α, in terms of y. Ans. $\alpha = 4(4-y)$ cm.p.s.p.s.



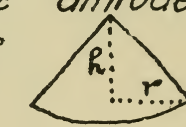
D-50 In the case described in D-41 find the rate at which the visible surface of the earth is increasing at the time mentioned. Area $= 2\pi R x$

Ans. 46.3 sq. mi. per sec.



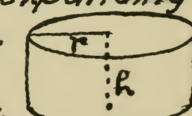
D-51 At what rate does the volume of a cone increase when the altitude and the radius of the base increase at the same rate, S cm. per sec. each?

Ans. $\frac{1}{3} S\pi(2rh + r^2)$ cu. cm. p. sec.

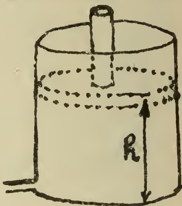


D-52 The dimensions of an expanding cylinder are $r = (16 + .0004t^2)$ cm. $h = (24 + .00015t^2 + .01t)$ cm., the t being $t°$ Cg. temperature. What is the increase in volume per degree at a temperature of 50° Cg.?

Ans. 128. cu. cm. per deg.

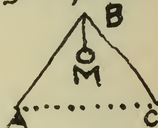


D-53 Oil is forced under a piston in a cylinder at a rate inversely proportional to the height, h, of the piston. If the piston is rising at the rate of $\frac{1}{7}$ cm. per sec., when $h = 5$ cm. find the rate at which oil is coming in when $h = 10$ cm. The piston's cross-section is 4 sq. cm.



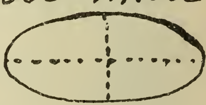
Ans. $\frac{2}{7}$ cu. cm. per sec.

D-54 A mast is being lifted by a pair of shears: $AB = BC = 25$ ft. How fast is M being raised when A and C are 20 ft. apart and being drawn together by a tackle at the rate of 6 inches per sec.?



Ans. 1.31 inches per sec.

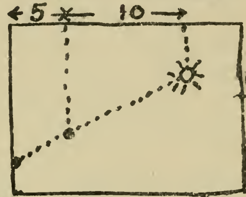
D-55 The area of an ellipse whose diameters are p and q in. is $\frac{1}{4}\pi pq$ sq. in. At what rate is its area increasing if, t being a time in seconds, $p = 2t^2$ and $q = 3t^4 - 1$?



Ans. $\pi t(9t^4 - 1)$ sq. in. per sec.

D-56 A white-hot bullet is dropped from

the ceiling of a dark room $\frac{1}{8}$ sec. after a cold bullet was dropped from another point on the ceiling 10 ft. away. The latter casts a shadow on the wall 5 ft. away. Take the distance fallen as 16 ft. × the square of the number of seconds of falling. Find the position, speed, acceleration, of the shadow when the hot body has fallen for one second. Ans. 22.37 ft. below the ceiling; speed, 38 ft. per sec. down; acceleration, 32 ft. per sec. per sec. down.



D-57 Show that if speed is proportional to [distance]^2, acceleration must be proportional to [speed]$^{\frac{3}{2}}$.

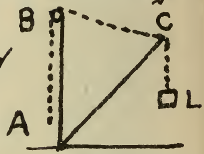
D-58 An irregular reservoir is being filled. The volume of water let in, V cu. ft., raises the height of the water, h ft. according to the formula

$$(h-2)^2 = 3(V \times 10^{-6} - 1.4)$$

Find the area of the water surface when six million cubic feet of water have been let in.

Ans. Nearly $2\frac{1}{2}$ million sq. ft.

D-59 *The rope ABC runs over a pulley at B and is fastened to the boom at C. The rope is wound in at A, $3\frac{1}{2}$ ft. per sec. How fast is the block, L, rising when BC is level? The dimensions of the derrick are: AB = 30 ft., AC = 35 ft.*



Ans. 2.075 ft. per sec.

D-60 *A train is* $[40t^2 - 5t^4]$ *miles from the starting point at the end of* t *hrs. Get its speed and acceler. in terms of* t.

D-61 *Find the time between stops of the train described in D-60.*

Ans. 2 hours.

D-62 *When does the D-60 train first begin to slow down after starting?*

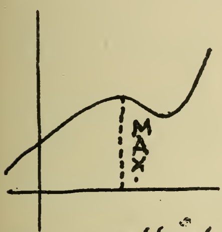
Ans. 1 hr. 9.3 min. after starting.

D-63 *An electrolytic cell deposits* 2×10^{-6} *cu. cm. p.s. on 3 cm. of wire* $\frac{1}{2}$ *m m. thick. How fast is the diameter increasing when the wire has become 1 mm. thick?*

Ans. $4\frac{1}{4}\ 10^{-6}$ cm. per sec.

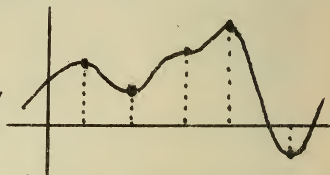
Maxima and Minima

When a function that has been increasing begins to decrease, the value at which it turns is called a MAXIMUM



— whether or not there may be still larger values of the function. The term maximum is used in a relative sense, the value so called being greater than the values on both sides of it, when the function is represented graphically or by a table. Similarly $f(x)$ is called a MINIMUM when $x=a$ if $f(a)$ is less than the values of $f(x)$ when x is a little less or greater than a.

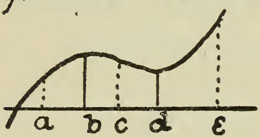
In elementary work we deal with continuous functions which have continuous derivatives. The graph of such a function is a continuous line without sharp turns like a contour line in a rolling country. On such a line the points whose altitudes (values of the function in the curve $y=f(x)$) are maxima or minima, will be at the tops and bottoms of the undulations. At these points the curve will be LEVEL, that is the slope of the tangent, $f'(\ )$, will $= 0$. (But the slope will also be $= 0$ at a TERRACE.)



If then we have a function, $f(x)$, and find All the points at which $f'(\ ) = 0$, this list will include all points at which $f(\ )$ is either maximun or minimum. These points are called CRITICAL POINTS. They may be sorted by whichever of the following three methods is most convenient.

FIRST METHOD OF TESTING CRITICAL ARGUMENTS

Calculate the value of $f(\ )$ at the critical point and compare with values for NEIGHBORING ARGUMENTS, less than and greater than the critical argument. What



is called the NEIGHBORHOOD of a cr. arg. is that range of arguments not separated from it by another cr. arg. In determining whether $f(d)$ is MIN., c and ε are neighboring arguments: comparing $f(d)$ with $f(a)$ would lead to no conclusion.

Second Method of Testing Critical Arguments

Find the slope of the graph (rate of increase of the function) for NEIGHBORING *arguments, chosen (as in the first method) greater and less than each critical value, one between each pair of them. Only the signs of $f'(\ )$ need be determined to show whether the curve runs up-level-down (maximum), down-level-up (minimum), or up-level-up or down-level-down, as at a terrace point.*

Third Method of Testing Critical Arguments

Substitute the critical arguments into the second derivative, $f''(\ )$, and if the result is PLUS *we have a valley-bottom, hence a minimum; if the result is* MINUS *we have a hill-top and hence a maximum: $\overset{+}{\smile}$ and $\underset{-}{\frown}$. If $f''(\ )$ comes out zero when a critical argument is substituted, try* NEIGHBORING *values.*

Graphical representation is of the greatest use in questions about maxima and minima and should never be neglected. Review pages 30 and 31.

Example: Examine $1+2x^3(x^2-1)$ for maximum and minimum values.

Solution:

Put $f(x) \equiv 1+2x^3(x^2-1)$ and differentiate:

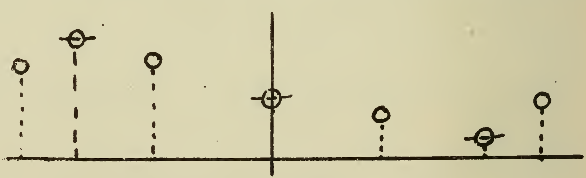
$f'(x) = 10x^2(x^2-\frac{3}{5})$ By solving $f'(x)=0$ the critical arguments are $x=0, \pm\sqrt{\frac{3}{5}}$

As this is an illustrative example each method of sorting out the maxima, etc., will be applied.

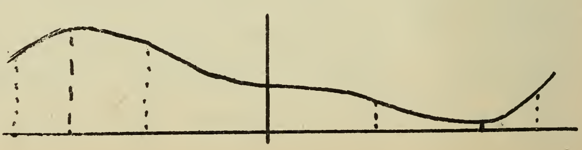
1st METHOD. $\sqrt{\frac{3}{5}} = .78$. Easy values of x in the intervals between critical values and beyond are $-1, -\frac{1}{2}, +\frac{1}{2}, +1$. Calculate a table, plotting results roughly:

Crit. Arg.		$-\sqrt{3/5}$		0		$+\sqrt{3/5}$	
Intermed.	-1		$-\frac{1}{2}$		$+\frac{1}{2}$		$+1$
f( )	1.	1.56	1.18	1.	.82	.44	1.
Verdict:		MAX.		Neither		MIN.	

This table gives this plot:



This then is the graph of the f(x)



2^{nd} METHOD. *Tabulate f′( ) for critical and neighboring values, and represent as slopes:*

Crit. Arg.		$-\sqrt{3/5}$		0		$+\sqrt{3/5}$	
Intermed.	−1		−½		+½		+1
f′( )	+	0	−	0	−	0	+
Slope	↗	→	↘	→	↘	→	↗
Verdict		MAX.		Neither		MIN.	
f( )		1.56				.44	

3^{rd} METHOD. *Obtain the second derivative*

$$f''(x) \equiv 40x(x^3 - 3/10)$$

substitute crit. arguments and tabulate;

Crit. Arg.	$-\sqrt{3/5}$	0	$+\sqrt{3/5}$
f″( )	−, ∴ ∩	0, ∴ try ±½	+, ∴ ∪
Verdict	MAX.	+ 0 −, ∴ ⌐	MIN.
f( )	1.56		.44

The fact that hills and valleys must alternate would show that x=0 is a terrace point since the critical points adjacent are a hill and a valley.

In doing the first dozen or so of the problems, practice the use of all three methods of sorting critical arguments in order to learn to select the most convenient one. Always plot the graph of the f(x) as a check on verdicts given.

When a concrete problem calls for the conditions under which a certain quantity, Q, will be a MAX. or a MIN., it is necessary first to note what second quantity can be varied whose value will CONTROL the value of Q. This latter is the INDEP. VAR. in the problem and Q must be expressed in terms of it. Then find the crit. args. and proceed according to one of the three methods described.

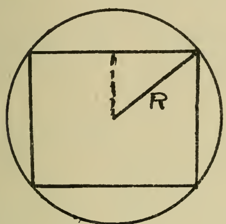
It frequently happens that the formula obtained for Q is a power, a root, or a reciprocal of some other quantity, M, which is easier to differentiate than Q. In such a case it is advisable to find what values of the independent variable make M a MAX. or a MIN. and then to infer from these results what are the MAX. and MIN. of Q.

Example: Find the greatest rectangle that can be inscribed in a circle.

Solution:

The circle must be regarded as a given

one, and its radius will be a constant, R. The size (Area) of the rectangle is controlled by the length taken for one side: then take one side as the



indep. var. and call it x. Express the other side in terms of x: it is

$$2\sqrt{R^2 - (x/2)^2}$$

The Area - which is the quantity to be maximized - is:

$$Q \equiv 2x\sqrt{R^2 - (x/2)^2}$$

Both Q and its square will have a maximum value for the same x, and

$$M \equiv Q^2 = 4R^2x^2 - x^4$$

is a simpler quantity to examine for maxima and minima than Q.

Put $f(x) \equiv M = 4R^2x^2 - x^4$,

$$f'(x) = 8R^2x - 4x^3$$

Find critical arguments by solving:

$$8R^2x - 4x^3 = 0$$

Critical x's are $x = 0$, $x = \pm R\sqrt{2}$. The values 0, and $-R\sqrt{2}$ can be ignored in solving this problem.

Examine $x = R\sqrt{2}$ by the 3rd METHOD.

$$f''(x) = 8R^2 - 12x^2$$

$f''(R\sqrt{2}) = -16R^2$, negative, indicates a MAX.

$\therefore x = R\sqrt{2}$ gives largest area (a square) $2R^2$.

MAXIMUM - MINIMUM PROBLEMS

It is not a sufficient solution of a problem in maxima and minima to find what critical arguments produce either; the MAX. or MIN. values of the function should be clearly stated together with the condition that produces each. The printed answers to such problems are abbreviated from the results that should be given in a proper solution.

INVESTIGATE FOR MAXIMA AND MINIMA

E-1 $y = (x^2 - 7x + 6)^2$

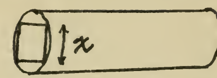
E-2 $y = x(x^2 - 3)$

E-3 $y = \frac{1}{(x-a)(b-x)}$

E-4 Q, if $(a - 2x)Q = (a - x)^3$

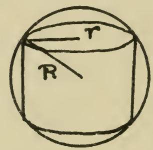
E-5 $y = (x-1)(x-2)(x-3)$

E-6 A beam of rectangular cross-section is to be cut from a cylindrical log 24 inches in diameter. Its strength is proportional to x^2y, its stiffness to x^3y, if x is its depth and y its breadth. Dimensions of strongest beam and of stiffest beam that can be cut, required.



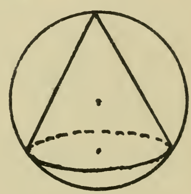
E-7 What cylinder inscribed in a given sphere has the greatest volume?

Ans. $r = R\sqrt{2/3}$

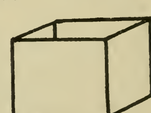


E-8 If the radius of a sphere is 12 inches, what is the height of a cone that can be turned down from it with the least waste?

Ans. 16 inches.



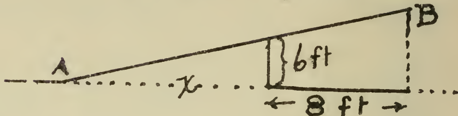
E-9 Three thousand two gallon measures with square bases are to be made of sheet copper. One gallon = 231 cu. in. What are the best dimensions? 9.74" × 9.74" × 4.61"



E-10 Find the least value of $x + \frac{1}{x}$

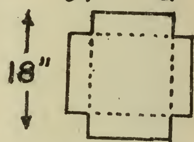
Ans. 2 when $x = 1$

E-11 What value of x will make AB least and how short will that value make it?



Ans. $AB = 19.6$ ft. $x = 6.6$ ft.

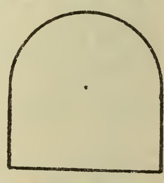
E-12 In making a box out of a square piece of tin the corners are cut out and wasted. Using 18" squares how shall one get greatest capacity for his money?



Ans. Make the box 3" high.

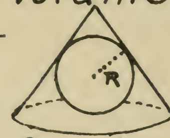
E-13 The cost of coal per hour for driving a steam ship varies as the cube of the speed. Show that against a current of 4 mi. per hr. the most economical speed is 6 mi. per hr.

E-14 A Norman window consists of a rectangle surmounted by a semi-circle. For a given perimeter determine what window will admit the most light.



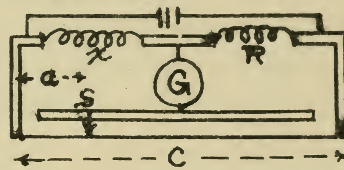
Ans. Make the radius = the perimeter $\div (\pi + 4)$

E-15 Find the altitude and volume of the cone of least volume circumscribed about a given sphere.



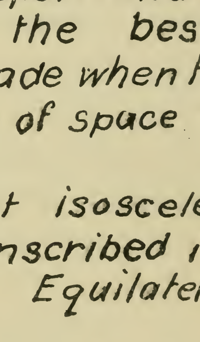
Ans. Alt. 4R, Vol. $\frac{8}{3}\pi R^3$.

E-16 In measuring resistance with a slide-wire bridge the percentage of error due to any error in setting the slider, s, is inversely proportional to $(ac-a^2)$. Show that the best measurements can be made when the slider, s, is near the middle of space c.



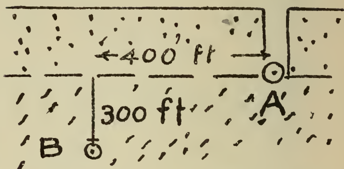
E-17 What is the greatest isosceles triangle that can be inscribed in a given circle? Ans. Equilateral.

E-18 Find the dimensions of the greatest rectangle that can be inscribed in a triangle, each of whose sides is 20 inches in length?



Ans. 10 inches × 8.65 inches.

E-19 A miner is to open a tunnel from A to B. On a level through A is a surface of separation between a soft upper layer through which a tunnel can be driven for \$10 a foot, and the solid rock where the cost would be \$30 a foot. What should it cost?



Ans. About \$12,500.

E-20 The hourly cost of fuel on a liner varies as the cube of her speed, being \$25 an hour for a speed of 10 mi. per hr. All other expenses come to \$145 an hour. In what time should she plan to make a 3000 mile trip?

Ans. In 8 days 17 hrs.

E-21 Divide a line into four parts so as to make the rectangle formed from them as large as possible.

Ans. 4 equal parts.

E-22 A box, square base and no cover, is to be made, using as little material as possible.

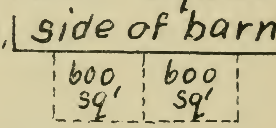
What dimensions should be used for a capacity of 108 cu."? 3" × 6" × 6".

E-23 In measuring current with a Tangent Galvanometer, the percentage of error due to a small error in the reading, x°, is proportional to

$$(\tan x^\circ + \cot x^\circ)$$

Show that it is least when $x^\circ = 45^\circ$. (Compare this problem with E-10.)

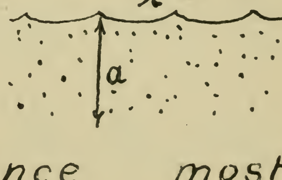
E-24 A farmer wants two equal rectangular hen-yards,



each to contain 600 sq. ft. Taking advantage of one side of the barn, what is the least cost of the job at $4\frac{1}{2}$ cents a foot for fencing put up?

Ans. \$5.40

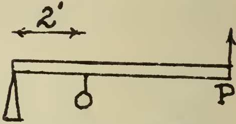
E-25 The speed of waves whose length is λ (lambda) is proportional to



$$\sqrt{\frac{\lambda}{a} + \frac{a}{\lambda}}$$

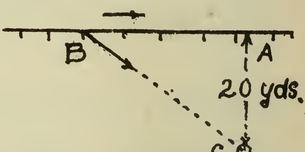
Show that they advance most slowly when $\lambda = a$.

E-26 A half-ton weight hangs two ft. from the end of a lever and is to be raised by lifting at P. If an iron lever is to be used weighing 10 lbs. to the foot, what length of lever will make the easiest lift?



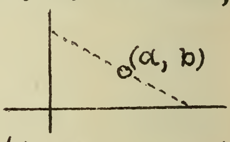
20 ft.

E-27. At what point, B, shall a passenger jump from his car, which goes at the rate of 8 mi. per hr., that he may reach c as quickly as possible? He can walk 4 miles per hr.



Ans. Make AB = 11.55 yds.

E-28 Through the point (a, b) a line, as short as possible, is drawn from one axis to the other. Show that the slope of this line must be $-\sqrt[3]{b/a}$ and its length $(a^{2/3} + b^{2/3})^{3/2}$

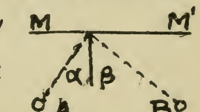


E-29 In the problem of the flow of air through a small opening, it is desired to get the maximum value for $x^{2/\gamma} - x^{1+1/\gamma}$, γ (gamma) being = 1.4 Ans. $x = 0.528$

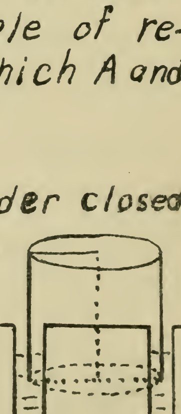
E-30 The power given to an external circuit by a 20 volt generator of internal resistance 1.8 ohms when the current is i amperes is $[20i - 1.8i^2]$ watts. With what current can this generator deliver the most power?

Ans. $55\frac{5}{9}$ watts with $5\frac{5}{9}$ amperes.

E-31 If the law of reflection were this: "The path from A to B by way of the mirror, MM', must be the shortest possible one", show that it follows that the angle of incidence, α, must equal the angle of reflection, β. Use a case in which A and B are equidistant from MM'.



E-32 A gas-holder is a cylinder closed at the upper end and open at the bottom where it sinks into water. What are the most economical dimensions for the cylinder?

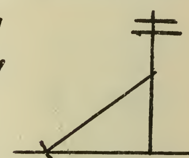


Ans. radius = height.

E-33 For a certain sum a man agrees to build a rectangular water tank and line it with lead. The base is to be a square and it is to hold 3000 cu. feet. What dimensions will make the cost of lining least?

Ans. 9.1 ft. x 18.2 ft. x 18.2 ft.

E-34 A telegraph pole at a bend in the road is protected from tipping over by a 20ft. stay fastened to the pole and to a stake. How far from the pole should the stake be driven in order that the tension in the stay-wire may have the greatest moment about the foot of the pole?



Ans. 14.14 ft.

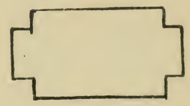
E-35 A man in a row boat is 3mi. from the nearest point, A, of a straight beach. He wishes to reach a point on the beach 5mi. from A. He can row 3 mi. per hr., and walk 4 mi. per hr. How shall he row?

Ans. So as to walk 1.6 miles.

E-36 What number exceeds its square by the greatest amount?

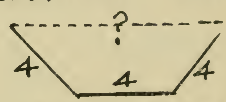
Ans. One-half.

E-37 I wish to make the most capacious box I can from a piece of cardboard 30" x 14". How big a square shall I cut from each corner before folding up?



Ans. 3 in. squares.

E-38 For what breadth across the top has this figure, symmetrical and 4 in. on each given side, the greatest area?



Ans. 8 inches.

E-39 What is the largest Box mailable in England, where the regulations forbid mailing if the sum of length and girth exceeds six feet?

Ans. 1 ft. x 1 ft. x 2 ft.

E-40 What is the largest cylindrical package mailable in England?

Ans. 2 ft. long, 4 ft. circum.

INTEGRATION

The process of finding the differential of one function leads to a second function. The process of retracing this process, from the differential to the function originally differentiated, is called INTEGRATION, and it is indicated by placing the INTEGRAL SIGN, $\int$, (an old-fashioned long-s), before the differential to be integrated. Therefore if

$$d\,F(x) \equiv \varphi(x)\,dx$$

then

$$\int \varphi(x)\,dx \equiv F(x)$$

But $F(x)$ is not the only quantity whose differential is $\varphi(x)\,dx$, for if A is any constant whatever:

$$d[\,F(x) + A\,] = \varphi(x)\,dx$$

Hence

$$\int \varphi(x)\,dx \equiv F(x) + A$$

The number, A, is called a CONSTANT OF INTEGRATION, and must be remembered as an ESSENTIAL part of every result obtained by integration.

Sometimes the arbitrary constant, (like the exponent one), is "understood" and not written, but this liberty the student should not permit himself at this stage of his studies.

The foregoing DEFINITION of INTEGRATION may be presented in this form:

$$\int du \equiv u + A$$

Five formulas for integration follow from the five formulas for differentials, which for this purpose are more conveniently put into the forms:

$$dc = 0$$
$$d(u+v) = du + dv$$
$$d(cu) = c\,du$$
$$d(vu) - u\,dv = v\,du$$
$$d\,\frac{u^{n+1}}{n+1} = u^n\,du$$

These are now to be changed to the integal form, making use of the law:

$$\int du \equiv u + A.$$

Note that if $n = -1$, there is no such quantity as $u^{n+1} \div (n+1)$ to differentiate.

$$\int 0 = c + A' = A$$
$$\int (du + dv) = u + v + A = \int du + \int dv$$
$$\int c\,du = cu + A = c\int du$$
$$\int v\,du = vu + A - \int u\,dv = v\int du - \int u\,dv$$
$$\int u^n du = \frac{u^{n+1}}{n+1} + A \quad \begin{cases}\text{Provided } n \text{ is} \\ \text{not equal to } -1.\end{cases}$$

These formulas may be restated in better form for use as follows:

I $\int 0 = A$. Add an ARBITARY CONSTANT to every integral.

II $\int(du + dv) = \int du + \int dv$. Integrate the TERMS of a SUM separately.

III $\int c\,du = c\int du$. A CONSTANT FACTOR may be moved from one side of the $\int$-sign to the other.

IV $\int v\,du = v\int du - \int u\,dv$. A VARIABLE FACTOR may not be moved past the $\int$-sign without introducing a certain new integral as an offset.

V $\int u^n du = \dfrac{u^{n+1}}{n+1} + A$, Provided $n \neq -1$. INCREASE the exponent of the base by one and DIVIDE by new exponent, unless the latter comes out = zero.

The use of IV is treated on page 150.

In using V make sure that the POWER is multiplied by the differential of its base. To provide for this a constant factor may often be introduced and offset: thus, $\int(2-5x^3)^n x^2 dx$
$= -\frac{1}{15}\int(2-5x^3)^n(-15x^2 dx) = -\frac{1}{15}\int(2-5x^3)^n d(2-5x^3)$

To verify an example in integration differentiate the result, and obtain the INTEGRAND, or quantity originally affected by the sign of integration.

Example: $\int(6x^2 + \frac{1}{\sqrt{x}} + [x^2+1]^3 x)\, dx$

$$= 6\int x^2 dx + \int x^{-\frac{1}{2}} dx + \tfrac{1}{2}\int(x^2+1)^3 2x\, dx$$
$$= 2x^3 + 2x^{\frac{1}{2}} + \tfrac{1}{2}\int(x^2+1)^3 d(x^2+1)$$
$$= 2x^3 + 2\sqrt{x} + \tfrac{1}{8}(x^2+1)^4 + A$$

Verification:

$$d(2x^3 + 2\sqrt{x} + \tfrac{1}{8}[x^2+1]^4 + A)$$
$$= (6x^2 + x^{-\frac{1}{2}} + 4\cdot\tfrac{1}{8}[x^2+1]^3 2x)\, dx$$
$= $ the integrand.

NOTE that $\int 2x(x^2+1)\, dx = 2\int(x^3+x)\, dx$
$$= \tfrac{1}{2}x^4 + x^2 + A$$
while $2x\int(x^2+1)\, dx = 2x(\tfrac{1}{3}x^3 + x + A)$
$$= \tfrac{2}{3}x^4 + 2x^2 + Ax$$

DRILL ON INTEGRALS

Two integrals of the same integrand may differ in the FORM of the constant of integration when worked by different methods. Thus $\int(x+1)^2\,dx = \int(x+1)^2\,d(x+1) = \frac{1}{3}(x+1)^3 + A$

$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + A$

or $= \int(x^2+2x+1)\,dx = \frac{1}{3}x^3 + x^2 + x + A'$

These results AGREE in meaning: $\frac{1}{3}x^3 + x^2 + x +$ SOME UNDETERMINED CONSTANT.

INTEGRATE: VERIFY RESULT IN EACH CASE:

F-1 $\int(x^2-1)^2\,dx$

F-2 $\int(x^2-1)\,x\,dx$

F-3 $\int\left(\frac{dx}{x^5} + dx\right)$

F-4 $\int\sqrt{s+1}\;ds$

F-5 $\int\sqrt{1-t}\;dt$

F-6 $\int\frac{dx}{(1-x)^3}$

F-7 $\int\frac{x^2\,dx}{(2x^3+1)^2}$

F-8 $\int(2m^3+1)\,dm$

F-9 $\int(1-q)\,dq$

F-10 $\int s\sqrt{s^2-1}\;ds$

F-11 $\int(2x^3+\sqrt{x} - x^{5/2} + 7x^{-3/8})\,dx$

F-12 $\int\frac{2\,dx}{(1+x)^2} + \frac{2x\,dx}{(1+x^2)^2} - \frac{6\,dx}{x^3} + ds$

Since INTEGRATION is the REVERSE of DIFFERENTIATION, questions like the preceding may be asked in these 3 different forms:

$$\int f(x)\,dx = ?$$

$$dy = f(x)\,dx, \qquad y = ?$$

$$dy/dx = f(x), \qquad y = ?$$

F-13 $\frac{dy}{dx} = 2x^2 - 3x^3 + \sqrt{x} \qquad y = ?$

F-14 $\int (x^2 + \sqrt{x})(2x - 1)\,dx = ?$

F-15 $dy = \left(2\sqrt{v} + \frac{1}{2\sqrt{v}}\right) dv \qquad y = ?$

F-16 $dy = \frac{2x^2 + 1}{(2x^3 + 3x - 4)^2}\,dx \qquad y = ?$

F-17 $\frac{dy}{dx} = \frac{2x - 3}{\sqrt{(x-1)(x-2)}} \qquad y = ?$

F-18 $dy = \sqrt{\frac{x}{x + \frac{1}{x}}}\,dx \qquad y = ?$

F-19 $\int (\sqrt{v}\,dv - s^2\,ds) = ?$

F-20 $\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{d^2 s}{dt^2} \qquad y = ?$

F-21 $\frac{dy}{dt} = 2(t^2 - s^2)(s\frac{ds}{dt} - t) \qquad y = ?$

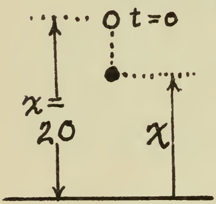
F-22 $\int \frac{d^2y}{dx^2}\,dx = \int \frac{d}{dx}\left(\frac{dy}{dx}\right) dx = \int d\left(\frac{dy}{dx}\right) = ?$

DETERMINATION OF CONSTANT

The constant of integration is arbitrary and unknown so far as the reversal of a differentiation is concerned. But some fact about the values of the variables involved may make it possible to determine the numerical value of the constant.

Example: A body is dropped from a height of 20 ft. Its speed is 32 ft. per sec. times the number of sec. Find a formula for its height in terms of the time.

Solution: Let t sec. = time of falling to a height of x ft.



Then

$$\text{DOWNWARD SPEED} \equiv -\frac{dx}{dt}$$

Hence $-\frac{dx}{dt} = 32t$

or $dx = -32t\,dt$

Integrating $x = \int dx = \int -32t\,dt$

$$= -16t^2 + A$$

When t *was zero,* x *was* 20, *hence*

$$20 = -16 \times 0^2 + A$$

$$A = 20$$

Accordingly height $= (20 - 16t^2)$ *ft.*

Integrate and determine the constant by means of the conditions given in each case.

F-23 $\frac{dy}{dx} = 2x^2$ and $y = 1$ when $x = 1$

F-24 $y = \int x^2\sqrt{x^3+1}\,dx$ and when $x = 2$, $y = 3$ Ans. $y = \frac{2}{9}(x^3+1)^{3/2} - 3$

F-25 $dx = \left(\frac{1}{2}t^2 + \sqrt{t}\right)dt$ and $t = 1$ makes $x = 0$. Ans. $x = \frac{1}{6}(\sqrt{t^3}+5)(\sqrt{t^3}-1)$

F-26 $ds = \frac{dy}{1-s^2}$ and $y = 0$ when $s = 0$. Ans. $3(s-y) = s^3$

F-27 $\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$ and $\frac{dy}{dx} = 2$ when $y = \frac{1}{2}$. Ans. $\frac{dy}{dx} = 2y + 1$.

F-28 $\frac{d^2y}{dx^2} = 6y\frac{dy}{dx}$ and when $x = 2$, $y = 1$, and $\frac{dy}{dx} = 3$. Ans. $y = \frac{1}{7-3x}$.

F-29 $\frac{d^2y}{dt^2} = 16t$ and when $t = 1$, $y = 0$, and $\frac{dy}{dt} = 2$. Ans. $3y = 8t^3 - 18t + 10$.

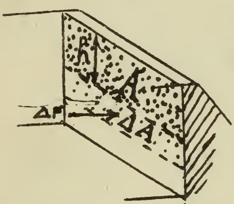
F-30 $dq = 2q^2\,dt$ and $t = 1$ when $q = 1$ Ans. $q(3-2t) = 1$.

F-31 $\left(\frac{dy}{dx}\right)^2 = 1$ always, and $x = 1$ when $y = 1$.

INTEGRAL RATE PROBLEMS.

As problems in rate of increase were solved by differentiation when the RATE (speed, acceleration, density, force, pressure, etc., see pp. 33, 158) was required so when a RATE is given the problem is solved by means of integration.

Example: The pressure on a dam at any depth is $62\frac{1}{2}$ lbs. per sq. ft. × the depth in feet. What is the whole pressure on a dam 20′ wide, 8′ deep?



The average pressure on a horizontal strip is Δ Force ÷ Δ Area

Hence the exact pressure

$$\equiv dF/dA = 62\tfrac{1}{2} h$$

Taking A sq. ft. as the area (VAR.) down to depth h ft. (h ≡ INDEP. VAR.) we get

$$A = 20h \quad \text{whence} \quad dA = 20\,dh$$

$$dF = 62\tfrac{1}{2}h\,dA = 1250h\,dh$$

$$\therefore F = \tfrac{1}{2}\,1250h^2 + \text{a constant}$$

When $h=0$, $F=0$, hence the const. $=0$

$$\therefore F = 625h^2$$

For the whole dam, $h = 8$ and Force is

625×8^2 lbs. or 20 Tons.

G-1. The speed of a body that starts from $x = 5$ ft. at the time $t = 2$ sec. is $8t(1+t)$ ft. per sec. Work out the formula for x in terms of t.

Ans. $x = \frac{1}{3}(8t^3 + 12t^2 + 97)$

G-2 The speed formula being $\frac{dx}{dt} = 3t^2$ ft. per sec., the body reaching $x = 12$ ft. when $t = 1\frac{1}{2}$ sec., find what x was when t was zero. Ans. $8\frac{5}{8}$ ft.

G-3 An elastic balloon is being filled with gas. It remains spherical, its radius, r ft., increasing at a rate

$$\frac{dr}{dm} = \frac{2m}{5\sqrt{1+m^2}} \text{ ft. per min.}$$

m being the number of minutes since r was zero. How long does it take to inflate it to a diameter of 20 ft? Ans. About 25 minutes.

G-4 $dy = \frac{x\,dx}{(1-x^2)^2}$, and when $x=0$, $y=0$. What is y in terms of x? Ans. $y = x^2 \div 2(1-x^2)$

G-5 Acceleration being constant and speed and dist. being zero when time is zero, prove distance varies as sq. of time.

G-6 At T o'clock Q is increasing at the rate of $(2T^2+T+1)$ units p.hr. Its value is 40 units at 3 o'clock. What is its value at quarter past four?

Ans. About 78 units.

G-7 A mine is deepened at the rate of $(20-y)^3 \times .015$ ft. per year, y being the number of years since the mine was opened. How deep is it at the end of 15 years?

Ans. 598. feet.

G-8 The cross-section of a river increases at the rate of about $\left(10+\frac{5x^4}{10^{12}}\right)$ sq. ft. p. mile at x miles from the source. What is the cross-section at the mouth of a river 1000 miles long?

Ans. 11,000. sq. ft.

G-9 The horizontal cross-sections of a hull are given by the formula:

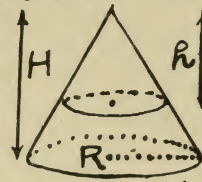
$$S=[(200-h)^3-169^3] \text{ sq. ft.}$$

h being the dist. below the water-line. Calculate total displacement if ship draws 30 ft.

Ans. About 40 million cu. ft.

G-10. Show that the cross-section of a right cone, height H cm., circular base, radius R cm., h cm. below the vertex, is

$$\pi h^2 R^2 \div H^2 \text{ sq. cm.}$$



Then by integration work out the volume of the cone.

G-11 At noon the tide's height was 6 ft. above mean sea level; m minutes later it was falling at the rate of $[\frac{1}{10} + 3(m+60)^3\, 10^{-7}]$ in. p. min. Find its height at 12·40 P.M. Ans. 5′ 1½″

G-12 The acceleration of a meteor is inversely as the sq. of its dist. from the center of the earth, being 32 ft. p. s. p. s. (or $\frac{1}{165}$ mi. per sec. per sec.) at the surface, 4000 mi. from the center. Its speed was ½ mi. p. s. 3000 mi. above the surface; with what speed does it strike?

Ans. About 4½ mi. per sec.

NOTE: To integrate $\frac{dv}{dt} = -k/x^2$ show that $v\,dt = dx$ and multiply corresponding sides of these two equations. Then integrate the $v\,dv = -kx^{-2}\,dx$

G-13 For a falling body, x ft. above ground, $\frac{d^2x}{dt^2} = -32.2$ f.p.s.p.s., if t is measured in seconds. If when $t=0$ the speed is zero and the height is H. ft. above ground, show by integrating and determining constants that at any instant denoted by t seconds

$$\begin{cases} \text{speed downward} = 32.2t \text{ ft. per sec.} \\ \text{height above ground} = (H - 16.1t^2) \text{ ft.} \end{cases}$$

G-14. During an explosion the gas in a cylinder is doing work by pushing on the piston at the rate of

$$[60{,}000. - 24 \times 10^6 (t - .05)^2] \text{ ft. lbs. per sec.}$$

at t sec. since the spark started the explosion. How much work is done against the piston in the first $\frac{1}{10}$ sec.?

Ans. 4000 ft. lbs.

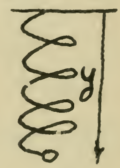
G-15 The slope of a curve equals the square of the ordinate, x, and the point $(2, 7)$ is on the curve: find its equation.

Ans. $3y = x^3 + 13.$

G-16 Brakes are gradually applied so that a train slows down at a rate proportional to the square of the time elapsed since application began. Work out formula for distance moved in first t seconds, and determine the proportionality factor if a train going 30 mi. p. hr. can be stopped in this way in 45 sec.

Ans. 3.55, units being mi.--sec.

G-17 For a ball hanging by a spring and vibrating $\frac{dv}{dt} = -k^2 y$, v being the speed in ft. p. sec., y being the dist. shown in feet, and t being a number of sec. Show that $v\,dt = dy$ and integrate as in G-12. Show that $v = k\sqrt{a^2 - y^2}$, a and k being constants.



G-18 The formula for the force needed to raise an hydraulic elevator x ft., is $F = (2.3 + .0025x)$ Tons.
Find the work done in raising the elevator 60 ft. About $142\frac{1}{2}$ ft.-tons.

G-19 Find f(x) if its derivative is $\sqrt{x-1}$ and $f(2) = 9$.

Ans. $f(x) \equiv \frac{2}{3}\left[\sqrt{(x-1)^3} + 12\frac{1}{2}\right]$

G-20 Assuming that the value of a mahogany tree over 40 yrs. old, say y years old, increases if left to grow at an annual rate of

$$\$[20(\sqrt[3]{y}-3)+3\sqrt{y}]$$

find the increase in value from the age of 100 years to the age of 200 years.

Ans. About $8,000.

G-21 A body falling down a hole from surface to center of earth, arrives with what speed, if its speed is v ft.p.s. on reaching x mi. below the surface where $v\frac{dv}{dx}=(32.-.008x)5280$?

Ans. 4.92 miles per sec.

G-22 The pay-roll of a prosperous concern increases daily, at the rate of $\$(10^4+10y^2)$ per year annually, y being the no. of yrs. after Jan 1, 1903. Find total amt. wages paid during the year 1907.

Ans. $10,204.00

G-23 When a bullet penetrates a target its speed is reduced at the rate $12\sqrt{1+x}$ ft. p. sec. p. inch. If it strikes with a speed of 2400 ft. p. sec. how far will it penetrate, x being the no. of inches penetrated at any instant?

Ans. About 44 inches.

G-24 If an iron ball falls in water with half the acceleration it has in air (the latter is 32.2 ft. p.s.p.s.), how long does it take to fall from a height of 1000 ft. above water to a depth of 500 ft. below its surface?

Ans. About $9\frac{3}{4}$ sec.

G-25 The radius of a sphere increases at a rate inversely proportional to its own length. Get a formula for the volume in terms of time. $V = K(t+A)^{3/2}$

G-26 In pulling a stake out of the ground, the resistance, R lbs., decreases as the stake gives, so that L, the no. of inches it has been raised, is related to R according to the formula:

$$R^2 (L+4)^3 = 10^6$$

calculate the work done in raising the stake the first five inches.

Ans. 27.8 ft. lbs.

TRANSCENDENTAL FUNCTIONS.

The functions which have been needed thus far in this book have all been formed from the argument by combinations of adding, subtracting, multiplying, dividing, raising to integral, fractional, or negative constant powers. A function whose formation calls for any other processes than a definite set of these is called **TRANSCENDENTAL**. The most familiar transcendental functions are the trigonometric functions, the circular (or inverse trigonometric) functions, the logarithms, and their inverses, the exponential functions (base with variable exponent)

The transcendental functions that are most conspicuous in elem. calculus are:

$$\log v$$
$$e^v$$
$$\sin v$$
$$\cos v$$
$$\tan v$$
$$\text{arc}\sin v$$
$$\text{arc}\tan v$$

The six steps described on page 18 will be followed in deducing the differentials of the functions:

$$\log v$$
$$\sin v$$

and from these results the differentials of the other five will flow.

Put $y \equiv \log v$

(steps 1 and 2:) $y + \Delta y = \log(v + \Delta v)$

(3) $\Delta y = \log(v + \Delta v) - \log v$

$= \log\left(1 + \frac{\Delta v}{v}\right)$

(4) $\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log\left(1 + \frac{\Delta v}{v}\right)$

$= \frac{1}{v} \frac{\Delta v}{\Delta x} \log\left(1 + \frac{\Delta v}{v}\right)^{v/\Delta v}$

(5)* $\frac{dy}{dx} = \frac{1}{v} \cdot \frac{dv}{dx} \cdot \log e$

(6) $dy \equiv d(\log v) = \frac{dv}{v} \log e$

---

* $\mathcal{L}_{\Delta v \doteq 0}\left(1 + \frac{\Delta v}{v}\right)^{v/\Delta v} \equiv \mathcal{L}_{h \doteq 0}(1 + h)^{1/h}$, which limit (see page 5) is the number 2.71828...., which is called e.

Put $z \equiv \sin v$

(1 and 2) $z + \Delta z = \sin(v + \Delta v)$

(3) $\Delta z = \sin(v + \Delta v) - \sin v$

$= 2\cos\frac{1}{2}(2v + \Delta v) \cdot \sin\frac{1}{2}\Delta v$

(4) $\frac{\Delta z}{\Delta x} = \cos\left(v + \frac{\Delta v}{2}\right) \cdot \frac{\Delta v}{\Delta x} \cdot \frac{\sin\frac{1}{2}\Delta v}{\frac{1}{2}\Delta v}$

(5)* $\frac{dz}{dx} = \cos v \cdot \frac{dv}{dx} \cdot \frac{2\pi}{n}$

(6) $dz \equiv$

$d(\sin v) = \cos v \cdot dv \cdot \frac{2\pi}{n}$

If $\log v$ is a COMMON LOGARITHM, (BASE 10), and the angle v is expressed in DEGREES, then the two MULTIPLIERS

---

* $\mathcal{L}_{\Delta v \doteq 0} \frac{\sin\frac{1}{2}\Delta v}{\frac{1}{2}\Delta v} = \mathcal{L}_{h \doteq 0} \frac{\sin h}{h}$ which limit (see page 5) is the number

$2\pi \div n$

the number n being such that n of the UNITS in which the ANGLE is measured = 1 REVOLUTION = 2π RADIANS = 360°. Therefore the MULTIPLIER, $2\pi \div n$, must be the RADIAN-EQUIVALENT of the ANGLE-UNIT used.

$\log e$ and $2\pi \div n$
have inconvenient numerical values:
.434294+ and .017453+

But if NAPERIAN (or NATURAL) logarithms (Base, $e = 2.718+$) and RADIANS are used, these multipliers take their simplest and most convenient values: namely, each is equal to the FACTOR ONE and so need not be written in.

For this reason e-LOGARITHMS and RADIAN-MEASURE are used wherever differentiation or integration is involved. Tables that will facilitate the use of these systems are given on pages 154-155. Throughout the remainder of this text it must be understood that logarithms are to the BASE e unless some other base is indicated, and that all angles are expressed in RADIANS unless some other unit is indicated.

$$\text{VI} \quad d \log_e u = \frac{du}{u}$$

$$\text{VII} \quad d \sin \theta \overset{rad.}{=} \cos\theta \, d\theta$$

From VI and VII are obtained:

$$\text{VIII} \quad d\,e^u = e^u\,du$$
$$\text{IX} \quad d\cos\theta^{rad.} = -\sin\theta\,d\theta$$
$$\text{X} \quad d\tan\theta^{rad.} = \sec^2\theta\,d\theta$$
$$\text{XI}^* \quad d\,\text{arc}\sin u = \frac{du}{\sqrt{1-u^2}}$$
$$\text{XII}^* \quad d\,\text{arc}\tan u = \frac{du}{1+u^2}$$

in the following manner:

VIII Put $y \equiv e^u$ Then $\log y = u \log e = u$

$$\frac{dy}{y} = u$$
$$dy = y\,du$$
$$d\,e^u = e^u\,du$$

IX
$$d\cos\theta = d(1-\sin^2\theta)^{\frac{1}{2}}$$
$$= \tfrac{1}{2}(1-\sin^2\theta)^{-\frac{1}{2}}(-2\sin\theta)\cos\theta\,d\theta$$
$$= -\sin\theta\,d\theta$$

---

* The functions ARC-SIN(u) and ARC-TAN(u) mean the NUMBER OF RADIANS in the angle whose sine or tangent is u. They do not mean the angles themselves and therefore may never be expressed as so many degrees.

X $d \tan\theta = d[\sin\theta \cdot (\cos\theta)^{-1}]$

$$= \cos\theta\, d\theta \cdot (\cos\theta)^{-1} + \sin\theta(-1)(\cos\theta)^{-2}(-\sin\theta\, d\theta)$$

$$= [1 + \tan^2\theta]\, d\theta$$

$$= \sec^2\theta\, d\theta$$

XI Put $\theta \equiv$ arc-sin u, Then $u = \sin\theta$

$$du = \cos\theta\, d\theta$$

$$d\theta = \frac{du}{\cos\theta} = \frac{du}{\sqrt{1 - \sin^2\theta}}$$

$$d \text{ arc-sin } u = \frac{du}{\sqrt{1 - u^2}}$$

XII Put $\varphi \equiv$ arc-tan u, Then $u = \tan\varphi$

$$du = \sec^2\varphi\, d\varphi$$

$$d\varphi = \frac{du}{\sec^2\varphi} = \frac{du}{1 + \tan^2\varphi}$$

$$d \text{ arc-tan } u = \frac{du}{1 + u^2}$$

Formulas for the differentials of a^v, $\sec\theta$, arc-cos u, etc., are often given but they are not often required and then are easily worked out by the methods used for VIII, IX, and XI.

DRILL ON TRANSCENDENTALS

Formulas VI to XII are converted into the integral form below. Observe that VI provides for the case to which exception is made under V.

$d u^n = n u^{n-1} du$ if $n \neq 0$ — V $\int u^n du = \frac{u^{n+1}}{n+1} + A$ if $n \neq -1$

$d \log u = \frac{du}{u}$ — VI $\int \frac{du}{u} = \log u + A$

$d \sin u = \cos u \, du$ — VII $\int \cos u \, du = \sin u + A$

$d e^u = e^u du$ — VIII $\int e^u du = e^u + A$

$d \cos u = -\sin u \, du$ — IX $\int \sin u \, du = -\cos u + A$

$d \tan u = \sec^2 u \, du$ — X $\int \sec^2 u \, du = \tan u + A$

$d \text{ arc-sin } u = \frac{du}{\sqrt{1-u^2}}$ — XI $\int \frac{du}{\sqrt{1-u^2}} = \text{arc-sin } u + A$

$d \text{ arc-tan } u = \frac{du}{1+u^2}$ — XII $\int \frac{du}{1+u^2} = \text{arc-tan } u + A$

The definition of logarithm is recalled:

If $N \equiv B^L$

Then $L \equiv \log_B N$

H-1 $d\, e^{2x}$

H-2 $d \cos 2s$

H-3 $d \log \sin^2 x$

H-4 $d \log(3v^2-1)$

H-5 $d \arctan \frac{x}{a}$

H-6 $\int \cos 3x\, dx$

H-7 $\int \frac{dy}{2y+1}$

H-8 $\int \frac{ds}{\sqrt{4-16s^2}}$

H-9 $\int \frac{x\, dx}{1-x^2}$

H-10 $\int \sin x \cdot \cos x\, dx$

H-11 $\int \frac{dv+dr}{r+v} + \int \frac{dr-dv}{(v-r)^2}$

H-12 $d(\sin^2\theta + \cos^2\theta)$ (two ways)

H-13 $d \log \sqrt[3]{\frac{x(1-x)}{1+x^2}}$ (split up first)

H-14 $\int \frac{x\, dx}{1+3x^2}$

H-15 $\int e^{-x}\, dx$

H-16 $d\, 10^x$

H-17 $\int \frac{\sin x}{\cos x}\, dx$

H-18 $\int d \sec\theta$

H-19 $d \log_{10} x$

H-20 $\int \frac{dx}{1+3x^2}$

H-21 $d \operatorname{arc\,sec} u$

H-22 $\int \frac{dx}{\sqrt{a^2-x^2}}$

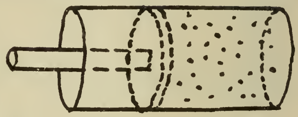
H-23 $d \cot\theta$

H-24 $\int \frac{3\, dx}{4+5x^2}$

H-25 $d \sin(x \text{ deg.})$

MISCELLANEOUS PROBLEMS

I-1 Air is being compressed in a closed cylinder. The outside air keeps a pressure of 15 lbs. per



sq. in. on one side of the piston and the compressed air resists with a pressure of 15 lbs. p. sq. in. x the quotient of the original volume divided by the instantaneous volume. What work must be done to reduce 16 cu. in. of air to $\frac{1}{8}$ its original volume? The piston area is 4 sq. in.

Ans. $72\frac{1}{4}$ inch lbs.

I-2 Weber showed that the number of heat-units necessary to raise one gram of diamond from 0° Cg. to t° Cg. is

$$.0947\,t + .000\,497\,t^2 - .000\,000\,12\,t^3$$

Find the specific heat (heat-units per degree rise in temperature) at the ordinary room temperature, 22° Cg.

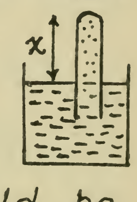
Ans. .1167

I-3 According to Newton's law of cooling the temp. of a hot bod falls at a rate proportional to its absolute temp. How long, then, does it take a body to cool from 5000° Abs. to 1000° Abs. if it begins to cool at the rate of 10° per sec.?

Ans. 13 min. $24\frac{1}{2}$ sec.

I-4 If the acceleration of a body varies directly as its speed, show that both speed and distance are exponential functions of the time.

I-5 A vapor under constant pressure is being used as a thermometer. If it expands at a rate per degree rise in temp. which is inversely proportional to its volume, and if its volume would be zero at ABS. ZERO, show that the temp. will be directly proportional to the square of the volume of the vapor.



I-6 The speed of a falling body is $44.27\sqrt{h}$ cm. p. sec., h cm. being the distance it has fallen. Show that the acceleration is constant, and is equal to 980 cm. p. sec. p. sec.

I-7 A signal can be transmitted by a submarine cable one cm. in diameter, insulated with an envelope of gutta percha t cm. thick, at a speed

$$\frac{1}{15}\, 10^7 \frac{1}{t^2} \log t$$

kilometers per sec. What is the maximum speed of signaling, and for what thickness of insulation?

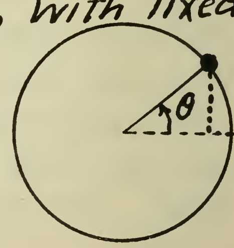
Ans. 123,000. Km. p. sec., 1.648 cm.

I-8 A guage records the height, y ft., of the tide in the form of a curve that from 11 A.M. to 2 P.M. satisfies the equation

$$y = x^2 - \sqrt{2}\, x + \tfrac{1}{2}$$

x being time after noon in hours, and y being measured up from a certain mark. Arrange a table to show the rate at which the tide rises or ebbs at the even half-hours from 11 A.M. to 2 P.M. inclusive.

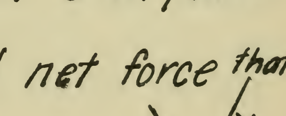
I-9 A wheel, radius R ft., with fixed center, turns at the rate of ω (omega) radians p. sec. Work out a formula for the height (above a level



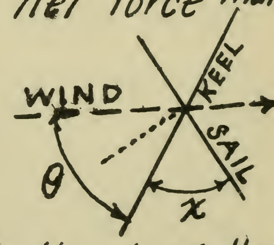
drawn through the hub) of a dot on the rim of the wheel – in terms of the angle θ, – and from it a formula for the speed of rising.

I–10 Work out a formula for the sideways motion and speed of a bug that lights on the rim of the wheel described in I–9, at a point just then level with the hub, and crawls along a spoke at the rate v ft.p. sec.

I–11 In this rail-spreader AH=AF=HB=HF =30 in. AB=50 in., and the screw shortenes HF $\frac{1}{2}$ in.p.s. At what rate are the rails spreading? Ans. $\frac{1}{3}$ in. p.sec.



I–12 To find the actual net force that sends the boat ahead, get the component of wind-pressure normal to the sail, then find the component of this along the direction of the keel. How had the sail best be set, and why?

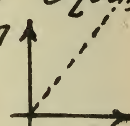


Ans. So that $x° = 90° + \frac{1}{2}\theta°$ because....

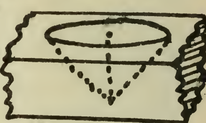
I–13 From the data given in I–8 find the time (to the nearest minute) at which the tide turned. Was it high or low tide?

Ans. Low at 12.42 P.M.

I–14 If $p\,dv$ and $v\,dp$ are equal, show that on a p-v-diagram the graph of the relation between p and v must be a straight line through the origin.



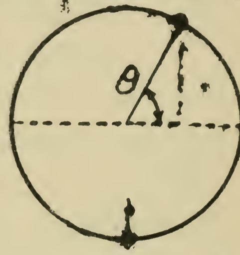
I–15 A counter-sink bores out a conical hole whose angle is 90°. If the area of the conical surface increases uniformly, show that the depth increases at a rate inversely as the depth, and the vol. at a rate directly as the depth.



I–16 The explosion of a meteor sends out a spherical wave, whose surface advances at a rate of 1500 ft. p. sec. in all directions. At what rate is the volume of air disturbed increasing after 10 sec?

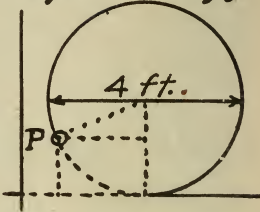
Ans. $4\frac{1}{4} \times 10^{12}$ cu. ft. per sec.

I-17 Force in dynes equals mass in grams × acceleration in cm. per sec per sec. Deduce a formula for centripetal force in the following manner. Get the formula for vertical height of a point moving around a circle, radius R cm, with a speed of v cm. per sec. From this find the vertical component of speed and the vertical acceleration. The acceleration toward the center may be found by taking the vertical acceleration at the instant the moving point is at the bottom of the wheel.



I-18 When a chip is placed in a current and released, it starts from rest and receives an acceleration proportional to the difference between its own speed and that of the current. If the speed of the current is 8 miles an hour, and after half a second the chip is going 6 miles an hour, show that the proportionality factor is 9979. for units in miles and hours.

I–19 A 4-ft. wheel is rolling along, 5 ft. per sec. Work out formulas for the x and the y of P, a point on the rim, which started at the origin. Find P's horizontal and vertical speeds and accelerations at the end of $1\frac{1}{2}$ seconds.

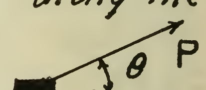


Ans.	speed	acceleration
Vertical	$4\frac{1}{10}$ f.p.s. right	$7\frac{1}{3}$ f.p.s.p.s. left.
Horizontal	$2\frac{1}{5}$ f.p.s. down	$10\frac{1}{4}$ f.p.s.p.s. down.

I–20 When gas blows out of a closed vessel into a vacuum, it blows at a rate proportional to the amount remaining. If this rate was 10 grams p.s. when 100 gms. remained, how long before only 50 gms. remain?

Ans. 6.93 sec.

I–21 When a box is pulled along the floor by a string the resistance is the product of the coefficient of friction (in this case, say $\frac{2}{3}$), by the difference between the weight of the box and the vertical component of the pull on the

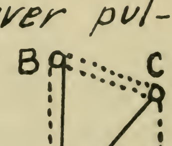


string. This is just balanced by the horiz. component of the pull. For what angle, θ, is this pull a minimum? Ans. About $33\frac{1}{3}°$.

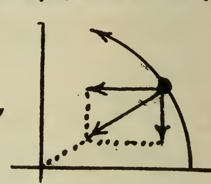
I-22 When a stone is whirled by a string that winds up on one's finger, the speed component normal to the string remains constant at 70 ft. per sec. The string, originally 8 ft. long, shortens $\frac{1}{10}$ ft. per revolution. How long does it take to wind up? Regard finger as a stationary point. Ans. 28.7 sec.

I-23 Find the polar coordinate equation of the path of the stone described in I-22.

I-24 The rope ABCB runs over pulleys at B and C and is made fast to the mast at B. If the rope is wound in at A, $3\frac{1}{2}$ ft. per sec., how fast is the end of the boom, C, rising when BC is level? Given AB = 30 ft., and AC = 35 ft. Ans. $1\frac{1}{4}$ in. per sec.

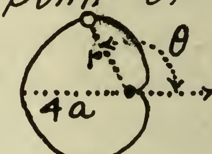


I-25 Deduce a formula for centripetal acceleration in this manner: Form expressions for the x and the y of a point revolving about the origin in a circle of radius, R cm., with a constant angular speed of ω (omega) radians per sec. Find the horizontal and vertical accelerations. Show that their resultant points toward the origin and that its magnitude is $R\omega^2$ c.p.s.p.s.



I-26 Locate the highest point of the CARDIOID, whose equation in polar coordinates is this:

$$r = 2a(1 - \cos\theta)$$

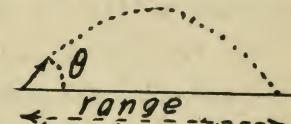


Ans. $r = 3a,\ \theta = 120°$

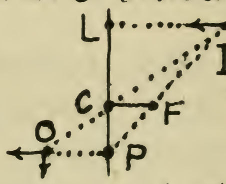
I-27 If W ft.lbs. is the work done in compressing a gas, and v cu. ft. is its volume when under p lbs. per sq. ft. pressure, then $dW = p \cdot dv$. But by BOYLE'S LAW $p \times v$ is constant. Find W in terms of v; also find W in terms of p.

I-28 *How near does the curve* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *come to the origin?* $.707a$

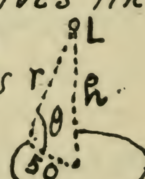
I-29 *The range of a projectile is* $16v^2 \sin 2\theta$ *miles if* v *is the speed at the muzzle in mi. per sec. How far can a gun shoot with a muzzle speed of 4000 ft.p.s.?* 12.65 *mi.*



I-30 (A *lens problem*) C, F, L *are fixed points.* CF = 5 *cm.* CL = 2 *cm.* I *is* 30 *cm. from* L *and approaching it at the rate of* 2 *cm.p.sec. At what speed is* O *moving and in what direction? Ans. To the left* $\frac{2}{25}$ *cm. per sec. and down* $\frac{4}{125}$ *cm. per sec.*



I-31 *How high a light,* L, *gives the best illumination for a path around a* 100 *ft. circular grass plot if the illumination is directly as* $\sin\theta \div r^2$ *at any point in the path?*



Ans. h = 35 *ft.* 4 *in.*

DEFINITE INTEGRALS

The integral, $\int dF(x) \equiv F(x) + A$, *is called an* **INDEFINITE** *integral on account of the unknown number A added as a constant of integration.*

A special and very convenient way of determining the value of the constant will now be developed. Given $\varphi(y)\,dy = f(x)\,dx$ *and* $y = b$ *when* $x = a$. *Required* $y = \beta = ?$ *if* $x = \alpha$

$$\text{Integrating} \quad \int \varphi(y)\,dy \equiv \Phi(y) + B$$
$$\int f(x)\,dx \equiv F(x) + A$$
$$\text{Substituting} \quad \Phi(b) + B = F(a) + A$$
$$\Phi(\beta) + B = F(\alpha) + A$$
$$\text{Subtracting} \quad \Phi(\beta) - \Phi(b) = F(\alpha) - F(a)$$

This result may be put in the form

$$\left[\int \varphi(y)\,dy\right]_{y=\beta} - \left[\int \varphi(y)\,dy\right]_{y=b}$$
$$= \left[\int f(x)\,dx\right]_{x=\alpha} - \left[\int f(x)\,dx\right]_{x=a}$$

or in this more compact form:

$$\int_b^\beta \varphi(y)\,dy = \int_a^\alpha f(x)\,dx$$

this NOTATION *being adopted*

$$\int_P^Q f(u)\,du \equiv \left[\int f(u)\,du\right]_{u=Q} - \left[\int f(u)\,du\right]_{u=P}$$

This new symbol:

$$\int_P^Q f(u)\,du$$

is called the DEFINITE *integral of* $f(u)\,du$ FROM P TO Q.

$f(u)\,du$ *is the* INTEGRAND
u " " VAR. OF INTEGRATION
P " " LOWER LIMIT
Q " " UPPER LIMIT

NOTE: *the* ORDER *in which the operations indicated by the* DEF. INT.

$$\int_P^Q f(u)\,du$$

must be performed is this:

1. *Integrate* $f(u)\,du$
2. *Substitute the upper limit*
3. *Substitute the lower limit*
4. *Subtract latter result from former*

The constant of integration, whatever its value, cancels out in evaluating a DEFINITE *integral.*

The result at the bottom of page 102 may now be stated thus:

Integrating both members of an equation by DEFINITE integrals, whose corresponding limits are CORRESPONDING values of the variables of integration, is equivalent to integrating and determining the constant of integration.

In the special case which occurs most frequently, $\varphi(y) \equiv 1$, that is, if we have:

Given $\frac{dy}{dx} = f(x)$ and $y = b$ when $x = a$
Required y in terms of x.

The corresponding UPPER limits are the GENERAL VALUES, y and x, which the required formula is to contain.

Hence $\int_b^y dy = \int_a^x f(x)\,dx$
or $y = b + \int_a^x f(x)\,dx$

NOTE that the x in the upper limit is a value to be substituted for the x in the integrand, which latter is the variable of integration. The latter does not appear in the result, and the equation might equally well be written

$$y = b + \int_a^x f(t)\, dt$$

The value of a Definite Integral depends only upon the LIMITS and the FORM of the INTEGRAND.

Example: The speed of a train is 20 mi.p.hr. at 2.10 P.M.; it is increasing at the rate $\frac{m}{25}$ mi.p.hr. at m min. past 2. Req.: General formula for speed

Solution: Use v for Speed in mi.p.hr.
m for minutes past 2 P.M.

$$\frac{dv}{dm} = \frac{m}{25} \quad \text{or} \quad dv = \frac{m}{25}\, dm$$

Integrating between corresponding limits

$$\int_{20}^{v} dv = \int_{10}^{m} \frac{m}{25}\, dm$$

$$v + A\Big]_{20}^{v} = \frac{m^2}{50} + A'\Big]_{10}^{m}$$

The HALF-BRACKETS with the LIMITS should be used to indicate that the last 3 steps are still to be performed

Finally: $v = (18 + \frac{m^2}{50})$ mi. per hr.

J-1 $\int_0^{\pi/2} \cos x\,dx = 1$

J-2 $\int_1^2 \frac{dv}{1+v} = .405$

J-3 $\int_0^1 \frac{dx}{\sqrt{x}} = 2.$

J-4 $\int_1^2 (x^2 + \frac{1}{x})^2 dx = 9.7$

J-5 $\int_{1/2}^1 \frac{dz}{\sqrt{1-z^2}} = 1.0472$

J-6 *Find y when $x=2$, if y increases $2x^2$ times as fast as x, and was one when x was one.* Ans. $5\frac{2}{3}$

J-7 $\int_{.72}^{2.38} \frac{dx}{4+x^2} = .52$

J-8 $dy = 2y\,dx$, $y=\frac{1}{2}$ when $x=2$, find y when $x=3$. Ans. 3.7

J-9 $\int_0^y y\,dy = \int_0^{\pi} \sin\theta\,d\theta$; solve. Ans. $y=\pm 2$

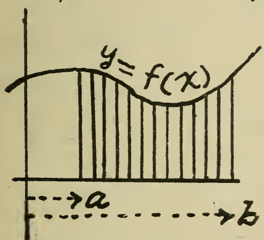
J-10 *Find the amount of gas burned in the whole town between 5.30 P.M. and 7.30 P.M. if the rate at which it is burned, R cu. ft. p. min., is, at t min. after 5 P.M., given by the formula* $(t-180)^2 = 60(530-R)$.

Ans. 45,000. cu. ft.

AREAS

A certain type of AREA is of great importance on account of its use in the representation of physical quantities.

Speaking of the x-y-plane of Analytical Geometry as having directions like a map, the type of area referred to is

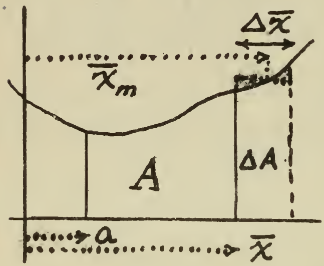


Bounded on the
NORTH by the curve, $y = f(x)$
SOUTH by the x-axis
WEST by a line, $x = a$
EAST by a line, $x = b$

To solve the problem of calculating this area, consider first the more general problem of an area like the above but bounded on the EAST by a VARIABLE vertical line whose x-coordinate will be called $\bar{x}$ to distinguish it from the x used as a running coordinate for the various points along the curve $y = f(x)$. Denote this VARIABLE AREA by A, and since it depends upon $\bar{x}$ we may put

$$A \equiv F(\bar{x})$$

Now determine the rate at which A increases as the eastern boundary is shifted along: that is find $dA/d\bar{x}$



There is a point, whose x-coordinate will be called $\bar{x}_m$, on the part of the curve over ΔA through which a level line can be drawn that will SQUARE OFF ΔA into a rectangle equivalent to ΔA in AREA. Its altitude is $f(\bar{x}_m)$, its width is $\Delta\bar{x}$.

Now follow out the remaining steps (see p. 18) for finding a differential:

(step 3) $\Delta A = f(\bar{x}_m)\cdot\Delta\bar{x}$

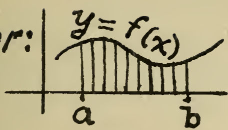
(4) $\dfrac{\Delta A}{\Delta\bar{x}} = f(\bar{x}_m)$

(5) $\dfrac{dA}{d\bar{x}} = f(\bar{x})$

(6) $dA = f(\bar{x})\,d\bar{x}$

We have $A=0$ when $\bar{x}=a$. Required $A=?$ when $\bar{x}=b$. Integrating between limits:

$$\int_0^A dA = \int_a^b f(\bar{x})\,d\bar{x}$$

or: 

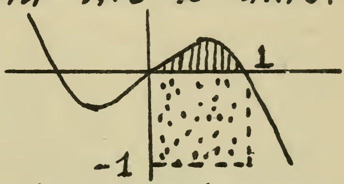
$$\left.\begin{array}{l}\text{Area under } y=f(x)\\ \text{from } x=a \text{ to } x=b\end{array}\right\} = \int_a^b f(x)\,dx$$

This formula is often written $\int_a^b y\,dx$, it being understood that the y is to be replaced by its value in terms of x from the equation of the upper boundary:

$$y = f(x)$$

Example: Find the area between the curve: $y = x(1-x^2)$ and the x-axis.

Solution.

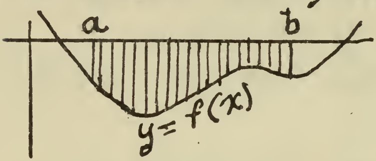


Plot the curve roughly and shade in the required area. On the diagram always indicate the vertical and horizontal scales used and draw a unit square (dotted) to show the AREA-UNIT in terms of which the result is to be expressed. The east and west boundaries are in this case mere points, $x=0$, and $x=1$. The shaded area is

$$\int_0^1 x(1-x^2)\,dx = -\frac{1}{4}(1-x^2)^2\Big]_0^1 = 0 + \frac{1}{4} = \frac{1}{4} \text{ dotted sq.}$$

If the UNIT SQUARE is of convenient size show the AREA-SCALE by indicating the area of a convenient rectangle.

If the curve runs below the x-axis the integral $\int_a^b y\,dx$ yields the negative of the actual area: for, on page 108, in the proof, the altitude will not be $f(\bar{x})$, which is negative, but $-f(\bar{x})$, which is positive.

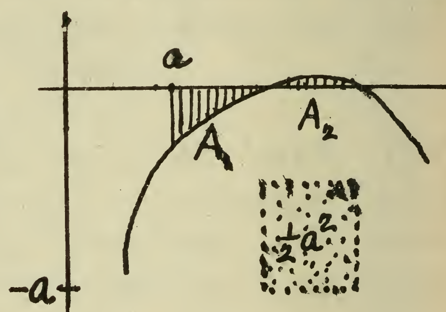


Example: Find the actual area bounded by the curve $10y(2x-a)=(x-2a)(3a-x)$, the x-axis, and the verticals at $x=a$ and at $x=3a$

Solution:

Plot

$$y=\frac{(x-2a)(3a-x)}{10(2x-a)}$$



$$A_1=-\int_a^{2a} y\,dx$$
$$A_2=+\int_{2a}^{3a} y\,dx$$

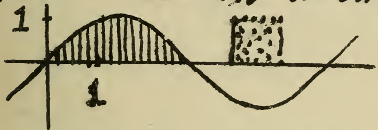
$$\int y\,dx=\int\frac{1}{20}\left[-x+\frac{9a}{2}-\frac{15}{2}\cdot\frac{a^2}{2x-a}\right]dx$$
$$=\frac{1}{80}\left[-2x^2+18ax-15a^2\log(2x-a)\right]$$

etc., etc., and finally

$$A_1+A_2=.056a^2+.004a^2=.060a^2$$

K-1 Find the area under the parabola, $y^2 = 2px$, from the origin to the vertical at $x = 18p$. Ans $72p^2$ sq. units

K-2 Find the area of one arch of the sinusoid:



$y = \sin x$

Ans. 2 sq. units

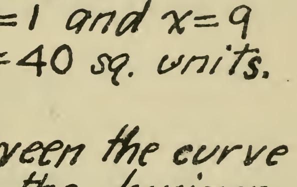
K-3 Find the actual area between the parabola, $y = x^2 - 8x + 12$, the x-axis, and verticals at $x = 1$ and $x = 9$

Ans. $2\frac{1}{3} + 10\frac{2}{3} + 27 = 40$ sq. units.

K-4 Find the area between the curve $y = 2x^3$, the y-axis, and the horizontals at $y = 2$ and $y = 4$.

Ans. 2.2797 sq units.

K-5 Find the area under the equilateral hyperbola, $xy = 1$, from the vertical at $x = 1$ to the vertical whose coordinate is x.



Ans. $[\log_e x]$ sq. units.

K-6 Find the actual area enclosed

by the curve, $y = x \div (x^2-2)$, the x-axis, and the line $x=1$.

Ans. .346 sq. units.

K-7 Find the area under the curve,

$$y = \frac{a}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$$

from $x=0$ to $x=a$. Ans. $\frac{e^2-1}{2e}a^2$ or $1.17a^2$

K-8 Find the area included between the parabolas whose equations are

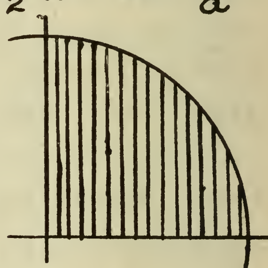
$y^2 = 8x$ and $x^2 = 8y$.

Ans. $21\frac{1}{3}$ sq. units.

K-9 By differentiating, verify this formula:

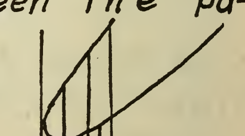
$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\text{arc sin}\frac{x}{a}$$

Then find the area of the quarter of the CIRCLE: $x^2+y^2=a^2$ which lies in the first quadrant.

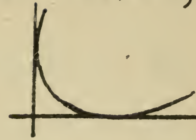


K-10 Find the area between the parabola, $x=(y-x-3)^2$, the x-axis, and the line $x=9$.

Answers. $85\frac{1}{2}$ or $49\frac{1}{2}$.



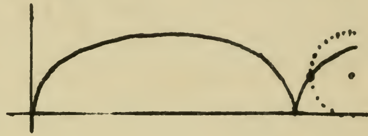
K-11 Find the area between the curve, $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and the two coordinate axes.



Ans. $\frac{1}{6}a^2$ sq. units.

K-12 Verify: $\int \cos^2\theta\, d\theta = \frac{1}{2}(\theta + \sin\theta \cdot \cos\theta)$

Then calculate the area of one arch of the cycloid whose parameter equations are:

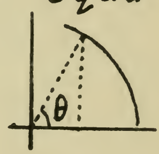


$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$

Ans. $3\pi a^2$

NOTE. The area of the GENERATING CIRCLE $= \pi a^2$

K-13 Get a pair of parameter equations for a circle, by expressing x and y in terms of θ. Using these, work out the area of the circle.



K-14 Find the area between the curve $xy = \log x^2$, the x-axis, and the lines $x=1$ and $x = e = 2.718...$ Ans. One UNIT.

K-15 Show that the entire area of the curve, $y^2 = x^2(a^2 - x^2)$ is $\frac{4}{3}a^3$ units.

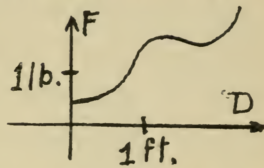
INTERPRETATION OF SLOPES AND AREAS.

The Problem Sets D, G, and I have suggested how various are the cases in which physical science is concerned with quantities whose derivatives or integrals have some important physical meaning.

The second method noted on page 3 is often the most practicable way of representing a function defined by physical phenomena. In representing a function graphically, the horizontal coordinate is used for the TIME or for the variable UNDER CONTROL—the INDEPENDENT variable; the dependent variable, or function is plotted as the vertical coordinate. On the axes of such a diagram it is necessary to note on each one not only the numerical scale but also the UNITS employed.

Such a diagram represents directly the values of the function: and in many cases the SLOPE of the graph and the AREA under it represent related physical quantities.

For example: on a FORCE-DISTANCE diagram, the graph shows directly how the force varies as distance is passed over. Now in any

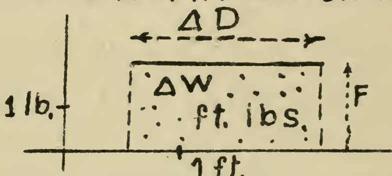


case where F remains constant as D varies, the force-distance work relation is

$F \times \Delta D = \Delta W$

If the force varies use the relation

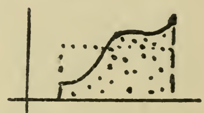
$[\text{Average}F] \cdot \Delta D \doteq \Delta W$



from which follows the difference quotient

$$\frac{\Delta W}{\Delta D} = \text{AVERAGE FORCE}$$

Then make the interval ΔD smaller so that $\Delta D \doteq 0$ and



we have AVG. FORCE $\doteq$ EXACT FORCE at the point toward which ΔD is shrinking.

$$F = \underset{\Delta D \doteq 0}{\mathcal{L}} [\text{AVG.} F] = \mathcal{L} \frac{\Delta W}{\Delta D} = \frac{dW}{dD}$$

We have then these two important exact relations in case of variable force

$$F = \frac{dW}{dD} \qquad W = \int F \, dD$$

Hence: on a WORK-DISTANCE DIAGRAM
FORCE is represented by SLOPE
and on a FORCE-DISTANCE DIAGRAM
WORK is represented by AREA.

In general a relation of the type

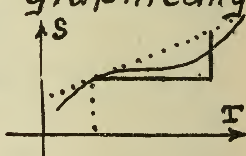
$$R = \Delta S \div \Delta T,$$

which is exact when R is constant, leads, by reasoning like the preceding, in case of variable R, to these two EXACT EQUATIONS:

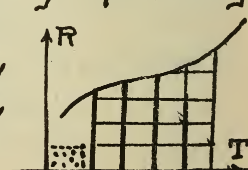
$$R = \frac{dS}{dT} \qquad S = \int R \, dT$$

Since many functions must be used for which no formula can be found, these relations are of great value in that they give GRAPHICAL means (the measurement of slopes and areas) of finding derivatives or integrals of any function whose graph can be put on paper.

SLOPE may be measured graphically by drawing a tangent and dividing the rise of a segment by its run.



AREA may be measured graphically by means of a planimeter, by counting "squares", by Simpson's Rule, and by various other rules.



K-16 The force necessary to compress a spring is $2(7-x)$ lbs., x inches being its length, variable during compression. Make a FORCE-DISTANCE DIAGRAM and find the WORK DONE graphically.

Ans. 14 inch lbs.

K-17 Draw, as accurately as you can free-hand, the HEIGHT-TIME DIAGRAM of a batted ball, knowing that its horizontal motion is uniform. From this diagram find the vertical component of its speed at a point half-way up. Indicate all data necessary and all units used carefully on the diagram.

K-18 Interpret SLOPE on a diagram where y represents WORK DONE and x represents time. Give the reasoning.

K-19 Interpret AREA on a diagram where x represents TIME and y represents RATE OF FLOW. Give reasoning.

K-20 On a PRESSURE-VOLUME DIAGRAM, AREA represents WORK. Boyle's Law is $P \times V = \text{Const.}$ Show that the WORK-VOLUME formula involves $\log_e V$.

SCALAR SLOPES AND AREAS

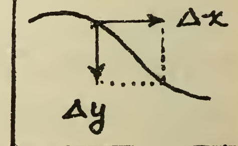
Many of the quantities used in Physical science are SCALARS, that is quantities which can be represented along a TWO-WAY SCALE like that of a thermometer; a certain value being designated as the ZERO, the others are designated as + or − according to the way in which they differ from the zero. TIME, TEMPERATURE, and WORK FURNISHED (the negative is work consumed), SLOPE, ALTITUDE,..., are SCALARS. Vectors, like distance, force, speed, acceleration,...., may be considered as scalars when only ONE DIRECTION and its OPPOSITE are to be considered.

Some quantities, density, specific gravity, absolute temp.,...., are pure numbers, being never associated with a direction or with a + or − sign.

SLOPE is SCALAR, being defined as

$$\mathcal{L}\frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx}$$

for Δy may be − when Δx is + and so make $\frac{dy}{dx}$ −.



AREA is SCALAR, being defined as

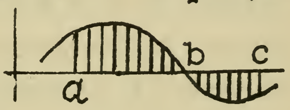
$$\int_a^b f(x)\,dx$$

since it is − if the graph is below the axis from a to b

SLOPE and Area are, then, so defined as to well represent physical scalars

A FORCE-DISTANCE DIAGRAM with a graph dipping below results from a case in which the working force is overcome and reversed by the resistances, or springs, so that work is being UNDONE from b to c. The total work that is DONE and STAYS DONE is

ACTUAL AREA, a to b − ACTUAL AREA, b to c



or $\int_a^b F.\,dx - \left[\text{negative of } \int_b^c F.\,dx\right]$

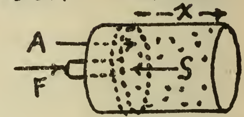
$$\equiv \int_a^b + \int_b^c \equiv \left[\int_{x=b} - \int_{x=a}\right] + \left[\int_{x=c} - \int_{x=b}\right] \equiv \left[\int_{x=c} - \int_{x=a}\right]$$

$$\equiv \int_a^c F.\,dx$$

Hence the DEFINITE integral from a to c takes proper account of signs and gives appropriate meaning to scalar area.

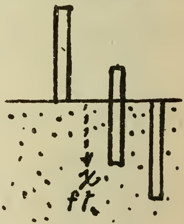
Account for all negative results in this set.

K-21 $S = F + A$ is an equation between forces measured in pounds. x is in ft. $Sx = 5000$. $A = 900$. Calculate the work done by F in reducing x from 1 ft. to 6 inches. Ans. -3015. ft. lbs.



K-22 A tide-mill pond contains 200,000 cu. ft. of water at noon. A gate is then opened admitting the sea at the rate of $[1000 \cos(\frac{2h}{7} + \frac{3}{4})$ rad.$]$ cu. ft. per hour, h being the number of hours after noon. How full is the pond at 9 P.M.? Ans. 197 000. cu. ft.

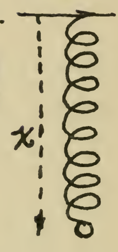
K-23 The force necessary to thrust a 40 ft. pile down into the water is $(\frac{1}{10}x - 3.2)$ tons. How much work is required for just submerging the entire pile. Ans -48 ft. tons.



K-24 Interpret AREA on a DENSITY-VOLUME DIAGRAM.

K-25 The upper end of a long spring is

worked by hand so as to give a ball attached to the lower end a vertical acceleration, which varies so as to be $(-32. + 60 \sin 3\pi t)$ in. p. sec. p. sec. upward at the end of t seconds. At the end of one second is the ball above or below its starting point and how far? Ans. 32 in. below.

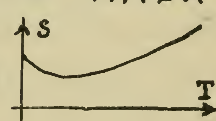


K-26 Interpret SLOPE on a SPACE-TIME DIAGRAM.

K-27 Interpret SLOPE and AREA on a SPEED-TIME DIAGRAM.

K-28 Interpret a negative AREA on an ACCELERATION-TIME DIAGRAM.

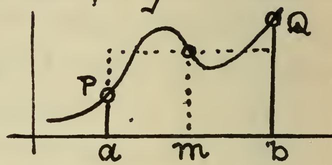
K-29 Interpret SLOPE and AREA on this diagram, where the horizontal represents a temperature and the corresponding vertical a specific heat



K-30 Find the value of $\int_0^1 \sqrt{a^2-x^2}\, dx$ by the help of a diagram.

THE LAW OF THE MEAN

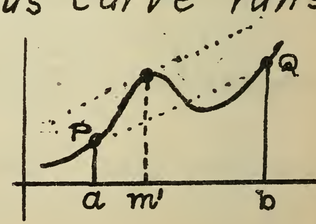
The statement is made on page 108 that there is a point on the curve PQ through which can be drawn a horizontal line by which the area PQba is SQUARED OFF into an EQUIVALENT rectangle.



The altitude of this point multiplied by the base, ab, gives the area of the figure PQab. This altitude is therefore called the average or MEAN altitude of the curve from P to Q The statement referred to above may now be put into this form:

> One of the values of an altitude as it varies continuously in an interval is the mean value of the altitude for that interval.

Again: if a continuous curve runs from P to Q without sharp corners, there is a point on the segment PQ at which a tangent can be



drawn that is parallel to the chord PQ.

The slope of this chord is the total net rise from P to Q divided by the run, and its slope is the average or MEAN slope of the curve from P to Q. The second statement may now be put into this form:

> One of the values of a slope as it varies continuously in an interval is the mean value of the slope for that interval.

These two statements are AXIOMATIC. They are summed up in this single, general AXIOM, called the LAW OF THE MEAN:

> One of the values of a FUNCTION as it varies continuously in an interval is the MEAN value of the function for that interval.

Regarding the diagrams as graphs of the function f(x), leads to these two forms of the law of the mean:

Integral form: $\int_a^b f(x)\,dx = f(m)\int_a^b dx$

Derivative form: $f(b)-f(a) = (b-a)f'(m)$

m between a and b

TAYLOR'S EXPANSION

Take any two numbers, and their sum:

$$S + P = G$$

any continuous function $f(x)$, and its derivatives, $f'(x)$, $f''(x)$; write down this equation in R

$$f(S) + P f'(S) + \frac{1}{\lfloor 2} P^2 f''(S) + \frac{1}{\lfloor 3} P^3 R - f(G) = 0$$

and consider it solved for R in terms of S, P, G.

Using this value of R, the functions $f(x), f'(x), f''(x)$, and the number G, form this new function:

$$\varphi(x) \equiv f(x) + (G-x) f'(x) + \frac{1}{\lfloor 2}(G-x)^2 f''(x) + \frac{1}{\lfloor 3}(G-x)^3 R - f(G).$$

The equation above shows that when x has the value $x = S$, this new function is equal to zero. It is zero when $x = G$ also. Differentiating the new function with respect to x:

$$\varphi'(x) = f'(x) + \begin{bmatrix} (G-x) f''(x) \\ - f'(x) \end{bmatrix} + \begin{bmatrix} \frac{1}{\lfloor 2}(G-x)^2 f'''(x) \\ -(G-x) f''(x) \end{bmatrix} - \frac{1}{\lfloor 2}(G-x)^2 R - 0.$$

$$= \frac{1}{\lfloor 2}(G-x)^2 \left[f'''(x) - R \right].$$

By the LAW OF THE MEAN, we may write

$$\varphi(G) - \varphi(S) = P\,\varphi'(m), \quad G \gtrless m \gtrless S$$

Substituting values just obtained for these:

$$0 - 0 = P \cdot \frac{1}{\lfloor 2}(G-m)^2 \left[f'''(m) - R \right]$$

Solve this for a new form of the value of R:

$$R = f'''(m) \qquad G \gtrless m \gtrless S$$

Substitute this in the original equation, transpose the last term, and obtain:

$$f(G) = f(S) + Pf'(S) + \frac{1}{\lfloor 2} P^2 f''(S) + \frac{1}{\lfloor 3} P^3 f'''(m).$$

which is the TAYLOR'S FORMULA FOR EXPANSION. G is any GIVEN number; S is a number to be SUBSTITUTED into $f(x)$ and its derivatives; $P (\equiv G-S)$ is the number whose POWERS appear in the expansion; the value of m is unknown except that it is between G and S. M is called the INTERMEDIATE ARGUMENT, and the term in which it appears is called the REMAINDER TERM.

This expansion is useful in cases where
I The REMAINDER is comparatively small.
II The REMAINDER can be closely computed.
Case I arises if G, S, and f( ) are such that
- P, ≡ G−S, is small
- the factor containing (m) is small
- the factorial divisor reduces the size of the remainder sufficiently.

Case II arises when the factor containing (m) has values near enough alike whether its argument is G or S or any number between G and S.

TAYLOR'S EXPANSION may be extended to any desired number of terms:

$$\varphi(G) = \varphi(S) + P\varphi'(S) + \frac{1}{\underline{|2}} P^2\varphi''(S) + \cdots$$
$$+ \frac{1}{\underline{|n}} P^n\varphi^N(S) + \frac{1}{\underline{|n+1}} P^{N+1}\varphi^{N+1}(m)$$

The proof follows the same course as when the Remainder is written after three terms.

The sign $\cong$ is used to indicate an

APPROXIMATE EQUALITY

In cases I and II, page 125, we may write:

$$\varphi(G) \cong \varphi(S) + P\varphi'(S) + \cdots + \frac{1}{\underline{|n}} P^n\varphi^N(S).$$

Example: Expand $\log_{10} 10\frac{1}{2}$ in powers of $\frac{1}{2}$

Solution:

We have $G \equiv 10\frac{1}{2}$, $P \equiv \frac{1}{2}$, $S \equiv G - P = 10$

Put $\varphi(x) \equiv \log_{10} x$. Then $\varphi(S) = 1$.

$\varphi'(x) = \frac{1}{x} \cdot \log_{10} e$, (see pages 85 & 87)

$= .434294\ldots x^{-1}$ Get $\varphi'(S)$

$\varphi''(x) = .434294\ldots(-x^{-2})$ Get $\varphi''(S)$

$\varphi'''(x) = .434294\ldots(+2x^{-3})$ Get $\varphi'''(S)$

$\varphi^{IV}(x) = .434294\ldots(-6x^{-4})$ $\varphi^{IV}(m) = ?$

Of all x's from 10 to $10\frac{1}{2}$, $x=10$ makes $\varphi^{IV}(x)$ greatest, $\cong 24 \times 10^{-5}$. Hence the remainder after the P^3 term does not exceed 10^{-9}. Then

$$\log_{10} 10\tfrac{1}{2} = 1 + .434294\ldots\left[\frac{1}{20} - \frac{1}{800} + \frac{1}{24\,000} + \text{REM.}\right]$$
$$\cong 1 + .434294\ldots[.05 - .00125 + .000\,04] = 1.02119$$

By an examination of the remainder the closeness of the approximation can be ascertained.

Example: Calculate $\sin 31^\circ$ from the value of $\sin 30^\circ$, expanding to the 2nd power term.
Solution: first note that the expansion process involves differentiation and during this part of the work use radians. (See page 87)
We have $S \equiv \pi/6 = .5236$, $P \equiv \frac{2\pi}{360} = .01745$;
Put $\varphi(x) \equiv \sin x$, so then $\varphi(S) = .5$

$$\varphi'(x) = \cos x, \qquad \varphi'(S) = .8660$$
$$\varphi''(x) = -\sin x, \qquad \varphi''(S) = -.5$$
$$\varphi'''(x) = -\cos x, \text{ and } \quad \varphi'''(m) \text{ is between}$$

$-\cos 30^\circ$ and $-\cos 31^\circ$, that is $-.8660$ and $-.8646$
Writing out the expansion with remainder:

$$\begin{aligned} \sin 31^\circ = \; & .5 \\ & + (.01745) \times .8660 \\ & + \tfrac{1}{2}(.01745)^2 \times (-.5) \\ & + \tfrac{1}{\underline{13}}(.01745)^3 \times \varphi'''(m) \\ = \; & .51\,503\,80 - \text{REMAINDER TERM.} \end{aligned}$$

and this remainder term is less than

$$\tfrac{1}{6}(.01745)^3 \times .8660 \quad \text{or} \quad .00\,000\,077$$

L-1 Expand $\cos(A+P)^{radians}$ to the term in P^5.

L-2 Calculate $\log_e(1.02)$ from $\log_e 1$, using expansion to 5th power. Estimate the remainder.

L-3 *Expand* $\tan x^{rad.}$ *from the term* $\tan 0$ *to the term in* x^5.

L-4 *Employ Taylor's Formula to expand* $(2+x)^3$ *in powers of* x.

L-5 *Expand* x^5-2x^2 *in powers of* $(x-1)$.

L-6 *Expand* e^{-x} *in powers of* x.

L-7 *Expand* $\log_{10}\cos x^\circ$, *using* $s\equiv 0^\circ$, *far enough to get two non-vanishing terms.*

L-8 *Show that the* BINOMIAL THEOREM *is a special case of Taylor's Expansion, developing* $(a+b)^n$ *in powers of* b.

L-9 *Ascertain, by means of a Taylor's expansion, whether* $(x-\sin x)$ *is more or less than* $(x^3\div 6)$ *when* x *is near zero.*

L-10 *Compare* $(e^x-e^{-x})^2$ *and* $2(1-\cos 2x)$ *when* $x=10^{-10}$ *by expanding in powers of* x.

L-11 *How accurately can* $\sqrt{e}$ $(e\doteq 2.718\ldots)$ *be found by means of an expansion in powers of* $\frac{1}{2}$ *from* e^0 *to the term in* $(\frac{1}{2})^3$?

L-12 Expand $\sec(\frac{1}{10}\text{ radian})$ starting with the term $\sec(0)$ and carrying the expansion to a Remainder term involving the third power of $\frac{1}{10}$.

L-13 How closely can $\sec(.1)$ be calculated from the expansion obtained in L-12?

L-14 Show that the expansion of $\frac{1}{1-x}$ in powers of x by Taylor's Formula is identical with that obtained by division.

L-15 Expand e^{kx} in powers of x.

L-16 Supply the proper expressions in the empty brackets of this expression:

$$F(x+\Delta x) = F(x) + \Delta x[\quad] + \tfrac{1}{2}[\quad] + [\quad].$$

L-17 Expand $e^{(2+3x)}$ in powers of x.

L-18 If $\bar{x} \equiv x+\Delta x$, how is $\hat{x}$ limited if

$$\varphi(\bar{x}) = \varphi(x) + \Delta x \cdot \varphi'(\hat{x}).$$

L-19 Expand x^3 in powers of $(x-k)$.

L-20 Expand $e^{(x^2)}$ to remainder in x^3.

TAYLOR'S FORMULA EXTENDED

Taylor's Formula for expansion can be extended to functions of several variables. In place of the notation G≡S+P *it will now be more convenient to use this notation:*

$$\bar{x} \equiv x + \Delta x, \quad \bar{y} \equiv y + \Delta y, \quad \ldots \quad etc.,$$

and the INTERMEDIATE ARGUMENT, *m*, *(see page 125) will be written:* $\hat{x}$, $\hat{y}$, *etc,*

$$x \gtrless \hat{x} \gtrless \bar{x}, \quad y \gtrless \hat{y} \gtrless \bar{y}, \quad \ldots \quad etc.$$

By the Law of the Mean:

$$f(\bar{x}) = f(x) + \Delta x \cdot \frac{d}{dx} f(\hat{x})$$

where the expression

$$\frac{d}{dx} f(\hat{x})$$

means: "differentiate f(x) with respect to x, then replace x by $\hat{x}$*," in short "*$f'(\hat{x})$*". The* $f'(\ )$ *notation does not serve if there are several arguments* (INDEP. VARIABLES) *as it does not indicate which one produces the increment (see page 18, steps 1 and 2) from which the derivative comes. Employ, therefore, this new symbol:*

$$\frac{\partial}{\partial x} \varphi(\ )$$

to mean: "differentiate $\varphi(\ )$*, regarding x as the independent and* ONLY *variable." It is called the* PARTIAL DERIVATIVE *with re-*

spect to x. The ∂ (a "round delta") is reserved for indicating partial derivatives.

$\varphi(\bar{x},\bar{y},\bar{z})$ may be regarded as a function of $\bar{x}$ alone for purposes of comparison with $\varphi(x,\bar{y},\bar{z})$, and so we have:

$$\varphi(\bar{x},\bar{y},\bar{z}) = \varphi(x,\bar{y},\bar{z}) + \Delta x \cdot \frac{\partial}{\partial x}\varphi(\hat{x},\bar{y},\bar{z})$$

By the same kind of reasoning we have:

$$\varphi(x,\bar{y},\bar{z}) = \varphi(x,y,\bar{z}) + \Delta y \cdot \frac{\partial}{\partial y}\varphi(x,\hat{y},\bar{z})$$

$$\varphi(x,y,\bar{z}) = \varphi(x,y,z) + \Delta z \cdot \frac{\partial}{\partial z}\varphi(x,y,\hat{z})$$

Combining these by substitution, we have

$$\varphi(\bar{x},\bar{y},\bar{z}) = \varphi(x,y,z) + \Delta x \frac{\partial}{\partial x}\varphi(\hat{x},\bar{y},\bar{z}) + \Delta y \frac{\partial}{\partial y}\varphi(x,\hat{y},\bar{z}) + \Delta z \frac{\partial}{\partial z}\varphi(x,y,\hat{z})$$

Writing φ for $\varphi(x,y,z)$ and $\Delta\varphi$ for $[\varphi(\bar{x},\bar{y},\bar{z}) - \varphi(x,y,z)]$ we have this approximation, if $\Delta x, \Delta y, \Delta z$ are small:

$$\Delta\varphi \cong \frac{\partial\varphi}{\partial x}\Delta x + \frac{\partial\varphi}{\partial y}\Delta y + \frac{\partial\varphi}{\partial z}\Delta z.$$

L-21 $\frac{\partial}{\partial x}(x^2 - 2xy) = ?$

L-22 $\frac{\partial}{\partial a}\sqrt{a^2+b^2} = ?$

L-23 $\frac{\partial}{\partial y}\sin\frac{\theta}{y} = ?$

L-24 $\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2}} = ?$

L-25 $\frac{\partial}{\partial x}\sin(x+y) = ?$

L-26 $\frac{\partial}{\partial x}\left[\frac{x+y}{x-y}\right] = ?$

Example: Obtain an approximation formula for the square of the cosine of an angle, in the neighborhood of 60°.
Solution:
Using radians at first *(see page 87)*
Put $F(x) \equiv \cos^2(x \text{ radians})$

$$F(\tfrac{\pi}{3}+\varepsilon) \cong F(\tfrac{\pi}{3}) + \varepsilon F'(\tfrac{\pi}{3})$$
$$\cong \cos^2 \tfrac{\pi}{3} - \varepsilon\, 2 \sin\tfrac{\pi}{3} \cos\tfrac{\pi}{3}$$
$$\cong .25 - .8660\,\varepsilon$$

Since $\delta^\circ = (\pi\delta \div 180)$ *radians, if* $\delta^\circ = \varepsilon^{rad}$ *we must put* $\varepsilon = \pi\delta \div 180$, *and we have*

$$\cos^2(60^\circ + \delta^\circ) \cong .25 - .8660(\pi\delta \div 180)$$
$$\cong .25 - .0155\,\delta$$

L-27 *Obtain an A.F. for the reciprocal of a number in the neighborhood of 25.*

L-28 *If* $l = \dfrac{T^2 g}{\pi^2}$ *get an A.F. for* l *when* T *is nearly 1, and* g *is nearly 980.*

L-29 *Obtain an A.F. for the area of a* $\triangle$ *in which* $a \cong 12$ *in.,* $b \cong 7$ *in., angle* $C \cong 90°$.

L-30 *Work out an A.F. for* $\log_{10} \sin x^\circ$ *when the angle is nearly a right angle.*

SMALL CORRECTIONS

In Physics we sometimes have given beside the formula for a quantity a certain set of STANDARD *conditions and values. Any other set of values is considered to be got by adding* CORRECTIONS *to the standard values. When the corrections are relatively small it is not convenient to calculate from the original formula, but rather from an approximation formula consisting of the standard value with an added term for each correction.*

For example: $V = \sqrt{.0141\,P(1 + \frac{t}{273}) \div c}$, *a sound-speed formula, the notation being:*

VARIABLE, UNITS, QUANTITY;	VALUE, STANDARD COND.
V meters p. sec., Speed of Sound,	V = 332.3
P dynes p. sq. cm., Atmosph. Press.,	P = 1,012,630.
t degrees Cg., Temperature,	t = 0
c gms. p. cu. cm., Density of Air,	c = .001293

The partial derivatives of V (or any function consisting of factors) are most easily calculated by splitting up $\log_e V$*, differentiating, and then multiplying by V.*

$$\log V = \tfrac{1}{2}\left[\log .0141 + \log P + \log\left(1 + \tfrac{t}{273}\right) - \log C\right]$$

$$\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{2P}; \qquad \frac{1}{V}\frac{\partial V}{\partial t} = \frac{1}{2(273+t)}; \qquad \frac{1}{V}\frac{\partial V}{\partial C} = -\frac{1}{2C}$$

The A.F. for a corrected value of V is (p.131)

$$V + \Delta V \cong V + \frac{\partial V}{\partial P}\cdot\Delta P + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial C}\Delta C$$

Calculate the coefficients, $\frac{\partial V}{\partial P}$, $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial C}$, using the standard values and substitute:

$$V + \Delta V \cong 332.3 + .000160\Delta P + .608\Delta t - 128000\Delta C.$$

L-31 Given $T = 2\pi\sqrt{\frac{l}{g}}$. For a seconds pendulum under standard conditions $T = 2$ sec., $l = 99.29$ cm., $g = 980$ cm. per sec. per sec. What corrections must be added to $T = 2$ in case of small corrections Δl and Δg?

L-32 Given $N = \frac{Pl}{fe}$. Get a formula for $N + \Delta N$, the standard set of values being $N = .0002$, $P = l = f = 1$, and $e = 5000$.

L-33 Given $Q = \frac{3xy^2}{z^3}$ and $Q = 1/4$ when $x = 2$, $y = 3$, and $z = 6$. What corrections must be added to $Q = 1/4$ to take account of corrections +.01 to x, +.02 to y, and +.04 to z?

Ans. $+\frac{1}{8}\cdot .01 + \frac{1}{6}\cdot .02 - \frac{1}{8}\cdot .04$ or $-.00042$

ABSOLUTE ERRORS

Calculations based upon measurements are liable to errors arising from the impossibility of measuring with absolute accuracy. The effect of a small error in a measurement upon the calculated result may be judged from an approximation formula.

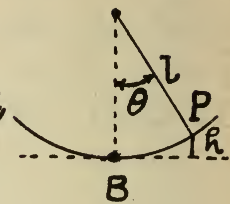
If $\varphi(\bar{x},\bar{y},\bar{z})$ is the quantity we wish to calculate, but the measurements of $\bar{x},\bar{y},\bar{z}$, give results too small by the unknown amounts $\Delta x, \Delta y, \Delta z$, what we really can calculate is not $\varphi(\bar{x},\bar{y},\bar{z})$ but $\varphi(x,y,z)$ which is too small by an amount

$$\Delta\varphi \cong \frac{\partial\varphi}{\partial x}\Delta x + \frac{\partial\varphi}{\partial y}\Delta y + \frac{\partial\varphi}{\partial z}\Delta z$$

The magnitude of the ERRORS, $\Delta x, \Delta y, \ldots$ etc., depends upon the LEAST COUNT of the measuring instruments. Thus if $x \cong 80$ cm, measured with a meter stick, Δx can hardly be more than .1 cm; the divisions on the protractor used for θ suggest the estimate for $\Delta\theta$; ...etc. The above formula assigns the absolute error, $\Delta\varphi$, to its separate sources, and enables us to compare the effects of the several data-errors.

Example: The height of P above B is calculated from the formula

$$h = l(1 - \cos\theta)$$



l is certainly correct to 1 m.m, and θ to 1°. How about h if $l = 107.21$ cm., $\theta = 24\frac{1}{3}°$?

Solution:

$$\Delta h \cong \Delta l(1 - \cos\theta) + \Delta\theta \cdot l \sin\theta$$

$$\cong \Delta l \times .0888 + \Delta\theta \times 44.2$$

We have $\Delta l < .1$; $\Delta\theta < .0174$ (radians = 1°)

$$\therefore \Delta h < .008 + .77 \quad \text{or} \quad .778 \text{ cm.}$$

Observe that a much rougher measurement of l might have been made, since Δl contributes comparatively little to the absolute error of the result, Δh.

L-34 Compare the effects upon $E = \frac{wv^2}{2g}$ of errors in the data if $w \cong 2$ poundals, $v \cong 10$ ft. per sec., and $g = 32.$ ft. per sec. per sec., and $\Delta w < .1$, $\Delta v < 2.$, $\Delta g < .2$

L-35 How seriously does a small error in measuring the diameter of a sphere affect its calculated volume?

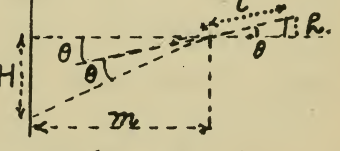
L-36 Discuss the comparative effect of errors in $x \cong 1$, $y \cong 1\frac{1}{2}$, $z \cong 3$, upon $xy^2 \div z^3$.

L-37 If g is calculated from $T = 2\pi\sqrt{l/g}$, and we take $\pi = \frac{22}{7}$, $l = 100$ cm., and $T = 2$, the errors being $\Delta\pi < .001$, $\Delta l < .1$, $\Delta T < .01$, what is the extreme possible error in g? Ans. $11\frac{1}{2}$

L-38 If we wish to determine $h = \frac{2T\cos\theta}{rmg}$ in a case where $T \cong .01$ cm., $\theta \cong 60°$, $r \cong .01$ cm., $m \cong 1.47$, and $g \cong 980$ cm. p. sec. p. sec., which of these quantities should be measured most accurately?

L-39 Given $E = \frac{1}{2}mv^2$. Find E by the correction method when $m = 1.001$ grams, and $v = 2.001$ cm. per sec.

L-40 Calculate h if $m = 3$ meters (within $\frac{1}{2}$ cm.), $l = 10$ cm. (within .01 cm.), and $H = 1.52$ cm. (within .02 cm.); and compare the effects of the errors in H, m, l, on h. (Optical Lever.)



L-41 Show that for two numbers nearly equal, P and Q, the geometric mean, $\sqrt{PQ}$, is nearly equal to the arithmetic mean, $\frac{1}{2}[P+Q]$. (Double weighing).

RELATIVE ERRORS

If $\Delta\varphi$ is the absolute error in φ, $\frac{\Delta\varphi}{\varphi}$ is the RELATIVE (or FRACTIONAL) error, and $100\,\frac{\Delta\varphi}{\varphi}\%$ is the PERCENTAGE of error. The formula connecting the absolute errors:

$$\Delta\varphi = \frac{\partial\varphi}{\partial x}\,\Delta x + \frac{\partial\varphi}{\partial y}\Delta y + \frac{\partial\varphi}{\partial z}\Delta z$$

is readily converted into a formula connecting the relative errors:

$$\frac{\Delta\varphi}{\varphi} = \frac{x}{\varphi}\,\frac{\partial\varphi}{\partial x}\,\frac{\Delta x}{x} + \frac{y}{\varphi}\,\frac{\partial\varphi}{\partial y}\,\frac{\Delta y}{y} + \frac{z}{\varphi}\,\frac{\partial\varphi}{\partial z}\,\frac{\Delta z}{z}$$

The comparative seriousness of the relative errors is judged from the magnitude of their coefficients, $\frac{x}{\varphi}\frac{\partial\varphi}{\partial x}$, etc. Observe that $\frac{x}{\varphi}\frac{\partial\varphi}{\partial x}$ is easily found from $x\frac{\partial}{\partial x}\log_e\varphi$

If the function, φ, is a product of powers the comparison of the effects of the relative errors is very easily made. For if $\varphi \equiv x^a \cdot y^b \cdot z^c \ldots$

$$\log\varphi = a\log x + b\log y + c\log z + \cdots$$

$$\frac{\partial}{\partial x}\log\varphi = \frac{a}{x}$$

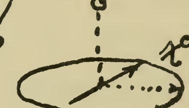
$$x\frac{\partial}{\partial x}\log\varphi = a\text{, the exponent of } x.$$

The relative error of each factor is, then, multiplied by the exponent of that factor. The high-power quantities should therefore be measured with greatest relative accuracy.

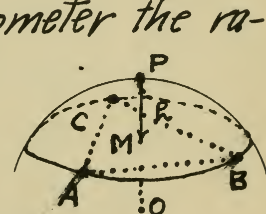
L-42 In calculating $s = \frac{1}{2}gt^2$, when $t = 5$ sec, and $g = 32.2$ ft.p.s.p.s., will an error of .1 sec. in t, or a 1% error in g be more serious in its effect upon s?

L-43 In the slide-wire bridge (see E16, page 61), $x : R :: a : c-a$, R and c being constants. Find the percentage of error in x due to a small error in a.

L-44 In a tangent galvanometer the current measured is proportional to the tangent of the angle $x°$. Find the percentage of error in the current due to a small error in reading x.



L-45 By means of a spherometer the radius of a sphere is found from these data: three points on the surface, A, B, C, form an equilateral Δ, side ≅ 5 cm.; PM, perpendicular to its plane at its center, ≅ 4 cm. Get an exact formula for the radius, OP, and an approximation formula for the errors, and judge which of the data should be most accurate.



INTEGRATION AS SUMMATION

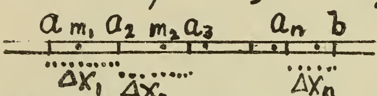
By the DEFINITION *given on page 103:*

$$\int_a^b f(x)\,dx \equiv F(b) - F(a) \quad \textit{where } F(x) \equiv \int f(x)\,dx$$

It follows that:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

If the RANGE *of values, a to b, be divided up into n parts, of any sizes, by a set of numbers:* $a, a_2, a_3, \ldots a_n, b,$ *we must have:*



$$\int_a^b f(x)\,dx = \int_a^{a_2} f(x)\,dx + \int_{a_2}^{a_3} f(x)\,dx + \cdots + \int_{a_n}^b f(x)\,dx$$

Using the Integral Form of the Law of the mean (see page 123) this becomes:

$$\int_a^b f(x)\,dx = f(m_1)\int_a^{a_2} dx + f(m_2)\int_{a_2}^{a_3} dx + \cdots + f(m_n)\int_{a_n}^b dx$$

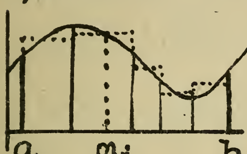
$$= f(m_1)\,\Delta x_1 + f(m_2)\,\Delta x_2 + \cdots + f(m_n)\,\Delta x_n$$

or, using the compact SUMMATION NOTATION:

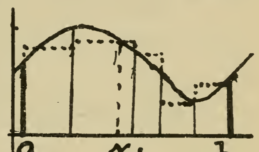
$$\int_a^b f(x)\,dx = \sum_1^n f(m_j)\,\Delta x_j$$

Consider now the sum $\sum_1^n f(x_j)\,\Delta x_j$, *which differs from the other only in that* x_j, *instead of being the coordinate of a* SQUARING OFF *point as* m_j *is, (see p.*

108) is any value whatever in the interval Δx_j. If the function is represented graphically each term in either sum can be represented by the area of a rectangle standing on the corresponding Δx, whose upper end touches the graph



of $f(x)$. The sums of these sets of rectangles may be quite different, but



if the number of parts into which the range, b...a, is divided is increased without limit in such a way that all the Δx-intervals approach zero, then will the limit of either sum be the AREA UNDER THE CURVE. As was shown on page 109, such an area is expressible as a definite integral. Hence the LIMIT of such a SUM as the second may be calculated by means of a definite integral. Symbolizing:

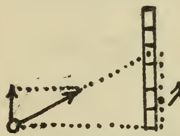
$$\underset{\Delta x \doteq 0}{\text{Limit}} \sum_a^b f(x)\,\Delta x = \int_a^b f(x)\,dx$$

The use of the $\int$-sign, (a long-s, initial of the word SUM), for the sign of integration originated in the property just proved to belong to the definite integral.

SUMMATION PROBLEMS

Example: Find the vertical component of attraction on a 3 gm. shot of a vertical 10 cm., 150 gm. wire, whose lower end is 20 cm. from the shot and level with it. Given that gravitational attractive force in dynes is $65 \times 10^{-9} \times$ product of the masses in grams ÷ sq. of their distance in cm.

Solution: Consider the wire divided into short lengths; let x cm. ≡ the height of any point in the wire. The mass of a chunk is $\frac{\Delta x}{10}$ 150 grams; its attractive force on the shot is



$$65 \times 10^{-9} \times \frac{\Delta x}{10} 150 \times 3 \div \left[\sqrt{400 + x^2}\right]^2$$

dynes, the x being the coordinate of that point the interval Δx toward which the attraction exerted by the chunk as a whole is directed.

Multiply this by the resolving cosine

$$x \div \sqrt{400 + x^2}$$

to get the amount contributed by this chunk toward the total VERTICAL force exerted upon the shot by the wire.

Then make a summation for all the chunks; the total vertical component is

$$\lim_{\Delta x \to 0} \sum_{x=0}^{x=10} \left[65 \times 10^{-9} \times \frac{150}{10} \times 3 \times \frac{x \, \Delta x}{(400 + x^2)^{3/2}} \right]$$

Compute this by means of the definite Integral:

$$\int_0^{10} 2925 \times 10^{-9} \frac{x \, dx}{(400 + x^2)^{3/2}}$$

$$= 2925 \times 10^{-9} \left[\frac{-1}{(400 + x^2)^{1/2}} \right]_0^{10}$$

$$= 2925 \times 10^{-9} \left[-.02\sqrt{5} + .05 \right]$$

$$= 155 \times 10^{-10} \text{ dynes.}$$

M-1 *Verify this formula:*

$$\int (a^2 + x^2)^{-3/2} dx = a^{-2} x (a^2 + x^2)^{-1/2}$$

Then calculate the HORIZONTAL *component of attraction in the example above.*

Ans. 654×10^{-10} *dynes.*

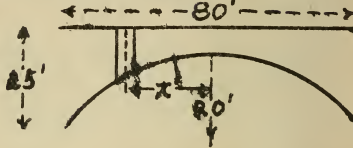
M-2 *If the density of a solution is* $[x^2 \div (1 + x)]$ *gms. per cu. cm. at a depth of* x *cm. below the surface, find by a summation process the mass in a jar of cross-section 15 sq. cm., 10 cm. deep.*

Ans. 635. *grams.*

M-3 *Solve* M-1 *if the wire's density varies directly with the height, first calculating the proportionality factor.*

Ans. 3, *gms. and cms.;* 618×10^{-10} *dynes.*

M-4 The weight, in tons per running foot, of a masonry arch is $(5 + \frac{x^2}{80})$ tons x ft. from the keystone. Find the total weight of masonry. Dimensions on figure.



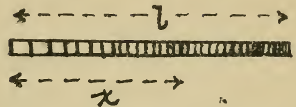
Ans. $9\frac{1}{3}$ hundred Tons.

M-5 The work in ft. lbs. done in raising the bricks for a wall equals the sum of the products of the weight of each brick multiplied by the distance it is raised, units being lbs. and ft. If a cubic foot of brick weighs 200 lbs and each brick has to be raised from the ground, find the total work done in raising brick to carry a wall 100 ft. long and 4 ft. thick to a height of 50 ft.

Ans. 10^8 ft. lbs.

M-6 Verify: $\int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + A$. Then find how much energy (= work done) is transferred in one second, beginning at $t=0$, by an alternating current, for which $I = 5 \cdot \sin(20t^{rad})$ amperes, and $E = 110 \cdot \sin(20t^{rad})$ volts, t being in seconds. Given that I amperes at E volts do $E \cdot I \cdot \Delta T$ joules of work in Δt seconds. Ans. 270 joules.

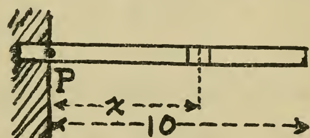
M-7 The density of a rod varies from point to point, so that at a dist. x cm. from one end it is



$[x^3 + (l-x)^2] \div 100$

grams p.cu.cm. Calculate its mass, its length, l, being 8cm, its cross-section being one sq.cm. Ans. $8\frac{2}{3}$ grams.

M-8 The weight of a beam being $3\frac{1}{2}$ lbs. per running inch, what is the moment about the point P of the weight of the projecting 10 feet of such a beam, one end of which is built firmly into a wall.



Ans. 2100 lbs. ft.

M-9 Kinetic Energy in gram-centimeters is equal to (Mass in grams)x(Speed in cm. per sec.)^2 Calculate the kinetic energy of a rod, 1200 gms. mass, 10cm. long, when it is whirling about one end at the rate of 3 revolutions per sec. 45×10^5 gm. cm.

M-10 Interpret $\int \left(\frac{dx}{dt}\right)^2 dm$, physically, if x measures distance, t measures time, and m measures mass. Assign suitable units.

SUMMATION ELEMENTS

In the summation-limit

$$\underset{\Delta x \doteq 0}{\mathcal{L}} \sum_{x=a}^{x=b} F(x)\cdot\Delta x$$

one of the single terms in the sum, $F(x)\cdot\Delta x$, is called a SUMMATION ELEMENT.

When the desired summation element can not be found exactly, it is possible, without any loss of accuracy whatever, to use an approximation for it that differs from it by higher powers of the increment involved, Δx. For, if $f(x)\cdot\Delta x$ is such an approximation:

$$F(x)\cdot\Delta x \equiv f(x)\cdot\Delta x + \varphi(x)\cdot(\Delta x)^2$$

and the summation limit above equals

$$\underset{\Delta x \doteq 0}{\mathcal{L}} \sum f(x)\cdot\Delta x + \underset{\Delta x \doteq 0}{\mathcal{L}} \sum \Delta x\cdot\varphi(x)\cdot\Delta x$$

The argument on page 141 shows that such a limit is unaltered by supposing all of the Δx's to be equal. In that case the second summation limit can be transformed thus:

(factoring) $$\mathcal{L}\sum \Delta x\cdot\varphi(x)\cdot\Delta x = \mathcal{L}\left[\Delta x\cdot\sum \varphi(x)\cdot\Delta x\right]$$

(by page 16) $$= \left[\mathcal{L}\,\Delta x\right] \times \left[\mathcal{L}\sum \varphi(x)\cdot\Delta x\right]$$

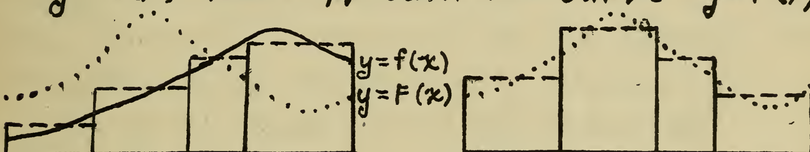
$$= 0 \times \int_a^b \varphi(x)\,dx = \text{ZERO}.$$

This equation is, therefore, EXACTLY true;

$$\underset{\Delta x \doteq 0}{\mathcal{L}} \sum_{x=a}^{x=b} F(x)\cdot\Delta x = \underset{\Delta x \doteq 0}{\mathcal{L}} \sum_{x=a}^{x=b} f(x)\cdot\Delta x$$

In neither of these summation limits does the equality of the Δx's make any difference, so that supposition can now be dropped.

The relation between these two sums is a simple one to show graphically. The functions $F(x)$ and $f(x)$ differ only by terms involving Δx's, and so when Δx's $\doteq 0$, the curve $y = f(x)$ must approach the curve $y = F(x)$



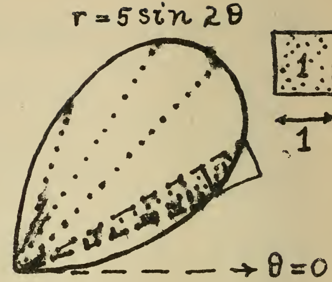
As each sum represents the area of a set of rectangles whose upper ends touch the corresponding curve, the summation-LIMITS in each case represent the area under the same curve, the limit-curve, $y = F(x)$.

Example: Find the area of the curve whose equation in polar coordinates is

$$r = [\cdot 5 \sin (2\theta \text{ radians}) \text{ inches}$$

Solution:

Plot the curve to a convenient scale:



Take as elements fan-shaped pieces. Call the angle of each piece $\Delta\theta$. Compare any piece with the inscribed and the circumscribed sectors. Its area is intermediate between these sectors' areas:

$$\tfrac{1}{2} r \cdot r \Delta\theta \qquad \text{and} \qquad \tfrac{1}{2}(r+\Delta r)\cdot(r+\Delta r)\Delta\theta$$

That these differ only by higher powers of $\Delta\theta$ appears on changing Δr, by the Law of the Mean, to a term in $\Delta\theta$, thus: (page 130)

$$\Delta r \equiv f(\theta+\Delta\theta) - f(\theta) = \Delta\theta \cdot \frac{d}{d\theta} f(\hat{\theta})$$

The required area is, therefore, exactly:

$$\underset{\Delta x \doteq 0}{\mathcal{L}} \sum_{\theta=0}^{\theta=\frac{\pi}{2}} \tfrac{1}{2} r^2 \Delta\theta$$

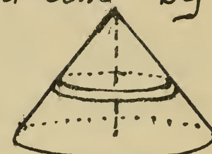
$$= \int_0^{\pi/2} \frac{25}{2} \sin^2 2\theta \, d\theta = \frac{25}{2} \int_0^{\pi/2} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{25}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{25}{4}\,\frac{\pi}{2} = 9.81 \text{ sq. in.}$$

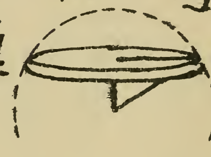
M-11 Work out the area of the circle $r = (a \sin\theta)$ cm., by a summation limit.

M-12 Find the area bounded by the lines: $r = \sec\theta$, $\theta = 0$, $\theta = \frac{\pi}{4}$. Ans. $\frac{1}{2}$ unit.

M-13 Work out the volume of a cone by means of a summation limit, using as elements slices made parallel to the base.

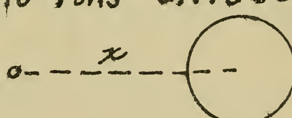


M-14 Work out the volume of a sphere by means of a summation limit, using as elements a set of parallel slices.



M-15 Work out the mass of a stick if its density varies as the square root of the distance from one end of the element considered, the density at the heavy end being $\frac{1}{30}$ lb. per cu. in., and the stick being 1 inch square and a yard long. Ans. $\frac{4}{5}$ lbs.

M-16 The force with which the sun pulls on a 1 lb. mass equals $1\frac{1}{2}\times 10^{9}$ tons divided by the square of its distance in miles from the sun's center. Find the work done by solar attraction in pulling in a 1 pound meteorite from the orbit of NEPTUNE, radius 28×10^{8} miles, to the surface of the sun, radius 43×10^{4} miles.



Ans. 3500 mile-tons.

INTEGRATION BY PARTS

Formula IV, page 70, leads to a fundamental method of transforming integrals.

IV $\int v\,du = v\int du - \int u\,dv$

A variable factor (represented above by v) may be moved outside the $\int$-sign, provided a certain integral is subtracted as an offset. This process is called INTEGRATION BY PARTS, the variable removed from under the $\int$-sign, v, and the integral of the remaining factor, u, being referred to as the PARTS.

This method is useful when the new integral, $\int u\,dv$, is {1. simpler than / 2. similar to} the given integral, $\int v\,du$.
The formula is often written: $\int v\,du = vu - \int u\,dv$.

1. Example of simpler new integral: $\int \log x\,dx$.

$$\int \log x\,dx = \log x \int dx - \int x\,d\log x$$
$$= x\log x - \int x\,\frac{dx}{x}$$
$$= x\log x - x + A$$

2. Example of similar new integral: $\int e^x \sin x\,dx$.

$$\int e^x \sin x\,dx = e^x \int \sin x\,dx - \int -\cos x\, e^x\,dx$$
$$= -e^x \cos x + \int e^x \cos x\,dx$$

Apply the same process to the new integral.

$$\int e^x \cos x\,dx = e^x \int \cos x\,dx - \int \sin x\, e^x\,dx$$
$$= e^x \sin x - \int e^x \sin x\,dx$$

This new integral is the original one. Substitute into the first equation, transpose the $\int e^x \sin x\,dx$, and divide by 2, obtaining:

$$\int e^x \sin x\,dx = \tfrac{1}{2} e^x (\sin x - \cos x)$$

N-1 $\int \text{arc}\sin s\,ds$

N-2 $\int x \cos x\,dx$

N-3 $\int \text{arc}\tan x\,dx$

N-4 $\int x^2 \sin x\,dx$

N-5 $\int e^{-x} \sin 2x\,dx$

N-6 $\int x e^x\,dx$

N-7 $\int x^2 \log x\,dx$

N-8 $\int x^3 e^x\,dx$

N-9 $\int \sqrt{a^2 - x^2}\,dx$. (Used in K-9). Use this transformation: $\int (a^2 - x^2)^{-1/2} x^2\,dx \equiv - \int (a^2 - x^2)^{+1/2}\,dx - a^2 \int (a^2 - x^2)^{-1/2}\,dx$

N-10 $\int (a^2 + x^2)^{-3/2}\,dx$ (Used in M-1) Begin with $\int (a^2 + x^2)^{-1/2}\,dx$, taking dx as du; transform the $(a^2 + x^2)^{-3/2} x^2$ into $(a^2 + x^2)^{-1/2} - a^2 (a^2 + x^2)^{-3/2}$; cancel the original integral and solve for $\int (a^2 + x^2)^{-3/2}\,dx$.

FINDING LISTS

If R is the rate of increase in S per unit of increase in T, (see pages 10, 33, 115, 116, 158),

$\Delta S \div \Delta T \equiv$ THE RATE, R, if R is a CONSTANT

$\Delta S \div \Delta T \equiv$ MEAN RATE, R, if R IS VARIABLE

$dS \div dT \equiv$ TRUE RATE, R, if R is EITHER

DEFINITIONS AND SYMBOLS

TABLES

Argument in Radians				NATURAL FUNCTIONS				Argument in Degrees			
I				+	+	+	+				I
	II			+	−	−	−			II	
		III		−	−	+	−		III		
			IV	−	+	−	+	IV			
				SIN	COS	TAN	SEC				
0	3.14	3.14	6.28	0.	1.	0.	1.	360	180	180	0
.09	3.05	3.23	6.20	.09	.99	.09	1.01	355	185	175	5
.17	2.97	3.32	6.11	.17	.98	.18	1.02	350	190	170	10
.26	2.88	3.40	6.02	.26	.96	.27	1.04	345	195	165	15
.35	2.77	3.49	5.93	.34	.94	.36	1.06	340	200	160	20
.43	2.71	3.58	5.85	.42	.91	.47	1.10	335	205	155	25
.52	2.62	3.67	5.76	.5	.87	.58	1.16	330	210	150	30
.61	2.53	3.75	5.67	.57	.82	.70	1.22	325	215	145	35
.70	244	3.80	5.59	.62	.77	.84	1.31	320	220	140	40
.79	2.36	3.93	5.50	.71	.71	1.	1.41	315	225	135	45
.87	2.27	4.01	5.41	.77	.62	1.19	1.56	310	230	130	50
.96	2.18	4.10	5.32	.82	.57	1.42	1.74	305	235	125	55
1.05	2.09	4.19	5.24	.87	.5	1.73	2.	300	240	120	60
1.13	2.01	4.28	5.15	.91	.42	2.14	2.36	295	245	115	65
1.22	1.92	4.36	5.06	.94	.34	2.75	2.92	290	250	110	70
1.31	1.83	4.45	4.97	.97	.26	3.73	3.86	285	255	105	75
1.40	1.75	4.54	4.89	.98	.17	5.67	5.76	280	260	100	80
1.48	1.65	4.63	4.80	.99	.09	11.43	11.47	275	265	95	85
1.57	1.57	4.71	4.71	1.	0.	NONE	NONE	270	270	90	90

RADIANS	CYCLES	DEGREES
6.283	1	360
12.566	2	720
18.850	3	1 080
25.133	4	1 440
31.416	5	1 800
37.699	6	2 160
43.982	7	2520
50.265	8	2880
56.549	9	3240
62.832	10	3600

Natural or e-Logarithms. Base: 2.718+

$N \equiv e^L \qquad L \equiv \log_e N$

N in the margin
L in the table

N	L
10	2.303
100	4.605
1000	6.908

N	L
.1	7.697–10
.01	5.395–10
.001	3.092–10

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.	0.	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2.	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3.	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.305	1.335	1.361
4.	1.386	1.411	1.435	1.459	1.482	1.504	1.526	1.548	1.569	1.589
5.	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6.	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7.	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8.	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9.	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

THE PLACE OF FRESHMAN CALCULUS IN THE CURRICULUM

At TUFTS COLLEGE the prescribed mathematics in the Engineering Department consists of the following courses;

1st. TERM, FRESHMAN YEAR.

Course 1. Computation. Use of trigonometric functions, logarithms, slide-rule, radicals and combination numbers. Plane triangles.

Course 2. Algebraic and graphical methods. Simultaneous and quadratic equations. Co-ordinates, graphs, straight lines, circles, and conic sections. Locus problems.

2nd. TERM, FRESHMAN YEAR

Course 3. This text-book is expressly intended for this course.

1st. TERM, SOPHOMORE YEAR

Course 4. The more advanced parts of trigonometry and elementary calculus, Review, arith

2nd. TERM, SOPHOMORE YEAR

Course 5. Three dimensional work in trigonometry, analytical geometry and calculus. Introduction to differential equations.

JUNIOR AND SENIOR YEARS

Privilege of electing higher mathematics given in the College of Letters.

While this text-book has been designed to fit a certain curriculum, its arrangement is such as to permit its use in a variety of cases. It may be taken up simultaneously with trigonometry and analytical geometry in the first term, since no transcendentals appear in the first 80 pages (save in E-23, p.63). It may be used in review in the later years since it deals so copiously with the important applications of the calculus in an engineering or scientific course. For such a purpose the chief omission that will be noted is that of centers of gravity and moments of inertia, and multiple integration.

The number of concrete problems is large, 196 out of the 417, and includes many problems of vital importance in the study of physics, and apt to have their mathematical features insufficiently considered in the science courses.

This text makes no pretension to completeness, but is intended to precede a calculus treating a greater range of topics. Only such topics are here presented as a Sophomore Engineer is likely to require.

Specially Named Rates

When R remains constant as S and T vary it is defined as the rate of increase of S in units of S per unit of increase of T by an equation of the form:

$$R = \Delta S \div \Delta T$$

Instances in this text are:

Speed $= \Delta$*(distance)* $\div \Delta$*(time)*
Acceleration $= \Delta$*(speed)* $\div \Delta$*(time)*
Power $= \Delta$*(work)* $\div \Delta$*(time)*
Current $= \Delta$*(volume)* $\div \Delta$*(time)*
Slope $= \Delta$*(rise)* $\div \Delta$*(run)*
Altitude $= \Delta$*(area)* $\div \Delta$*(base)*
Force $= \Delta$*(work)* $\div \Delta$*(distance)*
Cross-section $= \Delta$*(volume)* $\div \Delta$*(length)*
Mileage $= \Delta$*(fare)* $\div \Delta$*(distance)*
Pressure $= \Delta$*(force)* $\div \Delta$*(area)*
Density $= \Delta$*(mass)* $\div \Delta$*(volume)*
Specific heat $= \Delta$*(heat)* $\div \Delta$*(temperature)*

The units which precede and follow the "per" in the name of the unit of a rate whose unit is not specially named suggest the DEFINITION *of that rate: tensile strength is measured in lbs. per sq. in., hence is defined as* Δ*(force)* $\div \Delta$*(area).*