DIFFERENTIAL AND INTEGRAL CALCULUS

WITH APPLICATIONS

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REVISED

D. C. HEATH & CO., PUBLISHERS
BOSTON NEW YORK CHICAGO

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PREFACE.

This text-book is based upon the methods of "limits" and "rates," and is limited in its scope to the requirements in the undergraduate courses of our best universities, colleges, and technical schools. In its preparation the author has embodied the results of twenty years' experience in the class-room, ten of which have been devoted to applied mathematics and ten to pure mathematics.

It has been his aim to prepare a *teachable work for beginners*, removing as far as the nature of the subject would admit all obscurities and mysteries, and endeavoring by the introduction of a great variety of practical exercises to stimulate the student's interest and appetite.

Among the more marked peculiarities of the work the following may be enumerated: —

1. A large amount of explanation.
2. Clear and simple demonstrations of principles.
3. Geometric, mechanical, and engineering applications.
4. Historical notes at the heads of chapters giving a brief account of the discovery and development of the subject of which it treats.
5. Footnotes calling attention to topics of special historic interest.

6. A chapter on Differential Equations for students in mathematical physics and for the benefit of those desiring an elementary knowledge of this interesting extension of the calculus.

7. An arrangement of topics admitting of extensive eliminations without destroying the continuity of the subject.

8. A clear, open page.

The author desires to express here his acknowledgments to the friends who have aided him in his work. To Chas. M. Snelling, A.M., University of Georgia, and to T. H. Taliaferro, Ph.D., State College of Pennsylvania, the author's obligations are peculiarly great. Not only have they given valuable counsel, but they have been largely instrumental in freeing the work from typographical errors.

PREFACE TO REVISED EDITION

In presenting the revised edition of this work to the public, I wish to express my acknowledgment to L. W. Smith, A.M., Ph.D., Professor of Mathematics, Washington and Lee University, and to Colonel C. W. Watts, C.E., Professor of Mathematics, Virginia Military Institute. To these gentlemen the care of the revision of the work was submitted and to them is due full credit for all improvements.

E. W. NICHOLS.

LEXINGTON, VA.,
July 16, 1917.

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PART I.

DIFFERENTIAL CALCULUS.

DIFFERENTIAL CALCULUS.

CHAPTER I.

QUANTITIES. FUNCTIONS.

1. Quantity. That which can be increased, diminished, measured, or in general, anything to which mathematical processes are applicable is called **Quantity.**

Time, space, motion, velocity, force, and mass are examples, and with these, as with other quantities, we shall have more or less to do in illustrating and applying the principles which are to follow.

2. Classes of Quantity. In the abstract science of the Calculus, as in Analytic Geometry, quantities are divided into two general classes, viz., **Constants** and **Variables.**

3. Constants. A constant is a quantity whose value is fixed. Constants are usually represented by the first letters of the alphabet, a, b, c, etc., or by numbers.

4. Variables. A variable is a quantity which is, or is conceived to be, in an actual state of change. Variables are usually represented by the last letters of the alphabet, u, v, w, x, y, z, etc.

5. Illustrations. The usual algebraic expression of the law subject to which a point moves in generating a circle is $(x - a)^2 + (y - b)^2 = r^2$. If, then, we consider the generating point to be *actually in motion*, it is readily seen that its co-ordinates, x and y, are in a *state of variation*, and hence, by definition (§ 4), are *variables;* while the quantity r, which measures the distance of the generating point from the centre (a, b) of the circle, is *fixed in value*, and hence (§ 3) is a *constant.*

Again: A train leaves Jersey City for Philadelphia, and after the lapse of a certain time attains a uniform speed of 45 miles an hour, which it maintains until its destination is reached. The *distance* between the train and Jersey City, being at the instant under consideration in an *actual state of change* or *variation, is a variable*, while the *speed*, or *velocity* (45 miles per hour) *is a constant.*

Again: A meteor is falling to the earth: Both the *distance* between these bodies and their *mutual attraction are variables*, the latter varying inversely as the square of their distance apart.

Again: The volume of water in a cistern which is being filled or emptied by a continuous stream is a variable.

6. Functions. One variable is said to be a function of another when its value *depends* upon that of the latter. Thus: The area of a circle (πa^2) is a function of its radius (a); the area of a square is a function of its side; the expressions $x^2 + 1$, $\log x$, $\sin x$, $x^3 - \log x^2 + \tan x$, as well as all expressions which contain x only, are functions of x. In like manner a variable is said to be a function of two or more variables when its value depends upon their values. Thus, the area of an ellipse (πab) is a function of its semi-axes (a, b); the area of a rectangle is a function of its base and altitude; the volume of a rectangular parallelopiped is a function of its three dimensions; the expressions $x^2 + y^2 - a^2$, $\sin x + \tan y$, $x^3 - \log y^2$ are functions of x and y; the expressions $xy + z^3$, $z + \log xy$, xyz^2, etc. are functions of x, y, and z.

When a function of one or more variables involves no condition or conditions, the variables are *independent* of each other; that is, we may assign to each any value we please. Such is the case in all the illustrations given above. If, however, some condition is involved, as, for example, the equality to zero of any one of these expressions, then *one* variable at least must be dependent for its value upon the values assigned the others. Let us illustrate: In the function of x and y, $x^2 + y^2 - a^2$, x and y are independent variables — no condition being involved. But suppose we write $x^2 + y^2 - a^2 = 0$, then the range of values which may be assigned to x or y is at once limited, and we can no longer assign *arbitrarily* any values we please to both, but must assign values to one only, and ascertain from the equation the value of the other. The variable to which values are assigned is called the **independent** variable; the other, which now *represents* the function is called the **dependent** variable.

7. General Classes of Functions. Functions are either **Algebraic** or **Transcendental:**

I. **Algebraic Functions** are those which involve only the six fundamental operations of algebra, viz.: Addition, Subtraction, Multiplication, Division, Involution, and Evolution, with *constant* indices. Thus, $x + y$, $x^2 - \sqrt[3]{y}$, $\frac{xy}{z} + \sqrt{x^3}$ are algebraic functions.

II. **Transcendental Functions**, embracing all functions other than algebraic, are subdivided into various classes, the more important being:

1. **Trigonometric Functions**, such as $\sin x$, $\tan x$, $\sec x$, etc.;

2. **Circular**, or **Inverse Trigonometric Functions**, such as $\sin^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, etc.;

3. **Logarithmic Functions**, such as $\log x$, $\log (x + \sqrt{x^2 - y^2})$, etc.;

4. **Exponential, or Inverse Logarithmic Functions** such as a^x, x^y, $(u + v)^z$.

8. **Special Classes of Functions.** Both algebraic and transcendental functions are further subdivided, the subdivisions being dependent upon the standpoint from which they are viewed.

I. **Explicit and Implicit Functions.**

1. **The Explicit Function,** as $y = x^3 - ax + b$, or $y = \log x$, or $y = x^2z + \sin v$, where the simple fact that the equation is solved with respect to one of the variables which enters it, indicates *explicitly* that the first member (y) is a function of the variables which enter the second member.

2. **The Implicit Function,** as $y - x^3 + ax - b = 0$, or $y - \log x = 0$, or $y - x^2z - \sin v = 0$, where the fact that one of the variables is a function of the other is *implied* from the condition of equality to zero.

II. **Increasing and Decreasing Functions.**

1. **The Increasing Function,** as $y = sx + b$ (s = positive quantity), for as x *increases*, the function y *increases* also.

2. **The Decreasing Function,** as $y = -sx + b$ (s = positive quantity), for as x *increases*, the function y *decreases*.

It should be carefully observed that the terms *increase* and *decrease* are here used in an *algebraic* sense. In the common parabola $y = \pm\sqrt{2px}$ for example, y is an increasing or a decreasing function of x, according as we use the upper or lower sign before the radical; in other words, y is an increasing function of x in the first angle, and a decreasing function of x in the fourth.

III. **Continuous and Discontinuous Functions.**

1. **The Continuous Function** of a variable is a quantity that changes gradually, and passes through every intermediate value

from an initial to a final value, as the variable that enters it passes through every intermediate value from its initial to its final value.

Thus in the equation of the line, $y = sx + b$, y is a continuous function of x; for as x increases gradually, and passes through all intermediate values between $-\infty$ and $+\infty$, y also changes gradually, and passes through all intermediate values between $-\infty$ and $+\infty$. Again, in the circle $y = \pm\sqrt{a^2 - x^2}$, y is a continuous function of x between the limits $x = \pm a$.

2. **The Discontinuous Function** — as in the hyperbola $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$. Here y is a discontinuous function of x between the limits $x = \pm a$. For values of $x > \pm a$ numerically we readily see that y is a continuous function of x. In fact, functional forms frequently occur which are continuous between certain limits of the variable which enters it, and discontinuous between other limits of that variable. The Differential Calculus, however, has only to do with variables between their limits of continuity.

9. **Notation.*** The equation $y = f(x)$ is a symbolic expression of the sentence "y is an explicit function of x." Similarly $y = f(x, z, v)$ is to be read "y is an explicit function of x, z, and v." Implicit functions are also capable of general representation. Thus, $f(x, y) = 0$ means that x and y are implicit functions of each other.

If the same functional symbol occurs more than once in the same operation it is understood to refer to the same function. If the symbols are different, then the functions to which they refer are different. Thus $f(x)$ and $\phi(x)$ would, in the same operation, indicate that the functions of x to which reference was made were different.

* The notation $\phi(x)$ to indicate a function of x was introduced by John Bernouilli, the elder, in 1718; but the general adoption of symbols like f, F, ϕ, ψ, . . . to represent functions, was mainly due to Euler and Lagrange.

If a particular value is assigned a variable which enters a function, and we wish to express in general notation the resulting value of the function, we substitute for the variable this particular value. Thus, if we wish to indicate what $f(x)$ becomes when x is equal to a, or b, or o, we write $f(a)$, or $f(b)$, or $f(o)$, as the case may be. Another method, where the particular function with which we have to deal is given, is to place the value of the variable as a subscript to a semi-bracket placed on the right of the function, and equate this to the result obtained by substituting the value of the variable in the function. Thus

$$\left.\frac{x-1}{x+2}\right]_1 = 0; \qquad \left.\frac{x^2+2ax}{2x-a}\right]_a = 3a.$$

EXAMPLES.

1. If $f(x) = x^2 - 5x + 6$, show that
$f(1) = 2,\ f(-2) = 20,\ f(\frac{2}{3}) = 3\frac{1}{9},\ f(2) = 0,$
$f(3) = 0,\ f(x-1) = x^2 - 7x + 12,$
$f(2x) = 4x^2 - 10x + 6.$

2. If $f(x) = (x+1)(x-1)(x-2)$, show that
$f(-1) = f(1) = f(2), -3f(3) = 2f(-2).$

3. If $f(y) = e^y - e^{-y}$, show that $f(3y) = [f(y)]^3 + 3f(y)$.

4. If $\phi(x) = a^x$, show that $[\phi(x)]^2 = \phi(2x)$.

In the following implicit functions make y an explicit function of x.

5. $y^2 - 2xy + x^2 = 0$ $\qquad y = x$

6. $\log_a y = 2\log_a(a+x) - 1$ $\qquad y = \frac{(a+x)^2}{a}.$

7. Is y an increasing or decreasing function of x in the function given in Ex. 5? Is it continuous?

8. Show that the following equalities are true:

$$\left.\frac{x^2+x+1}{2x-1}\right]_1 = 3, \qquad \left.\frac{2\sin x\cos x}{\cos x - \sin x}\right]_0 = 0, \qquad \left.\frac{1}{\frac{\pi}{2}\csc^2 x}\right]_{\frac{\pi}{2}} = \frac{2}{\pi}.$$

9. In the equation $x(y-2)+y-c=0$, show that y is not a function of x when $c=2$.

10. Show that y is not a function of x in the equation

$$y = \frac{\sin x \sin \frac{1}{2}x + \cos \frac{1}{2}x \cos x}{\cos \frac{1}{2}x}.$$

11. If $f(x) = x^2 + \frac{1}{x}$, show that $f(a) = f\left(\frac{1}{a}\right)$.

12. If $f(x) = \sin x + \tan x$, find the values of $f\left(\frac{\pi}{2}\right)$; $f\left(\frac{\pi}{4}\right)$; $f(\pi)$.

13. If $f(x) = \log(x^3-1)$, find the values of $f(2)$; $f(3)$.

14. If $\psi(x) = \log x$, show that $\psi(abc) = \psi(a) + \psi(b) + \psi(c)$; also that $\psi\left(\frac{a}{b}\right) = \psi(a) - \psi(b)$.

15. If $\phi(x) = \sin x$, show that

$$\phi(x) + \phi(y) = 2\,\phi\left(\frac{x+y}{2}\right)\cdot\sqrt{1-\left[\phi\left(\frac{x-y}{2}\right)\right]^2}.$$

CHAPTER II.

FUNDAMENTAL PRINCIPLES.

10. Increment. *The increment of a variable is the amount of its change in any interval of time.*

We can always ascertain this amount by taking the algebraic difference between the values of the variable at the beginning and at the end of the interval — always subtracting the former from the latter. If the increment thus ascertained is positive, the variable is increasing; if negative, it is decreasing. See § 8, II. The increment of a variable is usually denoted by the symbol Δ placed before the variable. Thus Δx means 'the increment of x,' and is to be so read.

11. Uniform Change. Varied Change. When a variable so changes that its increment is numerically the same in all equal intervals of time, its change is said to be **uniform.**

In all other cases its change is said to be **Varied.**

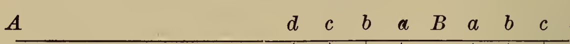


Fig. a.

Thus let AB represent graphically the state of a variable (u) at any instant, and let Ba, ab, bc, etc., represent its increments in *any* successive equal intervals of time; then if

$$Ba = ab = bc = \text{etc.},$$

the variable AB (u) is uniformly changing. Otherwise the variable is varied in its change. If u is an increasing variable, the increments Ba, ab, bc, etc., are positive; if decreasing, the increments are negative. § 10.

Again, suppose a bucket in the form of an inverted conical frustum is being filled from a hydrant in such a manner that the *depth* of the water increases by one inch in every second — and proportionately for any other interval of time — then the *depth* of the water is a variable in a state of *uniform change*, while the *volume* of the water is a variable in a state of *varied* change. Had we assumed a *cylindrical* bucket, then the volume as well as the depth of the water would have changed uniformly.

12. Uniform Motion. Varied Motion. A point is said to have *uniform motion* when the *distance* over which it passes, estimated from any point in its path, *is a variable in a state of uniform change*. When this *distance* is *varied* in its change, the *motion* of the point is also **Varied.**

Thus, Fig. *a*, let us suppose AB to be the path of a flowing point, which at the instant of consideration has reached the position B. Then if the *distance* AB is a uniformly changing variable, the *motion* of the point B is *uniform*. Otherwise the motion of B is *varied.*

COROLLARY. *The direction in which a distance is changing is determined at any instant by the direction of motion of the flowing point at that instant.*

13. Differential. *The differential of a variable is the increment it* WOULD *take on in any interval of time, if its change became uniform at the beginning, and continued so throughout that interval.* The differential of a uniformly changing variable is obviously the *actual* increment it takes on in any interval.

The usual notation for representing the differential of a variable is the letter d placed before the variable. Thus du, read 'differential of u,' indicates the *operation* of taking the differential of the variable u. It should be remembered that the symbol d before u *is not a coefficient*, but a *symbol of operation*, and is entirely analogous to the symbols *sin*, cos^{-1}, *log*, in the expressions $\sin u$, $\cos^{-1} u$, $\log u$.

COR. *The differential of an increasing variable is* POSITIVE, *and that of a decreasing variable is* NEGATIVE. (§ 10).

14. Illustrations. To illustrate the relation between an *increment* and a differential, as well as to secure a clear conception of each, let $y = f(x)$ be the equation of any locus such as *OPT*, Fig. 1, and let u represent the *variable area* bounded by the curve, the x — axis and the terminal ordinate *PM*, *as that ordinate moves uniformly to the right*, changing its length in obedience to the law expressed in the equation $y = f(x)$. Let $x = OM$, and let MN (Δx) be the increment of x in any interval of time, beginning at the instant when $x = OM$. Then

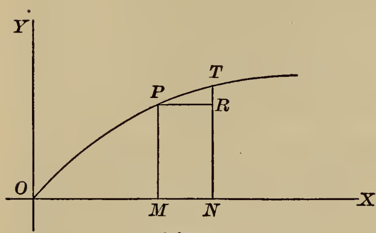


Fig. 1.

$$\Delta x = MN, \quad \Delta y = RT,$$

and $$\Delta u = \text{area } PMNT.$$

Since *PM* moves *uniformly* the distance $OM (= x)$ changes uniformly; hence $dx = \Delta x = MN$.

Again, du is *by definition* the *increment* that u (OPM) *would take on* if it became a uniformly changing variable, and so continued throughout the interval of time. But this supposition of uniform change in u (OPM) obviously requires the ordinate *PM* to remain constant in length throughout the interval. Hence

$$du = \text{area } PMNR.$$

But $$\text{area } PMNT - \text{area } PMNR = \text{area } PRT;$$

$$\therefore \quad \Delta u - du = \text{area } PRT.$$

15. Rate. The **Measure** of the **Rate** of **Change** of a variable or, more simply, its **Rate**, *is the increment it would take on in a* *unit of time if its change became uniform at the beginning and continued so throughout that unit.*

The rate of a *uniformly* changing variable is the *actual* increment it takes on in a *unit* of time.

Thus when we speak of a body falling at the rate of 50 feet per second, we mean that the variable distance through which the body has already fallen *would take on* the increment 50 feet in the next second, if its change became uniform at the beginning of the second, and continued so throughout.

Or, when we speak of a passing train as moving at the *rate* of 40 miles an hour, we mean that if its distance from some point in its path (say the last station) became at the instant of speaking a uniformly changing variable, and continued so for an hour, that it would take on the increment of 40 miles.

16. Relation Between a Differential and a Rate. It will be observed that the only difference between the definition of 'a differential' (§ 13) and that of 'a rate' (§ 15), is in the use of the term "interval of time" in the former, and the term "unit of time" in the latter. If, therefore, the "interval of time" is taken as the "unit of time," the rate of a variable and its differential are the same. If the "interval of time" is *not* taken as the "unit of time," let $dt =$ that interval (since time (t) changes uniformly, dt can represent any interval or increment of time), and let du be the corresponding differential of a variable u; then

$$r = \frac{du}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $r =$ rate of u. If $dt =$ unit of time, we have

$$r = du$$

as explained above, i.e., *the rate of a variable is its differential for a unit of time.*

Referring to the last illustration of § 15, suppose we say that the train will travel 160 miles in the next 4 hours at its present rate of travel, then

$$du = 160 \text{ miles and } dt = 4 \text{ hours,}$$

and $$r = \frac{du}{dt} = \frac{160}{4}\frac{m}{h} = 40 \text{ miles per hour}$$

as before.

17. Velocity. *Velocity is the rate of change of a distance.* Let s = variable distance traversed by a point, and let v = its velocity at any instant; then, § 16 (1),

$$v = \frac{ds}{dt} \quad \ldots\ldots\ldots\ldots \quad (2)$$

18. Relation *between the velocity of a point in its path and its component velocities in the direction of rectangular axes.*

Let $y = f(x)$ be the equation of any locus, as APB, when referred to rectangular co-ordinates. Let us further suppose that the point which generates this locus is at the instant of consideration at P, (x, y). We wish to compare the velocity of P in its path with its component velocities in the directions of X and Y. In other words, we wish to compare the rates of change of the *distances* AP $(= s)$, NP $(= x)$ and MP $(= y)$.

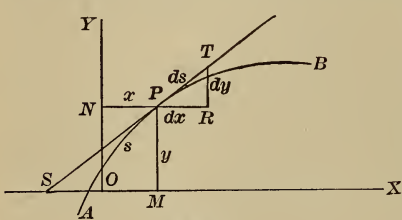


Fig. 2.

The *direction* of change of the varying distance AP $(= s)$ is at the instant of consideration *the direction of motion* of its generating point P, § (12) Cor.; but the direction of motion of P is at the instant in the direction of the tangent PT. Hence the direction of change of the distance AP is at the instant in the direction PT. Now, assuming the distance AP $(= s)$ to become a uniformly changing variable at the instant of reaching the value AP, lay off any distance PT, *in the direction of its change*, i.e., *along the tangent* PT, to represent the increment it would take on under this supposition in the interval of time dt; then, § 13, we have

$$ds = PT.$$

But if $AP\ (= s)$ becomes a uniformly changing variable in the direction PT, the co-ordinates of P, i.e., the *distances* NP $(= x)$, $MP\ (= y)$, also become uniformly changing variables, and would take on the increments PR and RT, respectively, in the same interval of time, dt. Hence

$$PR = dx, \quad RT = dy.$$

From the right triangle PTR we have

$$\overline{PT}^2 = \overline{PR}^2 + \overline{RT}^2;$$

i.e.,

$$(ds)^2 = (dx)^2 + (dy)^2 \quad \ldots \ldots \quad (3)$$

Hence

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2;$$

i.e., § 17, 2, *The square of the velocity of a point in its path is equal to the sum of the squares of its components in any two rectangular directions.*

Cor. Let $TSX = a$; then, since $TPR = TSX = a$, we have from the right triangle TPR the following important relations:

$$\frac{dx}{ds} = \cos a \quad \ldots \ldots \ldots \quad (4)$$

$$\frac{dy}{ds} = \sin a \quad \ldots \ldots \ldots \quad (5)$$

$$\frac{dy}{dx} = \tan a \quad \ldots \ldots \ldots \quad (6)$$

19. Signification of $\frac{dy}{dx}$.

I. *Geometric Signification.* Every relation between *two* variables which can be expressed in the form of an equation, $y = f(x)$, can in general be represented geometrically by a plane locus. Hence the ratio of the differentials of these variables $\left(\frac{dy}{dx}\right)$ ought to admit of geometric interpretation. We see from (6) of the preceding article that it does admit of such interpretation; for, Analytic Geometry, p. 25,

$$\tan a = s = \text{slope of tangent } TS.$$

But $\quad$ Slope of TS = slope of APB $[y = f(x)]$ at (x, y),

hence generally, $\quad \frac{dy}{dx} =$ *slope of* $y = f(x)$ *at* (x, y).

II. *Analytical Signification.*

since $\quad \frac{dy}{dt} = \text{rate of } y,$

and $\quad \frac{dx}{dt} = \text{rate of } x,$

and since $\quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx},$

we have $\quad \frac{dy}{dx} = \frac{\text{rate of } y}{\text{rate of } x}.$

Hence *the ratio of the differentials of two variables corresponding to the same interval of time is equal to the ratio of the rates of those variables at the beginning of that interval.*

Cor. If rate of x be taken as *unit rate.* Then

$$\frac{dy}{dx} = x\text{-rate of } y.$$

20. Remark.* The terms "unit of time" and "interval of time" as used in preceding articles do not refer to any *specific* portion of *time* — their values, whether great or small, not being considered. In the abstract science of the Calculus, time is a "foreign element"; but as all change occurs in time, it is *essential* to the *comparison* of the rates of variables related in any given way that the "unit of time," or "interval of time," used for this purpose should be understood to be the same.

Again, as any "interval of time" may be taken as a "unit of time" it will be found convenient to take

$$dt = \text{unit of time.}$$

Unless otherwise stated we shall so consider it in what ensues.

* Objection has frequently been made to Newton's method of fluxions, that it introduced a foreign idea, namely that of motion into geometry and analysis. This objection was answered by Newton when he stated that all his method contemplates is that one of the variables should increase uniformly (*aequabili fluxu*) as we conceive time to do.

CHAPTER III.

DIFFERENTIATION.

HISTORY. — It is not certain whether the Calculus was first discovered by Sir Isaac Newton (1642–1727) or simultaneously and independently by Newton and Gottfried Wilhelm Leibnitz (1646–1716). The facts — elicited after a bitter controversy extending throughout the eighteenth century — are briefly these:

1. Newton communicated his discovery to friends and in manuscript in 1669, although his method was not published until 1693.

2. Leibnitz published a memoir on the Calculus in 1684, yet the earliest use of its method in his note-books is dated 1675.

3. In 1849, Gerhard discovered among Leibnitz's papers a manuscript in Leibnitz's handwriting of extracts from one of Newton's papers, together with notes on their expression in the differential notation. A copy of this manuscript, it is known, had been sent to Tschirnhausen in May, 1675; and as he and Leibnitz were engaged on a piece of work at the time, it is possible that these extracts were made then. On the other hand, the extracts may have been made from the printed copy in 1704.

4. It is certain that Leibnitz enjoyed for fifteen years and unchallenged the honor of being the inventor of his Calculus. Newton himself rendered him that credit in the first two editions of his Principia.

The problem of the Calculus as stated by Newton was: Given the relation of the fluents (= variables) to find the relation of their fluxions (= rates). This is equivalent to differentiation. Leibnitz's *notation* being preferable to that of Newton has been generally adopted in treatises on the subject.

21. *The* **Differential Calculus** *is the science of rates, and its fundamental object is to determine and compare the rates of related variables.*

22. **Differentiation** *is the process of determining a differential.*

ALGEBRAIC FUNCTIONS.

23. *The differential of a constant is zero.*

Let c be any constant. By definition, § (3), a constant is a quantity whose value is *fixed;* hence, there can be no increment to its value in any unit of time.

Hence $$dc = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

24. *The differential of the sum of two variables is the sum of their differentials.**

To prove

$$d(u + v) = du + dv,$$

in which u and v are the values of two varying quantities at the *same* instant, and du and dv are their differentials.

Then
$$u + v,$$
$$u + du + (v + dv),$$
$$u + 2\,du + (v + 2\,dv),$$

are the values of the function $(u + v)$ at the instant and at the ends of two equal units of time following that instant *under the supposition of uniform change in u and v throughout the two units of time.* Subtracting the first value from the second, or the second value from the third, we find that the difference in either case is

$$du + dv.$$

But this difference is the *increment* taken on by the function $(u + v)$ in the same unit of time, and since it is the same for all equal units of time the function is a uniformly changing variable at the instant. Hence, § 13, this increment is its differential;

i.e., $$d(u + v) = du + dv \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Argobast (A.D. 1800) was the first to separate the symbol of operation from that of quantity. François in 1812, and Servois in 1814, were the first to give correct rules on the subject of operation.

Cor. 1. Since the differential for a unit of time is the rate of the variable, § 16, we may interpret (2) as follows: *The rate of the sum of two variables is the sum of their rates.*

Cor. 2. *The differential of any polynomial is the sum of the differentials of its terms.* Thus

$$d(u + v - w + z) = du + dv - dw + dz.$$

25. *The differential of the product of two variables is equal to the sum of the products arising from multiplying each variable by the differential of the other.*

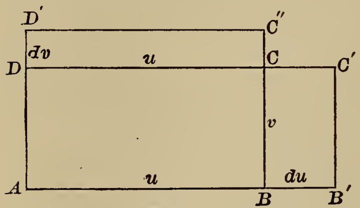


Fig. 3.

To prove

$$d(uv) = udv + vdu.$$

Let u and v be any two variables, and, *at the instant of consideration*, let their values be represented by the sides AB and AD of the rectangle $ABCD$. Then at the instant

$$uv = \text{area } ABCD.$$

Let $BB' = du$, $DD' = dv$, i.e., let BB' and DD' represent the increments that the variables u (AB) and v (AD) would take on in a unit of time if their changes became uniform at the instant of consideration and so continued for the unit of time. Now $d(uv)$ [$= d$ (area $ABCD$)] is the increment that the rectangle would take on in the same unit of time provided *its change became uniform* at the instant and continued so throughout the unit of time. The change in the rectangle $ABCD$ would obviously become uniform if it took on the increment

$$\text{area } DD'C''C + \text{area } BB'C'C.$$

Moreover, since we have supposed u and v to take on the increments du (BB') and dv (DD), the rectangle uv ($ABCD$) can change uniformly *in no other way.* Hence

$$d\,(\text{area } ABCD) = \text{area } DD'C''C + \text{area } BB'C'C;$$

i.e.,
$$d\,(uv) = udv + vdu \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

COR. 1. If in (3) we make $v = c =$ a constant, we have

$$d\,(uc) = udc + cdu.$$

Hence § 23 (1), $d\,(cu) = cdu$ . . . . (4)

i.e., *The differential of a constant multiplied by a variable is equal to the product of the constant and the differential of the variable.*

COR. 2. If in (3) we make $v = vw$, we have,

$$\begin{aligned} d\,(uvw) &= ud\,(vw) + vwdu \\ &= u\,(vdw + wdv) + vwdu \\ &= uvdw + uwdv + vwdu. \end{aligned}$$

Similarly we may prove

$$d(uvwz) = uvwdz + uvzdw + uwzdv + vwzdu \quad (5)$$

and so for any number of variables.

Hence, *The differential of the product of any number of variables is equal to the sum of the products arising from multiplying the differential of each variable by the product of all the others.*

26. *The differential of a fraction is equal to the denominator into the differential of the numerator minus the numerator into the differential of the denominator divided by the square of the denominator.*

To prove

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

Let $\frac{u}{v} = z$, then $u = vz$ and

$$du = vdz + zdv. \qquad \text{§ 25, (3)}$$

Replacing z by its value $\frac{u}{v}$, we have

$$du = vd\left(\frac{u}{v}\right) + \frac{u}{v}\,dv.$$

Hence, solving, $$d\left(\frac{u}{v}\right) = \frac{du - \frac{u}{v}\,dv}{v};$$

$$\therefore\ d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

COR. 1. If $v = c =$ a constant, formula (6) becomes

$$d\left(\frac{u}{c}\right) = \frac{c\,du - u\,dc}{c^2};$$

i.e., $$d\left(\frac{u}{c}\right) = \frac{du}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This is as it should be, since $\frac{u}{c} = \frac{1}{c}u$, which being differentiated by formula (4) gives $d\left(\frac{1}{c}.u\right) = \frac{1}{c}.du = \frac{du}{c}$.

COR. 2. If $u = c =$ a constant, then formula (6) becomes

$$d\left(\frac{c}{v}\right) = -c\frac{dv}{v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

27. *The differential of a variable with a constant exponent is equal to the product of the exponent, the variable with its exponent diminished by one and the differential of the variable.*

To prove

$$d(u^n) = nu^{n-1}\,du.$$

1. *Let n be a positive integer.*

Then $u^n = u.\ u.\ u.\ u.$ to n factors.

Hence,

$$d(u^n) = d(u.\ u.\ u.\ u \text{ to } n \text{ factors})$$
$$= u^{n-1}\,du + u^{n-1}\,du + u^{n-1}\,du \text{ to } n \text{ terms.} \qquad \S\ 25, (5)$$

Therefore

$$d(u^n) = nu^{n-1}\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

2. *Let n be a positive fraction and equal to $\frac{m}{p}$.*

Let $y = u^{\frac{m}{p}}$, then

$$y^p = u^m.$$

Hence, $$py^{p-1}\,dy = mu^{m-1}\,du. \qquad \text{Equa. (9)}$$

$$\therefore \qquad dy = \frac{m}{p} \cdot \frac{u^{m-1}}{y^{p-1}}\,du.$$

Substituting for y its value and reducing, we have

$$d\left(u^{\frac{m}{p}}\right) = \frac{m}{p}\,u^{\frac{m}{p}-1}\,du.$$

Hence the rule applies in this case.

3. *Let n be negative and equal to $-m$.*

Let $y = u^{-m} = \frac{1}{u^m}$, then, § 26. Cor. 2,

$$dy = -\frac{mu^{m-1}\,du}{u^{2m}};$$

i.e., $$du^{-m} = -mu^{-m-1}\,du.$$

Hence the rule applies in this case also.

4. Let n be incommensurable.

This case cannot be considered here as equa. (9) was deduced under the supposition that n was commensurable. The rule holds good in this case also, as will be shown in a subsequent article. See § 32.

28. Differential Equation. Differential Coefficient.

Let $y = x^m$, then by differentiation

$$dy = mx^{m-1}\,dx;$$

hence, $$\frac{dy}{dx} = mx^{m-1}.$$

The first of these equations is called The First Differential, or

The First Derived Equation, of the equation $y = x^m$, and the second is called the First Differential Coefficient, or First Derivative, of the same equation. Both of these equations, it will be observed, contain a new function of x, viz., mx^{m-1}. Hence, in general, if $y = f(x)$,

then $$dy = f'(x)\,dx,$$

and $$\frac{dy}{dx} = f'(x)$$

are, respectively, the *first differential equation* and the *first differential coefficient* of the equation $y = f(x)$.

Cor. Since dy and dx have the same or different signs, according as y and x are increasing or decreasing functions of each other, it follows that $\frac{dy}{dx}$ is positive or negative, according as y is an increasing or a decreasing function of x. § 13, Cor.

29. Formulas.

$$dc = 0 \tag{a}$$

$$d(cu) = c\,du \tag{b}$$

$$d(u + v) = du + dv \tag{c}$$

$$d(uv) = u\,dv + v\,du \tag{d}$$

$$d(uvw) = uv\,dw + uw\,dv + vw\,du \tag{e}$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \tag{f}$$

$$d(u^n) = nu^{n-1}\,du \tag{g}$$

Note.—These formulas are collected here for ease of reference. They should, however, be carefully committed to memory.

EXAMPLES.

Differentiate:

1. $5x + 6$.

$$d(5x + 6) = d(5x) + d(6) \qquad (c)$$

$$= 5\,dx. \quad Ans. \qquad (b)$$

2. $x^4 - nx^2 + c$.

$$d(x^4 - nx^2 + c) = d(x^4) - d(nx^2) + d(c) \qquad (c)$$
$$= 4x^3dx - nd(x^2) \qquad (g), (b)$$
$$= 4x^3dx - 2nxdx \qquad (g)$$
$$= 2x(2x^2 - n)\,dx. \quad \textit{Ans.}$$

3. $x^{\frac{1}{2}}(x^m - x)$.

$$d[x^{\frac{1}{2}}(x^m - x)] = x^{\frac{1}{2}}d(x^m - x) + (x^m - x)\,d(x^{\frac{1}{2}}) \qquad (d)$$
$$= x^{\frac{1}{2}}(mx^{m-1} - 1)\,dx + (x^m - x)\tfrac{1}{2}x^{-\frac{1}{2}}dx \quad (g), (c)$$
$$= \left[x^{\frac{1}{2}}(mx^{m-1} - 1) + \frac{x^m - x}{2\sqrt{x}}\right]dx.$$
$$= \frac{(2m + 1)x^m - 3x}{2\sqrt{x}}\,dx. \quad \textit{Ans.}$$

or, $$d[x^{\frac{1}{2}}(x^m - x)] = d(x^{m+\frac{1}{2}} - x^{\frac{3}{2}})$$
$$= (m + \tfrac{1}{2})\,x^{m+\frac{1}{2}-1}dx - \frac{3}{2}x^{\frac{1}{2}}dx \qquad (g), (c)$$
$$= \left[\frac{2m + 1}{2}x^{m-\frac{1}{2}} - \frac{3x^{\frac{1}{2}}}{2}\right]dx$$
$$= \frac{(2m + 1)x^m - 3x}{2\sqrt{x}}\,dx \quad \text{as before.}$$

4. $\dfrac{\sqrt[3]{x^2}(x^n + mx^2)}{x^{\frac{1}{2}}}$

$$d\left[\frac{\sqrt[3]{x^2}(x^n + mx^2)}{x^{\frac{1}{2}}}\right] = d[x^{\frac{1}{6}}(x^n + mx^2)]$$
$$= d(x^{\frac{6n+1}{6}} + mx^{\frac{13}{6}})$$
$$= \frac{6n + 1}{6}x^{\frac{6n+1}{6}-1}\,dx + \frac{13}{6}mx^{\frac{7}{6}}dx \quad (g), (c)$$
$$= \frac{(6n + 1)x^{\frac{6n-5}{6}} + 13mx^{\frac{7}{6}}}{6}\,dx$$
$$= [(6n + 1)x^n + 13mx^2]\frac{dx}{6\sqrt[6]{x^5}}. \quad \textit{Ans.}$$

As no special rule has been deduced for radical quantities, they must always be expressed as quantities affected with fractional exponents. It will be observed also that before differentiating in the above example, the expression was first *simplified.* This should always be done wherever practicable — not as a matter of principle, but with a view of simplifying the process.

5. $\dfrac{x^n (z+2)^p}{y^5}$

$$d\left\{\frac{x^n (z+2)^p}{y^5}\right\}$$

$$= \frac{y^5 d\{x^n (z+2)^p\} - x^n (z+2)^p d(y^5)}{y^{10}} \qquad (f)$$

$$= \frac{y^5\{x^n p (z+2)^{p-1} dz + (z+2)^p n x^{n-1} dx\} - x^n (z+2)^p 5 y^4 dy}{y^{10}}$$

$$= \frac{x^{n-1}(z+2)^{p-1}\{pxydz + (z+2)(nydx - 5xdy)\}}{y^6} \quad \textit{Ans.}$$

6. $x^2 + y^2 = a^2$, and find value of dy.

$$d(x^2 + y^2) = d(a)^2,$$

$$2xdx + 2ydy = 0,$$

$$\therefore dy = -\frac{x}{y}dx. \quad \textit{Ans.}$$

7. $y^2 = 4ax$, and find value of dy.

$$d(y^2) = d(4ax),$$

$$2ydy = 4adx,$$

$$\therefore dy = \frac{2a}{y}dx. \quad \textit{Ans.}$$

8. $a^2y^2 + b^2x^2 = a^2b^2$, and find value of dy.

$$\textit{Ans.} \quad dy = -\frac{b^2x}{a^2y}dx.$$

9. $xy = m$, and find value of dy.

$$\textit{Ans.} \quad dy = -\frac{y}{x}dx.$$

10. $y = (m + nx)\, x^3$. *Ans.* $dy = (4\, nx + 3\, m)\, x^2 dx$.

11. $y = \dfrac{x^m}{(1 + x)^m}$. *Ans.* $dy = \dfrac{mx^{m-1}}{(1 + x)^{m+1}} dx$.

This fraction may be placed in the form $x^m (1 + x)^{-m}$ and differentiated as a product. So with all fractions.

12. $y = (1 + x)\sqrt{1 - x}$. *Ans.* $dy = \dfrac{1 - 3\, x}{2\sqrt{1 - x}} dx$.

This product may be placed in the form $\dfrac{1 + x}{(1 - x)^{-\frac{1}{2}}}$ and differentiated as a fraction. Similarly for all products.

13. $y = x^{\frac{1}{3}} (x^{\frac{1}{2}} + 1)^{\frac{1}{2}}$. *Ans.* $dy = \dfrac{7\, x^{\frac{1}{2}} + 4}{12\sqrt[3]{x^2}\sqrt{x^{\frac{1}{2}} + 1}} dx$.

14. $y = x^5 (m + 3\, x)^3 (m - 2\, x)^2$.

Ans. $dy = 5\, x^4 (m + 3\, x)^2 (m - 2\, x)(m^2 + 2\, mx - 12\, x^2)\, dx$.

15. $y = \dfrac{x^m}{1 + x^m}$. *Ans.* $dy = \dfrac{mx^{m-1}}{(1 + x^m)^2} dx$.

16. $y = \sqrt{x^2 + a\sqrt{x}}$. *Ans.* $dy = \dfrac{4\sqrt{x^3} + a}{4\sqrt{x}\sqrt{x^2 + a\sqrt{x}}} dx$.

17. $y = \sqrt{ax} + \sqrt{c^2x^3}$. *Ans.* $dy = \dfrac{\sqrt{a} + 3\, cx}{2\sqrt{x}} dx$.

18. $y = \sqrt{\dfrac{1 - x}{1 + x}}$. *Ans.* $dy = -\dfrac{dx}{(1 + x)\sqrt{1 - x^2}}$.

19. $y = \dfrac{1}{x + \sqrt{1 + x^2}}$. *Ans.* $dy = \dfrac{x - \sqrt{1 + x^2}}{\sqrt{1 + x^2}} dx$.

20. $y = \dfrac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} - \sqrt{1 - x}}$. *Ans.* $dy = -\dfrac{1 + \sqrt{1 - x^2}}{x^2\sqrt{1 - x^2}} dx$.

Rationalize the denominator before differentiating.

21. $y = \dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$.

Ans. $dy = -\dfrac{2}{x^3}\left(1 + \dfrac{1}{\sqrt{1-x^4}}\right)dx.$

22. $y = \dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}-x}$.

Ans. $\dfrac{dy}{dx} = 2\left(\sqrt{1+x^2}+x\right)\left(\dfrac{x}{\sqrt{1+x^2}}+1\right).$

Write the first derivatives of the following:

23. $y = \dfrac{m}{(n^2+x^2)^3}$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{6\,mx}{(n^2+x^2)^4}$.

24. $y = \dfrac{2\sqrt{x}}{3+x^2}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{3\,(1-x^2)}{(3+x^2)^2\sqrt{x}}$.

25. $y = \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{1}{2\,(1+\sqrt{x})\sqrt{x\,(1-x)}}$.

26. $y = \dfrac{(x^3-2\,x)\,x^m}{x^2}$.

Ans. $\dfrac{dy}{dx} = (m+1)\,x^m - 2\,(m-1)\,x^{m-2}.$

27. $y = (a+bx^m)^n$. *Ans.* $\dfrac{dy}{dx} = mnb\,(a+bx^m)^{n-1}x^{m-1}.$

28. $y = \dfrac{1-x}{\sqrt{1+x^2}}$. *Ans.* $\dfrac{dy}{dx} = -\dfrac{1+x}{(1+x^2)^{\frac{3}{2}}}$.

29. $y = \dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{\sqrt{a}\,(\sqrt{x}-\sqrt{a})}{2\sqrt{x}\sqrt{a+x}\,(\sqrt{a}+\sqrt{x})^2}$.

30. $y = (2\sqrt{a}+\sqrt{x})\sqrt{\sqrt{a}+\sqrt{x}}$.

Ans. $\dfrac{dy}{dx} = \dfrac{4\sqrt{a}+3\sqrt{x}}{4\sqrt{x}\sqrt{\sqrt{a}+\sqrt{x}}}$.

TRANSCENDENTAL FUNCTIONS.

THE LOGARITHMIC FUNCTION.

30. *The differential of the logarithm of a variable is equal to the modulus of the system into the differential of the variable divided by the variable.*

Let u be the variable, and let m be the modulus of a system of logarithms whose base is a.

To prove $\qquad d(\log_a u) = m\dfrac{du}{u}\cdot$

Let c be any constant and let

$$u = cv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Differentiating, $\qquad du = cdv;$

or, $\qquad du = \dfrac{u}{v}dv.$

Hence $$\frac{du}{u} = \frac{dv}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Applying logarithms to (a)

$$\log_a u = \log_a v + \log_a c;$$

Differentiating, $\qquad d(\log_a u) = d(\log_a v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$

Dividing (c) by (b), we have

$$\frac{d(\log_a u)}{\dfrac{du}{u}} = \frac{d(\log_a v)}{\dfrac{dv}{v}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Now let us consider v at some one of its values, say v'.

Then the ratio $\qquad \left.\dfrac{d(\log_a v)}{\dfrac{dv}{v}}\right|_{v'} = m,$

where m is some constant. When $v = v'$ we have from (a) $u' = cv'$, i.e., some particular value of u. Since the ratios in (d) are always equal, we also have,

$$\left.\frac{d(\log_a u)}{\frac{du}{u}}\right]_{u' = cv'} = m.$$

But c is *any* constant, $\therefore$ u' $(= cv')$ is *any* value of u, $\therefore$, generally,

$$\frac{d(\log_a u)}{\frac{du}{u}} = m.$$

Hence,
$$d(\log_a u) = m\frac{du}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since the constant m depends only upon the constant a for its value, it is the *Modulus* of the system of logarithms whose base is a.

Cor. In the Napierian system the modulus is unity. Representing the base of that system by e, we have,

$$d(\log_e u) = \frac{du}{u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence, *The differential of the Napierian logarithm of a variable is equal to the differential of the variable divided by the variable.*

31. Remark.—The simplicity of equation (11) as compared with equation (10) explains the reason for the almost exclusive use of the Napierian system of logarithms in the higher analysis. We shall therefore restrict our attention to this system in the investigations which follow, unless the contrary is expressly indicated by the use of some subscript to the logarithmic symbol.

32. By means of the formula derived in article 30 for the differential of a logarithm we can now show that $d(u^n) = nu^{n-1}du$ when n is incommensurable. See § 27, Case 4.

Let $$y = u^n,$$

in which n is any incommensurable number.

Applying logarithms, $\log y = n \log u$;

$$\therefore \frac{dy}{y} = n\frac{du}{u},$$

$$\text{i.e.,}\ \frac{d(u^n)}{u^n} = n\frac{du}{u}.$$

Hence, $$d(u^n) = nu^{n-1}du.$$

Equation (9) is therefore true in all cases. The above method is of course applicable whatever the value of n.

THE EXPONENTIAL FUNCTIONS.

33. To prove

$$d(u^v) = vu^{v-1}du + u^v \log u dv,$$

in which u and v are variables.

Let $y = u^v$; then

$$\log y = v \log u.$$

$$\therefore \frac{dy}{y} = v\frac{du}{u} + \log u dv,$$

$$\text{i.e.,}\ \frac{d(u^v)}{u^v} = v\frac{du}{u} + \log u dv;$$

hence $$d(u^v) = vu^{v-1}du + u^v \log u dv \quad . \quad . \quad . \quad . \quad (12)$$

Cor. 1. Let $v = n =$ a constant; then formula (12) gives directly and generally

$$d(u^n) = nu^{n-1}du,$$

as previously determined. See §§ 27, 32.

Cor. 2. Let $u = a =$ a positive constant; then formula (12) becomes

$$d(a^v) = a^v \log a dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, *the differential of a constant affected with a variable exponent is equal to the constant affected with the same exponent into the logarithm of the constant into the differential of the exponent.*

COR. 3. If $u = e =$ base of the Napierian system, we have (since $\log e = 1$),

$$d(e^v) = e^v dv. \quad \text{. . . . .} \quad (14)$$

If we compare formulas (9) and (13) with formula (12) above, we see that the following rule may be given for the differential of a variable affected with a variable exponent:

The differential of a variable affected with a variable exponent is equal to the sum of the results obtained by differentiating, considering first one variable and then the other to be a constant.

34. Formulas.

$$d(\log_a u) = m\frac{du}{u}.$$

$$d(\log u) = \frac{du}{u}.$$

$$d(u^v) = vu^{v-1}du + u^v \log u\, dv.$$

$$d(a^v) = a^v \log a\, dv.$$

$$d(e^v) = e^v dv.$$

EXAMPLES.

Differentiate: *Ans.*

1. $\log x^2$. $\quad d(\log x^2) = \frac{2}{x}dx.$

2. $\log(3ax + x^3)$. $\quad d(\log(3ax + x^3)) = 3\frac{a + x^2}{3ax + x^3}dx$

3. $y = \log\sqrt{1 - x^3}$. $\quad dy = \frac{3}{2}\frac{x^2}{x^3 - 1}dx.$

4. $y = x^m e^x$. $\quad dy = x^{m-1}(m + x)e^x dx.$

5. $y = (x^2 - 2x + 2)e^x$. $dy = x^2e^x dx$.

6. $y = e^x(1 - x^3)$. $dy = (1 - 3x^2 - x^3)e^x dx$.

7. $y = \log(\log x)$. $dy = \dfrac{dx}{x \log x}$.

8. $y = x \log x$. $dy = (1 + \log x)\,dx$.

9. $y = x^n a^x$. $dy = a^x(n + x \log a)x^{n-1}dx$.

10. $y = \log_a x^3$. $dy = \dfrac{3m}{x}dx$.

11. $y = \dfrac{e^x}{1+x}$. $dy = \dfrac{e^x}{(1+x)^2}x dx$.

12. $y = e^x \log x$. $dy = e^x\left(\dfrac{1}{x} + \log x\right)dx$.

13. $y = x^x$. $dy = (\log x + 1)x^x dx$.

14. $y = x^{x^x}$. $dy = \left[\log x(\log x + 1) + \dfrac{1}{x}\right]x^{x^x}x^x dx$

15. $y = x^{\frac{1}{x}}$. $dy = (1 - \log x)x^{\frac{1}{x} - 2}dx$.

16. $y = a^{e^x}$. $dy = ye^x \log a dx$.

17. $y = e^{e^x}$. $dy = ye^x dx$.

18. $y = e^{x^x}$. $dy = yx^x(\log x + 1)\,dx$.

19. $y = \left(\dfrac{a}{x}\right)^x$. $dy = \dfrac{a^x}{x^x}\left(\log\dfrac{a}{x} - 1\right)dx$.

20. $y = e^{mx}\log x$. $dy = e^{mx}\left(\dfrac{1}{x} + m\log x\right)dx$.

21. $y = \log x . \log(\log x) - \log x$.

$$dy = \frac{\log(\log x)\,dx}{x}.$$

22. $y=(x^3-3x^2+6x-6)e^x$. $dy=x^3e^x dx$.

23. $y=\sqrt{x}-\log(1+\sqrt{x})$. $dy=\dfrac{dx}{2(1+\sqrt{x})}$.

24. $y=\log(1-x)+\dfrac{x}{1-x}\log x$. $dy=\dfrac{\log x}{(1-x)^2}dx$.

25. $y=\log^5 x$. $dy=5\log^4 x\dfrac{dx}{x}$.

26. $y=\log\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$. $dy=\dfrac{dx}{1-x^2}$.

27. $y=\log\dfrac{\sqrt{1-x^2}+x\sqrt{2}}{\sqrt{1-x^2}}$. $dy=\dfrac{dx\sqrt{2}}{(\sqrt{1-x^2}+x\sqrt{2})(1-x^2)}$.

28. $y=\log(\sqrt{x+m}+\sqrt{x-n})$. $dy=\dfrac{dx}{2\sqrt{(x+m)(x-n)}}$.

29. $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. $dy=\dfrac{4\,dx}{(e^x+e^{-x})^2}$.

It frequently happens that the process of differentiation of algebraic functions may be greatly simplified by applying logarithms before beginning the operation. Thus,

30. $y=\dfrac{x^m}{(1+x)^m}$ $\therefore\ \log y=m\log x-m\log(1+x)$

$$\therefore\ \frac{dy}{y}=m\left\{\frac{dx}{x}-\frac{dx}{1+x}\right\}$$

$$=m\frac{dx}{x(1+x)}$$

$$\therefore\ dy=my\frac{dx}{x(1+x)}$$

$$=m\frac{x^{m-1}}{(1+x)^{m+1}}dx.$$

Solve the following by this process:

Ans.

31. $y = \dfrac{x}{1 + x}$. $\qquad dy = \dfrac{dx}{(1 + x)^2}$.

32. $y = (1 + m^x)^2$. $\qquad dy = 2\,(1 + m^x)\,m^x \log m dx$.

33. $y = \dfrac{(1 + x^2)x}{\sqrt{1 - x^2}}$. $\qquad dy = \dfrac{1 + 3x^2 - 2x^4}{\sqrt{(1 - x^2)^3}}\,dx$.

34. $y = \sqrt{\dfrac{(x + a)^3}{x - a}}$. $\qquad dy = (x - 2a)\sqrt{\dfrac{x + a}{(x - a)^3}}\,dx$.

THE TRIGONOMETRIC FUNCTIONS.

35. *The differential of the sine of an angle is equal to the cosine of the angle into the differential of the angle.*

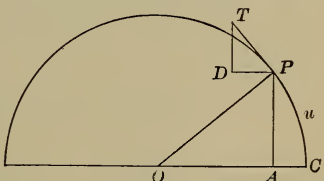


Fig. 4.

Let POC be any angle generated by the line OP, taken as the linear unit, revolving upward about O as an axis; then, in circular measure,

$$\text{Length of } PC = u = \text{measure of } POC.$$

If *length* u becomes a uniformly changing variable at the instant the generating point reaches the position P, then, § 18,

$$PT = du \text{ and } DT = d \sin u$$

(since $AP = \sin u$). From the right triangle DTP, we have,

$$DT = PT \cos DTP,$$

i.e., $$d\,(\sin u) = \cos u\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

36. *The differential of the cosine of an angle is equal to minus the sine of the angle into the differential of the angle.*

From the right triangle DTP, Fig. 4, we have

$$DP = PT \sin DTP,$$

i.e., $$-d \cos u = \sin u \, du,$$

or $$d(\cos u) = -\sin u du \quad . \quad . \quad . \quad . \quad . \quad (16)$$

since $OA = \cos u$ and OA is a decreasing variable, § (8), II.

Otherwise, thus: let $u = \frac{\pi}{2} - u$ in equa. (15), then

$$d \sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right),$$

i.e., $$d(\cos u) = -\sin u \, du.$$

37. *The differential of the tangent of an angle is equal to the square of the secant of the angle into the differential of the angle.*

To prove

$$d(\tan u) = \sec^2 u \, du.$$

From trigonometry,

$$\tan u = \frac{\sin u}{\cos u}.$$

Differentiating,

$$d(\tan u) = \frac{\cos u(\cos u \, du) - \sin u(-\sin u \, du)}{\cos^2 u}$$

$$= \frac{(\cos^2 u + \sin^2 u) \, du}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u} du,$$

$$\therefore d(\tan u) = \sec^2 u du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

38. *The differential of the cotangent of an angle is equal to minus the square of the cosecant into the differential of the angle.*

To prove

$$d(\cot u) = -\csc^2 u\, du.$$

We know that
$$\cot u = \frac{1}{\tan u},$$

$$\therefore\ d(\cot u) = -\frac{d(\tan u)}{\tan^2 u},$$

$$= -\frac{\sec^2 u}{\tan^2 u}du,$$

$$\therefore\ d(\cot u) = -\csc^2 u\, du \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Otherwise, thus; let $u = \left(\frac{\pi}{2} - u\right)$. Substituting in equation (17), we have

$$d\left[\tan\left(\frac{\pi}{2} - u\right)\right] = \sec^2\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right),$$

i.e.,
$$d(\cot u) = -\csc^2 u\, du.$$

The complementary functions which follow may be differentiated by the student in the same way.

39. *The differential of the secant of an angle is equal to the product of the secant, the tangent and the differential of the angle.*

To prove

$$d(\sec u) = \sec u \tan u\, du.$$

We know that
$$\sec u = \frac{1}{\cos u},$$

$$\therefore\ d(\sec u) = -\frac{d(\cos u)}{\cos^2 u}$$

$$= -\frac{(-\sin u\, du)}{\cos^2 u},$$

$$\therefore\ d(\sec u) = \sec u \tan u\, du \quad . \quad . \quad . \quad . \quad (19)$$

40. *The differential of the cosecant of an angle is equal to minus the product of the cosecant, the cotangent and the differential of the angle.*

To prove

$$d(\csc u) = -\csc u \cot u\, du.$$

We know that $\csc u = \frac{1}{\sin u}$,

$$\therefore\ d(\csc u) = -\frac{\cos u\, du}{\sin^2 u}$$

$$= -\csc u \cot u\, du. \quad . \quad . \quad . \quad (20)$$

41. *The differential of the versine of an angle is equal to the sine of the angle into the differential of the angle.*

To prove

$$d(\text{vers } u) = \sin u\, du.$$

We know that $\text{vers } u = 1 - \cos u$,

$$\therefore\ d(\text{vers } u) = -d \cos u$$

$$= \sin u\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

42. *The differential of the coversine of an angle is equal to minus the cosine of the angle into the differential of the angle.*

To prove

$$d(\text{covers } u) = -\cos u\, du.$$

We know that $\text{covers } u = 1 - \sin u$,

$$\therefore\ d(\text{covers } u) = -\cos u\, du \quad . \quad . \quad . \quad . \quad (22)$$

43. Formulas.

$$d \sin u = \cos u\, du$$

$$d \cos u = -\sin u\, du$$

$$d \tan u = \sec^2 u\, du$$

$$d \cot u = -\csc^2 u\, du$$

$$d \sec u = \sec u \tan u\, du$$

$$d \csc u = -\csc u \cot u\, du$$

$$d \text{ vers } u = \sin u\, du$$

$$d \text{ covers } u = -\cos u\, du$$

EXAMPLES.

1. $y = \sin 3x$. $\qquad dy = 3\cos 3x dx$.

2. $y = \sin^3 3x$. $\qquad dy = 9\sin^2 3x \cos 3x dx$.

3. $y = \cos mx$. $\qquad dy = -m \sin mx dx$.

4. $y = \tan^2 5x$. $\qquad dy = 10 \tan 5x \sec^2 5x dx$.

5. $y = \cot x^2$. $\qquad dy = -2x csc^2 x^2 dx$.

6. $y = \sec 4x$ $\qquad dy = 4 \sec 4x \tan 4x dx$.

7. $y = \sec^2 nx$. $\qquad dy = 2n \sec^2 nx \tan nx dx$.

8. $y = \log \sin x$. $\qquad dy = \cot x dx$.

9. $y = x^{\sin x}$. $\qquad dy = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) dx$.

10. $y = \frac{\tan x - \tan^3 x}{\sec^4 x}$. $\qquad dy = \cos 4x dx$.

11. $y = \sin(\log x)$. $\qquad dy = \frac{\cos(\log x)}{x} dx$.

12. $y = csc^n x$. $\qquad \frac{dy}{dx} = -n csc^n x \cot x$.

13. $y = \sin(1 + x^2)$. $\qquad \frac{dy}{dx} = 2x \cos(1 + x^2)$.

14. $y = \sin(\sin x)$. $\qquad \frac{dy}{dx} = \cos x \cos(\sin x)$.

15. $y = \cos mx \cos nx$.

$$\frac{dy}{dx} = -(m \cos nx \sin mx + n \cos mx \sin nx).$$

16. If $\sin 2x = 2 \sin x \cos x$ prove by differentiation that $\cos 2x = \cos^2 x - \sin^2 x$, and conversely.

17. If $2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$ prove by differentiation that

$$\cos^2 x - \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$ and conversely.

18. If $\sin(x+y) = \sin x \cos y + \cos x \sin y$ prove by differentiation that $\cos(x+y) = \cos x \cos y - \sin x \sin y$, and conversely.

19. If $\sin(x-y) = \sin x \cos y - \cos x \sin y$ prove by differentiation that $\cos(x-y) = \cos x \cos y + \sin x \sin y$, and conversely.

20. If $\sin \frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ prove that $\cos \frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$.

THE CIRCULAR FUNCTIONS.

44. *The differential of an angle is equal to the differential of its sine divided by the square root of one minus the square of the sine.*

To prove

$$d(\sin^{-1} u) = \frac{du}{\sqrt{1-u^2}}.$$

Let $v = \sin^{-1} u$. Then $\sin v = u$,

$$\therefore \cos v dv = du,$$

$$\therefore dv = d(\sin^{-1} u) = \frac{du}{\cos v};$$

but $\cos v = \sqrt{1-\sin^2 v} = \sqrt{1-u^2}$.

Hence $$d(\sin^{-1} u) = \frac{du}{\sqrt{1-u^2}} \quad . \quad . \quad . \quad . \quad (23)$$

The student may deduce this as well as all the following formulae by solving the formulae for the differential of the trigonometric functions for dv. Thus

Art. (35), Equa. (15), $d(\sin v) = \cos v dv$,

$$\therefore dv = \frac{d(\sin v)}{\cos v} = \frac{d(\sin v)}{\sqrt{1-\sin^2 v}}.$$

45. *The differential of an angle is equal to minus the differential of its cosine divided by the square root of one minus the square of the cosine.*

To prove $$d(\cos^{-1}u) = -\frac{du}{\sqrt{1-u^2}}.$$

Let $v = \cos^{-1}u$. Then $\cos v = u$,

$$\therefore\ -\sin v dv = du,$$

$$\therefore\ dv = d(\cos^{-1}u) = -\frac{du}{\sin v};$$

but $$\sin v = \sqrt{1-\cos^2 v} = \sqrt{1-u^2}.$$

Hence $$d(\cos^{-1}u) = -\frac{du}{\sqrt{1-u^2}} \quad . \quad . \quad (24)$$

46. *The differential of an angle is equal to the differential of its tangent divided by one plus the square of the tangent.*

To prove $$d(\tan^{-1}u) = \frac{du}{1+u^2}.$$

Let $v = \tan^{-1}u$. Then $\tan v = u$,

$$\therefore\ \sec^2 v dv = du,$$

$$\therefore\ dv = d(\tan^{-1}u) = \frac{du}{\sec^2 v};$$

but $$\sec^2 v = 1 + \tan^2 v = 1 + u^2.$$

Hence $$d(\tan^{-1}u) = \frac{du}{1+u^2} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

47. *The differential of an angle is equal to minus the differential of its cotangent divided by one plus the square of the cotangent.*

To prove $$d(\cot^{-1}u) = -\frac{du}{1+u^2}.$$

Let $v = \cot^{-1}u$. Then $\cot v = u$,

$$\therefore\ -\csc^2 v dv = du,$$

$$\therefore\ dv = d(\cot^{-1}u) = -\frac{du}{\csc^2 v}.$$

But $\csc^2 v = 1 + \cot^2 v = 1 + u^2$.

Hence $$d(\cot^{-1} u) = -\frac{du}{1+u^2}. \quad . \quad . \quad . \quad . \quad (26)$$

48. *The differential of an angle is equal to the differential of its secant divided by the secant into the square root of the square of the secant minus one.*

Let $v = \sec^{-1} u$. Then $\sec v = u$.

Hence $$d(\sec^{-1} u) = \frac{du}{u\sqrt{u^2-1}}. \quad . \quad . \quad . \quad . \quad (27)$$

The exercise is left for the student.

49. *The differential of an angle is equal to minus the differential of its cosecant divided by the cosecant into the square root of the square of the cosecant minus one.*

To prove $$d(\csc^{-1} u) = -\frac{du}{u\sqrt{u^2-1}}. \quad . \quad . \quad . \quad . \quad (28)$$

The exercise is left for the student.

50. *The differential of an angle is equal to the differential of its versine divided by the square root of twice the versine minus the square of the versine.*

To prove $$d(\text{vers}^{-1} u) = \frac{du}{\sqrt{2u-u^2}}. \quad . \quad . \quad . \quad . \quad (29)$$

The exercise is left for the student.

51. *The differential of an angle is equal to minus the differential of its coversine divided by the square root of twice the coversine minus the square of the coversine.*

To prove $$d(\text{covers}^{-1} u) = -\frac{du}{\sqrt{2u-u^2}}. \quad . \quad . \quad . \quad . \quad (30)$$

The exercise is left for the student.

52. Formulas.

$$d\,(\sin^{-1} u) = \frac{du}{\sqrt{1-u^2}}.$$

$$d\,(\cos^{-1} u) = -\frac{du}{\sqrt{1-u^2}}.$$

$$d\,(\tan^{-1} u) = \frac{du}{1+u^2}.$$

$$d\,(\cot^{-1} u) = -\frac{du}{1+u^2}.$$

$$d\,(\sec^{-1} u) = \frac{du}{u\sqrt{u^2-1}}.$$

$$d\,(\csc^{-1} u) = -\frac{du}{u\sqrt{u^2-1}}.$$

$$d\,(\text{vers}^{-1} u) = \frac{du}{\sqrt{2u-u^2}}.$$

$$d\,(\text{covers}^{-1} u) = -\frac{du}{\sqrt{2u-u^2}}$$

EXAMPLES.

1. $y = \sin^{-1}(3x-1)$. $\qquad dy = \dfrac{3\,dx}{\sqrt{6x-9x^2}}.$

2. $y = \sin^{-1}\dfrac{x}{a}$. $\qquad dy = \dfrac{dx}{\sqrt{a^2-x^2}}.$

3. $y = x\sin^{-1}x$. $\qquad dy = \left(\sin^{-1}x + \dfrac{x}{\sqrt{1-x^2}}\right)dx.$

4. $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$. $\qquad dy = \dfrac{dx}{\sqrt{1-x^2}}.$

5. $y = \tan^{-1}\dfrac{2x}{1-x^2}$. $\qquad dy = \dfrac{2\,dx}{1+x^2}.$

6. $y = \tan^{-1}\dfrac{2x}{1+x^2}$. $\qquad dy = \dfrac{2(1-x^2)\,dx}{1+6x^2+x^4}$.

7. $y = \cos^{-1}\dfrac{x}{a}$. $\qquad dy = -\dfrac{dx}{\sqrt{a^2-x^2}}$.

8. $y = \cos^{-1}\dfrac{x^{2n}-1}{x^{2n}+1}$. $\qquad dy = -\dfrac{2nx^{n-1}}{x^{2n}+1}\,dx$.

9. $y = \sec^{-1} mx$. $\qquad dy = \dfrac{dx}{x\sqrt{m^2x^2-1}}$.

10. $y = \sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}}$. $\qquad dy = \dfrac{dx}{\sqrt{a^2-x^2}}$.

11. $y = \sin^{-1}\sqrt{\sin x}$. $\qquad dy = \dfrac{\sqrt{1+\csc x}}{2}\,dx$.

12. $y = \text{vers}^{-1}\dfrac{8x}{9}$. $\qquad dy = \dfrac{2\,dx}{\sqrt{9x-4x^2}}$.

13. $y = (a^2+x^2)\tan^{-1}\dfrac{x}{a}$. $\qquad \dfrac{dy}{dx} = 2x\tan^{-1}\dfrac{x}{a} + a$.

14. $y = (x+a)\tan^{-1}\dfrac{\sqrt{x}}{\sqrt{a}} - \sqrt{ax}$. $\qquad \dfrac{dy}{dx} = \tan^{-1}\sqrt{\dfrac{x}{a}}$.

15. $y = \cos^{-1}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2}{e^x+e^{-x}}$.

16. $y = e^{\tan^{-1}x}$. $\qquad \dfrac{dy}{dx} = \dfrac{e^{\tan^{-1}x}}{1+x^2}$.

17. $y = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

18. $y = \sin^{-1}\dfrac{x+1}{\sqrt{2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-2x-x^2}}$.

19. $y = \sec^{-1}\dfrac{1}{2x^2-1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$.

20. $y = x\sqrt{a^2-x^2} + a^2\sin^{-1}\dfrac{x}{a}$. $\qquad \dfrac{dy}{dx} = 2\sqrt{a^2-x^2}$.

21. $y = \tan^{-1}\frac{x}{a} + \log\sqrt{\frac{x-a}{x+a}}$. $\frac{dy}{dx} = \frac{2ax^2}{x^4 - a^4}$.

22. $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$. $\frac{dy}{dx} = \sec x$.

23. $y = \log\tan^{-1}x$. $\frac{dy}{dx} = \frac{1}{(1+x^2)\tan^{-1}x}$.

24. $y = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$. $\frac{dy}{dx} = \frac{3}{1+x^2}$.

25. $y = \tan^{-1}x + \frac{1}{x}$. $\frac{dy}{dx} = -\frac{1}{x^2(1+x^2)}$.

26. $y = x - \sqrt{1-x^2}\sin^{-1}x$. $\frac{dy}{dx} = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$.

27. $y = (2x^2 - 1)\sin^{-1}x + x\sqrt{1-x^2}$.

$$\frac{dy}{dx} = 4x\sin^{-1}x.$$

28. $y = \tan^{-1}\sin^{-1}x$.

$$\frac{dy}{dx} = \frac{1}{(1 + (\sin^{-1}x)^2)\sqrt{1-x^2}}.$$

29. $y = x^2 + (\sin^{-1}x)^2 - 2\sin^{-1}x \,.\, x\sqrt{1-x^2}$.

$$\frac{dy}{dx} = \frac{4x^2\sin^{-1}x}{\sqrt{1-x^2}}.$$

30. $y = \tan^{-1}\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$. $\frac{dy}{dx} = \frac{1}{2}$.

31. $y = \log\sqrt{\frac{1-\cos x}{1+\cos x}}$. $\frac{dy}{dx} = \csc x$.

32. $y = \cos^{-1}\frac{3 + 5\cos x}{5 + 3\cos x}$. $\frac{dy}{dx} = \frac{4}{5 + 3\cos x}$.

33. $y = \frac{(1-x^2)^{\frac{3}{2}}\sin^{-1}x}{x}$.

$$\frac{dy}{dx} = \frac{1-x^2}{x} - \frac{1+2x^2}{x^2}(1-x^2)^{\frac{1}{2}}\sin^{-1}x.$$

34. $y = \log \dfrac{\sqrt{1 - x\sqrt{2} + x^2}}{\sqrt{1 + x\sqrt{2} + x^2}} + \tan^{-1} \dfrac{x\sqrt{2}}{1 - x^2}.$

$$\frac{dy}{dx} = \frac{2x^2\sqrt{2}}{1 + x^4}.$$

35. $y = \log \dfrac{1 + x}{1 - x} + \dfrac{1}{2} \log \dfrac{1 + x + x^2}{1 - x + x^2} + \sqrt{3} \tan^{-1} \dfrac{x\sqrt{3}}{1 - x^2}.$

$$\frac{dy}{dx} = \frac{6}{1 - x^6}.$$

CHAPTER IV.

LIMITS.

HISTORY. — What is known as the "method of limits" in the Calculus is founded on the following lemma in the first book of Newton's *Principia* (1687):

"Quantities and ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."

53. In deriving the differential formulae in Chapter III., we have taken as the basis of the operation what is known among mathematicians as the "**Method of Rates.**" We shall consider in this chapter another method, known as the "**Method of Limits,**" and show how by this method all the foregoing differential forms may be derived.

54. Limit. *The limit of a variable is a fixed value from which it can be made to differ by less than any assignable quantity but which it never reaches.* Thus,

Limit of x = Limit (.66666 . . .) = $\frac{2}{3}$, as the figure 6 is annexed an indefinite number of times.

Limit of x = Limit $(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots) = 2$, as the number of terms indefinitely increase.

The limits of the area and of the perimeter of a regular inscribed polygon are respectively, the area and circumference of the circumscribing circle as the number of sides of the polygon indefinitely increase.

Limit $\dfrac{\sin x}{\tan x} = 1$, as the value of x indefinitely diminishes, i.e., as x approaches the value zero.

Cor. It is evident from the definition that the *difference between a variable and its limit is a variable whose limit is zero.*

55. The student should be careful to distinguish the *limit of a variable* as above defined from the term limit as ordinarily used. Thus in the circle $x^2 + y^2 = a^2$, or $y = \pm\sqrt{a^2 - x^2}$, we are accustomed to say that the *limits* of the values of x are $\pm a$. By the term limit as thus used we mean that beyond these values there are no corresponding values of y, i.e., there are no points on the locus.

56. Principles. I. *If two variables are always equal and each approaches a limit, their limits are equal.*

Let $u = v$, and let limit of $u = a$ and limit of $v = b$; since $u = v$, we have $a - u = a - v$. But a is the limit of u; hence a is also the limit of v, § 54, Cor. But b is the limit of v; hence

$$a = b.$$

II. *The limit of the sum of any number of variables is the sum of their limits.*

Let $u, v, w, \ldots$ be any number of variables whose limits are $a, b, c, \ldots$, respectively, then

$$(a - u) + (b - v) + (c - w) + \ldots,$$

or

$$(a + b + c + \ldots) - (u + v + w + \ldots)$$

is a quantity whose limit is zero. § 54, Cor.

Hence Limit $(u + v + w \ldots) = (a + b + c \ldots)$.

III. *The limit of the product of any number of variables is the product of their limits.*

Let u and v be variables whose limits are a, and b, respectively.

Let $$a - u = x, \text{ and } b - v = y,$$

in which x and y are variables whose limits are zero. § 54.

Hence $$u = a - x,\ v = b - y,$$

hence $$uv = ab - bx - ay + xy;$$

$$\therefore \quad uv - ab = -bx - ay + xy.$$

But the limit of the second member of this equation is zero;

Hence $$\text{Limit } (uv) = ab.$$

And so for any number of variables.

57. Notation. In order to express the fact that a given function approaches a certain limit as the variable which enters it approaches a certain other limit, it is convenient to adopt some form of notation. The form in common use is illustrated by the following:

$$\text{Limit} \left[\frac{\sin x}{x}\right]_{x \doteq 0} = 1.$$

This expression is equivalent to the sentence 'the limit of $\frac{\sin x}{x}$, as x approaches zero as its limit, is 1.'

58. *The* **Differential Coefficient** *or* **First Derivative** *of a function is the limit of the ratio of the increment of the function to the increment of its variable as the increment of the variable approaches zero as its limit.*

Thus let $y = f(x)$ be any function of x, and let Δx be an increment of x and Δy be the corresponding increment of the function. Let the *complex symbol* $\frac{dy}{dx}$ represent the differential coefficient, or first derivative of $y = f(x)$; then, by definition

$$\text{Limit} \left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \frac{dy}{dx}.$$

59. REMARK. — The student should carefully observe that the symbol $\frac{dy}{dx}$ as here used is a symbol representing *the limiting value of a ratio.* It is *not* therefore a fraction, the numerator and denominator being dy and dx, respectively. We shall show in a subsequent article, see § 68, that the *fraction* $\frac{dy}{dx}$ is equal to Limit $\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0}$.

EXAMPLES.

Find the first derivative of the following:

1. $y = x^2$.

Let y' = value of y when $x = x + \Delta x$; that is, when x has taken on the increment Δx; then

$$y' = (x + \Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2,$$

$$\therefore\ y' - y = (2x + \Delta x)\,\Delta x;$$

But $$y' - y = \Delta y,$$

$$\therefore\ \frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Hence, Limit $\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = 2x$,

i.e., $$\frac{dy}{dx} = 2x.$$

2. $y = x^3 + 3$.

$$\therefore\ y' = (x + \Delta x)^3 + 3 = x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 3,$$

$$\therefore\ y' - y = \Delta y = 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3,$$

$$\therefore\ \frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2.$$

Hence, $$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = 3x^2,$$

i.e., $$\frac{dy}{dx} = 3x^2.$$

3. $$y = (2x - 1)(x + 2)$$
$$= 2x^2 + 3x - 2,$$
$$\therefore\ y' = 2(x + \Delta x)^2 + 3(x + \Delta x) - 2$$
$$= 2x^2 + 3x - 2 + 4x\Delta x + 3\Delta x + 2\Delta x^2,$$
$$\therefore\ \Delta y = y' - y = (4x + 3 + 2\Delta x)\Delta x,$$
$$\therefore\ \frac{\Delta y}{\Delta x} = 4x + 3 + 2\Delta x.$$

Hence, $$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = 4x + 3;$$

i.e., $$\frac{dy}{dx} = 4x + 3.$$

4. $$y = \frac{m}{x^2},$$
$$\therefore\ y' = \frac{m}{(x + \Delta x)^2},$$
$$\therefore\ \Delta y = \frac{m}{(x + \Delta x)^2} - \frac{m}{x^2} = -m\frac{(2x + \Delta x)\Delta x}{x^2(x + \Delta x)^2},$$
$$\therefore\ \frac{\Delta y}{\Delta x} = -m\frac{2x + \Delta x}{x^2(x + \Delta x)^2}.$$

Hence, $$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = -m\frac{2x}{x^4},$$
$$\therefore\ \frac{dy}{dx} = -\frac{2m}{x^3}.$$

5. $y = \dfrac{x + a}{2x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a}{2x^2}$.

6. $y = \sqrt{x^2 + a}$. $\qquad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + a}}$.

7. $y = a\sqrt{x^3}$. $\qquad \dfrac{dy}{dx} = \dfrac{3}{2}a\sqrt{x}$.

8. $y = \dfrac{m}{\sqrt{x^2 + 2}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{mx}{\sqrt{(x^2 + 2)^3}}$.

9. $y = \frac{a + bx + cx^2}{x}$. $\qquad \frac{dy}{dx} = c - \frac{a}{x^2}$.

10. $y = f(x)$. $\qquad \frac{dy}{dx} = \text{Limit}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]_{\Delta x \doteq 0}$.

60. The foregoing examples illustrate the meaning of the term 'differential coefficient' and explain the process by which it may be derived in any given case. The process, however, is lengthy and tedious, and, in a large majority of cases, very difficult. In the practical application of the 'method of limits' to the derivation of differential coefficients it is usual to derive a system of rules by aid of which the operation is greatly simplified. These rules have been derived already by the principles of the '*method of rates*' (Chapter III). We shall now show how they may be derived by the 'method of limits.'

61. To prove

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Let $y = u + v$, in which y, u and v are *functions* of x, and let $y = y'$, $u = u'$, $v = v'$ when $x = x + \Delta x$.

Then $$y' = u' + v',$$

and $$\Delta y = y' - y = u' + v' - (u + v)$$
$$= u' - u + (v' - v);$$

i.e., $$\Delta y = \Delta u + \Delta v.$$

Hence $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Therefore § 56, II.

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0} + \text{Limit}\left[\frac{\Delta v}{\Delta x}\right]_{\Delta x \doteq 0};$$

i.e., § 58, $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Replacing y in the first member by its value $u + v$, we have

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \qquad \text{Compare § (24), 2.}$$

62. To prove

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Let $\qquad y = uv;$

then, $\qquad y' = u'v',$

hence $\qquad y' - y = u'v' - uv.$

Adding and subtracting uv' in the second member, we have

$$y' - y = u(v' - v) + v'(u' - u);$$

that is, $\qquad \Delta y = u\,\Delta v + v'\,\Delta u.$

$$\therefore \frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v'\frac{\Delta u}{\Delta x}.$$

Hence, § 56, II., III.,

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[u\frac{\Delta v}{\Delta x}\right]_{\Delta x \doteq 0} + \text{Limit}\left[v'\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0};$$

i.e., $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

since v is the limit of v' as Δx approaches the limit zero.

Hence, since $y = uv$,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \qquad \text{Compare § (25), 3.}$$

63. To prove

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Let $$y = \frac{u}{v};$$

then, $$y' = \frac{u'}{v'};$$

$$\therefore\ y' - y = \frac{u'}{v'} - \frac{u}{v}$$

$$= \frac{u'v - v'u}{v'v}$$

$$= \frac{v(u' - u) - u(v' - v)}{v'v}.$$

Hence, $$\Delta y = \frac{v\,\Delta u - u\,\Delta v}{v'v}.$$

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{v}{v'v}\frac{\Delta u}{\Delta x} - \frac{u}{v'v}\frac{\Delta v}{\Delta x};$$

$\therefore$ § 56, II., III.,

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[\frac{v}{v'v}\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0} - \text{Limit}\left[\frac{u}{v'v}\frac{\Delta v}{\Delta x}\right]_{\Delta x \doteq 0};$$

i.e., $$\frac{dy}{dx} = \frac{v}{v^2}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx};$$

(since limit of v' as Δx approaches zero is v)

or $$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$ Compare § (26), 6.

64. To prove

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$$

in which n is a positive integer.

Let $$y = u^n;$$

then, $$y' = u'^n;$$

$$\therefore\ \Delta y = u'^n - u^n$$

$$= (u' - u)(u'^{n-1} + u'^{n-2}u + u'^{n-3}u^2 + \ldots u^{n-1})$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}\left(u'^{n-1} + u'^{n-2}u + u'^{n-3}u^2 + \ldots u^{n-1}\right).$$

Hence, § 56, III.,

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[u'^{n-1}\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0} + \text{Limit}\left[u'^{n-2}u\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0}$$
$$+ \ldots \text{ to } n \text{ terms.}$$

$$\therefore \frac{dy}{dx} = u^{n-1}\frac{du}{dx} + u^{n-1}\frac{du}{dx} + \ldots \text{ to } n \text{ terms,}$$

since the limit of u' as Δx approaches zero as a limit is u.

Hence, $$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$ Compare § (27), 9.

65. To prove

$$\frac{d(\log_a u)}{dx} = \frac{m}{u}\frac{du}{dx}.$$

Let $$y = \log_a u;$$

then, $$y' = \log_a u';$$

$$\therefore \Delta y = \log_a u' - \log_a u = \log_a \frac{u'}{u}$$

$$= \log_a\left(\frac{u + \Delta u}{u}\right)$$

$$= \log_a\left(1 + \frac{\Delta u}{u}\right)$$

$$= m\left(\frac{\Delta u}{u} - \frac{\Delta u^2}{2u^2} + \frac{\Delta u^3}{3u^3} - \ldots\right)$$

$$= \frac{m}{u}\Delta u\left(1 - \frac{\Delta u}{2u} + \frac{\Delta u^2}{3u^2} - \ldots\right);$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{m}{u}\frac{\Delta u}{\Delta x}\left(1 - \frac{\Delta u}{2u} + \frac{\Delta u^2}{3u^2} - \ldots\right).$$

Hence, § 56, III.,

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[\frac{m}{u}\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0} \times$$

$$\text{Limit}\left[\left(1 - \frac{\Delta u}{2u} + \frac{\Delta u^2}{3u^2} - \dots\right)\right]_{\Delta x \doteq 0}$$

i.e., $$\frac{dy}{dx} = \frac{d(\log_a u)}{dx} = \frac{m}{u}\frac{du}{dx}.$$ Compare § (30), 10.

Since the limit of Δu as Δx approaches zero as its limit is zero.

66. To prove

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}.$$

Let $y = \sin u$;

then, $y' = \sin u'$;

$$\therefore \Delta y = \sin u' - \sin u$$

$$= 2 \cos \tfrac{1}{2}(u' + u) \sin \tfrac{1}{2}(u' - u)$$

$$= 2 \cos\left(u + \frac{\Delta u}{2}\right) \sin \frac{\Delta u}{2}; \quad (\text{Since } u' = u + \Delta u)$$

$$\therefore \frac{\Delta y}{\Delta x} = \cos\left(u + \frac{\Delta u}{2}\right) \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}} \frac{\Delta u}{\Delta x}.$$

Hence, § 56, III.,

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[\cos\left(u + \frac{\Delta u}{2}\right)\right]_{\Delta x \doteq 0} \times$$

$$\text{Limit}\left[\frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}}\right]_{\Delta x \doteq 0} \times \text{Limit}\left[\frac{\Delta u}{\Delta x}\right]_{\Delta x \doteq 0};$$

i.e., $$\frac{dy}{dx} = \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}.$$ Compare § (35), 15.

Since Limit $\left[\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}\right]_{\Delta x \doteq 0} = 1.$

67. To prove

$$\frac{d(\sin^{-1}u)}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.\quad \text{Compare § (44), 23.}$$

Let $\sin^{-1} u = v$, or $u = \sin v$;

then, $\frac{du}{dx} = \cos v \frac{dv}{dx}$;

$$\therefore \frac{dv}{dx} = \frac{\frac{du}{dx}}{\cos v} = \frac{\frac{du}{dx}}{\sqrt{1-\sin^2 v}},$$

i.e., $$\frac{d(\sin^{-1}u)}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

68. *The limiting value of the ratio of the increment of a function to the corresponding increment of the variable, as the increment of the variable approaches zero as its limit, is the ratio of the differential of the function to the differential of the variable.*

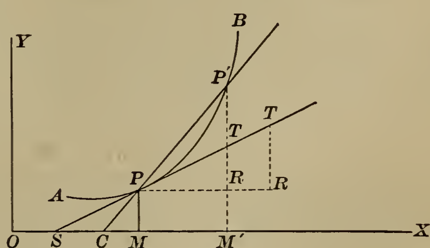


Fig. 5.

Let $y = f(x)$ be the equation of the locus AB, Fig. 5. Let $PP' = \Delta s$ and the *length* $AP = s$; then

$PR = \Delta x =$ corresponding increment of x (OM), and
$RP' = \Delta y =$ corresponding increment of y (MP).

Draw the secant PP' and the tangent TS at P;

then
$$\frac{\Delta y}{\Delta x} = \tan P'PR.$$

If we now suppose Δx to approach zero as its limit the point P' will approach P and the secant PP' will approach the tangent TS; hence

$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \text{Limit}\left[\tan P'PR\right]_{\Delta x \doteq 0} = \tan TSX.$$

But § 18, Cor., the ratio of dy to dx is equal to tan TSX, i.e.,

$$\frac{dy}{dx} = \tan TSX.$$

Hence,
$$\text{Limit}\left[\frac{\Delta y}{\Delta x}\right]_{\Delta x \doteq 0} = \frac{dy}{dx}.$$

CHAPTER V.

ANALYTICAL APPLICATIONS.

1. (*a*) Compare the rates of the ordinate and abscissa of the generating point of a circle whose radius is 5. (*b*) What does the ratio of the rates become at the points (− 3, 4), (0, 5), (5, 0)? (*c*) If the ordinate is increasing at the rate of 8 feet a second at the point whose abscissa is 4, what is the rate of the abscissa and what the velocity of the point in its path? (*d*) In which angles is the ordinate an increasing function of x?

(*a*) Referring the circle to its center and axes, we have $x^2 + y^2 = 25$ for its equation;

$$\therefore\ 2\,xdx + 2\,ydy = 0,$$

$$\therefore\ dy = -\frac{x}{y}\,dx,$$

i.e., the ordinate changes $-\frac{x}{y}$ times as fast as the abscissa.

(*b*) At the point (− 3, 4), $\frac{dy}{dx} = -\frac{x}{y} = \frac{3}{4}$; at (5, 0), $\frac{dy}{dx} = -\frac{5}{0}$ $= -\infty$; at (0, 5), $\frac{dy}{dx} = -\frac{0}{5} = 0$.

(*c*) The ordinates corresponding to $x = 4$ are found from the equation of the circle to be $y = \pm 3$, ∴ since $dy = 8$ ft. a second by hypothesis, we have

$$8 = -\frac{4}{\pm 3}\,dx = \mp\frac{4}{3}\,dx,$$

$$\therefore\ dx = \mp 6 \text{ feet a second.}$$

From § (18), 3, we have $ds = \sqrt{dx^2 + dy^2}$.

Substituting values found above we have

$$ds = \sqrt{36 + 64} = 10 \text{ feet a second.}$$

(*d*) Since $dy = -\frac{x}{y}dx$ it will be negative unless x and y have different signs, hence dy will be positive when the moving point is generating the second and fourth quadrants; therefore, § (8), II., y is an increasing function of x in those quadrants. It is obviously a decreasing function of x in the first and third quadrants.

2. (*a*) Compare the rates of the ordinate and abscissa of the parabola $y^2 = 2px$. (*b*) At what point is the rate of y equal to the rate of x? (*c*) Between what limits of x is the rate of y greater than that of x? (*d*) Between what limits is it less? (*e*) If the parameter of the curve is 8, and the generating point so moves that its abscissa is uniformly increasing at the rate of 10 feet a second, what is the velocity of the point in its path and the rate of the ordinate when $x = 8$? (*f*) What is the rate of increase of the area of the parabolic segment at the instant that $x = 8$?

(*a*) $dy = \frac{p}{y}dx$, i.e., the rate of $y = \frac{p}{y}$ times rate of x.

(*b*) By condition $dy = dx$, $\therefore 1 = \frac{p}{y} \therefore y = p$. Substituting in the equation of the parabola we find $x = \frac{p}{2} \therefore \left(\frac{p}{2}, p\right)$ or the extremity of the latus rectum is the required point. See "Analytic Geometry," p. 86.

(*c*) In the differential equation, $dy = \frac{p}{y}dx$, $dy > dx$ when $p > y$. But from the equation of the curve $p > y$ when $x < \frac{p}{2}$, $\therefore dy > dx$ between limits $x = 0$ and $x = \frac{p}{2}$.

(*d*) Similarly we can prove $dy < dx$ between limits $x = \frac{p}{2}$ and $x = \infty$.

(*e*) The equation becomes $y^2 = 8x$; $\therefore dy = \frac{4}{y} dx$. By hypothesis $dx = 10$ feet a second, and the point whose abscissa $x = 8$ has $y = 8$ for its ordinate, $\therefore dy = \frac{4}{8} 10 = 5$ feet a second. Also,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{125} = 11. + \text{ feet a second.}$$

(*f*) The area of a parabolic segment is $z = \frac{2}{3} xy$, $\therefore dz = \frac{2}{3} (xdy + ydx)$. Substituting values found under (*e*) above, we have

$$dz = \tfrac{2}{3} (40 + 80) = 80 \text{ sq. ft. a second.}$$

3. Compare the rates of the ordinates and abscissas in the following curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $xy = m$.

4. An elliptical metal plate is expanded by heat or pressure. What is the rate of change of its area when the semi-axes are 4 and 6, and each is increasing at the rate .1 in. a second? Let x and y be the semi-conjugate and semi-transverse axes, and let z be its area; then $z = \pi xy$. See "Analytic Geometry," p. 136.

$\therefore dz = \pi (xdy + ydx) = \pi (10) .1 = \pi$ sq. in. a second.

5. Steam is admitted by a valve into a circular cylinder, one end of which is closed by a piston. If the diameter of the base of the cylinder is 1 foot, and the steam is admitted at the rate of 10 cu. ft. a second, at what rate is the piston moving?

Let y = volume and x = altitude of cylinder at any instant; then

$$y = \pi\left(\frac{1}{2}\right)^2 x, \therefore dy = \frac{\pi}{4} dx,$$

$$\therefore dx = \frac{40}{\pi} \text{ feet a second.}$$

6. Gas is introduced into a thin elastic spherical film at the rate of 10 cu. ft. a second. At what rate is the radius increasing when the volume is $\frac{4000\pi}{3}$ cu. ft.?

Let y = volume and x variable radius; then

$$y = \frac{4}{3}\pi x^3, \therefore dy = 4\pi x^2 dx,$$

$$\therefore dx = \frac{dy}{4\pi x^2} = \frac{10}{400\pi} = \frac{1}{40\pi} \text{ feet a second.}$$

7. A man 6 feet in height, walking at the rate of 2 miles an hour, passes under an electric light 18 feet above the pavement. Assuming the pavement to be horizontal, find (*a*) the rate at which the man's shadow is lengthening; (*b*) the velocity of the end of his shadow; (*c*) the rate at which he is receding from the light when 15 feet from its foot.

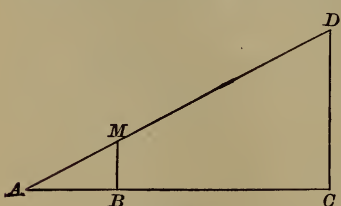


Fig. 6.

Let D be the light, and BM the position of the man at any instant.

(*a*) Let $AB = y$, $BC = x$; then, from similar triangles,

$$\frac{y}{6} = \frac{y+x}{18}, \therefore y = \frac{1}{2}x;$$

hence, $$dy = \frac{1}{2}dx = 1 \text{ mile per hour.}$$

(*b*) Let $AC = y$ and $BC = x$; then

$$\frac{y}{18} = \frac{y - x}{6}, \therefore y = \frac{3}{2}x;$$

hence, $$dy = \frac{3}{2}dx = 3 \text{ miles per hour.}$$

(*c*) Let A be the man's position, and let $AD = y$ and $AC = x$; then,

$$y^2 = x^2 + 324;$$

hence, $$dy = \frac{x}{y}dx = \frac{15}{24}2 = 1\tfrac{1}{4} \text{ miles per hour, nearly.}$$

8. Two ships start from Sandy Hook at the same time, one going N. 30° E. at the rate of 10 miles an hour, the other going due east at the rate of 12 miles an hour. At what rate are they separating at the end of two hours?

Let y = distance of ships apart at the instant, and let u and v be the varying distances of the vessels from Sandy Hook; then

$$y^2 = u^2 + v^2 - 2\,uv\cos 60° = u^2 + v^2 - uv;$$

$$\therefore\ 2\,ydy = 2\,udu + 2\,vdv - udv - vdu;$$

$$\therefore\ dy = \frac{(2\,u - v)\,du + (2\,v - u)\,dv}{2\sqrt{u^2 + v^2 - uv}} = 2\sqrt{31} \text{ miles an hour.}$$

9. The altitude of an equilateral triangle is 2 feet, and is increasing at the rate of 2 inches a minute. At what rate is the area increasing? At what rate is the perimeter increasing?

Ans. $32\sqrt{3}$ sq. in. a minute, $4\sqrt{3}$ in. a minute.

10. The apothegm of a regular hexagon is 2 feet, and is increasing at the rate of one inch a second. At what rate is the area increasing? *Ans.* $96\sqrt{3}$ sq. in. a second.

11. The altitude of a cone is constantly equal to twice the diameter of the base. If the altitude is 3 feet, and increasing one inch a second, at what rate is the volume changing? At what rate is its convex surface changing?

Ans. 81π cu. in. a second, $\frac{9}{2}\pi\sqrt{17}$ sq. in. a second.

12. A reservoir in the form of an inverted conical frustum, radius of smaller base = 100 ft. and elements inclined 45° to horizon, is used to supply an adjacent town with water. If the depth of the water at any instant is 10 feet, and is decreasing at the rate of two feet a day, at what rate is the town being supplied?

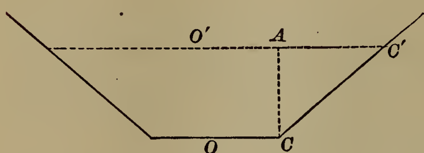


Fig. 7.

Let $AC = x$, $OC = a$; then $O'C' = a + x$. Let y = volume; then

$$y = \frac{\pi x}{3}\{(a+x)^2 + a^2 + (a+x)a\}$$

$$= \frac{\pi}{3}(3a^2x + 3ax^2 + x^3).$$

$$\therefore \quad dy = \frac{\pi}{3}(3a^2 + 6ax + 3x^2)\,dx$$

$$= 24200\,\pi \text{ cu. ft. a day.}$$

13. Under the action of internal forces a circular cylinder is changing. When the diameter is 24 in., and increasing at the rate of 1 in. a second, the altitude is 48 in., and decreases at the rate of 2 in. a second. At what rate is the volume changing? At what rate is the convex surface changing?

Ans. 288π cu. in. a second. Not changing.

14. (*a*) Compare the rates of the ordinate and abscissa of the generating point of the logarithmic curve, $y = \log_a x$. (*b*) At what point are the rates equal? (*c*) How do the rates compare when the moving point crosses the x-axis?

(*a*) $dy = \frac{m}{x} dx$, i.e., the rate of y is $\frac{m}{x}$ times the rate of x.

(*b*) At $x = m$.

(*c*) When $y = 0$, $x = 1$, $\therefore\ dy = mdx$.

15. Which increases more rapidly, a number or its logarithm? Let x = number, then $d \log_a x = \frac{m}{x} dx$; hence, if $m > x$ the logarithm increases more rapidly; if $m < x$ the number increases more rapidly, and if $m = x$, the rates are the same. In the Napierian system $m = 1$, $\therefore\ dx = xd \log x$, i.e., the number increases more rapidly or more slowly than its logarithm according as $x >$ or $<$ than 1.

16. (*a*) How much more rapidly is the number 342 increasing than its common logarithm? (*b*) How much more rapidly than its Napierian logarithm? (*c*) If the number increases by 1, how much will its common logarithm increase?

(*a*) $$d \log_{10} x = \frac{m}{x} dx = \frac{.434295}{342} dx,$$

since $m = .434295$ in common system.

Hence $$dx = \frac{342}{.434295} d \log_{10} x = 788\ d \log_{10} x,$$

i.e., the number increases 788 times as fast as its logarithm.

(*b*) Since $m = 1$ in Napierian system,

$$dx = 342\ d \log x;$$

i.e., the number increases 342 times as fast as its logarithm in the Napierian system.

(*c*) Since $dx = 1$, we have

$$d \log_{10} x = \frac{.434295}{342} \times 1 = .00126,$$

i.e., the logarithm of 343 is greater than the logarithm of 342 by .00126. This decimal is the tabular difference corresponding to the number 342 as given in tables of common logarithms.

17. What should be the tabular difference in tables of common logarithms for numbers between 6342 and 6343?

Ans. .000068.

18. Assuming that the ratio of the rates of change of a number and its logarithm remains constant while the number changes from 245 to 245.15, find the logarithm of the latter, assuming the logarithm of the former to be 2.389166.

Ans. 2.389432.

19. Prove by logarithms that $d(uvw) = uvdw + uwdv + vwdu$. Applying logarithms we have

$$\log (uvw) = \log u + \log v + \log w;$$

$$\therefore \frac{d(uvw)}{uvw} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w};$$

$$\therefore d(uvw) = uvdw + uwdv + vwdu.$$

20. Prove by logarithms that $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$.

21. Show that the ratio of the rates of a^x and x is equal to $a \log a$ when $x = 1$; of e^x and x is equal to e when $x = 1$; of x^x and x is equal to 1 when $x = 1$.

22. (*a*) Which increases more rapidly, an arc or its sine? (*b*) Where are their rates of increase the same? (*c*) Where is the rate of the arc twice that of the sine? (*d*) When the arc is 30° what is the ratio of the rates?

(*a*) $d(\sin x) = \cos x\,dx$. As the cosine is in general less than 1, the rate of the arc is in general greater than the rate of the sine.

(*b*) By hypothesis $d \sin x = dx \therefore \cos x = 1 \therefore x = 0$, i.e., when the arc is zero its rate and that of its sine are the same.

(*c*) By hypothesis $d(\sin x) = \frac{1}{2} dx \therefore \cos x = \frac{1}{2} \therefore x = 60°$.

(*d*) $d(\sin x) = \cos 30° \, dx = \frac{\sqrt{3}}{2} dx \therefore \frac{d(\sin x)}{dx} = \frac{\sqrt{3}}{2}$.

23. Which increases more rapidly, an arc or its tangent? At what value of the arc are the rates the same? At what value of the arc is its rate $\frac{1}{2}$ that of the tangent? At what value is its rate $\frac{1}{4}$ that of the tangent?

24. Assuming that the ratio of the rates of change of an arc and its cosine remains constant while the arc changes from 62° 42′ to 62° 42′ 25″ find the natural cosine of the latter arc, given the natural cosine of the former as .45865.

Ans. .45854.

25. A fly-wheel connected with a stationary engine is revolving uniformly at the rate of 2 turns a second. Compare the velocity of a point 1 foot from the axis with its horizontal velocity.

Dropping a perpendicular from the point in any position to the horizontal line through the axis, we readily see that the horizontal velocity of the point is the same thing as the rate of change of $\cos x$, where $x =$ arc already described, estimated from the origin of arcs. Hence

$$d(\cos x) = -\sin x dx,$$

i.e., the horizontal velocity is $\sin x$ times the velocity in its path. Since $dx = 2\pi . 2. = 4\pi$.

$$d(\cos x) = -4\pi \sin x;$$

$x = 0$, then $d(\cos x) = 0;$

$x = 30$, then $d(\cos x) = -2\pi$ ft. a second $= \frac{1}{2}$ velocity of point;

$x = 90°$, then $d(\cos x) = -4\pi$ ft. a second $=$ velocity of point;

$x = 150°$, then $d(\cos x) = -2\pi$ ft. a second $= \frac{1}{2}$ velocity of point;

$x = 270°$, then $d(\cos x) = 4\pi$ ft. a second $=$ velocity of point.

26. The crank of a steam engine is one foot in length and the coupling-rod is 6 feet; find the velocity of the piston per second when the crank revolves uniformly at the rate of 5 turns per second.

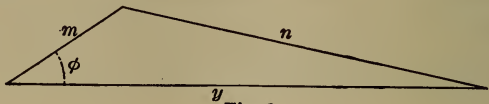


Fig. 8.

Let the length of crank $= m$ and length of coupling-rod $= n$. Let $\phi =$ varying angle described by the crank and $y =$ varying distance, the rate of change of which equals the velocity of the piston.

Then $$n^2 = m^2 + y^2 - 2my\cos\phi;$$

$$\therefore\ y^2 - 2my\cos\phi + m^2\cos^2\phi = n^2 - m^2 + m^2\cos^2\phi;$$

$$\therefore\ y = m\cos\phi + \sqrt{n^2 - m^2\sin^2\phi}.$$

Hence $$dy = -\left(m\sin\phi + \frac{m^2\sin 2\phi}{2\sqrt{n^2 - m^2\sin^2\phi}}\right)d\phi.$$

But $m = 1, n = 6, d\phi = 10\pi\ \therefore\ dy = -\left(\sin\phi + \dfrac{\sin 2\phi}{2\sqrt{36 - \sin^2\phi}}\right)10\pi.$

When $\phi = 0°$,

$$dy = 0.$$

When $\phi = 45°$,

$$dy = -\left(\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{36 - \frac{1}{2}}}\right)10\pi$$

$$= -\left(1 + \frac{1}{\sqrt{71}}\right)5\pi\sqrt{2} \text{ feet per second.}$$

When $\phi = 90°$, $dy = -10\pi$ feet a second.
When $\phi = 270°$,

$dy = 10\pi$ feet a second.

27. Compare the velocity of a train moving along a horizontal tangent with the velocity of a point at the base of the flange of one of the wheels. Compare also the vertical and horizontal components of the velocity of the flange point.

The velocity of the train is the same as the velocity of the center of one of the axles. Call this velocity v. The point on the flange of the wheel describes a cycloid whose equations are

$$x = a\theta - a\sin\theta,$$
$$y = a - a\cos\theta,$$

(" Analytical Geometry," p. 206) in which a = radius of wheel and

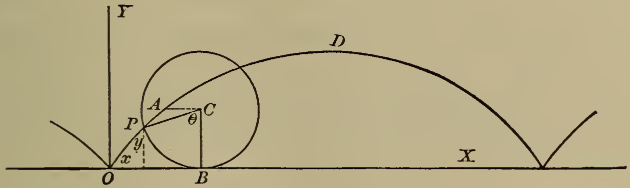


Fig. 9.

θ = angle through which the wheel has rolled at the instant of consideration.

Let P, Fig. 9, be the position of the flange point at the instant; then $PCB = \theta$. We are to compare the velocities of P and C.

Differentiating the equations of the curve, we have

$$dx = a(1 - \cos\theta)\,d\theta,$$
$$dy = a\sin\theta d\theta.$$

Since C is always vertically over the point B the velocity of C = rate of change of distance OB; but $OB = PB = a\theta$;
$\therefore\ v = ad\theta.$

Hence, $$dx = v(1 - \cos\theta),$$
$$dy = v\sin\theta.$$

From § 18, (3), we have

$$ds = \sqrt{dx^2 + dy^2}.$$

Hence, $$ds = \sqrt{v^2(1-\cos\theta)^2 + v^2\sin^2\theta} = v\sqrt{2(1-\cos\theta)}$$
$$= 2v\sin\frac{\theta}{2}.$$

Now at O, $\theta = 0°$ $\therefore$ $dx = 0$, $dy = 0$, $ds = 0$, i.e., at the instant the point touches the rail it is at rest.

At A, $\theta = 90°$ $\therefore$ $dx = v$, $dy = v$, $ds = v\sqrt{2}$, i.e., when the point is in a horizontal line with the center C both the horizontal and vertical components of its velocity are equal to the velocity of C, i.e., of the train, while the velocity in its path is equal to the square root of the sum of the squares of its component velocities. At D, $\theta = 180°$ $\therefore$ $dx = 2v$, $dy = 0$. $ds = 2v$, i.e., the vertical component of its velocity is zero while its horizontal component = velocity in its path is twice the velocity of the train.

Again: since $ds = 2v\sin\frac{\theta}{2} = \frac{2av}{a}\sin\frac{\theta}{2}$, we have

$$\frac{ds}{v} = \frac{2a}{a}\sin\frac{\theta}{2}.$$

But $$2a\sin\frac{\theta}{2} = \text{chord } PB \text{ and } a = BC;$$

$$\therefore \frac{ds}{v} = \frac{\text{chord } PB}{BC};$$

i.e., the velocities of P and C are proportional to the distances of the points from B; hence B is the instantaneous axis about which the wheel revolves.

28. Assuming the rectangular equation of the cycloid, deduce the same results as in Ex. 27.

The equation is $x = a \text{ vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}$.

See "Analytic Geometry," p. 207. Hence

$$dx = \frac{ydy}{\sqrt{2ay - y^2}}.$$

Since $PB = a \text{ vers}^{-1}\frac{y}{a}$, we have

$$d(PB) = da \text{ vers}^{-1}\frac{y}{a} = \frac{ady}{\sqrt{2ay - y^2}} = v;$$

$$\therefore\ dy = \frac{v}{a}\sqrt{2ay - y^2}.$$

Hence, $$dx = \frac{y}{a}v.$$

Hence, $$ds = v\sqrt{\frac{2y}{a}}.$$

At O, $y = 0$ $\therefore$ $dy = 0$, $dx = 0$, $ds = 0$.

At A, $y = a$ $\therefore$ $dy = v$, $dx = v$, $ds = v\sqrt{2}$.

At D, $y = 2a$ $\therefore$ $dy = 0$, $dx = 2v$, $ds = 2v$.

Also, $$\frac{ds}{v} = \frac{\sqrt{2ay}}{a} = \frac{\text{chord } PB}{BC}.$$

NOTE. — The cycloid enjoys the mechanical properties of being the curve of *quickest descent* and of *equal times*. The problem of determining the line of *quickest descent* under gravity was proposed by John Bernouilli in 1696. The origin of the calculus of variations may be traced to this problem. Pascal applied the Method of Indivisibles of Cavalieri with eminent success to the investigation of the properties of the cycloid-

CHAPTER VI.

GEOMETRIC APPLICATION.

CARTESIAN CURVES.

69. Tangent. Let $y=f(x)$ be the equation of any curve MN and let $P(x', y')$ be any point on this curve. Let AB represent the tangent and CD the normal at P.

Now the equation of *any* line through $P(x', y')$ is ("Analytical Geometry," p. 38)

$$y-y'=s(x-x')$$

in which s represents the slope of the line.

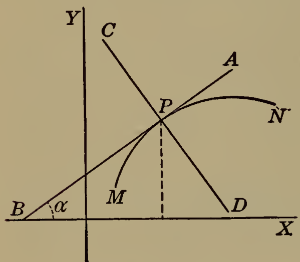


Fig. 9, a.

We have seen (§ 19) that $\frac{dy}{dx}$ ($=\tan\alpha$), as derived from the equation $y=f(x)$, represents the slope of the tangent to the curve at the point (x, y). If, therefore, we let $\frac{dy'}{dx'}(=s)$ represent the value of $\frac{dy}{dx}$ at the point (x', y') we have

$$y-y'=\frac{dy'}{dx'}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the general equation of the tangent to any plane curve.

70. Normal. Since the normal at $P(x', y')$ Fig. (9 a) is perpendicular to the tangent at that point, its slope is minus the reciprocal of that of the tangent. Hence

$$y-y'=-\frac{dx'}{dy'}(x-x') \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is the general equation of the normal to any plane curve.

EXAMPLES.

1. Find the equation of the tangent and normal to the circle
$x^2 + y^2 = a^2$.

Differentiating, $\frac{dy}{dx} = -\frac{x}{y} \therefore \frac{dy'}{dx'} = -\frac{x'}{y'}$;

hence, $$y - y' = -\frac{x'}{y'}(x - x');$$

or, reducing, $$\frac{xx'}{a^2} + \frac{yy'}{a^2} = 1$$

is the equation of the tangent, and

$$y - y' = \frac{y'}{x'}(x - x'),$$

or, $$\frac{y}{y'} = \frac{x}{x'}$$

is the equation of the normal.

2. Find the equations of the tangent and normal to the parabola $y^2 = 2px$.

$$\frac{dy}{dx} = \frac{p}{y}; \therefore \frac{dy'}{dx'} = \frac{p}{y'}.$$

Hence $$y - y' = \frac{p}{y'}(x - x'),$$

or, $$yy' = p(x + x')$$

is the equation of the tangent.

Substituting the value of $\frac{dy'}{dx'}$ in equa. (2) § 70, we have

$$y - y' = -\frac{y'}{p}(x - x')$$

for the equation of the normal.

3. Find the equations of the tangent and normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$Ans. \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1; \; y - y' = \frac{a^2y'}{b^2x'}(x - x').$$

4. Find the equations of the tangent and normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$Ans. \quad \frac{xx'}{a^2} - \frac{yy'}{b^2} = 1; \; y - y' = -\frac{a^2y'}{b^2x'}(x - x').$$

5. Given the equation $3a^2y = x^3 - 3ax^2 + b$; find (*a*) the direction of the curve at the points whose abscissas are $x = 0$ and $x = a$, and (*b*), the abscissas of the points where the curve is parallel to the x-axis.

(*a*) The direction of a curve at any point is that of its tangent at that point; hence, differentiating the equation, we have

$$\frac{dy}{dx} = \frac{x^2 - 2ax}{a^2}.$$

At $x = 0$, $\frac{dy}{dx} = 0$ $\therefore$ the tangent is $\parallel$ to the x-axis.

At $x = a$, $\frac{dy}{dx} = -1$ $\therefore$ the tangent makes an angle of $135°$ with x-axis.

(*b*) Equating the value of $\frac{dy}{dx}$ to zero, we have

$$\frac{x^2 - 2ax}{a^2} = 0;$$

hence,

$$x(x - 2a) = 0,$$

$$\therefore x = 0 \text{ and } x = 2a$$

are the abscissas of the points where the curve is $\parallel$ to x-axis.

6. Find the equations of the tangent and normal to the circle $(x-4)^2+(y+3)^2=25$ at the point $(7, 1)$.

7. Find the equations of the tangents to the hyperbola $4x^2-9y^2+36=0$ which are perpendicular to the line $2y+5x=10$.

Ans. $2x-5y\pm 8=0$.

8. Find the equation of the tangent to the cissoid $y^2=\dfrac{x^3}{2a-x}$.

Ans. $y-y'=\pm\dfrac{3ax'^{\frac{1}{2}}-x'^{\frac{3}{2}}}{(2a-x')^{\frac{3}{2}}}(x-x')$.

9. What angles do the cissoid $y^2=\dfrac{x^3}{2a-x}$ and the circle $x^2+y^2-8ax=0$ make with each other at their points of intersection?

Ans. At one point, 90°; at two others, 45°.

10. At what angle does the cissoid cut its base circle?

Ans. $\tan^{-1} 2$.

11. What is the equation of the tangent to the parabola $y^2=20x$ which makes an angle of 45° with the axis of x?

Ans. $y=x+5$.

12. Find the angles at which the curve $y^3=x^2-7x$ cuts the line $y=2$.

71. Length of Subtangent. Length of Tangent.

Let PT, Fig. 10, be the tangent to $MS\,(y=f(x))$ at the point $P(x', y')$.

Then, Subtangent $= AT = PA \cot ATP = \dfrac{PA}{\tan ATP} = \dfrac{y'}{\dfrac{dy'}{dx'}}$; § 19.

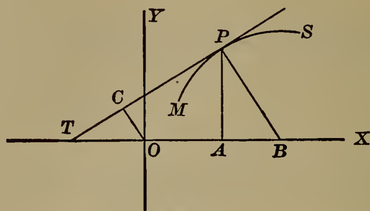


Fig. 10.

$$\therefore \text{ Subtangent} = y' \frac{dx'}{dy'}.$$

Again, Tangent $= PT = \sqrt{\overline{AP}^2 + \overline{AT}^2} = \sqrt{y'^2 + y'^2\left(\dfrac{dx'}{dy'}\right)^2}$;

$$\therefore \text{ Tangent} = y'\sqrt{1 + \left(\frac{dx'}{dy'}\right)^2}.$$

72. Length of Subnormal. Length of Normal. Perpendicular to Tangent.

Let PB be the normal to the curve at P, then

$$\text{Subnormal} = AB = AP \tan APB = AP \tan ATP.$$

$$\therefore \text{ Subnormal} = y' \frac{dy'}{dx'}.$$

Again, Normal $= PB = \sqrt{\overline{AP}^2 + \overline{AB}^2} = \sqrt{y'^2 + y'^2\left(\dfrac{dy'}{dx'}\right)^2}$,

$$\therefore \text{ Normal} = y'\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}.$$

Draw $OC \perp PT$, then

Perpendicular $= OC = \dfrac{OT}{\csc ATP} = \dfrac{OA - AT}{\sqrt{1 + \cot^2 ATP}} = \dfrac{x' - y'\dfrac{dx'}{dy'}}{\sqrt{1 + \left(\dfrac{dx'}{dy'}\right)^2}}$,

$$\therefore \text{ Perpendicular} = \frac{x'dy' - y'dx'}{(dy'^2 + dx'^2)^{\frac{1}{2}}}$$

EXAMPLES.

1. Find the lengths of the subtangents and the subnormals of the conic sections.

	Circle.	*Parabola.*	*Ellipse.*	*Hyperbola.*
Subtangents	$\frac{a^2-x'^2}{x'}$	$2x'$	$\frac{x'^2-a^2}{x'}$	$\frac{x'^2-a^2}{x'}$
Subnormals	x'	p	$\frac{b^2x'}{a^2}$	$\frac{b^2x'}{a^2}$

As lengths only were required the signs of the values are omitted. If we take T and A, Fig. 10, as points of reference, the signs will show whether the subtangents and subnormals are measured to the right or to the left.

2. Find the length of the normal to the catenary $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$.

Ans. $\frac{y'^2}{a}$.

3. Deduce the equation of the tangent to the hypocycloid of four cusps, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, and show that the portion of the tangent included between the coördinate axes is constant and equal to the radius of the base circle.

4. Find the lengths of the subtangent and subnormal to the cissoid $y^2=\frac{x^3}{2a-x}$. *Ans.* $\frac{(2a-x')x'}{3a-x'}, \frac{(3a-x')x'^2}{(2a-x')^2}$.

5. Show that the subtangent of the hyperbola $xy=m$ is equal to the abscissa of the point of tangency.

6. Show that the subtangent of the logarithmic curve $y=ae^{\frac{x}{c}}$ is constant and equal to c.

7. Show that the values of the normal and subnormal of the cycloid, $x=a\ \text{vers}^{-1}\frac{y}{a}-\sqrt{2ay-y^2}$, are $\sqrt{2ay'}$ and $\sqrt{(2a-y')y'}$, respectively, and from these values show that the line joining

the generating point and the foot of the vertical diameter of the rolling circle is always normal to the curve. See Ex. 27, p. 69.

8. Find the length of the perpendicular let fall from the origin to the tangent of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ans. $\sqrt[3]{ax'y'}$.

73. Rectilinear Asymptote. Equations of the Asymptote. The limiting position of a tangent to a curve as the point of tangency recedes to an infinite distance is called the *rectilinear* asymptote of the curve.

Of course curves with infinite branches only can have asymptotes.

Assuming the general equation of the tangent to any plane curve, § 69,

$$y - y' = \frac{dy'}{dx'}(x - x'),$$

and making successively $y = 0$, and $x = 0$, we obtain

$$I_x = x' - y'\frac{dx'}{dy'},$$

$$I_y = y' - x'\frac{dy'}{dx'},$$

for the intercepts of the tangent on the X-axis and the Y-axis. Now, if either of these intercepts approach a finite limit as either coördinate, x' or y', of the point of tangency approaches an infinite value there is an asymptote whose equation may be determined in either of the following ways:

1., By ascertaining the limits of I_x and I_y, i.e., by determining the *two* points in which the asymptote cuts the axes; or,

2., By ascertaining the limit of one of the variable intercepts of the tangent and the limit of its slope, $\frac{dy'}{dx'}$, as the tangent point recedes infinitely.

It frequently happens in the effort to ascertain the limit of I_x, or I_y, or $\frac{dy'}{dx'}$ that the values assume an indeterminate form. If so the process of evaluation is determined by principles explained in Ch. IX.

First Method. By ascertaining the limits I_x and I_y.

Let a and b be the limits I_x and I_y as x' or y' approaches an infinite value; then $$\frac{x}{a}+\frac{y}{b}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
is the equation of the asymptote in its symmetrical form.

Second Method. By ascertaining a or b, and the limit of $\frac{dy'}{dx'}$.

1. If we determine b; and the limit $\left[\frac{dy'}{dx'}\right]_{y'\doteq\infty}^{x'\doteq\infty}=s$, we have
$$y=sx+b \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
for the slope equation of the asymptote.

2. If we determine a; and the limit $\left[\frac{dy'}{dx'}\right]_{y'\doteq\infty}^{x'\doteq\infty}=s$, we have
$$x=\frac{1}{s}y+a \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
for the equation of the asymptote.

Cor. 1. If the limit of $I_x=0$, necessarily limit of $I_y=0$, and if limit of $I_y=0$, necessarily limit of $I_x=0$ and the asymptote passes through the origin. ∴ the equation takes the form $y=sx$.

Cor. 2. If the limit of $I_x=\infty$ and the limit of $I_y=\infty$, there is no asymptote.

Cor. 3. If the limit of $I_x=\infty$ and the limit of $I_y=b$, then the equation (1) becomes $$y=b$$
or if $$\text{Limit} \left[\frac{dy'}{dx'}\right]_{y'\doteq\infty}^{x'\doteq\infty}=0,$$
any limit $I_y=b$ some finite quantity, then equation (2) becomes
$$y=b$$
that is, in either of these cases the asymptote is ‖ to x-axis.

COR. 4. If the limit of $I_y = \infty$ and the limit of $I_x = a$, then equation (1) becomes

$$x = a,$$

or, if the limit $\left[\frac{dy'}{dx'}\right]_{y'=\infty}^{x'=\infty} = \infty$ and the limit of $I_x = a$, then equation (3) becomes

$$x = a.$$

That is, in either of these cases the asymptote is ‖ to Y-axis. As the second of the two methods explained above for determining an asymptote is usually the simpler, we shall adopt it in the following.

EXAMPLES.

1. Examine the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for asymptotes.

Here
$$\frac{dy'}{dx'} = \frac{b^2x'}{a^2y'} = \pm\frac{b}{a}\left(1 + \frac{b^2}{y'^2}\right)^{\frac{1}{2}};$$

$$\therefore \text{ Limit}\left[\frac{dy'}{dx'}\right]_{y' \doteq \infty}^{x' \doteq \infty} = \text{Limit}\left[\pm\frac{b}{a}\left(1 + \frac{b^2}{y'^2}\right)^{\frac{1}{2}}\right]_{y' \doteq \infty} = \pm\frac{b}{a}.$$

$$y' = \text{Limit}\left[y' - x'\frac{dy'}{dx'}\right]_{y' \doteq \infty}^{x' \doteq \infty} = \text{Limit}\left[y' - x'\frac{b^2x'}{a^2y'}\right]_{y' \doteq \infty}^{x' \doteq \infty}$$

$$= \text{Limit}\left[-\frac{b^2}{y'}\right]_{y' \doteq \infty} = 0.$$

Substituting these values in equation (2) § 73, we have,

$$y = \pm\frac{b}{a}x$$

for the equations of the asymptotes.

2. Examine the parabola $y^2 = 2px$ for asymptotes.

Here
$$\frac{dy'}{dx'} = \frac{p}{y'};$$

$$\therefore \text{ Limit}\left[\frac{dy'}{dx'}\right]_{y' \doteq \infty} = 0;$$

$\therefore$ if there is an asymptote it is ‖ to the x-axis; Cor. 3, § 73.

$$y' = \text{Limit}\left[y' - x'\frac{p}{y'}\right]_{y' \doteq \infty}^{x' \doteq \infty} = \text{Limit}\left[\frac{y'}{2}\right]_{y' \doteq \infty} = \infty.$$

Therefore there is no asymptote to the parabola.

3. Examine the curve $y^3 = x^3 + a^2x$ for asymptotes.

The equation solved for y shows that as $x \doteq \infty$, $y \doteq \infty$ and as $x \doteq -\infty$, $y \doteq -\infty$; $\therefore$ the curve has infinite branches in the first and third angle.

Here
$$\frac{dy}{dx} = \frac{x^2}{y^2} + \frac{a^2}{3y^2} = \frac{x^2}{(x^3 + a^2x)^{\frac{2}{3}}} + \frac{a^2}{3(x^2 + a^3x)^{\frac{2}{3}}}$$
$$= \frac{1}{\left(1 + \frac{a^2}{x^2}\right)^{\frac{2}{3}}} + \frac{a^2}{3(x^3 + a^2x)^{\frac{2}{3}}}$$

$$\therefore \text{Limit of } \left.\frac{dy'}{dx'}\right]_{x' \doteq \infty} = s$$
$$= \text{Limit}\left[\frac{1}{\left(1 + \frac{a^2}{x'^2}\right)^{\frac{2}{3}}} + \frac{a^2}{3(x'^3 + a^2x')^{\frac{2}{3}}}\right]_{x' \doteq \infty} = 1.$$

$\therefore$ the asymptote makes an angle of 45° with x-axis.

$$I_y = y' - x'\frac{dy'}{dx'} = \frac{2a^2x'}{3y^2} = \frac{2a^2x'}{3(x'^3 + a^2x')^{\frac{2}{3}}} = \frac{2a^2}{3\left(x'^{\frac{3}{2}} + \frac{a^2}{x'^{\frac{1}{2}}}\right)^{\frac{2}{3}}};$$

$$\therefore \text{Limit } I_y\Big]_{x' = \infty} = b = \text{Limit}\left[\frac{2a^2}{3\left(x'^{\frac{3}{2}} + \frac{a^2}{x'^{\frac{1}{2}}}\right)^{\frac{2}{3}}}\right]_{x' \doteq \infty} = 0.$$

$\therefore$ equation (2) § 73 $y = x$ is the equation of the asymptote.

4. Examine $y^3 = x^3 + ax^2$ for asymptotes. *Ans.* $y - x = \dfrac{a}{3}$.

5. Examine $y^3 = x^3 - ax^2$ for asymptotes. *Ans.* $y = x - \frac{1}{3}a$.

6. Examine $x^3 - x^2y + y = 0$ for asymptotes.

Ans. $y = x$.
$x = \pm 1$.

7. Examine $y^2 = \dfrac{x^3 + 4x^2}{x - 4}$ for asymptotes.

8. Examine $y^2 = \dfrac{x^3 - 4x^2}{x + 4}$ for asymptotes.

74. Asymptotes by Inspection. The limiting position of the tangent to a curve as the point of tangency recedes infinitely is evidently a straight line which the curve continually approaches but never reaches. Taking this view of a rectilinear asymptote we are frequently able to determine the equation of the asymptote by simply inspecting the equation of the curve.

EXAMPLES.

1. Determine the asymptote of the cissoid $y^2 = \dfrac{x^3}{2a - x}$.

We see from the equation that as x approaches the value $2a$, y approaches an infinite value, $\therefore$ the curve continually approaches but never reaches the line $x = 2a$.

$$\therefore x = 2a$$

is an asymptote.

2. Examine the witch $y^2 = \dfrac{4a^2x}{2a - x}$ for asymptotes.

We see that $x = 2a$ is an asymptote and as the curve has infinite branches in the first and fourth angles only there is no other asymptote.

3. Examine the conchoid $x^2y^2 = (b^2 - y^2)(a + y)^2$ for asymptotes.

Here $$x = \pm \frac{a + y}{y}\sqrt{b^2 - y^2}.$$

As $y \doteq 0$, $x \doteq \pm\infty$, $\therefore y = 0$, i.e., the x-axis is an asymptote.

4. Examine the curve of tangents for asymptotes.

We see from the equation $y = \tan x$ that as $x \doteq \frac{\pi}{2}$, or $\frac{3\pi}{2}$, or $\frac{5\pi}{2}$, or etc., that $y \doteq \infty$, or $-\infty$; $\therefore$

$$x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2} \text{ and } x = \frac{5\pi}{2}, \text{ etc.,}$$

are asymptotes. Similarly we may show that

$$x = -\frac{\pi}{2},\ x = -\frac{3\pi}{2},\ x = -\frac{5\pi}{2},\ \text{etc.},$$

are also asymptotes to the curve.

5. Examine $(x - 2a)y^2 = x^3 - a^3$ for asymptotes.

By inspection we find $x = 2a$ is an asymptote.

By analysis we find two others $y = x + a$ and $y + x + a = 0$.

6. Examine $a^2y - x^2y = a^3$ for asymptotes. *Ans.* $x = \pm a.$
$x = 0.$

7. Examine $y = \dfrac{a^2x}{(x-a)^2}$ for asymptotes. *Ans.* $x = a.$
$y = 0.$

75. Asymptotes by expansion. Where an equation can be readily solved for one of the variables and the second member can be expanded into a series the asymptote may frequently be more readily detected than by pursuing the general course as explained in § 73.

EXAMPLES.

1. To find by expansion the asymptotes of the hyperbola

From the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, we obtain

$$y = \pm\frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}}$$

$$= \pm\frac{bx}{a}\left(1 - \frac{1}{2}\frac{a^2}{x^2} - \frac{1}{8}\frac{a^4}{x^4} - \frac{1}{16}\frac{a^6}{x^6} - \cdots\right).$$

NOTE. — Binomial Formula.

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{\underline{|2}}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}b^3 + \cdots.$$

As x increases indefinitely, the curve approaches nearer and nearer the lines represented by the equations

$$y = \pm \frac{bx}{a};$$

$\therefore$ these equations represent the asymptotes to the curve.

2. To find by expansion the asymptote of $x^3 - xy^2 + ay^2 = 0$.

Here $$y = \pm\left(\frac{x^3}{x-a}\right)^{\frac{1}{2}} = \pm x\left(1 - \frac{a}{x}\right)^{-\frac{1}{2}}.$$

$$\therefore y = \pm x\left(1 + \frac{a}{2x} + \frac{3a^2}{8x^2} + \frac{5a^3}{16x^3} + \cdots\right).$$

$$\therefore y = \pm\left(x + \frac{a}{2}\right)$$

are equations of asymptotes. By *inspection* we see also that $x = a$ is the equation of an asymptote.

3. Find by expansion the asymptotes of the curve represented by the equation $y^2 - 2xy - x^2 + 2 = 0$.

Ans. $y = (1 \pm \sqrt{2})x$.

4. Find by expansion the asymptotes to the curves in Exercises 3, 4, 5, § 73.

POLAR CURVES.

76. *To find the slope of the tangent to a polar curve.*

From § 19, we have

$$\tan \alpha = \frac{dy}{dx}$$

for the slope of a curve when referred to rectangular coördinates. If we assume the pole coincident with the origin and the initial line coincident with the x-axis we have, "Analytical Geometry," Art. 34, (3),

$$x = r \cos \theta, \qquad y = r \sin \theta.$$

Hence, $$\frac{dy}{dx} = \frac{d(r \sin \theta)}{d(r \cos \theta)} = \frac{r \cos \theta \, d\theta + \sin \theta \, dr}{\cos \theta \, dr - r \sin \theta \, d\theta}$$

is the required expression.

Cor. from § 18 (3), we have

$$ds = \sqrt{dx^2 + dy^2};$$

$$\therefore \; ds = \sqrt{\{d(r \cos \theta)\}^2 + \{d(r \sin \theta)\}^2} = \sqrt{dr^2 + r^2 \, d\theta^2}.$$

77. Length of Subtangent. Length of Tangent.

Let MS be any curve referred to O as pole and OX as initial line. Let $P(r, \theta)$ be any point of the curve at which a tangent PB and a normal PA are drawn; then OB and OA, segments of the perpendicular to the radius vector OP, are respectively the subtangent and subnormal corresponding to the point $P(r, \theta)$.

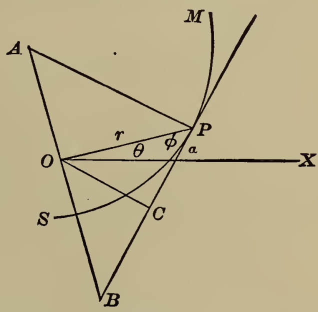


Fig. 11.

1. *To find a value for OB, the subtangent.*

From the triangle OPB

$$OB = r \tan \phi; \text{ but}$$

$$\tan \phi = \tan(\alpha - \theta) = \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta}$$

$$= \frac{\dfrac{r \cos \theta \, d\theta + \sin \theta \, dr}{\cos \theta \, dr - r \sin \theta \, d\theta} - \dfrac{\sin \theta}{\cos \theta}}{1 + \dfrac{r \cos \theta \, d\theta + \sin \theta \, dr}{\cos \theta \, dr - r \sin \theta \, d\theta} \cdot \dfrac{\sin \theta}{\cos \theta}} \qquad \text{Art. (76)},$$

$$\therefore \; \tan \phi = \frac{r d\theta}{dr};$$

$$\therefore \; \text{Subtangent} = r^2 \frac{d\theta}{dr}.$$

2. *To find a value for PB, the tangent.*

From the triangle OPB

$$PB = \sqrt{r^2 + \overline{OB}^2} = \sqrt{r^2 + r^4\frac{d\theta^2}{dr^2}};$$

$$\therefore \text{ Tangent} = r\sqrt{1 + r^2\frac{d\theta^2}{dr^2}}.$$

78. Length of Subnormal. Length of Normal. Perpendicular to Tangent.

From triangle OAP, Fig. 11,

$$OA = \frac{OP}{\tan OAP} = \frac{r}{\tan\phi} = \frac{r}{r\dfrac{d\theta}{dr}};$$

$$\therefore \text{ Subnormal} = \frac{dr}{d\theta};$$

also,
$$PA = \sqrt{\overline{OP}^2 + \overline{OA}^2} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}};$$

$$\therefore \text{ Normal} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}.$$

From triangle OCB (OC being perpendicular to BP), we have

$$OC = \frac{OB}{\sec COB} = \frac{r^2\dfrac{d\theta}{dr}}{\sqrt{1 + \tan^2\phi}} = \frac{r^2\dfrac{d\theta}{dr}}{\sqrt{1 + \dfrac{r^2 d\theta^2}{dr^2}}};$$

$$\therefore \text{ Perpendicular} = \frac{r^2}{\sqrt{\dfrac{dr^2}{d\theta^2} + r^2}}.$$

EXAMPLES.

1. A circle whose diameter is a is referred to the lower extremity of its vertical diameter as a pole, and to the tangent at that point as an initial line. Find the relation between ϕ, α, and θ.

The equation of the curve is evidently $r = a \sin\theta$,

$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{r}{\dfrac{dr}{d\theta}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta;$$

$$\therefore \phi = \theta.$$

Fig. 11, $\alpha = \phi + \theta = 2\,\theta = 2\,\phi.$

2. Show that the logarithmic spiral $r = a^\theta$ is an equiangular spiral, i.e., that the tangent makes a constant angle with the radius vector.

$$\tan\phi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{a^\theta}{a^\theta \log a} = \frac{1}{\log a}.$$

If $a = e$, then $\tan\phi = 1, \quad \therefore \phi = 45°.$

3. Show that the perpendicular from the pole to the tangent of the lemniscate $r^2 = a^2\cos 2\,\theta$ varies with r^3.

$$p = \frac{r^2}{\sqrt{r^2 + \dfrac{dr^2}{d\theta^2}}} = \frac{r^3}{\sqrt{a^4(\cos^2 2\,\theta + \sin^2 2\,\theta)}} = \frac{r^3}{a^2}.$$

Show also that $\phi = \frac{\pi}{2} + 2\,\theta.$

4. In the spiral of Archimedes $r = c\theta$ show that the product of the subnormal and subtangent is always equal to the square

of the radius vector. Find also the value of the tangent normal and perpendicular.

$$\text{Subtangent} = r^2\frac{d\theta}{dr} = \frac{r^2}{c}. \qquad \text{Subnormal} = \frac{dr}{d\theta} = c.$$

$$\therefore \text{Subtangent} \times \text{Subnormal} = r^2.$$

$$\text{Tangent} = \frac{r}{c}\sqrt{c^2 + r^2}.$$

$$\text{Normal} = \sqrt{r^2 + c^2}.$$

$$\text{Perpendicular} = \frac{r^2}{\sqrt{r^2 + c^2}}.$$

5. In the hyperbolic spiral $r\theta = c$ show that the polar subtangent is constant and equal to the circumference of the measuring circle.

6. In the following curve find the values for the length of the tangent, subtangent, normal, subnormal and perpendicular:

$$r = 2\,a\cos\theta.$$

79. Asymptotes.

Let $r = f(\theta)$ be the equation of MS, Fig. 12, and let AB be an asymptote. Draw $OC \parallel$ to AB and $OB \perp OC$; then OB is $\perp$ to AB. Now $OC = r$ is the radius vector of the infinite tangent point, and OB is the subtangent corresponding to that point. If, therefore, θ' be the limiting value of θ as r approaches ∞ as its limit, we have

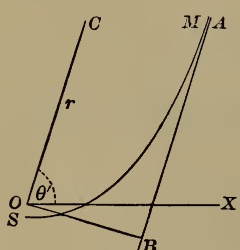


Fig. 12.

$$\text{Limit}\,[f(\theta)]_{\theta \doteq \theta'} = \infty,$$

and

$$\text{Limit}\left[\frac{r^2 d\theta}{dr}\right]_{\theta \doteq \theta'} = OB = \text{a finite value},$$

as the conditions for an asymptote to a polar curve.

Looking in the direction of the infinite radius vector OC, the distance

$$OB = \text{limit}\left[\frac{r^2 d\theta}{dr}\right]_{\theta \doteq \theta'}$$

is laid off to the right or left according as OB is positive or negative.

EXAMPLES.

1. Examine the curve of tangents $r = c \tan\theta$ for asymptotes.

Here, $\quad \text{Limit}\,[f(\theta)]_{\theta \doteq \theta'} = \text{Limit}\,[c \tan\theta]_{\theta \doteq \pm\frac{\pi}{2}} = \infty;$

also, $\quad \text{Limit}\left[r^2 \frac{d\theta}{dr}\right]_{\theta \doteq \theta'} = \text{Limit}\,[c \sin^2\theta]_{\theta \doteq \pm\frac{\pi}{2}} = c.$

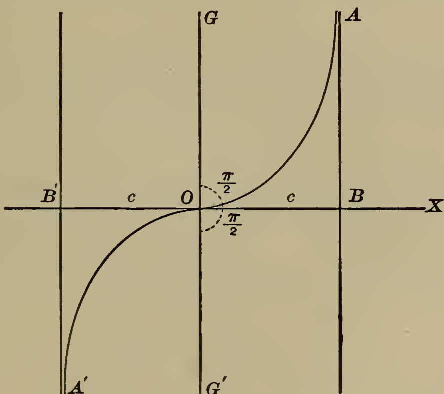


Fig. 13.

Hence drawing the curve and the radii vectores corresponding to the vectorial angles $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, we see that $BA \perp$ to OX at a distance c from O and $B'A' \perp$ to OX and at the same distance c from O are asymptotes to the curve.

2. Examine the lituus $r^2\theta = c$ for asymptotes.

Here $$\text{Limit } [f(\theta)]_{\theta \doteq \theta'} = \text{Limit } \left[\left(\frac{c}{\theta}\right)^{\frac{1}{2}}\right]_{\theta \doteq 0} = \infty,$$

and $$\text{Limit } \left[r^2 \frac{d\theta}{dr}\right]_{\theta \doteq \theta'} = \text{Limit } [-2\,r\theta]_{\theta \doteq 0} = 0.$$

Hence, the initial line is an asymptote.

3. Examine the hyperbolic spiral $r\theta = c$ for asymptotes.

$$\text{Limit } [f(\theta)]_{\theta \doteq \theta'} = \text{Limit } \left[\frac{c}{\theta}\right]_{\theta \doteq 0} = \infty,$$

$$\text{Limit } \left[r^2 \frac{d\theta}{dr}\right]_{\theta \doteq \theta'} = \text{Limit } [-c]_{\theta' \doteq 0} = -c;$$

$\therefore$ a line $\parallel$ to the initial line and at a distance c above it is an asymptote.

4. Taking the left-hand focus as a pole, examine the hyperbola for asymptotes. See "Analytical Geometry," Art. 110, Equa. 3.

5. Examine the following curves for asymptotes,

$r = 2\,a \tan\theta \sin\theta$. $\qquad r^2 \cos\theta = a^2 \sin 3\,\theta$.

$r = a \sec\theta \pm b$. $\qquad r \cos 2\,\theta = a\,(1 + \sin 2\,\theta)$.

CHAPTER VII.

SUCCESSIVE DIFFERENTIATION.

80. Successive Differential Equations. — If we differentiate the expression $y = x^4$ we obtain

$$dy = 4\,x^3dx.$$

Since dy is a function of x it can be differentiated. Regarding x as equicrescent, i.e., as changing uniformly, we have dx = a constant. Hence differentiating again, and representing $d(dy)$ by d^2y, we have

$$d^2y = 12\,x^2dx^2,$$

in which $dx^2 = (dx)^2$. Differentiating again, and representing $d(d^2y)$ by d^3y, we have

$$d^3y = 24\,xdx^3,$$

in which $dx^3 = (dx)^3$. The differentiation may obviously be continued until the second member vanishes.

Hence, in general, if

$$y = f(x), \text{ we have}$$

$$dy = f'(x)\,dx \quad \ldots\ldots\ldots \quad (a)$$

$$d^2y = f''(x)\,dx^2 \quad \ldots\ldots\ldots \quad (b)$$

$$d^3y = f'''(x)\,dx^3 \quad \ldots\ldots\ldots \quad (c)$$

$$d^4y = f^{\text{iv}}(x)\,dx^4 \quad \ldots\ldots\ldots \quad (d)$$

$$\cdots\cdots\cdots$$

$$d^ny = f^n(x)\;dx^n \quad \ldots\ldots\ldots \quad (e)$$

in which (a), (b), (c), (d) . . . (e) are called respectively the First, the Second, the Third, the Fourth . . . the nth Differential Equation of the equation $y = f(x)$.

81. Successive Differential Coefficients or Derivatives.

If we divide equations (*a*), (*b*), (*c*), (*d*), ... (*e*) by dx, dx^2, dx^3, dx^4, . . . dx^n, respectively, we have

$$\frac{dy}{dx} = f'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

$$\frac{d^2y}{dx^2} = f''(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

$$\frac{d^3y}{dx^3} = f'''(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c')$$

$$\frac{d^4y}{dx^4} = f^{\text{iv}}(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d')$$

. . . .

$$\frac{d^ny}{dx^n} = f^n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e')$$

These equations are called respectively, The First, The Second, the Third, the Fourth, . . . the *n*th Differential Coefficient or Derivative of the equation $y = f(x)$.

Cor. As $f^n(x)$ is the first derivative of $f^{n-1}(x)$ it follows § (28) Cor. that $f^n(x)$ is positive or negative according as $f^{n-1}(x)$ is an increasing or decreasing function of x.

EXAMPLES.

1. Write the successive differential equations and coefficients of $y = x^5$,

$$dy = 5\,x^4dx \quad \therefore \quad \frac{dy}{dx} = 5\,x^4.$$

$$d^2y = 20\,x^3dx^2 \quad \therefore \quad \frac{d^2y}{dx^2} = 20\,x^3.$$

$$d^3y = 60\,x^2dx^3 \quad \therefore \quad \frac{d^3y}{dx^3} = 60\,x^2.$$

$$d^4y = 120\,xdx^4 \quad \therefore \quad \frac{d^4y}{dx^4} = 120\,x.$$

$$d^5y = 120\, dx^5 \qquad \therefore \quad \frac{d^5y}{dx^5} = 120.$$

$$d^6y = 0 \qquad \therefore \quad \frac{d^6y}{dx^6} = 0.$$

2. Write the successive differentials of $y = 2x^3 - 3x^2 + 7x$.

$$dy = (6x^2 - 6x + 7)\, dx,$$
$$d^2y = (12x - 6)\, dx^2,$$
$$d^3y = 12\, dx^3,$$
$$d^4y = 0.$$

Find the differential coefficient indicated by the answers in the following:

3. $y = \dfrac{x^3}{1 - x}$. $\qquad \dfrac{d^4y}{dx^4} = \dfrac{24}{(1 - x)^5}$.

4. $y = x^4 \log x$. $\qquad \dfrac{d^6y}{dx^6} = -\dfrac{\lfloor 4}{x^2}$.

5. $y = \tan x + \sec x$. $\qquad \dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1 - \sin x)^2}$.

6. $y = e^{-x} \cos x$. $\qquad \dfrac{d^4y}{dx^4} = -4e^{-x} \cos x$.

7. $y = (x^2 + a^2) \tan^{-1}\dfrac{x}{a}$. $\qquad \dfrac{d^3y}{dx^3} = \dfrac{4a^3}{(a^2 + x^2)^2}$.

8. $y = \log (\sin x)$. $\qquad \dfrac{d^3y}{dx^3} = \dfrac{2 \cos x}{\sin^3 x}$.

9. $x^2 + y^2 = a^2$. $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{a^2}{y^3}$.

10. $a^2y^2 + b^2x^2 = a^2b^2$. $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

11. $y^2 = 2px$. $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{p^2}{y^3}$.

12. $a^2y^2 - b^2x^2 = -a^2b^2$. $\qquad \dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

By examining the successive derivatives we can frequently express the n^{th} derivative of a function.

13. $y = a^x$. $\qquad \dfrac{d^ny}{dx^n} = (\log a)^n a^x.$

14. $y = e^{ax}$. $\qquad \dfrac{d^ny}{dx^n} = a^n e^{ax}.$

15. $y = \log x$. $\qquad \dfrac{d^ny}{dx^n} = \dfrac{\underline{|n-1}\,(-1)^{n-1}}{x^n}.$

16. $y = \dfrac{1-x}{1+x}$. $\qquad \dfrac{d^ny}{dx^n} = \dfrac{2\,\underline{|n}\,(-1)^n}{(1+x)^{n+1}}.$

APPLICATIONS.

82. Definition. *Acceleration is the rate of change of velocity.* Let a = acceleration and v = velocity; then

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \qquad \left(\text{since } v = \frac{ds}{dt}, \quad \S\ 17\right).$$

1. A body, originally at rest, falls in a vacuum near the earth; find its velocity at any instant, and the acceleration of that velocity.

From Mechanics we have

$$s = \tfrac{1}{2} g t^2$$

for the distance s fallen by a body in the time t.

Differentiating successively we have, if we do not regard dt = unit of time,

$$ds = gtdt \quad \therefore \quad v = \frac{ds}{dt} = gt;$$

$$d^2s = gdt^2 \quad \therefore \quad a = \frac{d^2s}{dt^2} = g.$$

Hence the velocity of a falling body is a variable, while the acceleration of that velocity is constant.

2. A projectile thrown obliquely upward at an angle θ describes a parabola whose equation is $y = x \tan\theta - \frac{g}{2v^2\cos^2\theta}x^2$. Assuming the initial velocity $(= v)$, and the horizontal component of this velocity $(dx = v\cos\theta = \text{a constant})$, find the velocity of the projectile in its path, and the acceleration of its velocity vertically.

Here

$$dy = \tan\theta\, dx - \frac{g}{v^2\cos^2\theta}xdx,$$

$$= v\sin\theta - \frac{g}{v\cos\theta}x = \text{vertical component.}$$

$$d^2y = -g = \text{acceleration of velocity vertically.}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{v^2 - 2gy} = \text{velocity in its path.}$$

3. The distance described by a point whose initial velocity was u is given by the equation $s = ut + \frac{1}{6}ct^3$; find its velocity at any instant and the acceleration of the velocity.

Ans. $v = u + \frac{1}{2}ct^2$, $a = ct$.

4. The generating point of the parabola $y^2 = 2px$ moves with a constant velocity v'; find the velocities and accelerations in the direction of the axes.

Here $\quad dy = \frac{p}{y}dx.$

Since $\quad v' = ds = \sqrt{dy^2 + dx^2} = \frac{\sqrt{p^2 + y^2}}{y}dx,$

we have $\quad dx = \frac{y}{\sqrt{p^2 + y^2}}v' = \text{velocity in direction of } x;$

hence, $\quad dy = \frac{p}{\sqrt{p^2 + y^2}}v' = \text{velocity in direction of } y.$

Differentiating these values, we obtain

$$d^2x = \frac{p^3}{(p^2+y^2)^2}v'^2 \quad = \text{acceleration in direction of } x.$$

$$d^2y = -\frac{p^2y}{(p^2+y^2)^2}v'^2 = \text{acceleration in direction of } y.$$

Hence, since d^2x is always positive, its function, the velocity in the direction of x, is an increasing function; since d^2y is positive in the fourth angle and negative in the first, the velocity in the direction of y is increasing in the fourth and decreasing in the first angle. The student should bear in mind that the terms *increase* and *decrease* are used in an algebraic sense.

5. A point describes a circle of radius r with uniform velocity v. Show that the resultant acceleration at any position in its path is $\frac{v^2}{r}$.

6. The generating point of the cycloid, $x = a \text{ vers}^{-1}\frac{y}{a} - \sqrt{2\,ay - y^2}$, so moves that the component of its velocity in direction of x is constant and $= m$; find the velocities and accelerations in the direction of y and in its path.

Here
$$dy = \frac{\sqrt{2\,ay-y^2}}{y}dx = \frac{\sqrt{2\,ay-y^2}}{y}m,$$

$$ds = m\sqrt{\frac{2\,a}{y}},$$

$$d^2y = -\frac{a}{y^2}m^2,$$

$$d^2s = -\frac{m^2}{y^2}\sqrt{\frac{a}{2}(2\,a-y)}.$$

83. Theorem of Leibnitz.* To find an expression for the nth differential coefficient of the product of two variables which are functions of a third variable.

* Leibnitz published this theorem in 1710.

Let u and v be functions of x, and let u_1, u_2, u_3, etc., and v_1, v_2, v_3, etc., represent the successive derivatives of u and v. Let $y = uv$; then, § 25, 3,

$$\frac{dy}{dx} = uv_1 + vu_1.$$

Hence, $$\frac{d^2y}{dx^2} = uv_2 + u_1v_1 + v_1u_1 + vu_2 = uv_2 + 2\,u_1v_1 + vu_2.$$

Hence, $$\frac{d^3y}{dx^3} = uv_3 + u_1v_2 + 2\,(u_1v_2 + v_1u_2) + v_1u_2 + vu_3$$
$$= uv_3 + 3\,u_1v_2 + 3\,u_2v_1 + vu_3.$$

We observe that the coefficients in the second derivative are the same as those in the expansion $(v + u)^2 = v^2 + 2\,uv + u^2$, and that those in the third derivative are the same as in the expansion $(v + u)^3 = v^3 + 3\,uv^2 + 3\,u^2v + u^3$. We further observe that the subscripts in the derivatives are the same as the exponents in the expansions, except that u enters the first term and v enters the last term. It appears, therefore, that the values of the successive derivatives follow the laws of the Binomial Theorem both as regards the coefficients and subscripts, except in so far as indicated as to the subscripts. Assuming that the law holds for the nth derivative, we have

$$\frac{d^ny}{dx^n} = uv_n + nu_1v_{n-1} + \frac{n\,(n-1)}{\lfloor 2}\,u_2v_{n-2} + \text{etc.}, \ldots$$
$$+ nu_{n-1}v_1 + vu_n \quad \ldots \quad (a)$$

Differentiating again and collecting, we have

$$\frac{d^{n+1}y}{dx^{n+1}} = uv_{n+1} + (n+1)\,u_1v_n + \frac{(n+1)\,n}{\lfloor 2}\,u_2v_{n-1} + \text{etc}, \ldots$$
$$+ (n+1)\,u_nv_1 + vu_{n+1} \quad \ldots \quad (b)$$

Hence the law still holds. If n is any integer formula (b) shows that the law holds for the next higher integer. But we have shown that the law holds for the integer 3; hence it holds for the integer 4; hence it holds for the integer 5, etc. Hence the formula (a) holds for all positive integers.

In the usual notations formula (a) reads

$$\frac{d^n(uv)}{dx^n} = u\frac{d^nv}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{\underline{|2}}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}}$$

$$+ \text{etc.}, \ldots n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + v\frac{d^nu}{dx^n}.$$

EXAMPLES.

1. Find by Leibnitz Formula the third derivative of x^2e^{ax}.

Here $\quad v = e^{ax}, \qquad u = x^2,$

$$\frac{dv}{dx} = ae^{ax}, \qquad \frac{du}{dx} = 2x,$$

$$\frac{d^2v}{dx^2} = a^2e^{ax}, \qquad \frac{d^2u}{dx^2} = 2,$$

$$\frac{d^3v}{dx^3} = a^3e^{ax}. \qquad \frac{d^3u}{dx^3} = 0.$$

$$\therefore \quad \frac{d^3(x^2e^{ax})}{dx^3} = x^2 \cdot a^3e^{ax} + 3 \cdot 2x \cdot a^2e^{ax} + \frac{3 \cdot 2}{\underline{|2}} \cdot 2 \cdot ae^{ax}$$

$$= ae^{ax}(a^2x^2 + 6ax + 6).$$

2. Write the nth derivative in the example above.

$$\frac{d^n(x^2e^{ax})}{dx^n} = x^2a^ne^{ax} + 2nxa^{n-1}e^{ax} + \frac{n(n-1)}{\underline{|2}} 2 \cdot a^{n-2}e^{ax}$$

$$= a^{n-2}e^{ax}(a^2x^2 + 2nax + n(n-1)).$$

3. Write the nth derivative of x^2a^x.

$$\frac{d^n(x^2a^x)}{dx^n} = (\log a)^{n-2}a^x\,\{(n + x\log a)^2 - n\}.$$

4. $y = x\log x$; find nth derivative.

$$\frac{d^ny}{dx^n} = \frac{\underline{|n-2}}{x^{n-1}}(-1)^n.$$

5. $y = e^{-x}\cos x$; find 4th derivative.

$$\frac{d^4y}{dx^4} = -4e^{-x}\cos x.$$

84. *To find the values of the successive derivatives when neither variable is equicrescent.*

In the preceding articles and examples we have derived the successive derivatives under the supposition that x was equicrescent, i.e., that dx was constant and therefore $d^2x = 0$, $d^3x = 0$, etc. If the variables are not equicrescent, then both dx and dy are variables.

Let $y = f(x)$; then

$$\frac{dy}{dx} = f'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the first derivative whether x or y, or neither is equicrescent. Differentiating again and remembering that $\frac{dy}{dx}$ is a fraction with a variable numerator and denominator, we have,

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{dxd^2y - dyd^2x}{dx^3} = f''(x) \quad . \quad . \quad . \quad (2)$$

for the second derivative when neither x nor y is equicrescent. Differentiating again and collecting, we have

$$\frac{d\left(\frac{d\left(\frac{dy}{dx}\right)}{dx}\right)}{dx} = \frac{(d^3ydx - d^3xdy)\,dx - 3\,(d^2ydx - d^2xdy)\,d^2x}{dx^5}$$
$$= f'''(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for the third derivative when neither x nor y is equicrescent; and so we may continue the process until any desired derivative is reached.

Cor. 1. If x is equicrescent, we have from the equations above, since $d^3x \doteq 0$ and $d^2x = 0$,

$$\frac{dy}{dx} = f'(x).$$

$$\frac{d^2y}{dx^2} = f''(x).$$

$$\frac{d^3y}{dx^3} = f'''(x).$$

Cor. 2. If y is equicrescent we have from the same equations, since $d^2y = 0$ and $d^3y = 0$,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = f'(x),$$

$$-\frac{d^2xdy}{dx^3} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3} = f''(x),$$

and $$\frac{3(d^2x)^2dy - d^3xdydx}{dx^5} = \frac{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{d^3x}{dy^3}\frac{dx}{dy}}{\left(\frac{dx}{dy}\right)^5} = f'''(x).$$

Scholium. It frequently becomes convenient in applying the principles of the calculus to change the equicrescent variable in a differential expression, thus converting the expression into an equivalent one under another form.

Thus (1) if we desire to change a differential expression deduced under the assumption that x was equicrescent into an equivalent expression in which y is to be considered equicrescent, we merely substitute the successive derivatives under Cor. 2 for those under Cor. 1.

If (2) we desire an equivalent expression in which neither x nor y is considered equicrescent we substitute for the successive derivatives under Cor. 1 the expressions (2), (3), etc., above.

If (3) we desire to introduce a third variable z, a function of x or y, as the equicrescent variable we first proceed as under (2) and then substitute for the variables and their successive derivatives their values drawn from the given functional relation.

EXAMPLES.

1. If $y = \sqrt{\frac{1+x}{1-x}}$ show that $\frac{dx}{dy} = \frac{4y}{(y^2+1)^2}$.

Here
$$\frac{dy}{dx} = \frac{1}{(1-x)\sqrt{1-x^2}},$$
$$\therefore \frac{dx}{dy} = (1-x)\sqrt{1-x^2} = \frac{4y}{(y^2+1)^2}.$$

2. If $\frac{d^2y}{dx^2}\left(3\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2\right) - \frac{dy}{dx}\frac{d^3y}{dx^3} = 0$, show that $\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0$.

Replacing the derivatives by those given in § 84, Cor. 2, we have,

$$-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\left\{3\left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right] - \frac{1}{\left(\frac{dx}{dy}\right)^2}\right\} - \frac{1}{\frac{dx}{dy}}\,\frac{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx}{dy}\frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^5} = 0,$$

$$\therefore \frac{3\left(\frac{d^2x}{dy^2}\right)^2 + \frac{d^2x}{dy^2}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2 + \frac{dx}{dy}\frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^6} = 0,$$

$$\therefore \frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0.$$

3. Given $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0$. Show that $\frac{d^2y}{dz^2} + y = 0$ where $z = \tan^{-1} x$.

Here we proceed as explained in (3), § (84), scholium.

Hence, $$\frac{dxd^2y - dyd^2x}{dx^3} + \frac{2\tan z}{1+\tan^2 z}\frac{dy}{dx} + \frac{y}{(1+\tan^2 z)^2} = 0.$$

Since $x = \tan z$, we have

$$dx = \sec^2 z dz \text{ and } d^2x = 2\sec^2 z \tan z dz^2.$$

Hence, $$\frac{\sec^2 z\,(d^2y dz - 2\,dy dz^2 \tan z)}{\sec^6 z dz^3} + \frac{2\tan z dy}{\sec^4 z dz} + \frac{y}{\sec^4 z} = 0,$$

$$\therefore\ \frac{d^2y}{dz^2} - \frac{2\tan z dy}{dz} + \frac{2\tan z dy}{dz} + y = 0,$$

$$\therefore\ \frac{d^2y}{dz^2} + y = 0.$$

4. If $x^2 = 4z$, show that $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} + y = 0$ becomes $z\dfrac{d^2y}{dz^2} + \dfrac{dy}{dz} + y = 0.$

5. If $x = \cos\theta$, show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$ becomes $\dfrac{d^2y}{d\theta^2} = 0.$

6. If $x = ye^z$, show that $x\dfrac{d^2y}{dx^2} - \dfrac{x}{y}\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0$ becomes $y\dfrac{d^2z}{dy^2} + \dfrac{dz}{dy} = 0.$

7. If $x = a\cos\theta$, and $y = b\sin\theta$, show that

$$\frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}}{ab\sin^2\theta}.$$

8. Given $x = r\cos\theta$ and $y = r\sin\theta$, find the equivalent of

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \rho,$$ (1) when θ is equicrescent, (2) when r is equicrescent:

I. *When θ is equicrescent:*

Replacing the derivatives by their equivalents in § 84, (1) and (2), we have

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx^3}}{\frac{dxd^2y - dyd^2x}{dx^3}} = \frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y - dyd^2x} = \rho$$

for the general value of ρ when neither x nor y is equicrescent.

Differentiating the values of x and y, θ being equicrescent, we have

$$dx = \cos\theta dr - r\sin\theta d\theta,$$
$$d^2x = \cos\theta d^2r - 2\sin\theta d\theta dr - r\cos\theta d\theta^2,$$
$$dy = r\cos\theta d\theta + \sin\theta dr,$$
$$d^2y = \sin\theta d^2r + 2\cos\theta d\theta dr - r\sin\theta d\theta^2.$$

Hence, substituting and reducing, we have

$$\frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} = \rho.$$

II. *When r is equicrescent.*

Differentiating the values under this supposition, we have

$$d^2x = -r\sin\theta d^2\theta - 2\sin\theta drd\theta - r\cos\theta d\theta^2$$
$$d^2y = r\cos\theta d^2\theta + 2\cos\theta drd\theta - r\sin\theta d\theta^2.$$

Substituting these values together with those of dx and dy deduced under I. as they are unaltered by the new supposition, we have

$$\frac{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + r^2\frac{d\theta^3}{dr^3} + 2\frac{d\theta}{dr}} = \rho.$$

9. Show that when the equicrescent variable is changed from x to y in $\rho = \dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ we have

$$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$$

10. Change the equicrescent variable from x to y in the expression $(a^2 - x^2)\dfrac{d^2z}{dx^2} - \dfrac{a^2}{x}\dfrac{dz}{dx} - z = 0$, given $x^2 + y^2 = a^2$.

Ans. $x^2\dfrac{d^2z}{dy^2} - z = 0.$

CHAPTER VIII.

SERIES.

History. — Previous to the 17th century infinite series rarely occurred in mathematics. During the latter part of this century, and in the 18th, they came into very general use. It was supposed at this time that all higher calculations could be made to depend upon them. No universal criterion for determining the question of convergency or divergency was known at the time, nor, indeed, is one known to-day. James Gregory (1638–1675) was the first to draw a distinction between convergent and divergent series; yet Cauchy and Abel, distinguished mathematicians of the 19th century, were the *first* to question results based upon them. Newton's first mathematical discovery — the binomial theorem — was the first important contribution to this subject. The first rigorous proof of this theorem was given by Abel (1802–1829). Taylor's theorem, published in 1715, was the first *general* theorem on series published. The first correct proof of this theorem is due to Cauchy (1789–1857).

Maclaurin's Theorem, published in 1742, was admittedly founded on Taylor's. This theorem had in fact been previously published by Sterling, in 1717.

85. A **Series** is a number of terms which follow each other in obedience to some law.

Series are either **Finite** or **Infinite**; and *infinite* series are either **Convergent** or **Divergent**.

86. Finite Series. A series is finite when the number of its terms is finite. Thus, in the expansion,

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1,$$

the second member is a finite series.

87. Infinite Series. A series is infinite when the number of its terms is infinite. Thus, in the expansion,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots x^{n-1} + \cdots,$$

the second member is an infinite series.

88. A **Convergent Series** is an infinite series, the sum of the first n terms of which is a variable whose limit is finite.

All other infinite series are *divergent.*

Thus, in the geometrical progression given above, we have, from algebra, for the sum of the first n terms,

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots x^{n-1} = \frac{1-x^n}{1-x}$$

$$= \frac{1}{1-x} - \frac{x^n}{1-x}.$$

Hence, if $x < 1$, we have

$$\text{Limit}\left[1 + x + x^2 + x^3 + \cdots x^{n-1}\right]_{n \doteq \infty} = \text{Limit}\left[\frac{1-x^n}{1-x}\right]_{n \doteq \infty}$$

$$= \frac{1}{1-x} = \text{a finite quantity.}$$

Therefore it is a convergent series.

If $x > 1$,

$$\text{Limit}\left[1 + x + x^2 + x^3 + \cdots x^{n-1}\right]_{n \doteq \infty}$$

$$= \text{Limit}\left[\frac{1-x^n}{1-x}\right]_{n \doteq \infty} = \infty.$$

Therefore it is a divergent series.

89. Definitions. — The *sum of a finite series* is the sum of its terms. The *sum of a convergent series* is the limit to which the sum of the first n terms approaches as n is indefinitely increased.

The **Remainder** after n terms is the difference between the sum of the series and the sum of its first n terms.

Obviously, in a convergent series, this remainder is a variable whose limit is zero as n is indefinitely increased.

Thus, in the progression given above,

$$\frac{x^n}{1-x} = \text{Remainder after } n \text{ terms};$$

and if $x < 1$ the series is convergent, since

$$\text{Limit}\left[\frac{x^n}{1-x}\right]_{n \doteq \infty} = 0.$$

90. The development of a function consists in finding a series, the sum of whose terms is equal to the given function. Since any given function is necessarily finite, the sum of any equivalent series must be finite; hence the term *development* applies only to finite and convergent series.

91. Methods of Development:

I. By algebraic processes.

1. By division, as

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots x^n + \cdots$$

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots y^{n-1}.$$

2. By involution, as

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

3. By evolution, as

$$\sqrt[3]{a^2 + 2ax + x^2} = a^{\frac{2}{3}} + \tfrac{2}{3}a^{-\frac{1}{3}}x - \tfrac{1}{9}a^{-\frac{4}{3}}x^2 + \cdots$$

II. By general formulae:

1. Maclaurin's formula,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + f'''(0)\frac{x^3}{\underline{|3}} + \cdots \text{etc.}$$

2. Taylor's formula,

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\lfloor 2} + f'''(x)\frac{y^3}{\lfloor 3} + \cdots \text{ etc.}$$

NOTE. — There are various other formulae more general than those just given. The limits of this work preclude their discussion here. It may be remarked in passing that as yet no perfectly general method of distinguishing convergent and divergent series has been discovered. Reference will again be made to the subject at the end of the chapter.

92. Maclaurin's Theorem. — *The object of Maclaurin's Theorem is to develop a function of a single variable into a series arranged according to the ascending powers of the variable.*

Let $$f(x) = A + Bx + Cx^2 + Dx^3 + \cdots \tag{1}$$

be the proposed development, in which A, B, C, D, etc., are finite constants, whose values are independent of x. It is required to find the values of these constants. Writing the successive derivatives of $f(x)$, we have

$$f'(x) = B + 2\,Cx + 3Dx^2 + \cdots,$$
$$f''(x) = 2\,C + 2.3.Dx + \cdots,$$
$$f'''(x) = 2.3.D + \cdots, \text{ etc.}$$

Since the constants are independent of x, their values will be unaffected if we make $x = 0$. Hence, making $x = 0$ in the functions, and its derivatives, we have

$$A = f(0),$$
$$B = f'(0),$$
$$C = \frac{f''(0)}{\lfloor 2},$$
$$D = \frac{f'''(0)}{\lfloor 3}, \text{ etc.}$$

Substituting the values of the constants in (1), we have

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + \cdots \tag{2}$$

for Maclaurin's formula.

93. Remark. — Maclaurin's formula fails to develop a function of a single variable in the following cases:

(1) When it leads to a divergent series.

(2) When the function, or any one of its derivatives, becomes infinite for $x = 0$.

EXAMPLES.

1. Develop $(a + x)^5$

Here

$$\begin{aligned}
f(x) &= (a + x)^5 & \therefore\quad f(0) &= a^5,\\
f'(x) &= 5\,(a + x)^4 & \therefore\quad f'(0) &= 5\,a^4,\\
f''(x) &= 4.5.\,(a + x)^3 & \therefore\quad f''(0) &= 4.5\,a^3,\\
f'''(x) &= 3.4.5\,(a + x)^2 & \therefore\quad f'''(0) &= 3.4.5\,a^2,\\
f^{iv}(x) &= 2.3.4.5\,(a + x) & \therefore\quad f^{iv}(0) &= 2.3.4.5\,a,\\
f^{v}(x) &= 2.3.4.5 & \therefore\quad f^{v}(0) &= 2.3.4.5,
\end{aligned}$$

Since $$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + \cdots$$

we have

$$(a + x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$$

2. Develop $\dfrac{1}{1 - x} = (1 - x)^{-1}$.

Here

$$\begin{aligned}
f(x) &= (1 - x)^{-1} & \therefore\quad f(0) &= 1,\\
f'(x) &= (1 - x)^{-2} & \therefore\quad f'(0) &= 1,\\
f''(x) &= 2\,(1 - x)^{-3} & \therefore\quad f''(0) &= \lfloor 2,\\
f'''(x) &= 2.3.\,(1 - x)^{-4} & \therefore\quad f'''(0) &= \lfloor 3,\\
f^{iv}(x) &= 2.3.4\,(1 - x)^{-5} & \therefore\quad f^{iv}(0) &= \lfloor 4.\\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

Since $$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + \cdots$$

we have $$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots$$

3. Develop $\sin x$.

Here
$$\begin{aligned}
f(x) &= \sin x & \therefore\quad f(0) &= \sin 0 = 0,\\
f'(x) &= \cos x & \therefore\quad f'(0) &= \cos 0 = 1,\\
f''(x) &= -\sin x & \therefore\quad f''(0) &= -\sin 0 = 0,\\
f'''(x) &= -\cos x & \therefore\quad f'''(0) &= -\cos 0 = -1,\\
f^{iv}(x) &= \sin x & \therefore\quad f^{iv}(0) &= \sin 0 = 0,\\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

Since $f(x) = f(0) + f'(0)\,x + f''(0)\dfrac{x^2}{\lfloor 2} + f'''(0)\dfrac{x^3}{\lfloor 3} + \dots$

we have,
$$\sin x = 0 + x + 0 - \frac{x^3}{\lfloor 3} + 0 + \frac{x^5}{\lfloor 5} + 0 - \frac{x^7}{\lfloor 7} + \dots$$
$$= x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \dots$$

It will be observed that $f^{iv}(x) = \sin x = f(x)$; hence the 5th, 6th, 7th, and 8th derivatives will be the same as the 1st, 2d, 3d, and 4th respectively. We are therefore enabled to extend the development indefinitely without further differentiation. This development, together with those for $\cos x$, $\sin^{-1} x$, $\cos^{-1} x$ were given by James Gregory in 1667.

4. Prove $\cos x = 1 - \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} - \dfrac{x^6}{\lfloor 6} + \dots$

5. Develop $\log_a (1 + x)$, m being the modulus of the system.

$$\begin{aligned}
f(x) &= \log_a (1 + x) & \therefore\quad f(0) &= \log_a 1 = 0,\\
f'(x) &= m(1 + x)^{-1} & \therefore\quad f'(0) &= m,\\
f''(x) &= -m(1 + x)^{-2} & \therefore\quad f''(0) &= -m,\\
f'''(x) &= 2m(1 + x)^{-3} & \therefore\quad f'''(0) &= 2m,\\
f^{iv}(x) &= -6m(1 + x)^{-4} & \therefore\quad f^{iv}(0) &= -6m,\\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

$$\therefore \log_a (1 + x) = m\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right). \quad (1)$$

If $a = e$, $m = 1$,

$$\therefore \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \quad (2)$$

(1) and (2) are called the logarithmic series. Nicholas Mercator published this series in his Logarithmotechnia in 1668. It was the *first* series published.

6. Develop a^x.

Here $$a^x = 1 + \log a \cdot x + \log^2 a \frac{x^2}{\lfloor 2} + \log^3 a \frac{x^3}{\lfloor 3} + \ldots \quad (3)$$

If $a = e$, we have

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots \quad (4)$$

Series (3) and (4) are called the *Exponential Series.*

If in this last equation we make $x = 1$, we have,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots$$

$$\therefore e = 2.718281,$$

i.e., the base of the Napierian or Hyperbolic System of logarithms.

7. Develop $\log x$.

Here $$f(x) = \log x \quad \therefore \quad f(0) = -\infty.$$

It is unnecessary to proceed further. The function cannot be developed by the theorem. See § 93 (2).

8. Show that the following functions cannot be developed by Maclaurin's Theorem:

$$\cot x, \ \csc x, \ x^{\frac{3}{2}}, \ a^{\frac{1}{x}}$$

9. Develop $\sin^{-1} x$.

$$f(x) = \sin^{-1} x \quad \therefore \quad f(0) = 0,$$

$$f'(x) = (1 - x^2)^{-\frac{1}{2}} = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + \frac{15}{48}x^6 + \ldots \therefore f'(0) = 1,$$

$$f''(x) = x + \frac{3}{2}x^3 + \frac{15}{8}x^5 \cdots \quad \therefore \quad f''(0) = 0,$$

$$f'''(x) = 1 + \frac{9}{2}x^2 + \frac{75}{8}x^4 \cdots \quad \therefore \quad f'''(0) = 1.$$

etc. etc.

$$\therefore \sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots$$

It will be observed that the process of arriving at the successive derivatives in this example is simplified by expanding the first derivative by the Binomial Theorem.

10. Develop $\tan^{-1}x$.

$$f(x) = \tan^{-1}x \quad \therefore \quad f(0) = 0,$$
$$f'(x) = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \therefore \quad f'(0) = 1,$$
$$f''(x) = -2x + 4x^3 - 6x^5 + 8x^7 - \cdots \quad \therefore \quad f''(0) = 0,$$
$$f'''(x) = -2 + 12x^2 - 30x^4 + \cdots \quad \therefore \quad f'''(0) = -2,$$
$$f^{iv}(x) = 24x - 120x^3 + \cdots \quad \therefore \quad f^{iv}(0) = 0,$$

$$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

If $x = 1$, we have,

$$\tan^{-1}1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = .785398$$

$$\therefore \pi = 3.1416.$$

Derive the following:

11. $\sin 2x = 2x - \frac{8x^3}{\underline{|3}} + \frac{32x^5}{\underline{|5}} - \frac{128x^7}{\underline{|7}} + \cdots.$

12. $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots.$

13. $\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{x}{\sqrt{2}} - \frac{x^2}{2\sqrt{2}} + \frac{x^3}{6\sqrt{2}} + \cdots.$

14. $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots.$

15. $(1 + x)^{\frac{1}{3}} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \cdots.$

16. $(1 + x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots.$

94. Euler's Exponential values of sine and cosine.

If in the exponential series, Ex. 6,

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots$$

we substitute for x, $x\sqrt{-1}$ and $-x\sqrt{-1}$ successively, we have

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots + \sqrt{-1}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\right\}$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots - \sqrt{-1}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots\right\}$$

$\therefore$ Examples 3 and 4.

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x,$$

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x.$$

Hence
$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}},$$

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

95. Taylor's Theorem. **Taylor's Theorem** *has for its object the expansion of a function of the algebraic sum of two variables into a series arranged according to the ascending powers of one of the variables.*

Let $f(x+y) = A + By + Cy^2 + Dy^3 + \text{etc.},$

be the proposed development in which A, B, C, D, etc., are functions of x and the constants which enter the function. It is required to find the values of A, B, C, D, etc. Since the proposed development must be true for *all* values of x and y it will be true for *any* value a of x. Let A', B', C', D', etc., be the values of the coefficients for $x = a$; then

$$f(a+y) = A' + B'y + C'y^2 + D'y^3 + \text{etc.} \quad . \quad . \quad (a)$$

Writing the successive derivatives with respect to y, we have,

$$f'(a+y) = B' + 2\,C'y + 3\,D'y^2 + \text{etc.},$$
$$f''(a+y) = 2\,C' + 6\,D'y + \text{etc.},$$
$$f'''(a+y) = 6\,D' + \text{etc.},$$
$$\text{etc.}$$

Since the original function as well as its derivatives must be true for all values of y, they are true when $y = 0$; hence

$$f(a) = A',$$
$$f'(a) = B',$$
$$\frac{f''(a)}{2} = C',$$
$$\frac{f'''(a)}{\lfloor 3} = D',$$
$$\text{etc.}$$

Substituting these values in (a), we have,

$$f(a+y) = f(a) + f'(a)y + f''(a)\frac{y^2}{\lfloor 2} + f'''(a)\frac{y^3}{\lfloor 3} + \text{etc.},$$

for the proposed development when $x = a$. But a is *any* value of x; hence, generally,

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\lfloor 2} + f'''(x)\frac{y^3}{\lfloor 3} + \text{etc.} \quad . \quad (1).$$

COR. 1. If in (1) we make $x = 0$ and change y to x, we have,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + f'''(0)\frac{x^3}{\lfloor 3} + \text{etc.},$$

which is Maclaurin's formula. Hence Maclaurin's formula is a special case of Taylor's more general formula.

96. REMARK. Taylor's Formula fails to develop a function of the sum of two variables in the following cases:

(1) When it leads to a divergent series.

(2) When the function or any one of its derivatives becomes infinite for a value, or values, of one of the variables, it fails for that value or those values.

EXAMPLES.

1. Develop $(x + y)^5$.

Here

$$f(x) = x^5,$$
$$f'(x) = 5x^4,$$
$$f''(x) = 4.5.x^3,$$
$$f'''(x) = 3.4.5.x^2,$$
$$f^{iv}(x) = 2.3.4.5.x,$$
$$f^{v}(x) = 2.3.4.5.$$

Since, $f(x + y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\lfloor 2} + f'''(x)\frac{y^3}{\lfloor 3} + \cdots$

we have, $(x + y)^5 = x^5 + 5x^4y + 4.5\,x^3\frac{y^2}{\lfloor 2} + 3.4.5\,x^2\frac{y^3}{\lfloor 3}$

$$+ 2.3.4.5\,x\frac{y^4}{\lfloor 4} + 2.3.4.5\frac{y^5}{\lfloor 5}.$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

2. Develop a^{x+y}.

$$f(x) = a^x. \qquad f'''(x) = \log^3 a.a^x.$$
$$f'(x) = \log a.a^x. \qquad f^{iv}(x) = \log^4 a.a^x.$$
$$f''(x) = \log^2 a.a^x. \qquad \cdots\cdots$$
$$f^n(x) = \log^n a.a^x.$$

Since, $f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\underline{|2}} + f'''(x)\frac{y^3}{\underline{|3}} + \cdots$

we have, $$a^{x+y} = a^x + \log a.a^x y + \log^2 a.a^x.\frac{y^2}{\underline{|2}} + \log^3 a.a^x\frac{y^3}{\underline{|3}}$$
$$+ \cdots \log^n a.a^x.\frac{y^n}{\underline{|n}} + \cdots$$

3. Develop $\sin(x+y)$ and prove that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Here $$f(x) = \sin x. \qquad f'''(x) = -\cos x.$$
$$f'(x) = \cos x. \qquad f^{iv}(x) = \sin x.$$
$$f''(x) = -\sin x.$$

$$\therefore \sin(x+y) = \sin x + \cos x.y - \sin x\frac{y^2}{\underline{|2}} - \cos x\frac{y^3}{\underline{|3}} + \cdots$$
$$= \sin x\left(1 - \frac{y^2}{\underline{|2}} + \frac{y^4}{\underline{|4}} - \cdots\right) + \cos x\left(y - \frac{y^3}{\underline{|3}} + \frac{y^5}{\underline{|5}} - \cdots\right).$$

Hence, Exs. 3 and 4, p. 110,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Prove similarly the following trigonometric relations:

4. $\sin(x-y) = \sin x \cos y - \cos x \sin y$.

5. $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

6. $\cos(x-y) = \cos x \cos y + \sin x \sin y$.

7. Develop $\log_a(x+y)$ and show that m is the modulus of the system whose base is a.

$$\log_a(x+y)=\log_a x+m\left(\frac{y}{x}-\frac{y^2}{2x^2}+\frac{y^3}{3x^3}-\frac{y^4}{4x^4}+\frac{y^5}{5x^5}-\cdots\right).$$

If $x=1$, we have

$$\log_a(1+y)=m\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\cdots\right) \quad . \quad . \quad (a)$$

an expression similar to one previously derived by Maclaurin's theorem. See Ex. 5, p. 110.

If $y=0$, the formula fails to develop the function, § 96, (2).

From the logarithmic series in algebra we have

$$\log_e(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\cdots \quad . \quad . \quad . \quad (b)$$

Dividing (a) by (b) we have

$$\frac{\log_a(1+y)}{\log_e(1+y)}=\frac{m}{1}, \text{ or } \log_a(1+y)=m\log_e(1+y)$$

That is, m is the modulus of the system, and if $a=e$, $m=1$. See § 30, Cor.

Derive the following:

8. $e^{x+y}=e^x\left(1+y+\frac{y^2}{\underline{|2}}+\frac{y^3}{\underline{|3}}+\cdots\right).$

9. $\log\sec(x+y)=\log\sec x+\tan x\,.\,y+\sec^2 x\frac{y^2}{2}$

$$+\sec^2 x\tan x\frac{y^3}{3}+\cdots$$

10. $\sin^{-1}(x+y)=\sin^{-1}x+y(1-x^2)^{-\frac{1}{2}}+x(1-x^2)^{-\frac{3}{2}}\frac{y^2}{\underline{|2}}$

$$+(1+2x^2)(1-x^2)^{-\frac{5}{2}}\frac{y^3}{\underline{|3}}+\cdots$$

If $x=1$, the formula fails to develop the function. § 96, 2.

97. Bernouilli's Series. Resuming Taylor's formula,

$$f(x+y)=f(x)+f'(x)y+f''(x)\frac{y^2}{\underline{|2}}+f'''(x)\frac{y^3}{\underline{|3}}+\text{ etc.},$$

and making $y = -x$ and transposing, we have,

$$f(x) = f(0) + xf'(x) - \frac{x^2}{\underline{|2}} f''(x) + \frac{x^3}{\underline{|3}} f'''(x) - \frac{x^4}{\underline{|4}} f^{iv}(x) + \cdots$$

By aid of this formula we are enabled to expand a function of a single variable into a series. Thus,

$$e^{-x} = 1 - xe^{-x} - \frac{x^2}{\underline{|2}} e^{-x} - \frac{x^3}{\underline{|3}} e^{-x} - \frac{x^4}{\underline{|4}} e^{-x}.$$

Dividing through by e^{-x} and transposing, we have,

$$\frac{1}{e^{-x}} = e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots$$

See Ex. 6, p. 111.

98. We have referred in a previous article (90) to the fact that the term development was inapplicable to a divergent series, and the student was cautioned not to accept an infinite series obtained by any of the foregoing methods as the development of the function, unless the question of its convergency or divergency had been previously settled. It remains to show how series are examined for divergency and convergency.

99. Lemma. *If* $f(x)]_a = 0$ *and* $f(x)]_b = 0$ *and* $f(x)$ *is continuous between these limiting values of* x, *then* $f'(x) = 0$ *for some value of* x *intermediate between* $x = a$ *and* $x = b$.

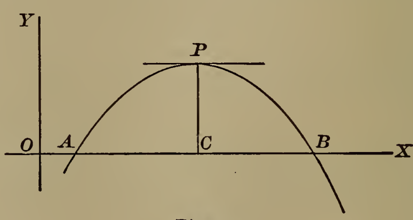


Fig. 14.

Let APB be the locus of $y = f(x)$, and let $OA = a$, and $OB = b$; then

$$f(x)]_a = 0, \text{ and } f(x)]_b.$$

Since, by hypothesis, $f(x)$ is continuous between the points $A(a, 0)$ and $B(b, 0)$, there must be some intermediate point (OC, CP) where the tangent to the curve is ∥ to the x-axis. But at such a point

$$\frac{dy}{dx} = f'(x) = 0. \quad \S\ 19.$$

Hence the proposition.

100. Lagrange's Theorem on the limits of Taylor's Theorem.[1]

Taylor's formula may be written,—

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\lfloor 2} + \cdots + f^{n-1}(x)\frac{y^{n-1}}{\lfloor n-1}$$
$$+ P\frac{y^n}{\lfloor n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which P is some function of x and y, and

$$P\frac{y^n}{\lfloor n} = \text{Remainder after } n \text{ terms.}$$

It is desired to find the value of P and thence the value of $P\frac{y^n}{\lfloor n}$.

Let $y = X - x$.

Substituting in (1) in every term, *including P*, and transposing we have

$$f(X) - f(x) - f'(x)(X-x) - f''(x)\frac{(X-x)^2}{\lfloor 2} - \cdots$$
$$- f^{n-1}(x)\frac{(X-x)^{n-1}}{\lfloor n-1} - P\frac{(X-x)^n}{\lfloor n} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which P is now a function of x and X.

Replacing x by z in every term in the preceding expression,

[1] This discovery of Lagrange placed Taylor's theorem on a satisfactory basis for the first time.

excepting P, and representing the resulting expression by $\phi(z)$, we have

$$\phi(z)=f(X)-f(z)-f'(z)(X-z)-f''(z)\frac{(X-z)^2}{\lfloor 2}-\cdots$$
$$-f^{n-1}(z)\frac{(X-z)^{n-1}}{\lfloor n-1}-P\frac{(X-z)^n}{\lfloor n} \quad . \ . \ . \ . \ . \ . \quad (3)$$

Now if $z=x$, we have $\phi(z)]_x=0$, since the second member of (3) reduces to the first member of (2).

Again, if $z=X$, we have $\phi(z)]_X=0$, since each term in the second member becomes zero.

Hence $\phi'(z)=0$, for some value of z between the limiting values x and X of z (§ 99). Let θ be a proper fraction. Then $z=x+\theta(X-x)$ represents any value of z between the values $z=x$ and $z=X$.

Differentiating (3) we have, after cancellation,

$$\phi'(z)=-f^n(z)\frac{(X-z)^{n-1}}{\lfloor n-1}+P\frac{(X-z)^{n-1}}{\lfloor n-1}.$$

Now, if $z=x+\theta(X-x)$, we have $\phi'(z)]_{x+\theta(X-x)}=0$,

hence, $$P=f^n(x+\theta(X-x)).$$

Multiplying both members by $\frac{y^n}{\lfloor n}$ and changing $X-x$ to y, we have

$$P\frac{y^n}{\lfloor n}=f^n(x+\theta y)\frac{y^n}{\lfloor n} \quad . \ . \ . \ . \ . \ . \ . \quad (4)$$

for the remainder after n terms. Substituting this value in (1), we have

$$f(x+y)=f(x)+f'(x)y+f''(x)\frac{y^2}{\lfloor 2}+\cdots+f^{n-1}(x)\frac{y^{n-1}}{\lfloor n-1}$$
$$+f^n(x+\theta y)\frac{y^n}{\lfloor n} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$

for the completed form of Taylor's theorem.

Had we assumed Py, instead of $P\frac{y^n}{\lfloor n}$, to represent the remainder after n terms at the outset, and followed the course of reasoning just concluded, we would have obtained

$$Py = f^n(x+\theta y)\frac{(1-\theta)^{n-1}}{\lfloor n-1}y^n \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as a second form of the remainder, and, consequently,

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{\lfloor 2} + \cdots + f^{n-1}(x)\frac{y^{n-1}}{\lfloor n-1}$$
$$+ f^n(x+\theta y)\frac{(1-\theta)^{n-1}}{\lfloor n-1}y^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

as a second completed form of Taylor's theorem.

If we make $x = 0$ in (5) and (7), and then change y to x, we have

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + \cdots f^{n-1}(0)\frac{x^{n-1}}{\lfloor n-1}$$
$$+ f^n(\theta x)\frac{x^n}{\lfloor n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\lfloor 2} + \cdots f^{n-1}(0)\frac{x^{n-1}}{\lfloor n-1}$$
$$+ f^n(\theta x)\frac{(1-\theta)^{n-1}}{\lfloor n-1}x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as the two completed forms of Maclaurin's theorem. Similarly from 4 and 6, we have

$$f^n(\theta x)\frac{x^n}{\lfloor n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$f^n(\theta x)\frac{(1-\theta)^{n-1}}{\lfloor n-1}x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for the two forms of the remainder after n terms in Maclaurin's theorem.

Since (§ 89) the remainder after any n terms, in any convergent series, is a variable whose limit is zero, the question of divergency or convergency can generally be ascertained by examining this remainder.

101. *If the n^{th} derivative is finite for all values of n in Taylor's and Maclaurin's expansions the function is developed.*

For the $(n-1)^{\text{th}}$ and n^{th} derivative in Taylor's expansion we have

$$f^{n-1}(x)\frac{y^{n-1}}{\lfloor n-1}, \quad f^{n}(x)\frac{y^{n}}{\lfloor n}.$$

Dividing the latter by the former, we have

$$\frac{f^{n}(x)}{f^{n-1}(x)}\cdot\frac{y}{n}.$$

Now $f^{n}(x)$ is, by hypothesis, finite, and, since the value of n is arbitrary, $f^{n-1}(x)$ is also finite; hence

$$\frac{f^{n}(x)}{f^{n-1}(x)} \text{ is finite.}$$

Again, since y is necessarily finite, $\frac{y}{n}$ becomes ultimately very small, and approaches zero as a limit as n approaches ∞.

$$\therefore \text{Limit} \quad \left[\frac{f^{n}(x)}{f^{n-1}(x)}\cdot\frac{y}{n}\right]_{n \doteq \infty} = 0.$$

Hence the values of the terms become infinitely small as n becomes infinitely great, and their sum, after n terms, approaches zero as a limit. Hence the series is convergent and the function developed.

The proof is similar for Maclaurin's expansions.

102. Let us now examine a few of the expansions previously determined, and ascertain if the term *development* used in connection with them has been properly used.

1. $$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$$ Ex. 3, p. 110.

Examining the derivatives, we see that

$$f^n(x) = \pm \sin x, \text{ or } \pm \cos x,$$

according as n is even or odd. Hence the n^{th} derivative is finite for all values of n, $\therefore$ (§ 101) the function is developed.

Similarly we may prove that

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \text{etc.}$$ Ex. 4, p. 110.

2. $$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$ Ex. 5, p. 110.

In this case, $$f^n(x) = (-1)^{n-1}\frac{\lfloor n-1}{(1+x)^n}.$$

Hence, $$f^n(\theta x)\frac{x^n}{\lfloor n} = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

Hence,

$$\text{Limit}\left[f^n(\theta x)\frac{x^n}{\lfloor n}\right]_{n \doteq \infty} = \text{Limit}\left[\frac{(-1)^{n-1}}{n}\cdot\left(\frac{x}{1+\theta x}\right)^n\right]_{n \doteq \infty} = 0,$$

if $x < +1$; $\therefore$ the function is developed for all positive values of x less than unity. By using the second form of the remainder we can show that the function is developed when x lies between 0 and -1.

It is to be noted that the logarithm of a number cannot be computed by developing the log x into a series (cf. Ex. 7, p. 111), and if the series obtained from the log $(1+x)$ were used, a large number of terms would be necessary in order to obtain accurate values.

If the series log $(1+x)$ and log $(1-x)$ are used, as explained below, logarithms may be obtained with accuracy and only a comparatively small number of terms used.

The series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1}$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \tag{2}$$

Subtract (2) from (1),

$$\log(1+x) - \log(1-x) = \log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \tag{3}$$

Let $\frac{1+x}{1-x} = \frac{n+1}{n}$, then $x = \frac{1}{2n+1}$, substitute this value in equation (3),

$$\log\left(\frac{n+1}{n}\right) = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right].$$

This series converges rapidly and by choosing $n = 1$, the log 2 may be obtained as follows:

$$\log\left(\frac{2}{1}\right) = 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right] = 0.693147.$$

Having obtained the log 2, the log 3 may be found by taking

$$n = 2.$$

$$\log\left(\frac{3}{2}\right) = \log 3 - \log 2 = 2\left[\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \cdots\right]$$

hence

$$\log 3 = \log 2 + 2\left[\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \cdots\right] = 1.09861.$$

By computing the logarithms of the prime numbers in this way, the logarithms of numbers of which these are factors may be found.

These logarithms are the Napierian or *natural* logarithms (base $= 2.71828$), and by multiplying by the modulus ($m = .4343$) the logarithms to the *base* 10 may be obtained.

3. $a^x = 1 + \log a \,.\, x + \log^2 a \dfrac{x^2}{\lfloor 2} + \cdots\cdot$ Ex. 6, p. 111.

Here, $\quad f^n(x) = a^x \log^n a.$

Hence, $\quad f^n(\theta x)\dfrac{x^n}{\lfloor n} = a^{\theta x}\log^n a \dfrac{x^n}{\lfloor n} = \dfrac{(x \log a)^n}{\lfloor n} a^{\theta x}$

Since $a^{\theta x}$ is finite and limit $\left[\dfrac{(x\log a)^n}{\lfloor n}\right]_{n \doteq \infty} = 0$, § 101,

we have,

$$\text{Limit}\left[\frac{(x \log a)^n}{\lfloor n} a^{\theta x}\right]_{n \doteq \infty} = 0.$$

Hence the function is developed.

4. $\sin^{-1} x = x + \dfrac{x^3}{6} + \dfrac{3}{24}\cdot\dfrac{x^5}{5} + \cdots$ Ex. 9, p. 111.

Here the ratio of the n^{th} term to the $(n-1)^{th}$ term is $\dfrac{x^2}{c}$ where c is some function of n and a finite quantity when n is finite. Necessarily $x > 1$, ∴ Limit $\left.\dfrac{x^2}{c}\right]_{n \doteq \infty} = 0$;

∴ the series is convergent.

5. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$

The reasoning is the same as in Ex. 4 when $x < 1$. When $x > 1$ the series is divergent.

CHAPTER IX.

ILLUSORY FORMS.

HISTORY. — The Marquis de St. Mesme (L'Hospital) published in his calculus (1696) a partial investigation of the limiting value of the ratio of functions which for a certain value of the variable take the form $\frac{0}{0}$.

John Bernouilli, the elder (1667–1748), was the first to solve the problem by aid of the calculus.

103. We are accustomed to consider the value of a fraction as *indeterminate* when for any given value or values of the variables which enter it, it assumes the form $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$, etc. It is our purpose to show (1), that such expressions are not necessarily indeterminate but are frequently *illusory*, and (2), to indicate a method by means of which their true values may in general be ascertained.

104. Definition. *The value of a fraction is the limit it approaches as its numerator, or denominator, or both, approach an assigned value or values.*

Thus, the fact that $\frac{x^2}{3x-1} = \frac{1}{2}$ when $x = 1$ may be expressed more generally, x being a variable, by writing

$$\text{Limit}\left[\frac{x^2}{3x-1}\right]_{x \doteq 1} = \frac{1}{2}.$$

105.* Evaluation of the forms $\frac{a}{0}, \frac{0}{a}, \frac{0}{0}$.

* Kepler introduced name and notion of *infinity* into geometry in 1615.

I. **Form** $\frac{a}{0}$.

Here, $$\text{Limit}\left[\frac{a}{x}\right]_{x \doteq 0} = \frac{a}{0} = \infty,$$

i.e., when the denominator of a fraction becomes infinitely small and approaches zero as its limit, the numerator being constant and finite, the value of the fraction becomes infinitely large.

II. **Form** $\frac{0}{a}$.

Here, $$\text{Limit}\left[\frac{x}{a}\right]_{x \doteq 0} = \frac{0}{a} = 0.$$

i.e., when the numerator of a fraction becomes infinitely small, and approaches zero as its limit, the denominator being constant and finite, the value of the fraction becomes infinitely small, and also approaches zero as its limit.

III. **Form** $\frac{0}{0}$.

This case frequently admits of evaluation,

(*a*) By Algebraic or Trigonometric reduction.

Thus $\dfrac{x^2 - 7x + 10}{x^2 - 4} = \dfrac{0}{0}$ when $x = 2$; but on factoring and cancelling, we have

$$\frac{x^2 - 7x + 10}{x^2 - 4} = \frac{(x-2)(x-5)}{(x-2)(x+2)} = \frac{x-5}{x+2} = -\frac{3}{4} \text{ when } x = 2.$$

Again: $\dfrac{x}{\sqrt{a+x} - \sqrt{a-x}} = \dfrac{0}{0}$ when $x = 0$; but multiplying both numerator and denominator by the complementary surd $\sqrt{a+x} + \sqrt{a-x}$ and reducing we have,

$$\frac{x}{\sqrt{a+x}-\sqrt{a-x}} = \frac{\sqrt{a+x}+\sqrt{a-x}}{2} = \sqrt{a} \text{ when } x \doteq 0.$$

Again: $\dfrac{\cos 2x}{\cot 2x} = \dfrac{0}{0}$ when $x = \dfrac{\pi}{4}$; but

$$\frac{\cos 2x}{\cot 2x} = \sin 2x = 1 \text{ when } x = \frac{\pi}{4}.$$

Obviously these methods are special in their character, and apply only to those algebraic and trigonometric forms which admit of ready reduction. We have, however, a general process afforded,

(*b*) By the Differential Calculus.

To deduce this process let $\left.\dfrac{\phi(x)}{\psi(x)}\right]_a = \dfrac{\phi(a)}{\psi(a)} = \dfrac{0}{0}$;

that is, let $\phi(a) = \psi(a) = 0$.

By Taylor's theorem, we have,

$$\frac{\phi(x+y)}{\psi(x+y)} = \frac{\phi(x) + \phi'(x)y + \phi''(x)\frac{y^2}{\lfloor 2} + \cdots}{\psi(x) + \psi'(x)y + \psi''(x)\frac{y^2}{\lfloor 2} + \cdots}.$$

Making $x = a$, we have (since $\phi(a) = \psi(a) = 0$ and y is a factor of both numerator and denominator),

$$\frac{\phi(a+y)}{\psi(a+y)} = \frac{\phi'(a) + \phi''(a)\frac{y}{\lfloor 2} + \phi'''(a)\frac{y^2}{\lfloor 3} + \cdots}{\psi'(a) + \psi''(a)\frac{y}{\lfloor 2} + \psi'''(a)\frac{y^2}{\lfloor 3} + \cdots}.$$

Hence,

$$\text{Limit}\left[\frac{\phi(a+y)}{\psi(a+y)}\right]_{y \doteq 0} = \text{Limit}\left[\frac{\phi'(a)+\phi''(a)\frac{y}{\lfloor 2}+\phi'''(a)\frac{y^2}{\lfloor 3}+\cdots}{\psi'(a)+\psi''(a)\frac{y}{\lfloor 2}+\psi'''(a)\frac{y^2}{\lfloor 3}+\cdots}\right]_{y \doteq 0} \quad (1)$$

i.e.;
$$\frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)}.$$

Hence
$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \left.\frac{\phi'(x)}{\psi'(x)}\right]_a.$$

Hence the general rule, *if* $\frac{\phi(x)}{\psi(x)} = \frac{0}{0}$ *for any value a of x the value of the fraction may in general be found by dividing the first derivative of the numerator by the first derivative of the denominator and then substituting a for x in this ratio.*

Thus
$$\frac{\phi(x)}{\psi(x)} = \frac{x^2 - 7x + 10}{x^2 - 4} = \frac{0}{0} \text{ when } x = 2,$$
$$\therefore \frac{\phi'(x)}{\psi'(x)} = \frac{2x - 7}{2x} = -\frac{3}{4} \text{ when } x = 2.$$

Hence, as before III (*a*), we have $-\frac{3}{4}$ for the true value of the fraction.

Cor. 1. If $\frac{\phi'(a)}{\psi'(a)} = \frac{0}{0}$ i.e., if $\phi'(a) = \psi'(a) = 0$, then the limits determined from equation (1), are $\frac{\phi(a)}{\psi(a)} = \frac{\phi''(a)}{\psi''(a)}$.

If $\phi''(a) = \psi''(a) = 0$, we have similarly
$$\frac{\phi(a)}{\psi(a)} = \frac{\phi'''(a)}{\psi'''(a)}.$$

Hence, generally,
$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \left.\frac{\phi'(x)}{\psi'(x)}\right]_a = \left.\frac{\phi''(x)}{\psi''(x)}\right]_a = \left.\frac{\phi'''(x)}{\psi'''(x)}\right]_a = \text{etc.,}$$

when these forms successively assume the illusory form $\frac{0}{0}$.

EXAMPLES.

1. Evaluate the fraction $\frac{e^x + e^{-x} - 2}{1 - \cos x}$ when $x = 0$.

Here
$$\left.\frac{\phi(x)}{\psi(x)}\right]_0 = \left.\frac{e^x + e^{-x} - 2}{1 - \cos x}\right]_0 = \frac{0}{0}.$$
$$\left.\frac{\phi'(x)}{\psi'(x)}\right]_0 = \left.\frac{e^x - e^{-x}}{\sin x}\right]_0 = \frac{0}{0}.$$
$$\left.\frac{\phi''(x)}{\psi''(x)}\right]_0 = \left.\frac{e^x + e^{-x}}{\cos x}\right]_0 = 2.$$
$$\therefore \left.\frac{e^x + e^{-x} - 2}{1 - \cos x}\right]_0 = 2.$$

Evaluate the following:

2. $\left.\dfrac{x - 2}{(x-1)^n - 1}\right]_2.$ *Ans.* $\dfrac{1}{n}$.

3. $\left.\dfrac{x^3 - 1}{x - 1}\right]_1.$ *Ans.* 3.

4. $\left.\dfrac{a^x - c^x}{x}\right]_0.$ *Ans.* $\log\dfrac{a}{c}$.

5. $\left.\dfrac{x - \sin x}{x^3}\right]_0.$ *Ans.* $\dfrac{1}{6}$.

6. $\left.\left(\dfrac{\sin nx}{x}\right)^m\right]_0.$ *Ans.* n^m.

7. $\left.\dfrac{1 - \sin x + \cos x}{\sin x + \cos x - 1}\right]_{\frac{\pi}{2}}.$ *Ans.* 1.

8. $\left.\dfrac{a^{\sin x} - a}{\log \sin x}\right]_{\frac{\pi}{2}}.$ *Ans.* $a \log a$.

9. $\left.\dfrac{e^x - e^{-x}}{\log(1 + x)}\right]_0.$ *Ans.* 2.

10. $\left.\dfrac{x^x - x}{1 - x + \log x}\right]_1.$ *Ans.* -2.

11. $\left.\dfrac{e^x - 2\sin x - e^{-x}}{x - \sin x}\right]_0.$ *Ans.* 4.

12. $\left.\dfrac{x^4 - 2x^3 + 2x - 1}{x^6 - 15x^2 + 24x - 10}\right]_1.$ *Ans.* .1

106. Evaluation of the forms $\frac{a}{\infty}$, $\frac{\infty}{a}$, $\frac{\infty}{\infty}$.

I. **Form** $\frac{a}{\infty}$.

Here, evidently, Limit $\left[\frac{a}{x}\right]_{x \doteq \infty} = 0.$

II. **Form** $\frac{\infty}{a}$.

$$\text{Limit}\left[\frac{x}{a}\right]_{x \doteq \infty} = \infty$$

III. **Form** $\frac{\infty}{\infty}$.

Let
$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \frac{\phi(a)}{\psi(a)} = \frac{\infty}{\infty}.$$

i.e., let $\phi(a) = \psi(a) = \infty$.

Taking the reciprocal of the numerator and denominator we may write,

$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \left.\frac{\frac{1}{\psi(x)}}{\frac{1}{\phi(x)}}\right]_a = \frac{0}{0}.$$

By § 105, III. (*b*), we have,

$$\left.\frac{\frac{1}{\psi(x)}}{\frac{1}{\phi(x)}}\right]_a = \left.\frac{\frac{-\psi'(x)}{[\psi(x)]^2}}{\frac{-\phi'(x)}{[\phi(x)]^2}}\right]_a = \left[\frac{\phi(x)}{\psi(x)}\right]^2 \cdot \left.\frac{\psi'(x)}{\phi'(x)}\right]_a;$$

hence,
$$\frac{\phi(a)}{\psi(a)} = \left[\frac{\phi(a)}{\psi(a)}\right]^2 \cdot \frac{\psi'(a)}{\phi'(a)}; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e.,
$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \left.\frac{\phi'(x)}{\psi'(x)}\right]_a.$$

Hence, the rule for evaluating the fraction $\frac{\phi(x)}{\psi(x)}$ that takes the form $\frac{\infty}{\infty}$ for any value a of x is the same as when it takes the form $\frac{0}{0}$. See § 105, III., *b*.

Thus, $$\left.\frac{\phi(x)}{\psi(x)}\right]_0 = \left.\frac{\log \sin x}{\cot x}\right]_0 = \frac{\infty}{\infty}.$$

$$\left.\frac{\phi'(x)}{\psi'(x)}\right]_0 = \left.\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}\right]_0 = -\sin x \cos x\Big]_0 = 0.$$

$$\therefore \left.\frac{\log \sin x}{\cot x}\right]_0 = 0.$$

107. Remark. — In deriving equation (2) Art. 106 from equation (1) of the same article we divided through by $\left[\frac{\phi(a)}{\psi(a)}\right]^2$. Obviously if the real value of this fraction is either 0 or ∞ the generality of equation (2) is not established. Let us examine these cases.

I. When $$\frac{\phi(a)}{\psi(a)} = 0.$$

Let c be any constant, then

$$\frac{\phi(a)}{\psi(a)} + c = \frac{\phi(a) + c\psi(a)}{\psi(a)} = \frac{\infty}{\infty}.$$

since $\phi(a) = \psi(a) = \infty$. But the real value of this expression is c; hence equa. (2) § 106 applies and we have

$$\frac{\phi(a)}{\psi(a)} + c = \frac{\phi(a) + c\psi(a)}{\psi(a)} = \frac{\phi'(a) + c\psi'(a)}{\psi'(a)} = \frac{\phi'(a)}{\psi'(a)} + c.$$

$$\therefore \frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)} = 0.$$

Hence Equation (2) is true in this case.

II. When $$\frac{\phi(a)}{\psi(a)} = \infty.$$

Then $$\frac{1}{\frac{\phi(a)}{\psi(a)}} = \frac{\psi(a)}{\phi(a)} = 0.$$

Hence, by I.,

$$\frac{\psi(a)}{\phi(a)} = \frac{\psi'(a)}{\phi'(a)} = 0;$$

$$\therefore \frac{\phi(a)}{\psi(a)} = \frac{\phi'(a)}{\psi'(a)} = \infty.$$

and equation (2) holds in this case. Hence, equation (2) is generally true for the illusory form $\frac{\infty}{\infty}$.

We may write, therefore, generally,

$$\left.\frac{\phi(x)}{\psi(x)}\right]_a = \left.\frac{\phi'(x)}{\psi'(x)}\right]_a = \left.\frac{\phi''(x)}{\psi''(x)}\right]_a = \left.\frac{\phi'''(x)}{\psi'''(x)}\right]_a = \text{etc.}$$

when the fractions assume successively the illusory form $\frac{\infty}{\infty}$.

EXAMPLES.

1. Evaluate $$\frac{\log x}{\frac{1}{x}} \text{ when } x = 0.$$

$$\left.\frac{\phi(x)}{\psi(x)}\right]_0 = \left.\frac{\log x}{\frac{1}{x}}\right]_0 = \frac{\infty}{\infty}.$$

$$\left.\frac{\phi'(x)}{\psi'(x)}\right]_0 = \left.\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right]_0 = -x\Big]_0 = 0.$$

Evaluate the following:

2. $\left.\dfrac{\log x}{\cot x}\right]_0.$ *Ans.* 0.

3. $\left.\dfrac{\frac{1}{x}}{\cot x}\right]_0.$ *Ans.* 1.

4. $\left.\dfrac{\sec x}{\sec 3x}\right]_{\frac{\pi}{2}}.$ *Ans.* -3.

It will be found simpler to transform this fraction so as to make it take the form $\frac{0}{0}$.

5. $\left.\dfrac{\log x}{x}\right]_\infty.$ *Ans.* 0.

108. Evaluate the forms $0 \cdot \infty$ and $\infty - \infty$.

Expressions which assume these forms can be readily reduced to expressions that assume either of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EXAMPLES.

1. Evaluate $x \log x$ when $x = 0$.

Here $[x \log x]_0 = 0(-\infty)$.

But $[x \log x]_0 = \left[\dfrac{\log x}{\frac{1}{x}}\right]_0 = \frac{\infty}{\infty}.$

Hence, Ex. 1, p. 132, we have

$$x \log x = 0 \text{ when } x = 0.$$

2. $[\sec x - \tan x]_{\frac{\pi}{2}}.$

Here $[\sec x - \tan x]_{\frac{\pi}{2}} = \infty - \infty$.

But $[\sec x - \tan x]_{\frac{\pi}{2}} = \left.\dfrac{1 - \sin x}{\cos x}\right]_{\frac{\pi}{2}} = \frac{0}{0}.$

$$\therefore \quad \frac{\phi'(x)}{\psi'(x)}\Big]_{\frac{\pi}{2}} = \frac{-\cos x}{-\sin x}\Big]_{\frac{\pi}{2}} = \cot x\Big]_{\frac{\pi}{2}} = 0.$$

$$\therefore \quad \Big[\sec x - \tan x\Big]_{\frac{\pi}{2}} = 0.$$

3. $\Big[2x\tan x - \pi\sec x\Big]_{\frac{\pi}{2}}$. *Ans.* -2.

4. $\Big[\sec 2x(1 - \tan x)\Big]_{\frac{\pi}{4}}$. *Ans.* 1.

5. $\left[\dfrac{1}{\log x} - \dfrac{1}{x-1}\right]_1$. *Ans.* $\frac{1}{2}$.

6. $\left[\dfrac{1}{\log x} - \dfrac{x}{\log x}\right]_1$. *Ans.* -1.

7. $\left[(1-x)\tan\dfrac{\pi x}{2}\right]_1$. *Ans.* $\dfrac{2}{\pi}$.

8. $\Big[e^{-x}\log x\Big]_\infty$. *Ans.* 0.

9. $\Big[\sec 3x\cos 7x\Big]_{\frac{\pi}{2}}$. *Ans.* $\frac{7}{3}$.

109. Evaluate the forms 0^0, ∞^0, 1^∞, 0^∞, ∞^∞.

Let $y = u^v$, in which u and v are functions of x.

Applying logarithms, we have

$$\log y = v\log u.$$

For some value a of x, let us suppose

(1) $u^v = 0^0$, i.e., $u = v = 0$; then $\log y = -0\cdot\infty$.

(2) $u^v = \infty^0$, i.e., $u = \infty$, $v = 0$; then $\log y = 0\cdot\infty$.

(3) $u^v = 1^\infty$, i.e., $u = 1$, $v = \infty$; then $\log y = \infty\cdot 0$.

(4) $u^v = 0^\infty$, i.e., $u = 0$, $v = \infty$; then $\log y = -\infty\cdot\infty = -\infty$, $\therefore y = 0$.

(5) $u^v = \infty^\infty$, i.e., $u = \infty$, $v = \infty$; then $\log y = \infty\cdot\infty = \infty$, $\therefore y = \infty$.

It appears, therefore, that forms (4) and (5) are not, properly speaking, illusory, and that the *logarithms* of forms (1), (2), and (3) may be evaluated by the method explained in the last article.

Thus $(1 + x)^{\frac{1}{x}} = 1^{\infty}$ when $x = 0$. Let $y = (1 + x)^{\frac{1}{x}}$, and, applying logarithms, we have

$$\log y = \frac{1}{x} \log (1 + x) = \infty \cdot 0 \text{ when } x = 0.$$

But $$\left.\frac{1}{x} \log (1 + x)\right]_0 = \left.\frac{\log (1 + x)}{x}\right]_0 = \tfrac{0}{0}.$$

Hence, $$\left.\frac{\phi'(x)}{\psi'(x)}\right]_0 = \left.\frac{\frac{1}{1 + x}}{1}\right]_0 = 1.$$

$$\therefore \log y = 1. \qquad \therefore y = e.$$

$$\text{i.e., } \left.(1 + x)^{\frac{1}{x}}\right]_0 = e.$$

Again, $\left.x^x\right]_0 = 0^0$. Let $y = x^x$, then

$\left.\log y\right]_{x=0} = \left.x \log x\right]_0 = -0 \cdot \infty = 0.$ See Ex. 1, p. 133.

$$\therefore \log y = 0. \qquad \therefore y = \left.x^x\right]_0 = 1.$$

110. Evaluation of compound illusory forms $\frac{0}{0}\cdot\frac{0}{0}$, $\frac{0}{0}\cdot\frac{\infty}{\infty}$, etc.

Such forms arise from factors in the function that take illusory forms for the given value of the variable.

We may evaluate the function by evaluating the factors separately. Thus

$$\left.\frac{\sin^2 x(\sec x - 1)}{x^3}\right]_0 = \left(\frac{\sin x}{x}\right)^2 \cdot \left.\frac{1 - \cos x}{x \cos x}\right]_0 = \frac{0}{0} \cdot \frac{0}{0}.$$

But $$\left.\frac{\sin x}{x}\right]_0 = \left.\frac{\cos x}{1}\right]_0 = 1. \quad \therefore \left.\left(\frac{\sin x}{x}\right)^2\right]_0 = 1;$$

and $$\left.\frac{1 - \cos x}{x \cos x}\right]_0 = \left.\frac{\sin x}{\cos x - x \sin x}\right]_0 = \tfrac{0}{1} = 0.$$

$$\therefore \left.\frac{\sin^2 x (\sec x - 1)}{x^3}\right]_0 = \left.\left(\frac{\sin x}{x}\right)^2 \left(\frac{1 - \cos x}{x \cos x}\right)\right]_0 = 0.$$

EXAMPLES.

Evaluate the following:

1. $\left.x^{\frac{1}{x}}\right]_{\infty}$. *Ans.* 1.

2. $\left.x^{\sin x}\right]_{0}$. *Ans.* 1.

3. $\left.\left(1+\frac{1}{x}\right)^{x}\right]_{\infty}$. *Ans.* e.

4. $\left.\left(1+\frac{1}{x^2}\right)^{x}\right]_{\infty}$. *Ans.* 1.

5. $\left.(1+ax)^{\frac{1}{x}}\right]_{0}$. *Ans.* e^a.

6. $\left.\frac{\tan x - x}{x - \sin x}\right]_{0}$. *Ans.* 2.

7. $\left.\frac{\sqrt{x}\tan x}{(e^x-1)^{\frac{3}{2}}}\right]_{0}$. *Ans.* 1.

8. $\left.\frac{e^x - e^{-x} - 2x}{(e^x-1)^3}\right]_{0}$. *Ans.* $\frac{1}{3}$.

9. $\left.\frac{x - \sin^{-1} x}{\sin^3 x}\right]_{0}$. *Ans.* $-\frac{1}{6}$.

10. $\left.x^m (\sin x)^{\tan x}\right]_{\frac{\pi}{2}}$ *Ans.* $\left(\frac{\pi}{2}\right)^m$

11. $\left.(\sin x)^{\tan x}\right]_{0}$. *Ans.* 1.

12. $\left.\frac{x \log(1+x)}{1-\cos x}\right]_{0}$. *Ans.* 2.

13. $\left.xe^{\frac{1}{x}}\right]_{0}$. *Ans.* ∞.

14. $\left.\frac{x^x - x}{1 - x + \log x}\right]_{1}$. *Ans.* -2

15. $\left.\frac{e^x - e^{-x} - 2x}{\tan x - x}\right]_{0}$. *Ans.* 1.

16. $\left.\frac{\tan(a+x) - \tan(a-x)}{\tan^{-1}(a+x) - \tan^{-1}(a-x)}\right]_{0}$. *Ans.* $\frac{1+a^2}{\cos^2 a}$.

CHAPTER X.

MAXIMA AND MINIMA.

History. — Kepler (1571–1630) was the first to observe that the increment of a variable was evanescent for values infinitely near a maximum or minimum value of the variable. This remark contains the germ of the rule given by Fermat (1601–1665).

The correct *theory* of Maxima and Minima was first given by Maclaurin in his Treatise of Fluxions (1742).

111. Definitions. — A **maximum value** of a function is a value greater than the values which immediately precede and follow it.

A **minimum value** of a function is a value smaller than the values which immediately precede and follow it.

From these definitions it appears that a *maximum* value of a function does *not* mean the *greatest* value of the function, nor does a *minimum* value mean the *smallest* value of a function. In fact, a given function may have several maximum values and several minimum values.

112. Conditions for Maxima and Minima.

Let $f(x)$ be a function of an increasing variable x. Then, by definition, $f(x)$ is an *increasing* function just before it reaches a maximum value, and a *decreasing* function immediately after it passes through that value. Hence, (Art. 13, Cor.), $f'(x)$ is *positive* $(+)$ *before* and *negative* $(-)$ *after* $f(x)$ attains a maximum value; hence,

$$f'(x) = 0, \text{ or } f'(x) = \infty$$

at a maximum point, since $f'(x)$ is continuous, and cannot, therefore, change sign from $+$ to $-$ without passing through one or the other of these values.

Again, by definition, $f(x)$ is a decreasing function of x just before it reaches a *minimum* value, and an increasing function of x immediately after it passes that value; hence, $f'(x)$ is *negative* $(-)$ before and positive $(+)$ after $f(x)$ attains a minimum value. Hence, at a minimum value,

$$f'(x) = 0, \text{ or } f'(x) = \infty.$$

It appears, therefore, that the *essential condition* for a maximum or minimum value of a function of a single variable is that its *first derivative shall change sign*, — in case of a maximum value, from $+$ to $-$; of a minimum value, from $-$ to $+$. It also appears that the value, or values, of x which render $f(x)$ a maximum or minimum will be found among the roots of the equations formed by equating the first derivative to zero or to infinity. These roots are called *critical* values of the variable, and must be separately examined, in order to ascertain which, if any, give rise to a maximum or minimum state of the function.

113. Illustration. — Since $f'(x)$, when considered geometrically, always represents the slope of the tangent to the curve $y = f(x)$ (§ 19), the principles of the preceding article may be graphically represented.

I. **Critical values which render $f'(x) = 0$.**

Let SM (Fig. 15) be the locus of the equation $y = f(x)$.

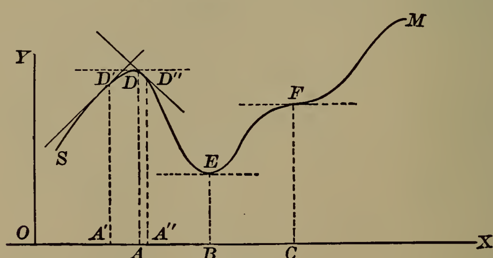


Fig. 15.

At the maximum and minimum points of the curve, D and E, and at such points as F where the direction of curvature changes, the tangents are ∥ to the x-axis;

hence, $$\frac{dy}{dx} = f'(x) = 0$$

for the critical values OA, OB, and OC, of x.

For a value of x a little less than OA (say OA') the tangent at the extremity of the corresponding ordinate $A'D'$ makes an *acute* angle with the X-axis; hence, $f'(x) = +$ quantity. For $x = OA, f'(x) = 0$. For $x = OA''$, $y = A''D''$, and the tangent makes an *obtuse* angle with the X-axis; hence $f'(x) = -$ quantity.

Hence at a *maximum* point, as D, $f'(x)$ passes through 0 from $+$ to $-$ direction. Similarly, to the left of the *minimum* point E the tangent makes an *obtuse* angle with X, while, to the right of it, the tangent makes an acute angle; hence $f'(x)$ at a minimum point passes through 0 from $-$ to $+$ direction.

At such a point as F, while $f'(x) = 0$, yet it does not change sign as x passes through the critical value OC, $f'(x)$ being positive $+$ (or negative $-$) on both sides of F; hence CF is neither a maximum nor a minimum value of y. This fact is also evident from the definitions, since CF is neither *greater* nor *smaller* than the ordinates which immediately precede and follow it.

The illustration further emphasizes the fact, that all the roots of the equation $f'(x) = 0$ do not necessarily correspond to maximum or minimum values of $f(x)$.

II. **Critical values which render $f'(x) = \infty$.**

Let SM (Fig. 16) be the locus of the equation $y = f(x)$.

At such points as D, E, F, where the tangents are perpendicular to the X-axis, we have

$$\frac{dy}{dx} = f'(x) = \infty.$$

OA, OB, OC are therefore critical values of x.

At D', $f'(x)]_{OA'} = +$ quantity; at D, $f'(x)]_{OA} = \infty$; at D'', $f'(x)]_{OA''} = -$ quantity; hence at a maximum point $f'(x)$ passes through ∞ from $+$ to $-$.

Similarly, at $E, f'(x)$ passes through ∞ from $-$ to $+$.

At F, $f'(x)]_{OC} = \infty$; but for values of x a little less and a little greater than OC we find $f'(x) =$ a positive quantity; hence $f'(x)$ does not change sign as x passes through the critical value OC; hence CF does not represent either a maximum or a minimum value of $f(x)$. This is also evident from the definition. It also appears that the roots of the equation $f'(x) = \infty$ do not *necessarily* render $f(x)$ either a maximum or a minimum.

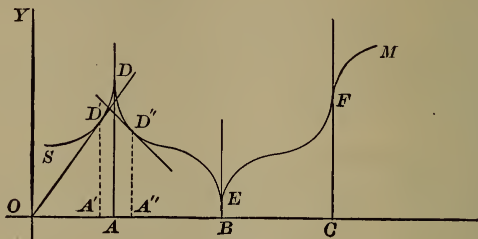


Fig. 16.

114. Methods of Investigation for Maximum and Minimum Values.

I. **By examining the given function.**

Let $f(x)$ be the given function, and let a be any one of the critical values found by equating $f'(x)$ to 0 or ∞, or both.

Then, by the definition, § 111, we have, h being a very small quantity,

For a maximum $\begin{cases} f(a) > f(a-h) \\ f(a) > f(a+h) \end{cases}$ . . . . (1)

For a minimum $\begin{cases} f(a) < f(a-h) \\ f(a) < f(a+h) \end{cases}$ . . . . (2)

Thus, let $$f(x) = \frac{x^3}{3} - 3x^2 + 8x;$$

then, $$f'(x) = x^2 - 6x + 8.$$

As no finite value of x will render $f'(x) = \infty$, we equate it to zero; hence

$$x^2 - 6x + 8 = (x-2)(x-4) = 0,$$
$$\therefore\ x = 2 \text{ and } x = 4,$$

are the critical values of x. Substituting values a little less and a little greater than $x = 2$ in the given function, $\frac{x^3}{3} - 3x^2 + 8x$, we have

$$f(x)]_1 = 5\tfrac{1}{3},$$
$$f(x)]_2 = 6\tfrac{2}{3},$$
$$f(x)]_3 = 6.$$

Hence,

$$\left.\begin{matrix} f(x)]_2 > f(x)]_1 \\ f(x)]_2 > f(x)]_3 \end{matrix}\right\} \therefore\ f(x)]_2 = \left(\frac{x^3}{3} - 3x^2 + 8x\right)\Big]_2 = 6\tfrac{2}{3} = \text{a maxi-}$$

mum value.

Substituting now values a little less and a little greater than $x = 4$, we have,

$$f(x)]_3 = 6,$$
$$f(x)]_4 = 5\tfrac{1}{3},$$
$$f(x)]_5 = 6\tfrac{2}{3}.$$

Hence,

$$\left.\begin{matrix} f(x)]_4 < f(x)]_3 \\ f(x)]_4 < f(x)]_5 \end{matrix}\right\} \therefore\ f(x)]_4 = \left(\frac{x^3}{3} - 3x^2 + 8x\right)\Big]_4 = 5\tfrac{1}{3} =$$

a minimum value.

Again let us consider the function,

$$\phi(x) = m - n(x-2)^{\frac{2}{3}}.$$

Here $$\phi'(x) = -\frac{2n}{3(x-2)^{\frac{1}{3}}}.$$

As no finite value of x can render this derivative $= 0$, we equate to ∞.

Hence $$-\frac{2n}{3(x-2)^{\frac{1}{3}}} = \infty,$$
$$\therefore\ x = 2$$

is the critical value. Hence

$$\phi(x)]_1 = m - n,$$
$$\phi(x)]_2 = m,$$
$$\phi(x)]_3 = m - n;$$

$$\therefore \left.\begin{array}{l}\phi(x)]_2 > \phi(x)]_1 \\ \phi(x)]_2 > \phi(x)]_3\end{array}\right\} \therefore \phi(x)]_2 = (m - n(x-2)^{\frac{2}{3}})]_2 = m =$$

a maximum value.

II. By examining the first derivative.

Let $f(x)$ be the given function, and let a be a critical value of x.

Let h be any very small quantity. Hence, § 112,

$$\text{For a maximum} \left\{\begin{array}{r} f'(a-h) = \text{a positive quantity} \\ f'(a) = 0 \text{ or } \infty \\ f'(a+h) = \text{a negative quantity} \end{array}\right\} \quad . \quad (3)$$

$$\text{For a minimum} \left\{\begin{array}{r} f'(a-h) = \text{a negative quantity} \\ f'(a) = 0 \text{ or } \infty \\ f'(a+h) = \text{a positive quantity} \end{array}\right\} \quad . \quad (4)$$

To illustrate let us resume the example

$$f(x) = \frac{x^3}{3} - 3x^2 + 8x.$$

Hence, $f'(x) = x^2 - 6x + 8 = 0, \therefore x = 2, x = 4,$

$$\therefore f'(x)]_1 = 3,$$
$$f'(x)]_2 = 0,$$
$$f'(x)]_3 = -1.$$

Hence, (3), $\frac{x^3}{3} - 3x^2 + 8x]_2 = 6\frac{2}{3}$ = a maximum as before. See I.

Taking the value $x = 4$, we have,

$$f'(x)]_3 = -1,$$
$$f'(x)]_4 = 0,$$
$$f'(x)]_5 = 3.$$

Hence, (4), $\left.\frac{x^3}{3} - 3x^2 + 8x\right]_4 = 5\frac{1}{3}$ = a minimum value. See I.

Taking the example

$$\phi(x) = m - n(x-2)^{\frac{2}{3}},$$

we have,

$$\phi'(x) = -\frac{2n}{3(x-2)^{\frac{1}{3}}} = \infty, \therefore x = 2,$$

$$\therefore \phi'(x)]_1 = \frac{2}{3}n,$$

$$\phi'(x)]_2 = \infty,$$

$$\phi'(x)]_3 = -\frac{2}{3}n;$$

$\therefore$ (3) $(m - n(x-2)^{\frac{2}{3}})]_2 = m$ = a maximum value. See I.

III. **By examining the second derivative.**

When $f(x)$ is a maximum, $f'(x)$ changes sign from + to −, § 112; hence, $f'(x)$ is a decreasing function of x; hence, $f''(x)$ must be *negative* for the critical value of x that renders $f(x)$ a maximum.

When $f(x)$ is a minimum, $f'(x)$ changes sign from − to +, § 112; hence, $f'(x)$ is an increasing function of x; hence, $f''(x)$ must be *positive* for the critical value of x that renders $f(x)$ a minimum.

Hence, if a be a critical value of x, then

For a maximum $f''(a)$ = a negative quantity.

For a minimum $f''(a)$ = a positive quantity.

Let us resume the example,

$$f(x) = \frac{x^3}{3} - 3x^2 + 8x.$$

Hence, $$f'(x) = x^2 - 6x + 8 = 0, \therefore x = 2, x = 4,$$

and $$f''(x) = 2x - 6,$$

$$\therefore f''(x)]_2 = -2, \therefore \left.\left(\frac{x^3}{3} - 3x^2 + 8x\right)\right]_2 = 6\tfrac{2}{3} = \text{a max. value.}$$

$$f''(x)]_4 = +2, \therefore \left.\left(\frac{x^3}{3} - 3x^2 + 8x\right)\right]_4 = 5\tfrac{1}{3} = \text{a min. value.}$$

Taking the example

$$\phi(x) = m - n(x-2)^{\frac{2}{3}},$$

we have,
$$\phi'(x) = -\frac{2n}{3(x-2)^{\frac{1}{3}}} = \infty, \quad \therefore x = 2;$$

also,
$$\phi''(x) = \frac{2n}{9(x-2)^{\frac{4}{3}}}.$$

Hence, $\phi''(x)]_2 = \infty$; hence, this method does not apply when the critical value renders the first derivative infinite.

For critical values that render $f'(x) = 0$, Method III. is usually the simplest. Frequently, however, the form of the given function is such as to render its *second derivative* difficult or tedious to obtain. In such cases Method II. should be employed.

It frequently happens, when Method III. is employed, that a critical value of x reduces $f''(x)$ to zero as well as $f'(x)$. How to proceed in such a case will be explained in the next article.

115. Maxima and Minima by Taylor's Theorem.

Resuming equation (1), § 114, and transposing, we have,

$$\left.\begin{array}{l} f(a-h) - f(a) < 0 \\ f(a+h) - f(a) < 0 \end{array}\right\} \quad \ldots \ldots \ldots \ldots \quad (b)$$

as essential conditions for a maximum value of $f(x)$ for the critical value a of x.

By Taylor's Theorem:

$$f(a-h) - f(a) = -f'(a) \cdot h + f''(a)\frac{h^2}{\lfloor 2} - f'''(a)\frac{h^3}{\lfloor 3} + \ldots \quad (c)$$

$$f(a+h) - f(a) = f'(a)\,h + f''(a)\frac{h^2}{\lfloor 2} + f'''(a)\frac{h^3}{\lfloor 3} + \ldots \quad (d)$$

If, therefore, $f(a)$ is a maximum value of $f(x)$ the second members of these equations, (c) and (d), must be less than zero, i.e., *negative*. If h is taken sufficiently small — and we can take it as small as we please — the first term in the second member, $f'(a)h$, can be made to exceed numerically the sum of all the other terms. But these terms have different signs; hence,

$$f'(a) = 0$$

is an essential condition to be fulfilled in order that the second members may be negative. It is therefore an essential condition for a maximum.

Making $f'(a) = 0$ in (c) and (d), we have,

$$f(a-h) - f(a) = +f''(a)\frac{h^2}{\lfloor 2} - f'''(a)\frac{h^3}{\lfloor 3} + \ldots \quad (e)$$

$$f(a+h) - f(a) = +f''(a)\frac{h^2}{\lfloor 2} + f'''(a)\frac{h^3}{\lfloor 3} + \ldots \quad (f)$$

Giving h such a value as to make the first terms numerically larger than the sum of all that follow them, we have, as a second essential condition for a maximum (since h^2 is positive),

$$f''(a) < 0.$$

If $f''(a)$ is zero for the critical value a of x, then a similar course of reasoning shows that

$$f'''(a) = 0, \text{ and } f^{iv}(a) < 0$$

are conditions for a maximum value. If $f^{iv}(a) = 0$, then

$$f^{v}(a) = 0, \text{ and } f^{vi}(a) < 0$$

are conditions for a maximum value.

By assuming equation (2) (§ 114), we can show, similarly, that

$$f'(a) = 0, \text{ and } f''(a) > 0$$

are essential conditions for a minimum value of $f(x)$, and if $f''(a) = 0$, that

$$f'''(a) = 0, \text{ and } f^{iv}(a) > 0$$

are essential conditions; and so on.

In general, therefore, if

$$f'(a) = 0, f''(a) = 0, f'''(a) = 0 \cdots f^{n-1}(a) = 0,$$

then, $f(a)$ is a maximum if n is even and $f^n(a) < 0$;

and $f(a)$ is a minimum if n is even and $f^n(a) > 0$;

and $f(a)$ is neither a maximum nor a minimum if n is odd and $f^n(a) > 0$ or < 0.

116. Practical Suggestions. — In the examples and problems which follow this article the following suggestions will be found of great service in simplifying the operations:

(1). The critical value that renders $f(x)$ a maximum or minimum will render $C + Df(x)$ a maximum or minimum.

(2). If $f(x)$ is positive then $[f(x)]^n$ is a maximum or minimum for a critical value that renders $f(x)$ a maximum or a minimum.

If $f(x)$ is negative and n is an odd integer, then $[f(x)]^n$ is a maximum or a minimum for a critical value that renders $f(x)$ a maximum or a minimum. If n is an even integer, then $[f(x)]^n$ is a maximum or minimum for a critical value that renders $f(x)$ a minimum or a maximum.

(3). The critical value that renders $f(x)$ a maximum or a minimum will render $\frac{1}{f(x)}$ a minimum or a maximum.

(4). The critical value that renders $f(x)$ a maximum or a minimum will render $\log_a f(x)$ a maximum or a minimum.

EXAMPLES.

1. Examine $mx^2 - 2nx + c$ for maximum and minimum values.

Here $$f(x) = mx^2 - 2nx + c.$$

$$f'(x) = 2mx - 2n = 0.$$

$$\therefore\ x = \frac{n}{m} \text{ is the critical value of } x.$$

Also $$f''(x) = 2m.$$

Since $f''(x)$ is positive, we have

$$mx^2 - 2nx + c\Big]_{\frac{n}{m}} = c - \frac{n^2}{m} = \text{a minimum value.}$$

2. Examine $16ax^3 - 60ax^2 + 48ax - 4a$ for maximum and minimum values.

As $4a$ is a constant factor we may omit it (§ 116, 1), and write

$$f(x) = 4x^3 - 15x^2 + 12x - 1.$$

Hence, $f'(x) = 12x^2 - 30x + 12 = 12(x^2 - \frac{5}{2}x + 1)$

$$= 12(x - \tfrac{1}{2})(x - 2) = 0.$$

$\therefore\ x = \frac{1}{2}$ and $x = 2$ are the critical values of x.

$$f''(x) = 24x - 30;$$

$$f''(x)\Big]_{\frac{1}{2}} = (24x - 30)\Big]_{\frac{1}{2}} = -18.$$

$\therefore\ f(x)$ is a maximum when $x = \frac{1}{2}$; also

$$f''(x)\Big]_2 = (24x - 30)\Big]_2 = 18.$$

$\therefore\ f(x)$ is a minimum when $x = 2$.

3. Examine $(x-4)^5(x+2)^4$ for maximum and minimum values.

Here $f(x) = (x-4)^5(x+2)^4$,

$$f'(x) = (x-4)^5\,4(x+2)^3 + (x+2)^4\,5(x-4)^4$$

$$= (x-4)^4(x+2)^3\{4(x-4) + 5(x+2)\}$$

$$= 9(x - \tfrac{2}{3})(x-4)^4(x+2)^3 = 0,$$

$\therefore\ x = \frac{2}{3},\ x = 4,\ x = -2$, are critical values.

As the work of obtaining the second derivative is tedious, let us use Method II. (§ 114), i.e., let us see how the first derivative passes through zero, as x passes through its critical values.

As x passes through the value $\frac{2}{3}$, $f'(x)$ changes sign from $-$ to $+$;

As x passes through the value 4, $f'(x)$ remains positive, i.e., does not change sign;

As x passes through the value -2, $f'(x)$ changes sign from $+$ to $-$.

Hence,

when $x = \frac{2}{3}$, $f(x)$ is a minimum;

when $x = 4$, $f(x)$ is neither a maximum nor a minimum;

when $x = -2$, $f(x)$ is a maximum.

4. Examine $b + (x - a)^{\frac{4}{3}}$ for maximum and minimum values.

Here $\qquad f'(x) = \frac{4}{3}(x - a)^{\frac{1}{3}} = 0; \quad \therefore x = a.$

As x passes through the critical value a, $f'(x)$ passes through zero from $-$ to $+$. $\therefore$ for $x = a$, $f(x)]_a = b$, a minimum value.

5. $f(x) = b + (x - a)^{\frac{5}{3}}$.

Here $\qquad f'(x) = \frac{5}{3}(x - a)^{\frac{2}{3}} = 0; \therefore x = a.$

As x passes through the critical value a, $f'(x)$ does *not* change sign; hence, $f(x)]_a = b$, neither a maximum nor a minimum.

6. $f(x) = 2x^3 - 9ax^2 + 12a^2x - 4a^3$.

$f(x)]_a = a^3$, a maximum; $f(x)]_{2a} = 0$, a minimum.

7. $f(x) = \dfrac{1 - x + x^2}{1 + x - x^2}$.

$f(x)]_{\frac{1}{2}} = \frac{3}{5}$, a minimum.

8. $f(x) = 2x^3 + 3x^2 - 36x + 12$.

$f(x)]_{-3} = 93$, a maximum; $f(x)]_2 = -32$, a minimum.

9. $f(x) = x^3 - 3x^2 - 9x + 5$.

$f(x)]_{-1} = 10$, a maximum; $f(x)]_3 = -23$, a minimum.

10. $f(x) = \dfrac{\log x}{x}$.

$f(x)]_e = \dfrac{1}{e}$, a maximum.

11. $f(x) = \dfrac{x^2 - 7x + 6}{x - 10}$.

$f(x)]_4 = 1$, a maximum; $f(x)]_{16} = 25$, a minimum.

12. $f(x) = x^3 + \dfrac{48}{x}$.

$f(x)]_{-2} = -32$, a maximum; $f(x)]_2 = 32$, a minimum.

13. $f(x) = \dfrac{x}{1 + x \tan x}$.

$f(x)]_{\cos x} = \dfrac{\cos x}{1 + \sin x}$, a maximum.

14. $f(x) = \sin x + \sin x \cos x$.

$f(x)]_{\frac{\pi}{3}}$, a maximum.

15. $f(x) = \dfrac{\sin x}{1 + \tan x}$.

$f(x)]_{\frac{\pi}{4}}$, a maximum.

117. For convenience of reference the following expressions for the measures of areas and volumes are introduced.

$$\text{Area of triangle} = \tfrac{1}{2}xy,$$
$$\text{Area of rectangle} = xy,$$
$$\text{Area of trapezoid} = \frac{x+z}{2}y,$$

in which x and z represent the bases and y represents the altitude. Let S = surface, V = volume, x = radius of circle or sphere, s = slant height.

$$V = \pi x^2 y = \text{volume of cylinder},$$
$$V = \frac{\pi x^2 y}{3} = \text{volume of cone},$$
$$V = \tfrac{4}{3}\pi x^3 = \text{volume of sphere},$$
$$S = 2\pi xy = \text{lateral surface of cylinder},$$
$$S = \pi xs = \text{lateral surface of cone},$$
$$S = 4\pi x^2 = \text{surface of sphere}.$$

PROBLEMS.

1. Divide a number a into two such parts that their product shall be a maximum.

Let $x =$ one part; then $a - x =$ the other; hence,

$$f(x) = (a - x)\,x,$$

$$\therefore f'(x) = a - 2x = 0, \quad \therefore x = \frac{a}{2}.$$

Also $$f''(x) = -2, \text{ a negative quantity.}$$

Hence $f(x)\Big]_{\frac{a}{2}} = \dfrac{a^2}{4}$, a maximum; hence the product of the two parts will be greatest when the parts are equal.

2. Divide a number a into two factors such that their sum shall be a minimum.

$f(x) = x + \dfrac{a}{x}$; $f'(x) = 1 - \dfrac{a}{x^2} = 0$, $\therefore x = \sqrt{a}$. Also $f''(x)$ $= \dfrac{2a}{x^3}$, which for $x = \sqrt{a}$ is *positive;* hence $f(x)\Big]_{\sqrt{a}}$, a minimum; hence the factors are equal when their sum is a minimum.

3. Divide the number 10 into two parts such that the square of one part multiplied by the cube of the other shall be a maximum.

$$f(x)\Big]_4, \text{ a maximum}, \therefore 4 \text{ and } 6 \text{ are the parts.}$$

4. Required the height at which a light should be placed above a table so that the page of an open book placed at a given horizontal distance (a) from the light may receive the greatest illumination.

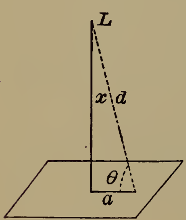


Fig. 17.

From optics we have the principle that the *intensity of the illumination varies directly as the sine of the angle of incidence and inversely as the square of the distance.*

Let i = intensity of the illumination, then from the principle and the figure, we have

$$i \propto \frac{\sin\theta}{d^2},$$

$$\therefore\ i = c\,\frac{\frac{x}{d}}{d^2} = \frac{cx}{d^3} = \frac{cx}{(a^2+x^2)^{\frac{3}{2}}},$$ (c = intensity at units distance from the light).

Hence,

$$f(x) = \frac{x}{(a^2+x^2)^{\frac{3}{2}}};\ f'(x) = \frac{(a^2+x^2)^{\frac{3}{2}} - 3x^2(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^3} = 0;$$

$$\therefore\ (a^2+x^2)^{\frac{1}{2}}\{a^2 - 2x^2\} = 0,$$

$$\therefore\quad x = \frac{a}{\sqrt{2}} \text{ and } x = a\sqrt{-1}.$$

As the second value is imaginary, the first value $\frac{a}{\sqrt{2}}$ only can satisfy the condition of the problem. Hence $\frac{a}{\sqrt{2}}$ is the required height. It frequently happens in the solution of problems that the critical value which satisfies the given conditions may be detected without analytical examination. Such, for instance, as the value $\frac{a}{\sqrt{2}}$ of x in the above example.

5. Find the line of shortest length that can be drawn through a given point (a, b) and terminate in the rectangular axes to which the point is referred. *Ans.* $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

6. Show that the area of the right triangle formed by a line through (a, b) and the co-ordinate axes is a minimum when the base of the triangle is $2a$ and its altitude is $2b$.

7. Show that of all rectangles of a given area the square has the least perimeter.

Let m = area, y = altitude, x = base, P = perimeter; then

$$P = 2x + 2y \text{ and } m = xy,$$

$$\therefore P = f(x) = 2x + \frac{2m}{x},$$

$$\therefore f'(x) = 2 - \frac{2m}{x^2} = 0, \therefore x = \sqrt{m}.$$

Hence, $$y = \frac{m}{x} = \sqrt{m},$$

$\therefore$ the square has the shortest perimeter of all equivalent rectangles.

8. Show that of all triangles of a given perimeter constructed on a given base (b) the isosceles triangle has the greatest area.

Here Area $= \sqrt{s(s-x)(s-y)(s-b)}$ in which $s = \frac{x+y+b}{2}$ and by condition $x + y + b = c$, a constant.

9. A box of maximum contents is to be made from a rectangular piece of tin $30'' \times 14''$; required the side of the square to be cut out of each corner of the tin sheet.

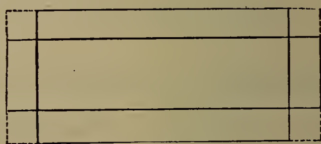


Fig. 18.

Here $$v = f(x) = x(14 - 2x)(30 - 2x)$$

Ans. $x = 3$.

10. Find the maximum rectangle that can be inscribed in a given circle.

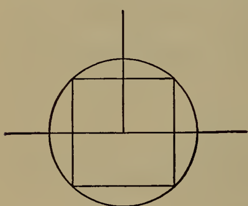


Fig. 19.

$$\text{Area} = 4xy \text{ and } x^2 + y^2 = a^2;$$

$$\therefore \text{Area} = 4\sqrt{a^2x^2 - x^4},$$

$$\therefore f(x) = a^2x^2 - x^4.$$

Hence, $f'(x) = 2a^2x - 4x^3 = 0$;

$$\therefore x = \frac{a}{\sqrt{2}}.$$

Hence, $$y = \sqrt{a^2 - x^2} = \frac{a}{\sqrt{2}}.$$

$\therefore$ The figure is a square and its

$$\text{Area} = 4xy = 2a^2.$$

11. Find the maximum right cylinder that can be inscribed in a sphere of radius a. *Ans.* Height $= \frac{2}{3} a\sqrt{3}$.

12. Find the right cylinder of maximum convex surface that can be inscribed in a given sphere. *Ans.* Height $= a\sqrt{2}$.

13. Find the greatest right cylinder that can be inscribed in a given right cone. *Ans.* Altitude $= \frac{1}{3}$ altitude of cone.

14. Find the right cylinder of greatest convex surface that can be inscribed in a given right cone. *Ans.* $S = \frac{\pi ab}{2}$.

15. Show that the cone of greatest volume and greatest convex surface that can be inscribed in a sphere of radius (a) has $\frac{4}{3} a$ for its altitude.

16. Show that the altitude of the cone of least volume that circumscribes the sphere is $4a$.

17. Of all cones of a given slant height show that the one, the ratio of whose altitude to the radius of its base is $\frac{1}{\sqrt{2}}$, is a maximum.

18. Of all circular sectors of a given perimeter show that the one whose arc = twice the radius is greatest.

Let $x =$ radius; then, $2x +$ arc $= p$, a constant, and

$$\text{Area} = \tfrac{1}{2} x (p - 2x).$$

19. A Norman window, consisting of a semicircle surmounting a rectangle, is to be of a given perimeter and so constructed that the light admitted shall be a maximum; required, the height and breadth of the window.

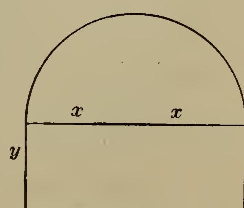


Fig. 20.

With the notation of the figure, we have,

$$\text{Area} = 2xy + \frac{\pi x^2}{2},$$

and $\text{Perimeter} = p = 2y + 2x + \pi x.$

$$\therefore \text{ Area} = f(x) = px - 2x^2 - \frac{\pi x^2}{2};$$

$$\therefore f'(x) = p - 4x - \pi x = 0.$$

$$\therefore x = \frac{p}{4+\pi}.$$

Hence, $$y = \frac{p}{4+\pi}.$$

Therefore the height of rectangle = radius of semicircle.

We have heretofore, where two variables occurred in the expression we desired to investigate for maximum and minimum values, substituted for one of the variables its value in terms of the other, as derived from given conditions. Thus, in $A = 2xy + \frac{\pi x^2}{2}$ above we substituted for y its value in terms of x, as determined by the condition $p = 2y + 2x + \pi x$. It frequently happens that this substitution may be made more conveniently after differentiation.

Thus, $$f(x) = 2xy + \frac{\pi x^2}{2}.$$

$$\therefore f'(x) = 2x\frac{dy}{dx} + 2y + \pi x.$$

And from $$p = 2y + 2x + \pi x$$

we have, $$0 = 2\frac{dy}{dx} + 2 + \pi.$$

$$\therefore \frac{dy}{dx} = -\frac{2+\pi}{2}.$$

Substituting now this value of $\frac{dy}{dx}$ together with that of y drawn from the value of p in the value of $f'(x)$, we have,

$$f'(x) = 2x\left(\frac{-2-\pi}{2}\right) + 2\left(\frac{p}{2} - x - \frac{\pi x}{2}\right) + \pi x$$

$$= p - 4x - \pi x, \text{ as before.}$$

20. A person in a boat 3 miles from shore wishes to reach a point 5 miles down the coast *in the shortest time.* Assuming that he can walk 5 miles an hour and row only 4 miles an hour, at what point must he land?

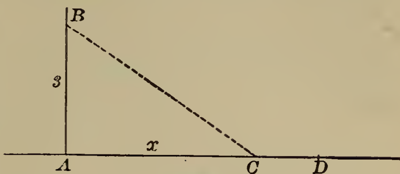


Fig. 21.

From the figure, we have (C being the landing-point),

$$CB = \sqrt{9 + x^2},\ CD = 5 - x.$$

∴ Total time of rowing and walking is

$$\frac{\sqrt{9+x^2}}{4} + \frac{5-x}{5} = f(x);$$

$$\therefore f'(x) = \frac{x}{4\sqrt{9+x^2}} - \frac{1}{5} = 0.$$

$$\therefore x = 4.$$

i.e., he must land one mile from the point he desires to reach.

21. What must be the dimensions of a square-based box, open at the top, whose volume is 108 cu. in., in order that the material of which it is made may be a minimum?

Here $S = x^2 + 4xy$, and $V = x^2y = 108$,

$$\therefore S = f(x) = x^2 + \frac{432}{x}.$$

$$\therefore x = 6, \text{ and } y = 3.$$

22. From a given quantity of material a circular cylindrical cup with open top is to be made; required its dimensions in order that the volume may be a maximum.

Here $V = \pi x^2 y$ and $S = \pi x^2 + 2\pi xy = c$, a constant,

$$\therefore V = f(x) = \pi x^2\left(\frac{c - \pi x^2}{2\pi x}\right) = \frac{cx - \pi x^3}{2}.$$

$$\therefore x = y = \sqrt{\frac{c}{3\pi}}.$$

23. Assuming the fact that the area of a segment of a parabola is $\frac{2}{3}$ the rectangle on the ordinate and abscissa, show that the greatest parabola that can be cut from a given right circular cone is one whose axis $= \frac{3}{4}$ the slant height of the cone.

CHAPTER XI.

PARTIAL AND TOTAL DIFFERENTIATION.

118. Partial Differentials. *The partial differential of a function of two or more variables is the differential obtained under the supposition that only one of the variables that enters it is changing.*

Thus let $u = x^2 + xz + \log y$ and let $\partial_x u$, $\partial_y u$, $\partial_z u$ represent the partial differentials of u; then

$$\partial_x u = (2x + z)\, dx,$$

$$\partial_y u = \frac{dy}{y},$$

$$\partial_z u = x dz.$$

119. Partial Derivatives. *The partial derivative of a function of two or more variables is the ratio of the partial differential of the function to the differential of the changing variable.*

Thus, in the example of the preceding article, we have,

$$\frac{\partial u}{dx} = 2x + z,$$

$$\frac{\partial u}{dy} = \frac{1}{y},$$

$$\frac{\partial u}{dz} = x.$$

It will be observed that in writing partial derivatives that the subscript in $\partial_x u$, $\partial_y u$, $\partial_z u$, may be omitted as the denominators of the derivatives indicate the variable that is supposed to be changing. In writing partial differentials it is frequently more convenient to use a notation in which the differential of the changing variable enters. Thus, instead of using $\partial_x u$, $\partial_y u$, $\partial_z u$ to represent the partial differentials, we may use

$$\frac{\partial u}{dx}dx,\ \frac{\partial u}{dy}dy,\ \frac{\partial u}{dz}dz.$$

Thus in the example selected we may write,

$$\frac{\partial u}{dx}dx = (2x + z)\,dx,$$

$$\frac{\partial u}{dy}dy = \frac{dy}{y},$$

$$\frac{\partial u}{dz}dz = xdz,$$

for the partial differentials.

EXAMPLES.

1, If $u = x^y$, show that $\partial_x u + \partial_y u = yx^{y-1}dx + x^y \log xdy$.

Here $\partial_x u = yx^{y-1}dx$ and $\partial_y u = x^y \log xdy$,

$\therefore\ \partial_x u + \partial_y u = yx^{y-1}dx + x^y \log xdy$.

2. $u = \log(e^x + e^y)$, show that $\frac{\partial u}{dx} + \frac{\partial u}{dy} = 1$.

Here $\frac{\partial u}{dx} = \frac{e^x}{e^x + e^y}$ and $\frac{\partial u}{dy} = \frac{e^y}{e^x + e^y}$,

hence, $\frac{\partial u}{dx} + \frac{\partial u}{dy} = \frac{e^x + e^y}{e^x + e^y} = 1$.

3. $u = \log(x + \sqrt{x^2 + y^2})$ show that $x\frac{\partial u}{dx} + y\frac{\partial u}{dy} = 1$.

4. $u = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, show $\frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy$.

5. $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\frac{\partial u}{dx} + \frac{\partial u}{dy} + \frac{\partial u}{dz}$

$= \frac{3}{x + y + z}$.

6. $u = \sin(xy)$, show that $\frac{\partial u}{dx} + \frac{\partial u}{dy} = (x + y)\cos(xy)$.

7. $u = \log(x + y)$, show that $\frac{\partial u}{dx} + \frac{\partial u}{dy} = \frac{2}{e^u}$.

8. $u = \tan^{-1}\left(\frac{x - y}{x + y}\right)^{\frac{3}{2}}$, show that $x\frac{\partial u}{dx} + y\frac{\partial u}{dy} = 0$.

120. Euler's Theorem of Homogeneous Functions.

Let $u = ax^m y^n + bx^{m'} y^{n'} + cx^{m''} y^{n''} + \text{etc.}$ . . . . . (1)
be a homogeneous function of x and y not involving fractions in which

$$m + n = m' + n' = m'' + n'' = \text{etc.} = p.$$

To prove that

$$x\frac{\partial u}{dx} + y\frac{\partial u}{dy} = pu.$$

Differentiating (1) partially, we have, after multiplying by x, and y,

$$x\frac{\partial u}{dx} = amx^m y^n + bm'x^{m'} y^{n'} + cm''x^{m''} y^{n''} + \text{etc.}$$

$$y\frac{\partial u}{dy} = anx^m y^n + bn'x^{m'} y^{n'} + cn''x^{m''} y^{n''} + \text{etc.}$$

Hence, by adding, we have,

$$x\frac{\partial u}{dx} + y\frac{\partial u}{dy} = (m+n)(ax^m y^n + bx^{m'} y^{n'} + cx^{m''} y^{n''} + \text{etc.}) = pu \quad . \quad (2)$$

The proposition may readily be proved to be true for a homogeneous function of any number of variables; also for homogeneous fractions.

EXAMPLES.

Verify Euler's Theorem in the following examples by partial differentiation.

1. $u = 4x^3y^2 + 3x^2y^3 - 2xy^4$.

Here u is a homogeneous function of x and y of the 5^{th} degree.

Hence, by (2) Art. 120,

$$x\frac{\partial u}{dx}+y\frac{\partial u}{dy}=5\,u.$$

By partial differentiation,

$$x\frac{\partial u}{dx}+y\frac{\partial u}{dy}=20\,x^3y^2+15\,x^2y^3-10\,xy^4=5\,u.$$

2. $u=x^3+y^3.$ $$x\frac{\partial u}{dx}+y\frac{\partial u}{dy}=3\,u.$$

121. Differentiation by Use of Partial Differentials. *The* **Total Differential,** *or simply the* **Differential,** *of a function of several variables is equal to the sum of its partial differentials.*

Let $u=f(x, y, z, \text{etc.})$; it is easily seen that

$$du=\frac{\partial u}{dx}dx+\frac{\partial u}{dy}dy+\frac{\partial u}{dz}dz+\text{etc.}$$

in which du *in the first member* represents the differential of u under the supposition that all the variables which enter it are changing. For an examination of all the differential forms, both of algebraic and transcendental functions, as derived in Chapter III., shows that only the *first powers* of the differentials of the variables enter the differential of their function. If, therefore, we differentiate $u=f(x, y, z, \text{etc.})$, and collect the coefficients of dx, dy, and dz, — these coefficients being usually functions of the variables which enter the original function, — we may write,

$$du=d[f(x, y, z, \text{etc.})]$$
$$=\phi(x,y,z,\text{etc.})\,dx+\psi(x,y,z,\text{etc.})\,dy+\omega(x,y,z,\text{etc.})\,dz+\text{etc.}$$

Now, if we differentiate partially the original functions, we obtain,

$$\frac{\partial u}{dx}dx=\phi\,(x, y, z, \text{etc.})\,dx,$$

$$\frac{\partial u}{dy}dy=\psi\,(x, y, z, \text{etc.})\,dy,$$

$$\frac{\partial u}{dz}dz = \omega(x, y, z, \text{etc})\, dz.$$

Hence, adding,

$$\frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{dz}dz = \phi(x, y, z, \text{etc.})\, dx + \psi(x, y, z, \text{etc.})\, dy$$
$$+ \omega(x, y, z, \text{etc.})\, dz + \text{etc.}$$

Hence, finally,

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{dz}dz + etc.$$

To illustrate, let us resume the example of § 118,

$$u = x^2 + xz + \log y.$$

Here,
$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{dz}dz$$
$$= (2x + z)\, dx + \frac{dy}{y} + xdz.$$

Obviously the example above may be differentiated by the rules deduced in Chapter III.; and, as a matter of fact, those rules are sufficient to enable us to differentiate any algebraic or transcendental function, whether explicit or implicit. In certain *forms* of expressions, however, it will be found more convenient to adopt the process here explained.

Cor. I. If $u = f(x, y) = c$, a constant, be an implicit function of y, then

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy = 0.$$

hence,
$$\frac{dy}{dx} = -\frac{\frac{\partial u}{dx}}{\frac{\partial u}{dy}}.$$

i.e., the first derivative of an implicit function is minus the ratio of its partial derivatives.

Let $u = a^2y^2 + b^2x^2 = a^2b^2$; then

$$\frac{\partial u}{dx} = 2\,b^2x, \quad \frac{\partial u}{dy} = 2\,a^2y.$$

Hence,
$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{dx}}{\dfrac{\partial u}{dy}} = -\frac{b^2x}{a^2y},$$

a result previously obtained by direct differentiation (see Ex. 8, p. 26). This method will also be found *convenient* in differentiating many cases of implicit functions.

122. Total Derivative. The total derivative of a function of several variables is the ratio of the total differential of the function to the differential of the independent variable that enters it.

Thus, if $u = f(x, y, z, \text{etc.})$, then (§ 121),

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{dz}dz + \text{etc.} \quad . \quad . \quad . \quad . \quad (a)$$

Now, if $x = \phi(v)$, $y = \psi(v)$, $z = \omega(v)$, etc.; then u is indirectly a function of v through x, y, and z.

Dividing both members of (a) by dv, we have,

$$\frac{du}{dv} = \frac{\partial u}{dx} \cdot \frac{dx}{dv} + \frac{\partial u}{dy} \cdot \frac{dy}{dv} + \frac{\partial u}{dz} \cdot \frac{dz}{dv} + etc. \quad . \quad (b)$$

To illustrate, let $u = x^3 - xy + \log z$, in which $x = v^2$, $y = \sin v$, and $z = e^v$; then

$$\frac{\partial u}{dx} = 3\,x^2 - y; \; \frac{dx}{dv} = 2\,v; \; \frac{\partial u}{dy} = -\,x; \; \frac{dy}{dv} = \cos v; \; \frac{\partial u}{dz} = \frac{1}{z};$$
$$\frac{dz}{dv} = e^v.$$

Substituting in the formula (b), we have

$$\frac{du}{dv} = 2\,(3\,x^2 - y)\,v - x\cos v + \frac{e^v}{z} = 6\,v^5 - 2\,v\sin v - v^2\cos v + 1$$

for the total derivative. The same result may be obtained by substituting the values of x, y, and z in the value of u and then differentiating; thus,

$$u = v^6 - v^2 \sin v + \log e^v.$$

Hence, $\dfrac{du}{dv} = 6\,v^5 - 2\,v \sin v - v^2 \cos v + 1$, as before.

COR. 1. If $u = f(x, y, z)$ in which $y = \phi(x)$, $z = \psi(x)$; then

$$\frac{du}{dx} = \frac{\partial u}{dx} + \frac{\partial u}{dy}\frac{dy}{dx} + \frac{\partial u}{dz}\frac{dz}{dx}.$$

COR. 2. If $u = f(x, y)$, in which $x = \phi(v)$ and $y = \psi(v)$; then,

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy,$$

and

$$\frac{du}{dv} = \frac{\partial u}{dx}\frac{dx}{dv} + \frac{\partial u}{dy}\frac{dy}{dv}.$$

COR. 3. If $u = f(x, y)$ and $y = \phi(x)$; then

$$\frac{du}{dx} = \frac{\partial u}{dx} + \frac{\partial u}{dy}\frac{dy}{dx}.$$

COR. 4. If $u = f(y)$ and $y = \phi(x)$, then,

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}.$$

EXAMPLES.

By aid of partial differentials differentiate the following, and verify the results by direct differentiation:

1. $u = x^y$, $\qquad du = yx^{y-1}dx + x^y \log x dy.$

2. $u = xyz$, $\qquad du = yzdx + xzdy + xydz.$

3. $u = \dfrac{x}{y}$, $\qquad du = \dfrac{ydx - xdy}{y^2}.$

4. $u = x^5 + \log \sin y + \cos z$, $du = 5\,x^4 dx + \cot y\, dy - \sin z dz.$

5. $u = x^{\log y},$ $\qquad du = x^{\log y}\left\{\frac{\log y}{x}dx + \frac{\log x}{y}dy\right\}.$

6. $u = a^x e^y,$ $\qquad du = a^x e^y \{\log a dx + dy\}.$

7. $u = \tan^{-1}\frac{y}{x},$ $\qquad du = \frac{xdy - ydx}{x^2 + y^2}.$

8. $u = \log \tan^{-1}\frac{x}{y},$ $\qquad du = \frac{ydx - xdy}{(x^2 + y^2)\tan^{-1}\frac{x}{y}}.$

Write the first derivative of each of the following implicit functions by use of partial derivatives, and verify results by the direct process:

9. $a^2y^2 - b^2x^2 = - a^2b^2.$

Let $u = a^2y^2 - b^2x^2 + a^2b^2.$

Then, § 121, Cor. 1, we have,

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{dx}}{\frac{\partial u}{dy}} = -\frac{-2b^2x}{2a^2y} = \frac{b^2x}{a^2y} = \frac{bx}{a\sqrt{x^2 - a^2}}.$$

Directly: $\qquad 2a^2ydy - 2b^2xdx = 0,$

$$\therefore \frac{dy}{dx} = \frac{b^2x}{a^2y}.$$

10. $x^y - y^x = 0,$ $\qquad \frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}.$

11. $\frac{ye^{ny}}{x^m} = a,$ $\qquad \frac{dy}{dx} = \frac{my}{x(1 + ny)}.$

12. $\sin(xy) + \tan(xy) = m,$ $\qquad \frac{dy}{dx} = -\frac{y}{x}.$

13. $\sin(xy) - ax = 0,$ $\qquad \frac{dy}{dx} = \frac{a - y\cos xy}{x\cos xy}.$

Find the total derivative of each of the following:

14. $u = x^3 + y^2 - xz$, in which $x = v^2$, $y = \sin v$, $z = \log v$.

Here $$\frac{du}{dv} = \frac{\partial u}{dx}\frac{dx}{dv} + \frac{\partial u}{dy}\frac{dy}{dv} + \frac{\partial u}{dz}\frac{dz}{dv}.$$ See Art. 122.

Also $$\frac{\partial u}{dx} = 3x^2 - z, \quad \frac{\partial u}{dy} = 2y, \quad \frac{\partial u}{dz} = -x,$$

$$\frac{dx}{dv} = 2v, \quad \frac{dy}{dv} = \cos v, \quad \frac{dz}{dv} = \frac{1}{v};$$

hence, $$\frac{du}{dv} = (3x^2 - z)\,2v + 2y\cos v - \frac{x}{v}$$

$$= (3v^4 - \log v)\,2v + 2\sin v\cos v - v$$
$$= 6v^5 - 2v\log v + \sin 2v - v.$$

To obtain this result directly w: substitute the values of x, y, and z in the value of u and obtain

$$u = v^6 + \sin^2 v - v^2 \log v.$$

Hence, $$\frac{du}{dv} = 6v^5 + 2\sin v\cos v - \left(v^2\frac{1}{v} + 2v\log v\right)$$
$$= 6v^5 + \sin 2v - v - 2v\log v,$$

as before.

15. $u = y^2 + z^4 + \sin xy$, $y = x^3$, $z = \tan x$.

Here $$\frac{du}{dx} = \frac{\partial u}{dx} + \frac{\partial u}{dy}\frac{dy}{dx} + \frac{\partial u}{dz}\frac{dz}{dx}.$$ See § 122, Cor. 1.

$$\therefore \frac{du}{dx} = 6x^5 + 4(\tan^3 x\sec^2 x + x^3\cos x^4).$$

16. $u = y^3 + zy + z^2$, $y = e^x$, $z = \sin x$.

Here $$\frac{du}{dx} = \frac{\partial u}{dy}\frac{dy}{dx} + \frac{\partial u}{dz}\frac{dz}{dx}.$$ See § 122, Cor. 2.

$$\therefore \frac{du}{dx} = 3e^{3x} + e^x(\sin x + \cos x) + \sin 2x.$$

17. $u = \tan^{-1}\frac{y}{x},\ y^2 = 2\,px.$

Here $$\frac{du}{dx} = \frac{\partial u}{dx} + \frac{\partial u}{dy}\frac{dy}{dx}.$$ See § 122, Cor. 3.

$$\therefore\ \frac{du}{dx} = -\frac{\sqrt{px}}{(x^2 + 2\,px)\sqrt{2}}.$$

18. $u = \log\frac{x}{y},\ y = \sin x.$ $\qquad \frac{du}{dx} = \frac{1}{x} - \cot x.$

19. $u = \tan y^2,\ y = \log x.$

Here $$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}.$$ See § 122, Cor. 4,

$$\therefore\ \frac{du}{dx} = \frac{2\log x \sec^2(\log x)^2}{x}.$$

20. $u = \sin\frac{z}{y},\ y = x^2,\ z = e^x.$ $\qquad \frac{du}{dx} = (x - 2)\frac{e^x}{x^3}\cos\frac{e^x}{x^2}.$

21. $u = \tan^{-1}\frac{z-y}{z+y},\ z = e^x,\ y = e^{-x}.$ $\qquad \frac{du}{dx} = \frac{2\,e^{2x}}{e^{4x}+1}.$

22. $u = \sin^{-1}(y - z),\ y = 3\,x,\ z = 4\,x^3.$ $\qquad \frac{du}{dx} = \frac{3}{\sqrt{1 - x^2}}.$

23. $u = \frac{x^4y^2}{4} - \frac{x^4y}{8} + \frac{x^4}{32},\ y = \log x.$ $\qquad \frac{du}{dx} = x^3(\log x)^2.$

SUCCESSIVE PARTIAL DIFFERENTIATION.

123. Successive Partial Differentials and Derivatives.

In general, if $u = f(x, y)$, then

$$\frac{\partial u}{dx}\,dx = \phi(x, y)\,dx \quad \ldots\ldots\ldots \quad (a)$$

and

$$\frac{\partial u}{dy}\,dy = \psi(x, y)\,dy \quad \ldots\ldots\ldots \quad (b)$$

i.e., the partial differentials of a function of x and y are, in general, functions of x and y. Hence, differentiating again, with respect to either x or y, regarded as equicrescent, we obtain a *second* partial differential. Thus from (a) we have,

$$\frac{\partial}{dx}\left(\frac{\partial u}{dx}dx\right)dx = \frac{\partial^2 u}{dx^2}dx^2 = \phi_1(x, y)\,dx^2,$$

$$\frac{\partial}{dy}\left(\frac{\partial u}{dx}dx\right)dy = \frac{\partial^2 u}{dydx}dxdy = \phi_2(x, y)\,dxdy;$$

and from (b),

$$\frac{\partial}{dy}\left(\frac{\partial u}{dy}dy\right)dy = \frac{\partial^2 u}{dy^2}dy^2 = \psi_1(x, y)\,dy^2,$$

$$\frac{\partial}{dx}\left(\frac{\partial u}{dy}dy\right)dx = \frac{\partial^2 u dydx}{dxdy} = \psi_2(x, y)\,dydx.$$

As the second partial differentials are also, in general, functions of x and y, we may again differentiate and obtain a set of third partial differentials, and so on. Thus, from the notation adopted above, we have

$$\frac{\partial^3 u}{dx^2dy}dx^2dy$$

for the symbol of the result obtained by *three* successive partial differentiations, *two* of these being with respect to x, and *one* with respect to y. Hence,

$$\frac{\partial^3 u}{dx^2dy}$$

is a symbol of the *third partial derivative* obtained by the same process. Other symbols of a third partial derivative are obviously,

$$\frac{\partial^3 u}{dx^3},\ \frac{\partial^3 u}{dy^3},\ \frac{\partial^3 u}{dxdy^2};$$

and similarly for other derivatives.

124. To prove,

$$\frac{\partial}{dx}\left(\frac{\partial u}{dy}\right) = \frac{\partial}{dy}\left(\frac{\partial u}{dx}\right),$$

i.e., that

$$\frac{d^2 u}{dx dy} = \frac{d^2 u}{dy dx}.$$

Let $u = f(x, y)$; then, regarding y as a constant, we have,

$$\frac{\Delta u}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

Ex. 10, p. 52. Now, regarding x as constant, we have,

$$\frac{\Delta\left(\frac{\Delta u}{\Delta x}\right)}{\Delta y} = \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y \Delta x}. \quad \ldots \quad (1)$$

Reversing the above order, we have,

$$\frac{\Delta u}{\Delta y} = \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

and

$$\frac{\Delta\left(\frac{\Delta u}{\Delta y}\right)}{\Delta x} = \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x, y)}{\Delta x \Delta y} \quad \ldots \quad (2)$$

Hence,

$$\frac{\Delta\left(\frac{\Delta u}{\Delta y}\right)}{\Delta x} = \frac{\Delta\left(\frac{\Delta u}{\Delta x}\right)}{\Delta y}.$$

Passing to limits, we have,

$$\frac{\partial\left(\frac{\partial u}{dy}\right)}{dx} = \frac{\partial\left(\frac{\partial u}{dx}\right)}{dy},$$

i.e.,

$$\frac{d^2 u}{dx dy} = \frac{d^2 u}{dy dx}.$$

COR. Similarly we may prove,

$$\frac{\partial^3 u}{dx^2 dy} = \frac{\partial^3 u}{dy dx^2},$$

$$\frac{\partial^n u}{dx^{n-3} dy^3} = \frac{\partial^n u}{dy^3 dx^{n-3}},$$

i.e., whatever the number of differentiations *the order of differentiation is immaterial.*

EXAMPLES.

1. $u = x^5 y - \sin y$; prove $\frac{\partial^2 u}{dxdy} = \frac{\partial^2 u}{dydx}$.

We have, $$\frac{\partial u}{dy} = x^5 - \cos y;$$

$$\therefore \frac{\partial^2 u}{dxdy} = 5x^4.$$

Also, $$\frac{\partial u}{dx} = 5x^4 y;$$

$$\therefore \frac{\partial^2 u}{dydx} = 5x^4.$$

Hence, $$\frac{\partial^2 u}{dxdy} = \frac{\partial^2 u}{dydx}.$$

2. $u = x \log (xy - 1)$; prove that $\frac{\partial^2 u}{dxdy} = \frac{\partial^2 u}{dydx}$.

3. $u = xy(x + y^2)$; show that $\frac{\partial^3 u}{dxdy^2} = \frac{\partial^3 u}{dy^2 dx}$.

4. $u = (x^2 + y^2)^{\frac{1}{3}}$; show that $3x\frac{\partial^2 u}{dxdy} + 3y\frac{\partial^2 u}{dy^2} + \frac{\partial u}{dy} = 0$.

5. $u = \cos(x + y)$; show that $\frac{\partial^2 u}{dxdy} = \frac{\partial^2 u}{dydx}$.

125. *To find the successive total differentials of a function of two independent variables.*

Let $u = f(x, y)$; then, (§ 121),

$$du = \frac{\partial u}{dx} dx + \frac{\partial u}{dy} dy.$$

Differentiating, remembering that $\frac{\partial u}{dx}$ and $\frac{\partial u}{dy}$ are, in general, functions of x and y, and that x and y, being independent, may be regarded as equicrescent, we have,

$$d^2 u = \frac{\partial^2 u}{dx^2} dx^2 + \frac{\partial^2 u}{dxdy} dxdy + \frac{\partial^2 u}{dydx} dydx + \frac{\partial^2 u}{dy^2} dy^2$$
$$= \frac{\partial^2 u}{dx^2} dx^2 + 2 \frac{\partial^2 u}{dxdy} dxdy + \frac{\partial^2 u}{dy^2} dy^2.$$

Similarly, we find,

$$d^3 u = \frac{\partial^3 u}{dx^3} dx^3 + 3 \frac{\partial^3 u}{dx^2 dy} dx^2 dy + 3 \frac{\partial^3 u}{dxdy^2} dxdy^2 + \frac{\partial^3 u}{dy^3} dy^3,$$

and so on. By observing the analogy between the exponents of

$$du, \ d^2 u, \ d^3 u, \ \ldots$$

and those of the development of

$$(x + a), \ (x + a)^2, \ (x + a)^3, \ \ldots$$

we are enabled to write the value of $d^n u$.

The student may apply this process to any example.

CHAPTER XII.

DIRECTION OF CURVATURE. POINTS OF INFLEXION.

CARTESIAN CURVES.

126. A curve is concave upward or convex upward at a point according as the tangent at the point lies below or above the curve.

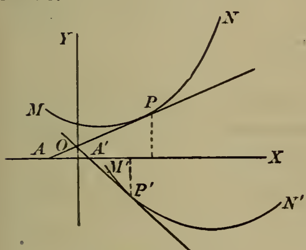


Fig. 22, a.

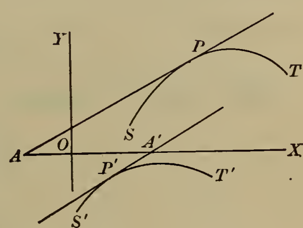


Fig. 22, b.

Thus, Fig. (a), the curves MN and $M'N'$ are concave upward; and, Fig. (b), the curves ST, $S'T'$ are convex upward at the points P, P'.

127. Investigation for Direction of Curvature.

Let $$y = f(x)$$

be the equation of any curve; then

$$\frac{dy}{dx} = f'(x)$$

is the slope of the tangent to the curve at the point (x, y). § 19.

It is readily seen from Fig. (*a*), by examining either of the curves MN or $M'N'$, that $\frac{dy}{dx}$ increases as x increases; hence $\frac{dy}{dx}[=f'(x)]$ is an increasing function of x; hence, § 28, Cor.,

$$\frac{d^2y}{dx^2} > 0.$$

On the other hand, the slopes of the tangents to the curves ST, $S'T'$, Fig. (*b*), are decreasing functions of x, i.e., $\frac{dy}{dx}[=f'(x)]$ decreases as x increases; hence, at such points as P, P'

$$\frac{d^2y}{dx^2} < 0.$$

Hence, in general, a curve $y=f(x)$ is concave upward or convex upward at the point (x, y) according as $\frac{d^2y}{dx^2} >$ or $<$ than 0.

128.* Point of Inflexion. The point at which the direction of the curvature changes is called a point of inflexion.

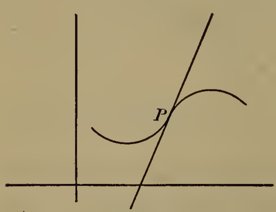


Fig. 23.

Such a point is P, Fig. 23. Since the curvature changes from concavity to convexity upward, or from convexity to concavity upward at a point of inflexion $\frac{d^2y}{dx^2}$ changes sign from $+$ to $-$, or from $-$ to $+$; hence at the point

$$\frac{d^2y}{dx^2} = 0 \text{ or } \infty.$$

To determine, therefore, whether any given curve has a point of inflexion we obtain the second derivative from its equation,

* Sluze, in 1659, pointed out a general method for determining points of inflexion by reducing it to a question of maxima and minima, viz., by investigating for a maximum or minimum intercept made by a tangent on any axis from a fixed point.

and equate the result to zero or infinity. The roots of this equation are the critical values. If for values of x a little less and a little greater than any one of these critical values $\frac{d^2y}{dx^2}$ changes sign, then for that critical value there is a point of inflexion.

To illustrate, let us examine the equation

$$6y = c(x-a)^3$$

for a point of inflexion.

Here $$\frac{d^2y}{dx^2} = c(x-a) = 0,$$

$\therefore\ x = a$ is a critical value; and as $\frac{d^2y}{dx^2}$ obviously changes sign as x passes through the value a the curve has a point of inflexion at $(a, 0)$.

EXAMPLES.

1. Determine the direction of curvature of the parabola

$$y^2 = 2px.$$

Here $$\frac{d^2y}{dx^2} = -\frac{p^2}{y^3}.$$

For negative values of y, $\frac{d^2y}{dx^2} > 0$; for positive values of y, $\frac{d^2y}{dx^2} < 0$; hence the parabola is concave upward below the x-axis and convex upward above the x-axis.

2. Show that the hyperbola $xy = m$ is concave upward in the first angle and convex upward in the third angle.

Here $$\frac{d^2y}{dx^2} = \frac{2m}{x^3}.$$

3. Show that $3a^2y - x^3 + 3ax^2 - 6a^3 = 0$ has a point of inflexion at $(a, \frac{4}{3}a)$, and that the curve is convex upward on the left of this point and concave upward on the right.

4. Examine the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ for points of inflexion.

Points of inflexion $\left(\frac{2}{\sqrt{3}}a, \frac{3}{2}a\right), \left(-\frac{2}{\sqrt{3}}a, \frac{3}{2}a\right)$.

5. Examine for direction of curvature:

(*a*) $x^2 + y^2 = a^2$.

(*b*) $x^2 = 2py$.

(*c*) $a^2y^2 - b^2x^2 = a^2b^2$.

6. Examine for points of inflexion:

(*a*) $y = 2x^3 - 3x^2 - 12x + 12$.

(*b*) $y = 2x^3 - 11x^2 + 12x + 10$.

(*c*) $y = x^3 - 3x^2 + 1$.

(*d*) $y = 2x^3 - 24x^2 + 2x - 1$.

POLAR CURVES.

129. A polar curve is convex or concave to the pole at a point according as the tangent to the curve at the point does, or does not, lie on the same side of the curve as the pole.

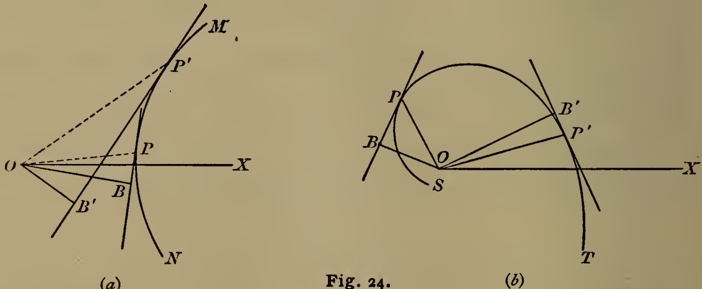


Fig. 24.

Thus, Figs. 24, the curve MN is convex, and ST is concave to the pole O.

130. Investigation for Direction of Curvature.

Let $$r = f(\theta)$$

be the equation of either of the curves *MN*, *ST*, Figs. 24, — *MN* and *ST* being *any* two curves referred to polar coördinates. Let *PB*, *P'B'*, be two tangents drawn at any two points *P*, *P'*; and let *OB*, *OB'* be perpendiculars let fall from the pole on these tangents. From Fig. 24 (*a*) we see that as r (*OP*) increases p (*OB*) decreases, and from Fig. 24 (*b*) that as r (*OP*) increases p (*OB*) increases; hence, in either case,

$$p = F(r),$$

and since p is a decreasing function of r when the curve is convex and an increasing function of r when it is concave, we have

$$\frac{dp}{dr} < 0 \text{ or } > 0,$$

according as the curve is convex or concave to the pole.

From § 78, we have

$$p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}};$$

hence to investigate any curve $r = f(\theta)$ for direction of curvature at a given point we first obtain $\frac{dr}{d\theta}$ from the equation of the curve and substitute the square of the value found in the above value of p, the first derivative of the resulting expression, $\left(\frac{dp}{dr}\right)$, in which the coördinates of the given point have been substituted, will determine by its sign whether the curve is concave or convex to the pole at the given point.

Thus let us examine the spiral of Archimedes $r = a\theta$ for the direction of curvature at the point (r, θ).

Here $$\frac{dr}{d\theta} = a,$$

hence, $$p = \frac{r^2}{\sqrt{r^2 + a^2}},$$

$$\therefore \frac{dp}{dr} = \frac{r^3 + 2a^2 r}{(r^2 + a^2)^{\frac{3}{2}}},$$

which is positive for all positive values of r; hence the curve is everywhere concave to its pole.

131. Point of Inflexion. Since the tangent to a curve at a point of inflexion *crosses* the curve (see Fig. 23), the curve is convex to the pole on one side of the point, and concave to the pole on the other side; hence, $\frac{dp}{dr}$ changes sign from $-$ to $+$, or from $+$ to $-$; hence, at the point,

$$\frac{dp}{dr} = 0 \text{ or } \infty.$$

To examine a polar curve, therefore, for a point of inflexion we obtain $\frac{dp}{dr}$ as in the preceding article, and ascertain what values of r, if any, reduces this derivative to 0 or ∞. If such values of r exist we then ascertain whether or not $\frac{dp}{dr}$ changes sign as r passes through this value; if it does, the critical value corresponds to a point of inflexion. Thus in the Spiral of Archimedes, discussed in the preceding article, we find by equating $\frac{dp}{dr}$ to 0 that

$$r = 0 \text{ and } r = a\sqrt{-2}$$

are critical values; and by equating it to ∞ that $r = a\sqrt{-1}$ is another. As two of these values are imaginary and the other $(r = 0)$ corresponds to the starting-point of the curve there is no point of inflexion.

EXAMPLES.

1. Examine the lituus, $r^2\theta = a$, for direction of curvature and for points of inflexion.

Here
$$\frac{dr}{d\theta} = -\frac{a}{2r\theta^2} = -\frac{r^3}{2a},$$
$$\therefore p = \frac{2ar}{\sqrt{4a^2 + r^4}}.$$

Hence,
$$\frac{dp}{dr} = \frac{8a^3 - 2ar^4}{(4a^2 + r^4)^{\frac{3}{2}}} = \frac{2a(4a^2 - r^4)}{(4a^2 + r^4)^{\frac{3}{2}}} = 0.$$
$$\therefore r^4 - 4a^2 = 0,$$
$$\therefore r = \sqrt{2a}.$$

Hence, when $r < \sqrt{2a}$, $\frac{dp}{dr} > 0$, $\therefore$ curve is concave to the pole; and when $r > \sqrt{2a}$, $\frac{dp}{dr} < 0$, $\therefore$ curve is convex to the pole. Since $\frac{dp}{dr}$ changes sign as r passes through the value $\sqrt{2a}$, that value of r corresponds to a point of inflexion.

2. Show that the logarithmic spiral $r = a^\theta$ has no point of inflexion, and that it is everywhere concave to the pole.

Examine for direction of curvature:

3. $r = a$.

4. $r = 2a\sin\theta$.

5. $r = 2a\cos\theta$.

6. $r\theta = c$.

7. $r = c\theta^2$.

CHAPTER XIII.

CURVATURE. CIRCLE AND RADIUS OF CURVATURE. EVOLUTE AND INVOLUTE.

HISTORY. — Huygens, in the third chapter of his *Horologium Oscillatorium* (1673), defines evolutes and involutes, proves some of their more elementary properties, and illustrates his method by finding the evolutes of the cycloid and the parabola.

132. *The* **Measure of the Curvature,** or more simply, **The Curvature** *of a curve, is the ratio of the rate of change of its direction to the rate of change of its length.*

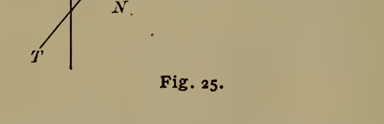


Fig. 25.

Let α be the angle which the tangent to the curve MN at P, Fig. 25, makes with the X-axis and let $NP = s$; then, since the direction of a curve at a point is the same as that of the tangent, we have $d\alpha$ = rate of change of direction of the curve, and ds = rate of change of the length of the curve. Hence, by definition,

$$\kappa = \frac{d\alpha}{ds}$$

in which κ represents the curvature of MN at the point P.

To show how this ratio measures the curvature let us suppose the curve MN to be generated by the point P moving with any velocity v and *carrying its tangent along with it* as it moves. Let us further suppose the curve SL (tangent to MN and PT at P) to be generated by a coincident point P moving with the same velocity v. Then, by definition, the curvature of SL at P is

$$\kappa' = \frac{d\alpha'}{ds'}$$

in which $d\alpha'$ is the rate of change of the direction of PT, *the tangent to* SL, at P, and ds' is the rate of change of the length $SP = s'$. But since the generating points move with *the same velocity*, we have

$$v = ds = ds' \qquad \S\ 17;$$

hence,

$$\frac{\kappa}{\kappa'} = \frac{d\alpha}{d\alpha'}$$

i.e., the curvature of two curves at any two points are to each other as the rates of change of their direction.

For example, let $d\alpha = 30°$ a second and $d\alpha' = 60°$ a second; then

$$\frac{\kappa}{\kappa'} = \frac{30°}{60°} = \frac{1}{2},$$

or

$$\kappa' = 2\,\kappa,$$

i.e., the curvature of one curve is *twice* that of the other.

133. Circle of Curvature. Radius of Curvature.

The circle tangent to a curve at a point, and having *the same curvature as the curve at the point, is called the* **Circle of Curvature,** *for that point.* The **Radius of Curvature** *is the radius of the circle of curvature*, and the **Center of Curvature** is the center of this circle.

Thus, Fig. 26, if the circle CPP' has the *same* curvature as the curve NM at the point P, CPP' is the circle of curvature for that point; OP, the radius of CPP', is the radius of curvature, and the center O, the center of curvature.

It is obvious that the circle of curvature and radius of curvature vary from point to point as the curvature of the curve changes.

Since, by definition, the circle of curvature CPP' and the curve NM have the same curvature at P, we have, § 132,

$$\kappa = \frac{d\alpha}{ds}$$

for the curvature of the circle at P. But if we suppose the length of the arc $PP'(= ds)$ to be the velocity with which the

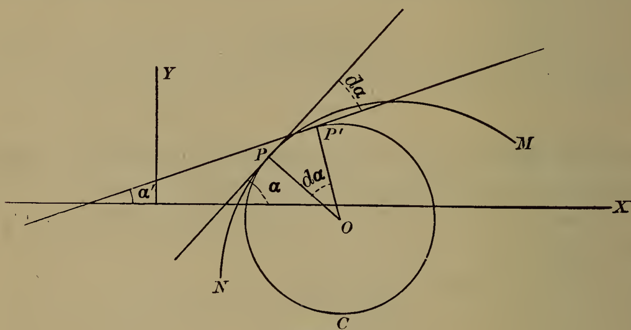


Fig. 26.

generating point passes through P, then $POP' = \alpha - \alpha'$ represents the angular velocity of the tangent, i.e., the rate of change of its direction. Hence,

$$POP' = d\alpha.$$

Let $OP = \rho$, then from the circle,

$$POP' = \frac{\text{arc } PP'}{\rho}$$

i.e.,

$$d\alpha = \frac{ds}{\rho}.$$

Hence,

$$\rho = \frac{ds}{d\alpha} = \frac{1}{\kappa}$$

i.e., *the radius of curvature is the reciprocal of the curvature.*

COR. 1. If $\rho' = \frac{ds'}{d\alpha'}$ be the radius of curvature at any other point of the same curve, or at some point of another curve,

we have
$$\frac{\rho}{\rho'} = \frac{\frac{ds}{d\alpha}}{\frac{ds'}{d\alpha'}};$$

i.e.,
$$\frac{\rho}{\rho'} = \frac{\frac{d\alpha'}{ds'}}{\frac{d\alpha}{ds}} = \frac{\kappa'}{\kappa}.$$

Hence, the curvatures of a curve at any two points are inversely as the radii of curvature at the points.

134.* Expressions for the Radius of Curvature.

1. **In Terms of Rectangular Coördinates.**

Remembering that $ds = (dx^2 + dy^2)^{\frac{1}{2}}$, § 18, and that $\alpha = \tan^{-1}\frac{dy}{dx}$, § 19, we have

$$\rho = \frac{ds}{d\alpha}$$

$$= \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{d\tan^{-1}\frac{dy}{dx}}$$

$$= \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{\frac{\frac{d^2y}{dx}}{1 + \frac{dy^2}{dx^2}}};$$

* Equation (1) § 134, was first given by John Bernoulli in 1701.

hence, after reduction $\rho = \dfrac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ . . . . . . (1)

If y is taken as the equicrescent variable, we have (Ex. 9, p. 104).

$$\rho = \frac{\left\{1 + \left(\dfrac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}$$

COR. If in (1) $\dfrac{d^2y}{dx^2} = 0$, $\rho = \infty$, $\therefore \kappa = \dfrac{1}{\rho} = 0$; i.e., at a point of inflexion the curvature is zero. See § 128.

2. **In Terms of Polar Coördinates.**

In this case $ds = (r^2d\theta^2 + dr^2)^{\frac{1}{2}}$, § 76; $\tan\phi = \dfrac{r\,d\theta}{dr} = \dfrac{r}{\dfrac{dr}{d\theta}}$.

$\therefore\ \phi = \tan^{-1}\dfrac{r}{\dfrac{dr}{d\theta}}$; $\alpha = \theta + \phi$. See § 77.

Hence, $\rho = \dfrac{ds}{d\alpha}$

$$= \frac{(r^2d\theta^2 + dr^2)^{\frac{1}{2}}}{d(\theta + \phi)} = \frac{(r^2d\theta^2 + dr^2)^{\frac{1}{2}}}{d\theta + d\phi}$$

$$= \frac{(r^2d\theta^2 + dr^2)^{\frac{1}{2}}}{d\theta + d\tan^{-1}\dfrac{r}{\dfrac{dr}{d\theta}}}$$

$$= \frac{(r^2d\theta^2 + dr^2)^{\frac{1}{2}}}{d\theta + \dfrac{\dfrac{\dfrac{dr}{d\theta}dr - r\dfrac{d^2r}{d\theta}}{\left(\dfrac{dr}{d\theta}\right)^2}}{1 + \dfrac{r^2}{\left(\dfrac{dr}{d\theta}\right)^2}}}.$$

Hence, reducing, we have

$$\rho = \frac{\left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If r is taken as the equicrescent variable, we have (Ex. 8, p. 103),

$$\rho = \frac{\left\{ 1 + r^2\left(\frac{d\theta}{dr}\right)^2 \right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + r^2\left(\frac{d\theta}{dr}\right)^3 + 2\frac{d\theta}{dr}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Cor. Since $\rho = \infty$ at a point of inflection (§ 134, Cor.), we have, from (1), $r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 0$ as a necessary condition for such a point.

135. *At a point of* **Maximum Curvature** *the circle of curvature lies entirely within the curve.*

For on either side of the point the curvature of the curve is *less* than at the point, while the curvature of the circle of curvature is the *same* on either side of the point of tangency as it is at that point. Hence, the circle of curvature lies entirely within the curve.

Similarly, we may show that at a point of minimum curvature the circle of curvature lies entirely *without* the curve.

Cor. Since, in general, the curvature of a curve at a given point is *less* than at the preceding consecutive point and *greater* than at the following consecutive point, or *vice versa*, we have, as a general proposition, that *the circle of curvature crosses its curve.*

EXAMPLES.

1. Show that the curvature of the circle $x^2 + y^2 = a^2$ is constant, and find its radius of curvature for the point (x, y).

Here,
$$\kappa = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

From the equation $x^2 + y^2 = a^2$, we obtain

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{a^2 - x^2}};$$

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3} = -\frac{a^2}{\sqrt{(a^2 - x^2)^3}}.$$

Hence,
$$\kappa = \frac{\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}}{\left\{1 + \frac{x^2}{a^2 - x^2}\right\}^{\frac{3}{2}}} = \frac{1}{a}.$$

Hence, the curvature of the circle is constant, as previously assumed, and is the reciprocal of its radius.

Again,
$$\rho = \frac{1}{\kappa}. \quad \therefore \rho = a,$$

i.e., the radius of the circle is its radius of curvature.

2. Find the radius of curvature of the logarithmic spiral $r = e^{a\theta}$.

Here,
$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

From the equation we have

$$\frac{dr}{d\theta} = ae^{a\theta}, \quad \frac{d^2r}{d\theta^2} = a^2e^{a\theta}.$$

$$\therefore \rho = \frac{\{e^{2a\theta} + a^2e^{2a\theta}\}^{\frac{3}{2}}}{e^{2a\theta} + 2a^2e^{2a\theta} - a^2e^{2a\theta}}.$$

$$= r\sqrt{1 + a^2}.$$

Find the radius of curvature of each of the following:

3. $y^2 = 4ax.$ $\qquad \rho = \dfrac{2\sqrt{(x+a)^3}}{\sqrt{a}}.$

4. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ $\qquad \rho = \dfrac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$

5. $r^2 = a^2 \cos 2\theta.$ $\qquad \rho = \dfrac{a^2}{3r}.$

6. $r = a(1 - \cos\theta).$ $\qquad \rho = \frac{2}{3}\sqrt{2ar}.$

7. $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}).$ $\qquad \rho = \dfrac{a}{4}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2.$

8. $x = a \operatorname{vers}^{-1}\dfrac{y}{a} - \sqrt{2ay - y^2}.$ $\qquad \rho = 2\sqrt{2ay}.$

9. $2xy = a^2.$ $\qquad \rho = \dfrac{(x^2 + y^2)^{\frac{3}{2}}}{a^2}.$

10. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$ $\qquad \rho = 3\sqrt[3]{axy}.$

11. $e^{\frac{y}{a}} = \sec\dfrac{x}{a}.$ $\qquad \rho = a \sec\dfrac{x}{a}.$

12. $r = a\sec^2\dfrac{\theta}{2}.$ $\qquad \rho = 2a\sec^3\dfrac{\theta}{2}.$

Find the radius of curvature of the following at point indicated:

13. $y = x^2 + 2$ at $(1, 3)$.

14. $y = x^3 - 3x^2 + 2$ at $(1, 0)$.

15. $3y = x^3 + 2$ at $(1, 1)$.

16. $x^2 + y^2 = 25$ at $(3, 4)$.

17. $x^2 - y^2 = 9$ at $(5, 4)$.

136. Evolute. *The* **Evolute** *of a curve is the locus of the centers of the circles of curvature of the curve.*

137. Involute. *The* **Involute** *is the curve whose centers of curvature form the Evolute.*

Thus, if the curve $N'O'S'$ is the locus of the center of curvature of the curve NPM, $N'O'S$ is the evolute of NPM, and NPM is the Involute. See Fig. 27.

138. Equation of the Evolute.

Let $y = f(x)$ be the equation of involute NPM (Fig. 27), and let O' be the center of curvature corresponding to any point P on the involute. Let (x, y), (x', y'), be the coördinates of P and O', respectively. It is required to determine (1st) the values of x' and y' in terms of x and y, and (2d) to determine from these values in conjunction with the equation of the involute, $y = f(x)$, a general relation between x' and y', i.e., *to determine the equation of the evolute.*

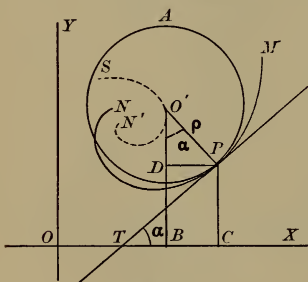


Fig. 27.

1. *To determine the coördinates of the center of curvature* (x', y').

Let $O'P = \rho$, and draw $PD \perp O'B$. Since $O'P$ is $\perp$ to the tangent PT, we have $DO'P = \alpha$. We have, therefore,

$$OB = OC - PD \quad \text{and} \quad O'B = PC + O'D.$$

But $OB = x'$, $OC = x$, $PD = \rho \sin \alpha$; and $O'B = y'$, $PC = y$, $O'D = \rho \cos \alpha$. Whence, substituting, we have

$$\left.\begin{aligned} x' &= x - \rho \sin \alpha \\ y' &= y + \rho \cos \alpha \end{aligned}\right\} \quad \dots\dots\dots \quad (a)$$

or since $\sin \alpha = \dfrac{dy}{ds} = \dfrac{dy}{\sqrt{dx^2 + dy^2}}$ and $\cos \alpha = \dfrac{dx}{ds} = \dfrac{dx}{\sqrt{dx^2 + dy^2}}$

these values together with the value of ρ substituted in (a) give

$$x' = x - \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y' = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. *To determine the equation of the evolute.*

If we now combine with (1) and (2) the equation of the involute,

$$y = f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

eliminating the variable coördinates x and y of the involute, we shall have a resulting equation in x' and y'. The equation thus obtained is obviously that of the evolute.

To illustrate, let us find the equation of the evolute of the parabola, $y^2 = 4\,ax$.

Differentiating we obtain

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}, \quad \frac{d^2y}{dx^2} = -\tfrac{1}{2}\sqrt{\frac{a}{x^3}}.$$

Hence, $$x' = x + \frac{\left\{1 + \frac{a}{x}\right\}\sqrt{\frac{a}{x}}}{\frac{1}{2}\sqrt{\frac{a}{x^3}}},$$

i.e., $$x' = 3\,x + 2\,a, \quad x = \frac{x' - 2\,a}{3}.$$

Also, $$y' = y - \frac{\left\{1 + \frac{a}{x}\right\}}{\frac{1}{2}\sqrt{\frac{a}{x^3}}},$$

i.e., $$y' = -\frac{y^3}{4\,a^2}, \quad y = -(4\,a^2y')^{\frac{1}{3}}.$$

Substituting these values in the equation of the parabola, we have,

$$(4a^2y')^{\frac{2}{3}} = 4a\frac{x'-2a}{3};$$

i.e., $$27ay'^2 = 4(x'-2a)^3$$

is the equation of the evolute.

The semi-cubic parabola CDE is the locus of this equation, the branch DE being the evolute of OB, and DC the evolute of OA.

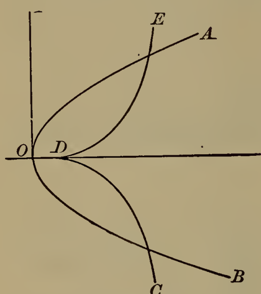


Fig. 28.

Since $y' = 0$, gives $x' = 2a$, the vertex D is at a distance from the origin equal to the semi-latus rectum of the parabola $y^2 = 4ax$.

An inspection of the figure shows that OD is also the radius of curvature of the curve at O, and since it is evidently a *minimum*, the curvature of the parabola is a *maximum* at the origin.

EXAMPLES.

Find the coördinates of the centers of the curvature of the following, also of Ex. 13–17, p. 187.

1. $y = x^2 - 8x + 15$ at $(1, 3)$.
2. $y = x^3 - 8x^2 + 4 + 6$ at $(1, 1)$.
3. $y = e^x$ at $(0, 1)$.
4. $y = \sin x$ at $\left(\frac{\pi}{2}\right), 1$.

139. Properties of the Evolute.

1. *A tangent to the evolute is normal to the involute.*

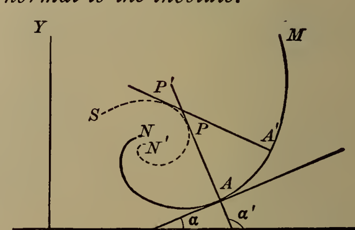


Fig. 29.

Let $N'S$, Fig. 29, be the evolute and MN the involute. Let PA be a tangent at any point $P(x', y')$ of the evolute, and TA a tangent to the involute at the point $A(x, y)$ where PA cuts the curve. We wish to show that PA is $\perp$ TA; i.e., that

$$\alpha' = \frac{\pi}{2} + \alpha.$$

From § 138, we have

$$y' = y + \rho \cos \alpha,$$
$$x' = x - \rho \sin \alpha.$$

Differentiating these equations, we have

$$dy' = dy - \rho \sin \alpha d\alpha + \cos \alpha d\rho,$$
$$dx' = dx - \rho \cos \alpha d\alpha - \sin \alpha d\rho.$$

But $\rho \sin \alpha d\alpha = \frac{ds}{d\alpha}\frac{dy}{ds} d\alpha = dy$, and $\rho \cos \alpha d\alpha = \frac{ds}{d\alpha}\frac{dx}{ds} d\alpha = dx$, §§ 133, 18; hence,

$$dy' = \cos \alpha d\rho \quad \ldots\ldots\ldots \quad (a)$$
$$dx' = - \sin \alpha d\rho . \quad \ldots\ldots\ldots \quad (b)$$

therefore, by division,

$$\frac{dy'}{dx'} = - \cot \alpha ;$$

i.e., § 19,
$$\tan \alpha' = \tan \left(\frac{\pi}{2} + \alpha\right);$$
$$\therefore \ \alpha' = \frac{\pi}{2} + a,$$

and the tangent to the evolute is normal to the involute.

2. *The difference between two radii of curvature is equal to the length of the arc of the evolute between the corresponding centers of curvature.*

Thus, Fig. 29, we wish to prove that

$$P'A' - PA = PP'.$$

Squaring and adding equations (a) and (b), we have

$$dy'^2 + dx'^2 = d\rho^2 (\cos^2 \alpha + \sin^2 \alpha);$$

hence,
$$ds'^2 = d\rho^2;$$
$$\therefore \ ds' = d\rho,$$

i.e., the rate of change of $N'P (= s')$ is equal to the rate of change of PA $(=\rho)$; but since any interval of time may be regarded as a unit of time, we have $ds' = PP'$ and $d\rho = P'A' - PA$; hence,

$$P'A' - PA = PP'.$$

140. The properties of the evolute demonstrated in the preceding article afford a method of describing the involute mechanically when the evolute is given. Take a string of any length, and wind it around the evolute $N'S$, Fig. 29, one end of the string being at N. If we place the point of a pencil in a loop of the string formed at N, and unwind the string, the pencil-point will describe the involute. For the string is always tangent to the evolute $N'S$ and normal to the involute NM; also, the *difference* between the unwound lengths in any two positions, say $P'A'$, PA, is evidently equal to the arc of the evolute (PP') between the points of tangency.

Since the curve $N'S$ is usually of indefinite length, there may be an infinite number of involutes of which it is the evolute; there can, however, be only one evolute of which it is the involute.

EXAMPLES.

1. Find the evolute of the circle, $x^2 + y^2 = a^2$.

Here, $$\frac{dy}{dx} = -\frac{x}{y},\quad \frac{d^2y}{dx^2} = -\frac{a^2}{y^3}.$$

Hence, $$x' = x - \frac{\left(1 + \frac{x^2}{y^2}\right)\frac{x}{y}}{\frac{a^2}{y^3}} = x - x = 0.$$

$$y' = y - \frac{1 + \frac{x^2}{y^2}}{\frac{a^2}{y^3}} = y - y = 0.$$

Hence, the evolute is a point.

2. Find the evolute of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Here, $$\frac{dy}{dx}=-\frac{b^2x}{a^2y},\ \frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

Hence, $$x'=\frac{(a^2-b^2)\,x^3}{a^4},\ y'=-\frac{(a^2-b^2)y^3}{b^4}$$

Substituting the values of x and y drawn from these values of x' and y' in the equation of the ellipse and reducing, we have

$$\frac{x'^{\frac{2}{3}}}{b^{\frac{2}{3}}}+\frac{y'^{\frac{2}{3}}}{a^{\frac{2}{3}}}=\frac{(a^2-b^2)^{\frac{2}{3}}}{a^{\frac{2}{3}}b^{\frac{2}{3}}}$$

for the equation of the evolute.

3. The hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. $\qquad \frac{x'^{\frac{2}{3}}}{b^{\frac{2}{3}}}-\frac{y'^{\frac{2}{3}}}{a^{\frac{2}{3}}}=\frac{(a^2+b^2)^{\frac{2}{3}}}{a^{\frac{2}{3}}b^{\frac{2}{3}}}$.

4. The tractrix, $x=a\log\frac{a+\sqrt{a^2-y^2}}{y}-\sqrt{a^2-y^2}$, has for its evolute the catenary, $y'=\frac{a}{2}(e^{\frac{x'}{a}}+e^{-\frac{x'}{a}})$.

5. The cycloid, $x=a\ \text{vers}^{-1}\frac{y}{a}-\sqrt{2ay-y^2}$.

$$x'=a\ \text{vers}^{-1}\left(-\frac{y'}{a}\right)-\sqrt{-2ay'-y'^2}.$$

6. The hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{a}\right)^{\frac{2}{3}}=1$.

$$\left(\frac{x'+y'}{a}\right)^{\frac{2}{3}}+\left(\frac{x'-y'}{a}\right)^{\frac{2}{3}}=2.$$

7. The equilateral hyperbola, $2xy=a^2$.

$$\left(\frac{x'+y'}{a}\right)^{\frac{2}{3}}-\left(\frac{x'-y'}{a}\right)^{\frac{2}{3}}=2.$$

CHAPTER XIV.

CONTACT OF CURVES. ENVELOPES.

History. — Envelopes may be said to have originated with the investigations of Huygens on evolutes and those of Tschirnhausen (1631–1708) on caustics.

Leibnitz laid the foundation of the theory in a memoir written in 1692.

141. Orders of Contact. Let $y = \phi(x)$ and $y = \psi(x)$ be the equations of two curves, and let $x = a$ be the abscissa of a common point; then

$$\phi(a) = \psi(a),$$

i.e., their corresponding ordinates are equal. If we suppose, moreover, that the curves touch each other, we have

$$\phi'(a) = \psi'(a),$$

i.e., the curves have a common tangent at the point. When these conditions are fulfilled for any two curves they are said to have a **Contact of the First Order.**

If $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, and, also,

$$\phi''(a) = \psi''(a),$$

then the curves not only touch each, but their *curvature* at the common point is the same since $c\left(= \frac{1}{\rho}\right)$ is a function of the first and second derivatives only. See § (134), 1. Under these conditions the curves are said to have a **Contact of the Second Order.**

If $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, $\phi''(a) = \psi''(a)$, and also,

$$\phi'''(a) = \psi'''(a),$$

the curves are said to have a **Contact of the Third Order**; and so on. Hence, generally, if

$$\phi(a)=\psi(a),\ \phi'(a)=\psi'(a),\ \phi''(a)=\psi''(a)\ \ldots\ \phi^n(a)=\psi^n(a)\ \ldots (a)$$

the curves have a **contact** *of the* **n**th **order.**

COR. 1. *A contact of the* n^{th} *order involves* $n+1$ *conditions.*

COR. 2. As only $n+1$ conditions can *in general* be imposed upon a locus whose equation contains $n+1$ arbitrary constants, the highest order of contact to such a curve is in general the n^{th}. Thus the straight lines $Ax+By+C=0$, having only two arbitrary constants, can have a contact of the 1st order; the circle $(x-a)^2+(y-b)^2=r^2$ having *three* arbitrary constants can have a contact of the second order.

142. *Two curves in contact do or do not cross each other at their common point according as their order of contact is even or odd.*

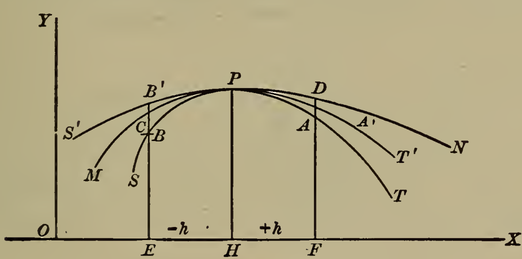


Fig. 30.

Let $y=\phi(x)$ and $y=\psi(x)$ be the equations of two curves having the n^{th} order of contact, and let $x=a$ be the abscissa of their common point P, Fig. 30. Let us add a small increment h to a, then

$$\phi(a+h)-\psi(a+h)$$

is the difference between the ordinates of the two curves corresponding to the same abscissa $x=a+h$. Expanding both terms in the expression by Taylor's Theorem, and collecting like derivatives, we have (since $\phi(a) = \psi(a)$)

$$\phi(a+h) - \psi(a+h) = \{\phi'(a) - \psi'(a)\} h + \{\phi''(a) - \psi''(a)\} \frac{h^2}{\lfloor 2}$$
$$+ \{\phi'''(a) - \psi'''(a)\} \frac{h^3}{\lfloor 3} + \cdots \cdots \quad (1)$$

1. *If n is even,*

then $\phi'(a) = \psi'(a)$, $\phi''(a) = \psi''(a)$, ... $\phi^n(a) = \psi^n(a)$,

and the terms in the second number of (1) successively vanish until the $(n+1)^{\text{th}}$ term is reached. As this term contains h affected with an *odd* exponent $(n+1)$ its *sign* will change with h, and if h is taken sufficiently small the numerical value of this term will exceed the sum of all the other terms. Hence the sign of the second member, that is, *the sign of the difference of the ordinates*, changes with the sign of h; hence the curves cross each other. In Fig. 30, the curves MN and $S'T'$ illustrate this case, P being the point of contact.

2. *If n is odd,*

then the first term that does not vanish contains h affected with an even exponent $(n+1)$. If this term is made to control in the second member by giving h a very small value, then the second member, and hence the difference of the ordinates, will not change sign with h; hence the curves do not cross each other.

The curves MN and ST illustrate this case.

Cor. 1. Since the *smaller* the difference between the ordinates $\phi(a+h)$ and $\psi(a+h)$ for any small value of h the *closer* the curves approach coincidence near P and since $\phi(a+h) - \psi(a+h)$ becomes smaller and smaller as the number of terms in the second member of (1) decreases, it follows that *as the order of contact increases, the closer the curves approach coincidence.*

143. *At a point of maximum or minimum curvature the circle of curvature has a contact of the third order.*

Let $$y = f(x) \qquad (a)$$
and $$(x-a)^2 + (y-b)^2 = r^2 \qquad (b)$$

be the equation of the curve and circle of curvature, respectively.

From § 134 (1), we have

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

By condition,

$$\frac{d\kappa}{dx} = 0. \qquad \text{§ 112.}$$

Hence,

$$\frac{d\kappa}{dx} = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\frac{3}{2}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} 2\frac{dy}{dx}\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3} = 0.$$

Solving, we have $$\frac{d^3y}{dx^3} = \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2}{1 + \frac{dy^2}{dx^2}}. \qquad (c)$$

Differentiating (b) successively, we have

$$x - a + (y-b)\frac{dy}{dx} = 0,$$

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0, \qquad (d)$$

$$(y-b)\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0.$$

Substituting the value of $y - b$ drawn from (d), and solving with respect $\frac{d^3y}{dx^3}$, we have

$$\frac{d^3y}{dx^3} = \frac{3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2}{1 + \frac{dy^2}{dx^2}} \qquad (e)$$

which is identical with (c). The third derivatives are therefore equal, and the contact is of the third order.

We may obtain the same result in the following very simple manner. Since at the point the curvature is a maximum or a minimum the circle of curvature *does not* cross the curve, §135; hence, § 142, 2, the order of contact is *odd*. But the expression

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

is the expression for both the curvature of the circle and the curve at the point of contact; hence, the *first* and *second* derivatives drawn from their equations are equal; hence, the order of contact of the curves is, *at least*, the *third*. This article and the one which follows explain the significance of the term "in general," used in § 141, Cor. 2. The general statement there given admits of exceptions at certain singular points of curves.

144. *The tangent to a curve at a point of inflexion crosses the curve.*

Let $y = f(x)$ . . . . (a)

and $Ax + By + C = 0$ . . . . (b)

be the equations of the curve and its tangent at a point of inflexion.

Then, we have from (a),

$$\frac{d^2y}{dx^2} = 0$$

as a condition for a point of inflexion, § 128. From (*b*) we have also,

$$\frac{d^2y}{dx^2} = 0;$$

hence, at a point of inflexion, the order of contact of the curve and its tangent is the *second;* hence the tangent and the curve cross each other, § 142, 1. See Fig. 23.

EXAMPLES.

1. Find the order of contact of the curves,

$$(a)\ y = 2x^2, \text{ and } (b)\ y = 3x - x^2.$$

Here (1, 2) is their common point. Differentiating and substituting, we have,

from $(a)\ \frac{dy}{dx} = 4x = 4,$ from $(b)\ \frac{dy}{dx} = 3 - 2x = 1.$

∴ The curves *intersect* at the point (1, 2).

2. Find the order of contact of

$$y = ax^3, \qquad y = 3ax^2 - 3ax + a.$$

Combining the equations we have (1, a) for the common point; then

$$\frac{dy}{dx} = 3ax^2 = 3a, \qquad \frac{dy}{dx} = 6ax - 3a = 3a.$$

$$\frac{d^2y}{dx^2} = 6ax = 6a, \qquad \frac{d^2y}{dx^2} = 6a.$$

$$\frac{d^3y}{dx^3} = 6a, \qquad \frac{d^3y}{dx^3} = 0.$$

Hence the curves have the second order of contact.

3. What order of contact has the circle, $(x - \frac{3}{4}a)^2 + (y - \frac{3}{4}a)^2 = \frac{a^2}{2}$, and the parabola, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, at $\left(\frac{a}{4}, \frac{a}{4}\right)$? *Ans.* Third.

4. Find the equation of the circle of curvature of the curve, $y^4 = 4a^2x^2 - x^4$.

5. What is the highest order of contact possible to two conics? *Ans.* Fourth.

6. What is the highest order of contact possible to the ellipse and parabola. *Ans.* Third.

7. Given $xy = 3x - 1$ and $y - x - 1 = a(x-1)^2$, find the value of a in order that the two curves may have a contact of the second order.

145. Families of Curves. *Curves whose equations differ only in the values of the constants which enter them are said to be of the same family.*

Thus the equation $(x-a)^2 + (y-b)^2 = r^2$ is the equation of a family of a circles whose positions and magnitudes depend upon the values of the constants a, b, r. Again, the equation $Ax + By + C = 0$ is the equation of a family of straight lines whose directions and positions with reference to the axes depend upon the values of A, B, C.

The constants which enter equations are called **Parameters,** and if one or more of these are supposed to vary, they are called **Variable Parameters.**

146. Envelope. *The Envelope of a family of curves is a curve tangent to each member of the family.*

Thus, if we assume a to be the variable parameter in the equation $(x-a)^2 + (y-b)^2 = r^2$, b and r remaining constant, we have (Fig. 31) a series of circles, all of whose centers are on the line MN, at a distance b from the x-axis. The envelopes of this family of circles are evidently the ‖ lines AB and EF, whose equations are

$$y = b \pm r.$$

If we assume b to vary, a and r remaining constant, we have a family of circles whose centers lie along the line LK at a distance a from the y-axis. The envelopes in this case being CD and HG, whose equations are

$$x = a \pm r.$$

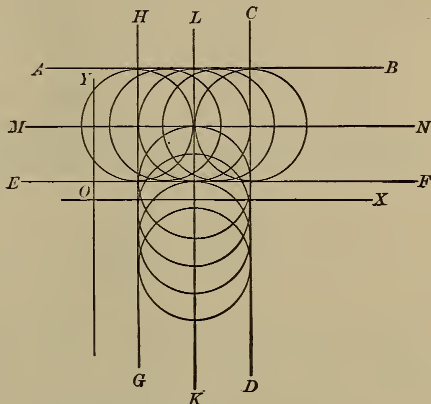


Fig. 31.

Similarly, we may suppose a and b to vary, r remaining constant, or a and r to vary, b remaining constant, etc.; or we may suppose all three to vary at the same time. In each case we have a family of circles, and a curve tangent to the members of that family is called the envelope.

It is evident, in this case, that if r alone varies there is no envelope.

147. To determine the equation of the envelope.

Let $u = f(x, y, a) = 0 \ldots (c)$ be the equation of *any* one (MN) of a family of curves, a being the variable parameter, and let $u = f(x, y) = 0 \ldots (d)$ be the equation of the envelope ST. Let P (x, y) be the point of tangency.

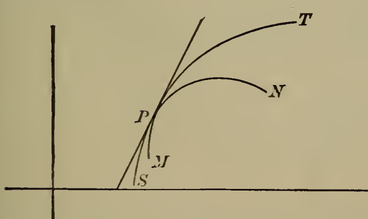


Fig. 32.

Since the curves are tangent to each other at (x, y) the first derivatives drawn from their equations must be equal; hence differentiating each we have,

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy + \frac{\partial u}{da}da = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{\dfrac{\partial u}{dx} + \dfrac{\partial u}{da}\dfrac{da}{dx}}{\dfrac{\partial u}{dy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Also, from (d), $du = \dfrac{\partial u}{dx}dx + \dfrac{\partial u}{dy}dy = 0.$

$$\therefore \frac{dy}{dx} = -\frac{\dfrac{\partial u}{dx}}{\dfrac{\partial u}{dy}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

But (e) and (f) are equal; hence equating and reducing, we have

$$\frac{\partial u}{da}\frac{da}{dx} = 0;$$

$$\therefore \qquad \frac{\partial u}{da} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

is a condition which the envelope must fulfill. If, then, we combine (g) and (c), eliminating a, we have an equation expressing the relation between x and y for *every* such point as P; hence the equation, thus ascertained, is the equation of the envelope.

To illustrate, let us find the envelope of the family $(x - a)^2 + (y - b)^2 = r^2$ in which a is the variable parameter.

Here $\qquad u = (x - a)^2 + (y - b)^2 - r^2 = 0;$

$$\therefore \frac{\partial u}{da} = -2(x - a) = 0;$$

$$\therefore \quad a = x.$$

Substituting this value of a in $(x - a)^2 + (y - b)^2 = r^2$, we have

$$y = b \pm r,$$

as before. See § 146.

Similarly we may show that

$$x = a \pm r$$

is the equation of the envelope when b is the variable parameter.

148. *The evolute of any curve is the envelope of its normals.*

We might readily infer this from § 140. We may prove the fact, however, as follows:

From § 70, we have

$$y - y' = -\frac{dx'}{dy'}(x - x'),$$

or,

$$(y - y')\frac{dy'}{dx'} + x - x' = 0 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for the equation of the normal to any plane curve whose equation is expressed in Cartesian coördinates. Taking x' as the variable parameter and differentiating, we have

$$(y - y')\frac{d^2y'}{dx'^2} - \frac{dy'^2}{dx'^2} - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Solving (b) with respect to y and substituting in (a), we have,

$$y = y' + \frac{1 + \left(\frac{dy'}{dx'}\right)^2}{\frac{d^2y'}{dx'^2}},$$

$$x = x' - \frac{\left\{1 + \left(\frac{dy'}{dx'}\right)^2\right\}\frac{dy'}{dx'}}{\frac{d^2y'}{dx'^2}},$$

for the coördinates of *any* point on the envelope. But these values are identical with the coördinates of the center of curvature as found in § 138. Hence the envelope of the normals is the evolute of the curve.

EXAMPLES.

1. Find the equation of the envelope of the lines, $y = sx + \dfrac{p}{2s}$, s being the variable parameter.

Here $$u = y - sx - \frac{p}{2s} = 0;$$

hence, $$\frac{\partial u}{ds} = -x + \frac{p}{2s^2} = 0; \therefore s^2 = \frac{p}{2x};$$

$$\therefore y = \sqrt{\frac{px}{2}} + \sqrt{\frac{px}{2}},$$

i.e., $$y^2 = 2px,$$

a parabola. The given equation $y = sx + \dfrac{p}{2s}$ will be recognized as the slope form of the equation of the tangent to the parabola $y^2 = 2px$. See Ana. Geom., p. 96. The parabola is, of course, the envelope of its tangents.

2. Find the envelope of the family of lines represented by each of the following equations:

$$y = sx \pm \sqrt{s^2a^2 + b^2}, \qquad y = sx \pm \sqrt{s^2a^2 - b^2}.$$

Ans. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

3. The slope form of the equation of the normal to the parabola $y^2 = 4ax$ is $y = (x - 2a)s - as^3$; find the equation of the evolute of the curve. By § 148, the evolute is the envelope of the normal. We are required, therefore, to find the envelope of the series of lines represented by the equation $y = sx - 2as - as^3$, s being the variable parameter.

Here $$\frac{\partial u}{ds} = -x + 2a + 3as^2 = 0;$$

$$\therefore \qquad s = \sqrt{\frac{x - 2a}{3a}};$$

$$\therefore \quad y = (x - 2a)\sqrt{\frac{x - 2a}{3a}} - a\sqrt{\left(\frac{x - 2a}{3a}\right)^3}$$

$$= 3a\sqrt{\left(\frac{x - 2a}{3a}\right)^3} - a\sqrt{\left(\frac{x - 2a}{3a}\right)^3}$$

$$= 2a\sqrt{\left(\frac{x - 2a}{3a}\right)^3}$$

$$\therefore \; 27\, ay^2 = 4\,(x - 2a)^3,$$

which is the desired equation. We have previously deduced this equation by the direct method. See § 138 and figure 28.

4. The hypotenuse of a right triangle changes its position, its length remaining unaltered; find its envelope.

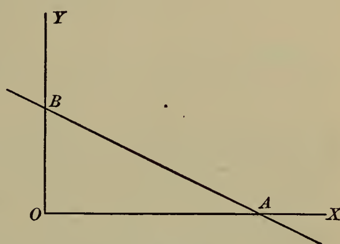


Fig. 33.

Let OBA be the triangle, BA being any one position of the hypotenuse.

Let $BA = c$, a constant, $OB = b$, $OA = a$. Then the equation of BA is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (m)$$

and by condition, $$a^2 + b^2 = c^2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (n)$$

Let us take a as the variable parameter. Ordinarily we would find the value of b in terms of a from the given condition, and substitute in the equation of the line, and then proceed as in preceding examples; but in this case, as in others with which we have had to deal, the simpler process is to substitute after differentiation.

Since b is a function of a, we have from (m),

$$\frac{\partial u}{da} = -\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \quad . \; . \; . \; . \; . \; . \quad (p)$$

from (n), $$2a + 2b\frac{db}{da} = 0;$$

$$\therefore \frac{db}{da} = -\frac{a}{b}.$$

This value in (p) gives, after reduction,

$$b^3x = a^3y \quad \ldots \ldots \ldots \quad (r)$$

We are now to eliminate a and b, having the relation (m), (n), and (r).

From (n) and (r) we find,

$$a = \frac{cx^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}, \quad b = \frac{cy^{\frac{1}{3}}}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}}.$$

These values in (m) give, after reduction,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

for the equation of the envelope. This curve is the four-cusped hypocycloid, and is generated by a point on the circumference of a circle as it rolls on the concave side of another circle whose diameter is four times that of the rolling circle. This problem was discussed by John Bernouilli in 1692.

5. Find the envelope of the lines $\frac{x}{a} + \frac{y}{b} = 1$, subject to the condition that $\sqrt{a} + \sqrt{b} = \sqrt{c}$, a constant.

Ans. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = c^{\frac{1}{2}}$.

6. Prove by direct process that the envelope of the lines $\frac{x}{a} + \frac{y}{b} = 1$, subject to the condition $ab = 2c$ is $2xy = c$, a hyperbola.

7. Find the envelope of all ellipses having a constant area (πc^2), the axes being coincident. *Ans.* $2xy = \pm c^2$.

CHAPTER XV.

SINGULAR POINTS.

HISTORY. — Joseph Saurin (1659–1737) was the first to show how the tangents at multiple points of curves could be determined by analysis.

Newton discussed double points in a plane and at infinity in his "Optics" (1704).

Rules for finding and discriminating multiple points were given by Maclaurin in his "Treatise of Fluxions" (1742).

149. Singular Points *are those points of a curve having some peculiar property not possessed by the other points of the curve.* Thus, the point of inflexion is *singular* in that it is the point where the direction of curvature changes, — a property not possessed by the other points of the curve. With this point we have already had to deal (§ 128). It is now our purpose to consider in order the more common of these singular points.

150. Multiple Points *are those points of a curve common to two or more of its branches.*

As the branches must either *pass through* the point or simply *meet* at the point, there are two classes:

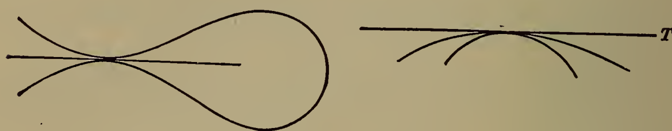
1. *The branches pass through the point.*

(1) (2)

Fig. 34.

(*a*) If the branches *intersect*, the point is called a **Point of Intersection**; and the point is a *double*, *triple*, or *quadruple* . . . point according as *two*, *three*, or *four* . . . branches pass through it. Figures 1 and 2 illustrate a *double* and a *triple* point. Since the curve has as many *tangents* at a point of intersection as there are branches, it is obvious that $\frac{dy}{dx}$ must have *two different values* at a double point; *three different values* at a triple point; and so on.

(*b*) If the branches *touch* as they pass through the point, it is called a **Point of Osculation**; and this point is of the *first species* or *second species*, according as the branches lie on *opposite* or on the *same* side of their common tangent.



(3) (4)

Fig. 34.

Thus, Figs. (3) and (4) are illustrations of osculating points of the first and second species. Here $\frac{dy}{dx}$ has *two equal* values.

2. *The branches meet at the point.*

(*a*) If the branches have a *common tangent* at the point, the point is a **Cusp** of the *first* or *second* species, according as the branches lie on *opposite* or on the *same* side of the tangent.

(5) (6)

Fig. 34.

Here, also, $\frac{dy}{dx}$ has *two equal* values.

(*b*) If the branches have *different* tangents at the point the point is called a **Point Saillant, or Shooting-Point.**

Here $\frac{dy}{dx}$ has *two different* values, It may be remarked, however, that shooting-points occur only in loci whose equations are transcendental. Such points are usually determined by inspection.

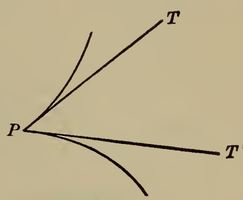


Fig. 34 (7).

151. Isolated *or* **Conjugate Points** are those points which are *isolated* from the curve, but whose coördinates satisfy its equation.

As the curve has *no* direction at an isolated point, $P(x, y)$, Fig. 8, it is obvious that $\frac{dy}{dx}$ has an imaginary value at such a point. But imaginary values arise from the presence of radicals with even indices; hence, if $\frac{dy}{dx}$ has one imaginary value it has necessarily *two* such values.

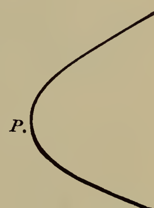


Fig. 34 (8).

152. The **Point d'Arrêt,** or **Stop Point,** is a point at which a branch of a curve stops. This point, peculiar to transcendental curves, is usually determined by inspection.

153. Investigation for Singular Points. Let $u = f(x, y) = 0$ be the *rationalized* equation of any plane locus; then § 121, Cor. 1,

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{dx}}{\frac{\partial u}{dy}}.$$

We have found (§§ 150, 151) that $\frac{dy}{dx}$ must have *more than one value* in all cases of multiple and isolated points. But, since differentiation of a *rational* equation cannot give rise to an *irrational* expression, $\frac{dy}{dx}$ can have only *one* value for any given point (x', y') unless

$$\frac{dy}{dx} = \frac{0}{0};$$

i.e., unless $$\frac{\partial u}{dx} = 0, \text{ and } \frac{\partial u}{dy} = 0.$$

Hence, since the point (x', y') satisfies the equation, $f(x, y) = 0$,

if $$u = 0, \quad \frac{\partial u}{dx} = 0, \quad \frac{\partial u}{dy} = 0,$$

for that point then it *may* be a multiple or isolated point; i.e., x' and y' are critical values which require investigation. Further investigation consists in evaluating the expression

$$\left.\frac{dy}{dx}\right]_{(x', y')} = -\left.\frac{\dfrac{\partial u}{dx}}{\dfrac{\partial u}{dy}}\right]_{(x', y')} = \frac{0}{0}.$$

Referring now to the figures and definitions of §§ 150, 151, we see:

(*a*) If $\frac{dy'}{dx'}$ has two or more *unequal* values, *and* y is real for $x = x' \pm h$, h being a small increment, the point is a *point of intersection*. See § 150, 1, (*a*).

(*b*) If $\frac{dy'}{dx'}$ has two or more *unequal* values, and y is real for $x = x' + h$ (or $x = x' - h$), and imaginary for $x = x' - h$ (or $x = x' + h$), the point is a *shooting-point.* § 150, 2, (*b*).

(*c*) If $\frac{dy'}{dx'}$ has two or more equal values, and y is real for $x = x' \pm h$, the point is an *osculating-point.* § 150, 1, (*b*).

(*d*) If $\frac{dy'}{dx'}$ has two or more equal values, and y is real for $x = x' + h$ (or $x = x' - h$) and imaginary for $x = x' - h$ (or $x = x' + h$), the point is a *cusp*. § 150, 2, (*a*).

To determine in the last two cases, (*c*) and (*d*), whether the point is of the *first* or *second species* (Figures (3), (4), (5), (6)), usually the simplest way is to write the equation of the tangent $y - y' = \frac{dy'}{dx'}(x - x')$, and compare the ordinates of the tangent for $x = (x' + h)$ or $x = x' - h$ with those of the curve for the same abscissa. If the ordinate of the tangent exceeds (or is less than) the corresponding ordinates of the curve, the point is of the *second* species; otherwise it is of the *first* species.

(*e*) If $\frac{dy'}{dx'}$ has an imaginary value, and y is real when $x = x'$, the point is *isolated*.

While the above statement is true, yet the converse, viz., that at an isolated point (x', y'), $\frac{dy'}{dx'}$ is imaginary, is not necessarily true. For at such a point y is necessarily imaginary, when $x = x' \pm h$ (h being some small quantity). Now, by Taylor's Theorem, we have

$$y = f(x' \pm h) = f(x') \pm f'(x')h + f''(x')\frac{h^2}{\lfloor 2} \pm \cdots$$

Hence, y is imaginary when *any* derivative in the second member is imaginary.

Hence, y may be imaginary while $\frac{dy}{dx}$ $(= f'(x'))$ may be real.

In examining, therefore, a given curve for isolated points the simplest and most satisfactory test is, after determining the critical values as above explained, to substitute values $(x' \pm h)$ a little less and a little greater than these in the original equation, and ascertain if they render y imaginary.

EXAMPLES.

1. Investigate the lemniscata for singular points.

Here, $$u = (x^2 + y^2)^2 - a^2(x^2 - y^2) = 0,$$

and $$\frac{\partial u}{dx} = 4x(x^2 + y^2) - 2a^2x = 0,$$

$$\frac{\partial u}{dy} = 4y(x^2 + y^2) + 2a^2y = 0.$$

Solving the last two equations, we find,

$$(0, 0), \left(\frac{a}{2}\sqrt{2}, 0\right), \left(-\frac{a}{2}\sqrt{2}, 0\right),$$

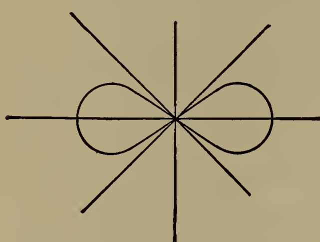


Fig. 35.

to be critical points. Of these, however, the first (0, 0) only satisfies the condition $u = 0$ (§ 153). We are, therefore, to evaluate the first derivative $\frac{dy}{dx}$ which takes the illusory form $\frac{0}{0}$ for the point (0, 0).

That is,

$$\left.\frac{dy}{dx}\right]_{0,0} = -\left.\frac{\frac{\partial u}{dx}}{\frac{\partial u}{dy}}\right]_{0,0} = -\left.\frac{4x^3 + 4xy^2 - 2a^2x}{4x^2y + 4y^3 + 2a^2y}\right]_{0,0} = \frac{0}{0};$$

hence,

$$\left.\frac{dy}{dx}\right]_{0,0} = -\left.\frac{12x^2 + 4\left(2xy\frac{dy}{dx} + y^2\right) - 2a^2}{4\left(x^2\frac{dy}{dx} + 2yx\right) + 12y^2\frac{dy}{dx} + 2a^2\frac{dy}{dx}}\right]_{0,0} = \frac{2a^2}{2a^2\frac{dy}{dx}};$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 1 \text{ when } x = 0 \text{ and } y = 0;$$

$$\therefore \frac{dy}{dx} = \pm 1.$$

Therefore the curve has two non-coincident tangents at the origin. Therefore, since the equation of the curve is algebraic, the origin is a double point of intersection. The values of the derivative (± 1) show that the tangents are inclined at angles of $45°$ and $135°$ to the x-axis.

2. Show that the curve $y^2 = x^4(1 - x^2)$ has a double point of osculation of the first species at the origin.

Here,
$$u = y^2 - x^4 + x^6 = 0,$$
$$\frac{\partial u}{dx} = 6x^5 - 4x^3 = 0,$$
$$\frac{\partial u}{dy} = 2y = 0.$$

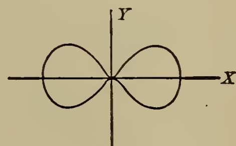


Fig. 36.

We see that the partial derivatives give (0, 0), a point of the curve, as a critical point;

hence,
$$\frac{dy}{dx} = -\left.\frac{6x^5 - 4x^3}{2y}\right]_{0,0} = \frac{0}{0};$$

$$\therefore \quad \frac{dy}{dx} = -\left.\frac{30x^4 - 12x^2}{2\dfrac{dy}{dx}}\right]_{0,0} = \frac{0}{2\dfrac{dy}{dx}}.$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 0. \qquad \therefore \frac{dy}{dx} = \pm 0 \text{ for the point } (0, 0).$$

Hence, the curve has two coincident tangents at the origin, which coincide with the x-axis. Hence the point (0, 0) is a double point of osculation, or a cusp.

Resuming the equation, $y^2 = x^4 - x^6$, we have,

$$y = \pm x^2\sqrt{1 - x^2},$$

which, for all values of x less, numerically, than 1, give real values for y; hence § 153, (c), (0, 0) is a point of osculation.

Again, since the equation shows that the curve is symmetrical, with respect to the x-axis, i.e., to its tangent at the origin, the origin is a point of osculation of the *first species.*

3. Show that the curve $y^2 = x^3(1-x)$ has a cusp of the first species at the origin.

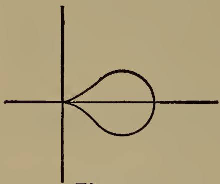


Fig. 37.

$$u = y^2 - x^3 + x^4 = 0,$$

$$\frac{\partial u}{dx} = -3x^2 + 4x^3 = 0,$$

$$\frac{\partial u}{dy} = 2y = 0.$$

$\therefore$ (0, 0) is a critical point.

$$\frac{dy}{dx} = -\left.\frac{\frac{\partial u}{dx}}{\frac{\partial u}{dy}}\right]_{0,0} = -\left.\frac{4x^3 - 3x^2}{2y}\right]_{0,0} = \frac{0}{0}.$$

$$\frac{dy}{dx} = -\left.\frac{12x^2 - 6x}{2\frac{dy}{dx}}\right]_{0,0} = \frac{0}{2\frac{dy}{dx}};$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 0. \quad \therefore \frac{dy}{dx} = \pm 0, \text{ at the origin.}$$

Therefore, at the origin (0, 0), the curve has two tangents coinciding with the x-axis.

From the equation of the curve, we have,

$$y = \pm\sqrt{x^3(1-x)}.$$

Hence, since x cannot be negative, the curve is situated in the first and fourth quadrants, and is symmetrical with respect to the x-axis. Hence the origin is a cusp of the first species.

4. Show that the cissoid $(2a - x)y^2 = x^3$ has a cusp of the first species at the origin.

5. Show that $(y - x^2)^2 = x^5$ has a cusp of second species at the origin.

6. Show that the semi-cubic parabola $ay^2 = x^3$ has a cusp of the first species at the origin.

7. Show that the cycloid $x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2\,ay - y^2}$ has an infinite number of cusps of the first species.

8. Show that the curve $a\,(x^2 + y^2) = x^3$ has a conjugate point at the origin.

Fig. 38

$$u = ax^2 + ay^2 - x^3 = 0.$$

$$\therefore \quad \frac{\partial u}{dx} = 2\,ax - 3\,x^2 = 0.$$

$$\frac{\partial u}{dy} = 2\,ay.$$

$\therefore$ (0, 0) is a critical point.

$$\frac{dy}{dx} = \frac{3\,x^2 - 2\,ax}{2\,ay}\Bigg]_{0.0} = \frac{6\,x - 2\,a}{2\,a\dfrac{dy}{dx}}\Bigg]_{0.0} = -\frac{1}{\dfrac{dy}{dx}}.$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = -1, \text{ or } \frac{dy}{dx} = \pm\sqrt{-1}.$$

Hence, the origin is a conjugate point. See (*e*) § 153.
Otherwise, thus: Solving the equation, we have,

$$y = \pm\,x\sqrt{\frac{x-a}{a}}.$$

This equation is satisfied for the point (0, 0). But y is imaginary for *any* negative value of x and for any positive value of x less than a; hence (0, 0) is isolated from the curve.

9. The curve $y^2 = x\,(x + a)^2$ has a conjugate point at $(-\,a, 0)$.

10. The origin is a conjugate point of the curve $y^2\,(x^2 - a^2) = x^4$.

11. Show that the point $(a, 0)$ is a conjugate point of the curve $ay^2 - x^3 + 4\,ax^2 - 5\,a^2x + 2\,a^3 = 0$.

12. Show that the curve $a^3y^2 - 2\,abx^2y - x^5 = 0$ has a double point of osculation at the origin and that one branch of the curve has a point of inflexion at that point.

13. Show that the curve $y = x\cot^{-1}x$ has a *point saillant* at the origin.

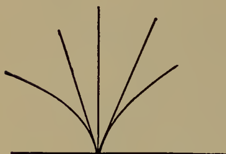


Fig. 39.

Since y is positive, and has only one value for any value of x, positive or negative, the curve lies in the first and second angles; and since $x = 0$ gives $y = 0$, the curve passes through the origin.

Here $$\frac{dy}{dx} = \cot^{-1}x - \frac{x}{1 + x^2}.$$

If we suppose x to approach 0 from the positive direction, we have,

$$\frac{dy}{dx} = \cot^{-1}x - \frac{x}{1 + x^2}\bigg]_0 = \cot^{-1}0 = \frac{\pi}{2} = 1.57.$$

If we suppose x negative and approaching 0, we have

$$\frac{dy}{dx} = \cot^{-1}(-x) + \frac{x}{1 + x^2}\bigg]_0 = \cot^{-1}(-0) = -\frac{\pi}{2} = -1.57.$$

Hence there are two non-coincident tangents to the curve at the origin. Hence the origin is a *point saillant.* § 150.

14. Show that the curve $y - x + ye^{\frac{1}{x}} = 0$ has a point saillant at the origin.

15. Show that the curve $y - e^{-\frac{1}{x}} = 0$ has a point d'arrêt at the origin.

16. Show that the curve $y = x\log x$ has a point d'arrêt at the origin.

17. $x^4 - ax^2y - axy^2 + a^2y^2 = 0.$

A conjugate point at (0, 0).

18. $x^4 + 2ax^2y - ay^3 = 0$.

A triple point of intersection at (0, 0).

19. $ay^2 = (x - a)^2 (x - b)$.

At $x = a$ there is a conjugate point, a double point or a cusp according as $a < b$, $a > b$ or $a = b$.

20. Examine the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ for cusps.

CHAPTER XVI.

LOCI.

154. In tracing curves in Analytic Geometry we usually solve the equation of the curve with respect to one of the variables that enter it; then assigning values to the variable in the second member we determine the values of the other. A smooth curve traced through the points thus determined we call the locus of the equation. This process is at best only approximate and is limited in its application to those curves whose equations are of lower degrees. In equations of higher degrees the difficulty is even greater as we can only determine approximately the *positions* of the series of points. By the aid of the Differential Calculus we are enabled to determine the *singularities* of the locus from its equation and from these to obtain a general idea of its form. We have seen in the preceding chapter, for example, how to investigate any locus for singular points; in § 127, how to determine the *direction* of curvature; in §§ 73, 79, how to determine whether or not the curve has asymptotes and if so, to determine their equations, etc. We propose to treat a few curves in this general manner, and to indicate an order of procedure that will enable the student to enter upon an intelligent investigation of any equation with which he may have to deal.

ALGEBRAIC EQUATIONS.

155. Suggestions.

1. Determine as far as possible the form and properties of the locus from its equation.

2 Deduce the first and second derivatives from the equation and investigate.

(*a*), for asymptotes. Cf. § 73.
(*b*), for maxima and minima points. Cf. § 114.
(*c*), for singular points. Cf. § 153.
(*d*), for direction of curvature. Cf. § 127.

EXAMPLES.

1. Trace the curve $y = \dfrac{a^2x}{(x-a)^2}$.

Here $x = 0$ gives $y = 0$; hence the curve passes through the origin. As x approaches the value a, y approaches an infinite value; hence $x = a$ is the equation of an asymptote to the curve, § 74. Again as x increases numerically, and approaches positive or negative infinity, y decreases and approaches 0 as a limit; hence $y = 0$, or the x-axis is an asymptote to the two branches of the curve, one extending infinitely in the first angle and the other infinitely in the third angle. From the given equation we have

$$\frac{dy}{dx} = -\frac{a^2(x+a)}{(x-a)^3} \quad . \; . \; . \quad (a); \quad \frac{d^2y}{dx^2} = \frac{2a^2(x+2a)}{(x-a)^4} \quad . \; . \; . \quad (b)$$

Here $\dfrac{dy}{dx} = 0$ gives $x = -a$, and this value in (*b*) gives $\dfrac{d^2y}{dx^2} = \dfrac{1}{8a}$, a positive quantity; hence at the point whose abscissa is $-a$, y is a minimum.

Making $x = 0$ in (*a*) we find $\dfrac{dy}{dx} = 1$; hence the tangent at the origin (since $x = 0$ gives $y = 0$) makes an angle of 45° with the x-axis.

Placing $\dfrac{d^2y}{dx^2} = 0$, we find $x = -2a$. Since (*b*) changes sign as x passes through this value, $x = -2a$ is the abscissa of a point of inflexion. Again, since $\frac{d^2y}{dx^2}$ is positive or negative according as x is algebraically greater or less than $-2a$, the curve is concave upward between the limits $x = -2a$ and $x = \infty$, and concave downward between the limits $x = -2a$ and $x = -\infty$. Reviewing the facts elicited we are enabled to trace the curve as in the figure. It may be remarked in passing that an asymptote to an algebraic curve is always approached by two infinite branches. The hyperbola affords a familiar illustration.

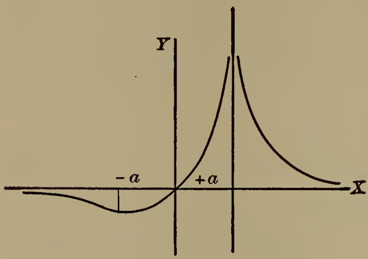


Fig. 40.

2. Trace the curve $y^2 = x^2(x - a)$.

Here $x = 0$ gives $y = 0$; hence the curve passes through the origin.

Again, all negative values of x, and all positive values of x less than a, render y imaginary; hence the origin is an isolated point.

All positive values of x greater than a give two values of y equal numerically with contrary signs; hence the curve is symmetrical with respect to the x-axis and extends indefinitely in the direction of positive abscissas from the limit $x = a$. When $x = a, y = 0$; hence the curve cuts the x-axis at the point $(a, 0)$.

Here

$$\frac{dy}{dx} = \frac{3x - 2a}{2\sqrt{x - a}} \quad . \quad . \quad . \quad (a); \qquad \frac{d^2y}{dx^2} = \frac{3x - 4a}{4\sqrt{(x - a)^3}} \quad . \quad . \quad . \quad (b)$$

When $x = a$, $\frac{dy}{dx} = \infty$; hence the tangent to the curve where it crosses the x-axis is perpendicular to that axis.

Since (a) does not change sign as x increases, the curve has no maximum or minimum points.

Since (b) changes sign as x passes through the value $\frac{4}{3}a$ there are points of inflexion corresponding to this value of x.

If we take the positive sign of the radical in the denominator of (b) we find that $\frac{d^2y}{dx^2}$ is negative between the limits $x = a$ and $x = \frac{4}{3}a$, and positive for all values of x greater than $\frac{4}{3}a$; if we take the negative value of the radical we find the reverse is true. Hence the curve is concave *toward* the x-axis between the limits $x = a$ and $x = \frac{4}{3}a$, and convex toward that axis when $x > \frac{4}{3}a$.

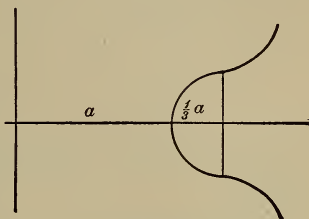


Fig. 41.

3. Trace the curve $a^3y^2 = 2\,a^2x^2y + x^5$.

Solving we have $y = \frac{x^2}{a^2}(a \pm \sqrt{a\,(x + a)})$.

$x = 0$ gives $y = 0$; hence the origin is a point of the curve.

For all *positive* values of x, y has two real values of opposite signs; hence the curve extends indefinitely in the first and fourth angle. $x = -a$, $y = a$, and for all negative values of x between the limits $x = 0$ and $x = -a$, y has *two positive* unequal values; for negative values of x greater numerically than $-a$ y is imaginary; hence the curve has a loop in the *second* angle.

Let $u = a^3y^2 - 2a^2x^2y - x^5 = 0$; then

$$\frac{dy}{dx} = \frac{4\,a^2xy + 5\,x^4}{2\,a^3y - 2\,a^2x^2} = \frac{0}{0} \text{ for the point } (0, 0).$$

Evaluating, $$\frac{dy}{dx} = \left.\frac{4\,a^2x\frac{dy}{dx} + 4\,a^2y + 20\,x^3}{2\,a^3\frac{dy}{dx} - 4\,a^2x}\right]_{0,\,0} = \frac{0}{2\,a^3\frac{dy}{dx}};$$

hence, $$\frac{dy}{dx} = \pm 0;$$

∴ The origin is a point of osculation, the x-axis being a

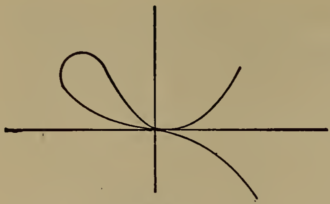


Fig. 42.

common tangent to the two branches. But we have seen above that one branch of the curve *crosses* the x-axis at the origin (i.e., the curve has been shown to pass from the *second* to the fourth angle through the origin); hence the origin is also a point of inflexion. Such a point is called a point of *oscul-*inflexion.

4. $y^3 = 2\,ax^2 - x^3$.

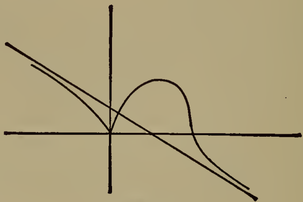


Fig. 43.

5. $y + x^2y = x$.

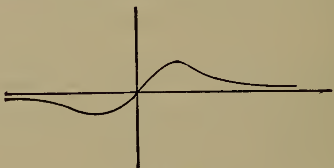


Fig. 44.

6. $x^3 - 2\,x^2y - 2\,x^2 - 8\,y = 0$.

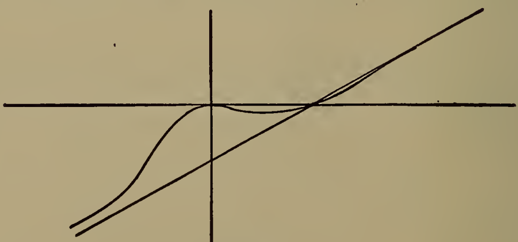


Fig. 45.

7. $y(x - a) = x(x - 2a)$.
8. $ay^2 = x^4 + x^5$.
9. $y(a - x) = x^2(a + x)$.
10. $y = x^2(1 - x^2)^3$.
11. $y^2(x^2 - a^2) = x^4$.
12. $y^3 = a^3 - x^3$.
13. $y(a^2 - x^2) = a^3$.

POLAR EQUATIONS.

156. Suggestions.

1. Determine as far as possible the form and properties of the curve from its equation.

2. Deduce the first derivative of r with respect to θ from the equation of the curve.

(*a*) Investigate for asymptotes. Cf. § 79.

(*b*) Investigate for maximum and minimum points. Cf. § 112.

(*c*) Investigate for points of inflexion. Cf. § 131.

(*d*) Investigate for direction of curvature. Cf. § 130.

EXAMPLES.

1. Trace the curve $r = a \sin 3\theta$.

$r = 0$, when $\theta = 0°, \theta = 60°, \theta = 120°$, $\theta = 180°$, etc.; hence, the curve repeatedly passes through the origin.

$r = a$ (a maximum value, since $\sin 3\theta$ cannot exceed unity) when $\theta = 30°$, $\theta = 150°$, $\theta = 270°$.

$r = -a$, a minimum value, when $\theta = -30°$, $\theta = -150°$, $\theta = -270°$.

As θ increases from 0° to 30°, r increases from 0 to a; as θ increases from 30° to 60°, r decreases from a to 0; hence, the curve has a loop in the first angle.

As θ increases from 60° to 90°, r decreases from 0 to $-a$; as θ increases from 90° to 120°, r increases from $-a$ to 0; hence, the curve has a similar loop to the first, situated partially in the third angle and partially in the fourth angle.

As θ increases from $120°$ to $150°$, r increases from 0 to a; as θ increases from $150°$ to $180°$, r diminishes from a to 0; hence, there is a loop in the second angle.

As θ increases from $180°$ to $360°$, the corresponding values of r are the same in magnitude and direction as those already indicated.

Here, $$\frac{dr}{d\theta} = 3a \cos 3\theta.$$

Since $\frac{dr}{d\theta} = 3a\cos 3\theta = 0$ when $\theta = 30°$, $\theta = 150°$, $\theta = 270°$ and since $\frac{dr}{d\theta}$ changes sign as θ passes through these values it follows that r is a maximum for these values of θ, — a fact already ascertained from the equation.

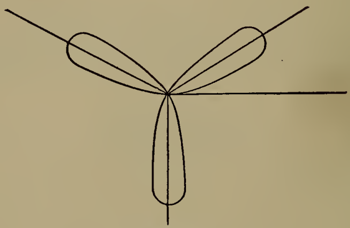


Fig. 46.

2. $r = a\sin 2\theta$.

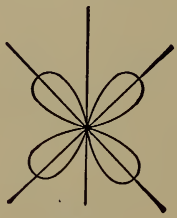


Fig. 47.

3. $r^3 = a^3 \cos^4 \frac{3}{4}\theta.$

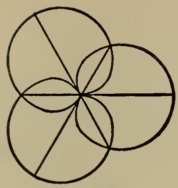


Fig. 48.

4. $r^4 = a^5 \cos^5 \frac{4}{5}\theta.$

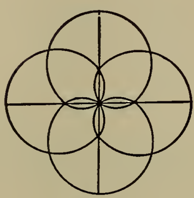


Fig. 49.

5. $r = a \sec \frac{\theta}{3}.$

From the equation we readily see that the curve is of the general form given in the figure.

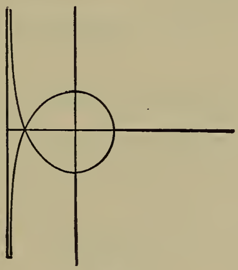


Fig. 50.

Here $$\frac{dr}{d\theta} = \frac{a}{3}\sec\frac{\theta}{3}\tan\frac{\theta}{3} = \frac{a}{3}\frac{\sin\frac{\theta}{3}}{\cos^2\frac{\theta}{3}};$$

$$\text{Subtangent} = r^2\frac{d\theta}{dr} = 3\,a\csc\frac{\theta}{3}.$$

When $\theta = 270°$, we have,

$$r = \infty,$$

and $$\text{Subtangent} = 3a;$$

hence, a line perpendicular to the initial line, and at a distance $3a$ to the left of the pole, is an asymptote to the curve.

6. $r = 2a \tan\theta \sin\theta$. (Cissoid).

Hence, $$r = 2a\frac{\sin^2\theta}{\cos\theta} = \frac{2a}{\cos\theta} - 2a\cos\theta.$$

$\theta = 0°$, $r = 0$; and as θ increases, r increases.

When $\theta = 90°$, $r = \infty$.

As θ decreases from $0°$ to $-90°$, r increases from 0 to ∞.

From the equation above, we have

$$\frac{dr}{d\theta} = \frac{2a\sin\theta(2\cos^2\theta + \sin^2\theta)}{\cos^2\theta}.$$

Hence,

$$\text{Subtangent} = r^2\frac{d\theta}{dr} = \frac{2a\sin^3\theta}{2 - \sin^2\theta}.$$

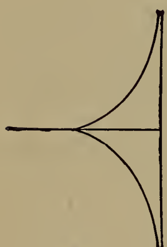


Fig. 51.

When $\theta = 90°$, $r = \infty$, and $r^2\frac{d\theta}{dr} = 2a$; hence, a line $\perp$ to the initial line, and at a distance $2a$ to the right of the pole, is an asymptote. Tracing the curve from the above data, we find it as in the figure.

7. $r = a\sec\theta \pm a$. (Conchoid.)

8. $r = a\sin\frac{\theta}{2}$.

9. $r = a\cos 2\theta$.

10. $r = a\cot\theta\cos\theta$.

INTEGRAL CALCULUS.

CHAPTER I.

TYPE FORMS.

HISTORY. — The Integral Calculus may be said to have taken its origin from methods employed by Cavalieri, Wallis, and others for the determination of quadrature of curves and cubature of solids. The processes thus employed were developed and reduced to a suitable notation by Newton and Leibnitz.

The term "integral" was first used by James Bernouilli (1654–1705).

157. The Integral Calculus is the inverse of the Differential; and its fundamental object is to determine the function — the relation between the rates or differentials of the variables which enter it being given.

158. Integral. Integration. A function is termed the *integral* of its differential, and the *process* by means of which it is derived is termed *integration*.

The process of integration is simply a reversion of the process of differentiation; hence no new philosophical principle is involved in the process. Thus, since

$3x^2dx$ is the differential of x^3,

x^3 is the integral of $3x^2dx$.

Again: $\frac{dx}{x}$ is the differential of $\log x$;

hence, $\log x$ is the integral of $\frac{dx}{x}$.

159. Notation. The operation of integration is denoted by the symbol $\int$, read "integral of."

Thus, in the examples of § 158, we write

$$\int 3x^2 dx = x^3,$$

$$\int \frac{dx}{x} = \log x.$$

Since integration and differentiation involve inverse operations the symbols $\int$ and d neutralize each other.

160. Indefinite Integrals. Constant of Integration.

Since

$$d(x^3 + 5) = 3x^2 dx,$$
$$d(x^3 - 3) = 3x^2 dx,$$
$$d(x^3 \pm c) = 3x^2 dx,$$

and, in general,

$$d(f(x) + c) = f'(x)\,dx,$$

it follows that,

$$\int 3x^2 dx = x^3 + c,$$

$$\int f'(x)\,dx = f(x) + c,$$

where c is some *indefinite* constant. The constant c is called the **Constant of Integration**; and as its value is in general unknown, the integral of which it forms a part is **indefinite.** While the process of integration gives, it seems, an indefinite result, yet in the *practical application* of the process the *data* of the problem will enable us to determine the value of c, or to eliminate it altogether, and thus enable us to render the result definite.

To avoid useless repetition we shall omit the constant of integration in what follows. The student must bear in mind, however, that it is to be *understood* as entering *every* integral expression.

161. Elementary principles.

1. *The integral of the sum of any number of differentials is the sum of their integrals.* Cf. § 24, (1).

Since $d(u \pm v) = du \pm dv \therefore \int d(u \pm v) = \int (du \pm dv)$;

i.e., $$\int (du \pm dv) = u \pm v.$$

But $$\int du \pm \int dv = u \pm v;$$

$$\therefore \int (du \pm dv) = \int du \pm \int dv \qquad (a)$$

The symbol of integration is, therefore, *distributive.*

2. *A constant factor may be placed before or after the integral sign.* Cf. § 25, Cor. 1.

Since $d(cu) = cdu \therefore \int d(cu) = \int cdu$;

i.e., $$\int cdu = cu.$$

But $$c\int du = cu;$$

$$\therefore \int cdu = c\int du \qquad (b)$$

162. Type Formulae.

1. *The integral of a variable with a constant exponent into the differential of the variable is the variable with an exponent increased by one divided by the increased exponent.* Cf. § 27, (9).

Since $d(u^{n+1}) = (n+1)u^n du \therefore \int d(u^{n+1}) = (n+1)\int u^n du$;

i.e., $$(n+1)\int u^n du = u^{n+1};$$

$$\therefore \int u^n\, du = \frac{u^{n+1}}{n+1} \qquad (1)$$

2. *The integral of a fractional expression in which the numerator is the differential of the denominator is the logarithm of the denominator.* Cf. § 33, (14).

Since $$d(\log u) = \frac{du}{u}$$

$$\therefore \int \frac{du}{u} = \log u \qquad (2)$$

Schol. Since

$$\int \frac{du}{u} = \int u^{-1}\,du = \frac{u^0}{0} = \infty,$$

we see that formula (1) does *not* apply when $n = -1$. Such expressions, therefore, as $\int u^{-1}\,du$ should be placed in a fractional form $\int \frac{du}{u}$ and formula (2) applied.

3. *The integral of a constant with a variable exponent into the differential of the variable is the constant affected with the same exponent divided by the logarithm of the constant.* Cf. § 33, (13).

Since $d(a^u) = a^u \log a du \;\therefore\; \int d(a^u) = \log a \int a^u du$;

i.e., $$\log a \int a^u du = a^u;$$

$$\therefore \int a^u\,du = \frac{a^u}{\log a} \qquad (3)$$

Cor. If $a = e$, we have from (3),

$$\int e^u\,du = e^u \qquad (4)$$

This also follows directly from the fact that $d(e^u) = e^u du$.

EXAMPLES.

The numbers and letters which follow the examples refer to the formulae of §§ 161, 162.

1. $\int x^3 dx = \frac{x^4}{4}$. (1)

2. $\int \frac{dx}{\sqrt{x}} = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2x^{\frac{1}{2}}$. (1)

3. $\int \frac{dx}{x} = \log x$. (2)

4. $\int (a + bx)^2 b dx = \frac{(a + bx)^3}{3}$. (1)

5. $\int (a + bx)^2 dx = \frac{1}{b}\int (a + bx)^2 b dx = \frac{(a + bx)^3}{3b}$. (b), (1)

6. $\int \frac{bdx}{a + bx} = \log (a + bx)$. (2)

7. $\int (x^2 + 1)^2 x dx = \frac{(x^2 + 1)^3}{6}$. (b), (1)

8. $\int a^{3x} dx = \frac{1}{3}\int a^{3x} 3 dx = \frac{a^{3x}}{3 \log a}$. (b), (3)

9. $\int e^{\frac{3}{2}x} dx = \frac{2}{3} e^{\frac{3}{2}x}$. (b), (4)

10. $\int (a + b)^x dx = \frac{(a + b)^x}{\log (a + b)}$. (3)

11. $\int \left\{ \frac{adx}{x} + \frac{bdx}{x^2} + \frac{cdx}{x^3} - \frac{5\, dx}{x^4} \right\} = a\int \frac{dx}{x} + b\int x^{-2} dx$

$+ c\int x^{-3} dx - 5\int x^{-4} dx = a \log x - \frac{b}{x} - \frac{c}{2x^2} + \frac{5}{3x^3}$.

12. $\int (1 + x^2)(1 + x)\,x\,dx = \int \{x + x^2 + x^3 + x^4\}\,dx$

$= \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}.$

13. $\int \frac{x^3 - 6x^2 + 12x - 8}{x^2}\,dx = \int \left\{x - 6 + \frac{12}{x} - \frac{8}{x^2}\right\} dx$

$= \frac{x^2}{2} - 6x + 12 \log x + \frac{8}{x}.$

14. $\int \frac{4x^3 dx}{\sqrt{m + x^4}} = \int (m + x^4)^{-\frac{1}{2}} 4x^3 dx = 2\sqrt{m + x^4}.$

15. $\int c(4mx + 3a^2x^3)^{\frac{4}{3}} (4m + 9a^2x^2)\,dx = \frac{3}{7} c(4mx + 3a^2x^3)^{\frac{7}{3}}.$

16. $\int (a^{2x} + e^{\frac{1}{2}x})\,dx = \frac{a^{2x}}{2 \log a} + 2e^{\frac{1}{2}x}.$

17. $\int (e^{ax} + e^{\frac{x}{a}})\,dx = \frac{e^{ax}}{a} + ae^{\frac{x}{a}}.$

18. $\int a^x e^x\,dx = \frac{a^x e^x}{1 + \log a}.$

19. $\int \frac{dx}{x \log x} = \int \frac{\frac{dx}{x}}{\log x} = \log(\log x).$ (2)

20. $\int \frac{\log x\,dx}{x} = \int \log x \frac{dx}{x} = \frac{\log^2 x}{2}.$ (1)

21. $\int \frac{(e^x + 1)^2}{\sqrt{e^x}}\,dx = \frac{2}{\sqrt{e^x}}\left(\frac{e^{2x}}{3} + 2e^x - 1\right).$

22. $\int \log^m x \frac{dx}{x} = \frac{(\log x)^{m+1}}{m + 1}.$

23. $\int \frac{5bx\,dx}{3m - 6bx^2} = -\frac{5}{12} \log(3m - 6bx^2) = \log \frac{1}{\sqrt[12]{(3m - 6bx^2)^5}}.$

24. $\int 3\, a^{x^2} x\,dx = \frac{3}{2}\,\frac{a^{x^2}}{\log a}.$

25. $\int m e^{e^x} e^x dx = m e^{e^x}.$

26. $\int (a^2 - x^2)^{\frac{3}{2}} x\,dx = -\frac{1}{2}\int (a^2 - x^2)^{\frac{3}{2}}(-2\,x\,dx) = -\frac{(a^2 - x^2)^{\frac{5}{2}}}{5}.$

27. $\int \frac{3\,x^2 + 2\,x + 1}{x^3 + x^2 + x + 1}\,dx = \log(x^3 + x^2 + x + 1).$

28. $\int \frac{(x^m - a^m)^2 dx}{x} = \frac{x^m}{2\,m}(x^m - 4\,a^m) + a^{2m}\log x.$

29. $\int \frac{a\,(x - a)^3}{bx^2}\,dx = \frac{a}{b}\left\{\frac{x^2}{2} - 3\,ax + 3\,a^2 \log x + \frac{a^3}{x}\right\}.$

30. $\int \frac{\frac{1}{3}\,x\,dx}{x^2 + \frac{3}{4}} = \frac{1}{6}\log(x^2 + \frac{3}{4}).$

31. $\int 2\,ax\left(\frac{x^2}{p^2} + 1\right)^{\frac{1}{2}} dx = \frac{2\,a}{3\,p}(p^2 + x^2)^{\frac{3}{2}}.$

32. $\int \frac{x^2 + 1}{x - 1}\,dx = \frac{x^2}{2} + x + 2\log(x - 1).$

33. $\int \frac{x^{n-1}dx}{(a + bx^n)^m} = \frac{(a + bx^n)^{1-m}}{bn\,(1 - m)}.$

34. $\int \frac{dx}{(mx)^{\frac{m-1}{m}}} = \frac{1}{m^{\frac{m-1}{m}}}\int x^{\frac{1-m}{m}}\,dx = (mx)^{\frac{1}{m}}.$

35. $\int \frac{x^{m-1} + 1}{(x^m + mx)^{\frac{1}{2}}}\,dx = \int (x^m + mx)^{-\frac{1}{2}}(x^{m-1} + 1)\,dx$

$= \frac{2\sqrt{x^m + mx}}{m}.$

36. $\displaystyle\int \frac{x^{m-1}+1}{x^m+mx}\,dx = \frac{\log(x^m+mx)}{m}.$

37. $\displaystyle\int \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}\,dx = x+\sqrt{x^2+4}.$

38. $\displaystyle\int \frac{(\sqrt{x^2+4}+x)^2}{\sqrt{x^2+4}}\,dx = \frac{(x+\sqrt{x^2+4})^2}{2}.$

39. $\displaystyle\int \frac{2x-1}{2x+3}\,dx = \int\left\{1-\frac{4}{2x+3}\right\}dx = \int dx - 2\int\frac{2\,dx}{2x+3}$
$= x - \log(2x+3)^2.$

40. $\displaystyle\int \frac{2\,a dx}{x\sqrt{2\,ax-x^2}} = \int (2\,ax-x^2)^{-\frac{1}{2}}\,2\,ax^{-1}dx$

$\displaystyle = \int (2\,ax^{-1}-1)^{-\frac{1}{2}}\,2\,ax^{-2}dx = -\frac{2\sqrt{2\,ax-x^2}}{x}.$

41. $\displaystyle\int \frac{xdx}{(2\,ax-x^2)^{\frac{3}{2}}} = \int (2\,a-x)^{-\frac{3}{2}}x^{-\frac{1}{2}}dx$

$\displaystyle = \int (2\,ax^{-1}-1)^{-\frac{3}{2}}x^{-2}dx = \frac{x}{a\sqrt{2\,ax-x^2}}.$

42. $\displaystyle\int \frac{3\,dx}{\sqrt{x^2+3}\,(\sqrt{x^2+3}-x)} = x+\sqrt{x^2+3}.$

43. $\displaystyle\int \frac{\left(x^{n-1}+\frac{n-1}{n}x^{n-2}\right)dx}{(x^n+x^{n-1})^{\frac{p}{q}}} = \frac{q\,(x^n+x^{n-1})^{\frac{q-p}{q}}}{n\,(q-p)}.$

44. $\displaystyle\int \frac{m\,(\log x)^n\,dx}{x} = \frac{m\,(\log x)^{n+1}}{n+1}.$

45. $\displaystyle\int \frac{x-2}{x\sqrt{x}}\,dx = 2\sqrt{x}+\frac{4}{\sqrt{x}}.$

46. $\int x\sqrt{x+a}\,dx = \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2}{3}a(x+a)^{\frac{3}{2}}.$

47. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \frac{2}{3a}\{(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\}.$

48. $\int \frac{a+bx}{a'+b'x}dx = \int \left\{\frac{b}{b'} + \frac{ab'-ba'}{b'(a'+b'x)}\right\}dx$

$= \frac{b}{b'}x + \frac{ab'-ba'}{b'^2}\log(a'+b'x).$

163. Type Formulae. (Continued.) Cf. § 43.

$$\int \sin u\,du = -\cos u \text{ or vers } u \qquad (5)$$

$$\int \cos u\,du = \sin u \text{ or } -\text{covers } u \qquad (6)$$

$$\int \sec^2 u\,du = \tan u \qquad (7)$$

$$\int \csc^2 u\,du = -\cot u \qquad (8)$$

$$\int \sec u \tan u\,du = \sec u \qquad (9)$$

$$\int \csc u \cot u\,du = -\csc u \qquad (10)$$

$$\int \tan u\,du = -\log\cos u = \log\sec u \qquad (11)$$

$$\int \cot u\,du = \log\sin u \qquad (12)$$

$$\int \sec u\,du = \log(\sec u + \tan u) \qquad (13)$$

$$\int \csc u\,du = \log(\csc u - \cot u) \qquad (14)$$

Formulae (5) to (10), inclusive, follow directly from the differential form (§ 35; *et seq.*). Formulae (11) to (14), inclusive, may be derived as follows:

$$\int \tan u\, du = -\int \frac{-\sin u}{\cos u} du = -\log \cos u = \log \frac{1}{\cos u}$$
$$= \log \sec u.$$

$$\int \cot u\, du = \int \frac{\cos u}{\sin u} du = \log \sin u.$$

$$\int \sec u\, du = \int \frac{(\sec u + \tan u) \sec u}{\sec u + \tan u} du$$
$$= \int \frac{\sec^2 u\, du + \tan u \sec u\, du}{\sec u + \tan u}$$
$$= \log (\sec u + \tan u).$$

$$\int csc\, u\, du = \int \frac{(csc\, u - \cot u)\, csc\, u\, du}{csc\, u - \cot u}$$
$$= \int \frac{csc^2 u\, du - csc\, u \cot u\, du}{csc\, u - \cot u}$$
$$= \log (csc\, u - \cot u).$$

EXAMPLES.

1. $\int \left(\sin \frac{x}{2} + \cos 2x\right) dx = -2\int -\sin \frac{x}{2}\frac{dx}{2} + \frac{1}{2}\int \cos 2x\, 2\, dx$
$= \frac{1}{2} \sin 2x - 2 \cos \frac{x}{2}.$

2. $\int \sin^3 x \cos x\, dx = \frac{\sin^4 x}{4}.$

3. $\int \frac{\sin \theta}{\cos^2 \theta} d\theta = -\int \cos^{-2} \theta (-\sin \theta\, d\theta) = -\frac{(\cos \theta)^{-1}}{-1}$
$= \sec \theta$, or thus, $\int \frac{\sin \theta}{\cos^2 \theta} d\theta = \int \sec \theta \tan \theta\, d\theta = \sec \theta.$

4. $\int \frac{\sin 3x}{\cos^2 3x} dx = \frac{\sec 3x}{3}.$

5. $\int \sec^2 (ax)\, dx = \frac{\tan (ax)}{a}.$

6. $\int \sec^2 (x^2)\, x dx = \frac{\tan x^2}{2}.$

7. $\int \frac{1 + \cos x}{x + \sin x} dx = \log (x + \sin x).$

8. $\int (\tan 2x - 1)^2 dx = \int \{\tan^2 2x + 1 - 2 \tan 2x\} dx$

$= \int \sec^2 2x dx - \int \tan 2x . 2\, dx = \frac{1}{2} \tan 2x + \log \cos 2x.$

9. $\int (\tan x + \cot x)^2 dx = \int \{\sec^2 x + \csc^2 x\} dx = \tan x - \cot x.$

10. $\int (1 + \sec 2\theta)^2 d\theta = \theta + \frac{1}{2} \tan 2\theta + \log(\sec 2\theta + \tan 2\theta).$

11. $\int \tan^3 (ax) \sec^2 (ax)\, dx = \frac{\tan^4 (ax)}{4a}.$

12. $\int e^{\sin (ax)} \cos (ax)\, dx = \frac{e^{\sin (ax)}}{a}.$

13. $\int (\csc 3x + 1)^2 dx = \int \csc^2 3x dx + 2 \int \csc 3x dx + \int dx$

$= x - \frac{1}{3} \cot 3x + \frac{2}{3} \log (\csc 3x - \cot 3x).$

14. $\int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \frac{dx}{2} = \log \tan \frac{x}{2}.$

15. $\int \frac{dx}{\sin x \cos x} = \log \tan x.$

16. $\int \frac{\tan x \, dx}{a + b \tan^2 x} = \int \frac{\sin x \cos x \, dx}{a \cos^2 x + b \sin^2 x}$

$= \frac{1}{2(b-a)} \log (a \cos^2 x + b \sin^2 x).$

17. $\int \sin^2 \theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta = \frac{\theta}{2} - \frac{1}{4} \sin 2\theta.$

18. $\int \cos^2 \theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta.$

19. $\int \frac{d\theta}{\cos \theta} = \log \tan \left(\frac{\pi}{4} + \frac{\theta}{2}\right).$ Since $\cos \theta = \sin \left(\frac{\pi}{2} + \theta\right).$

20. $\int \frac{\sec^2 \theta \, d\theta}{1 + 3 \tan \theta} = \frac{1}{3} \log (1 + 3 \tan \theta).$

164. Type Formulae. (Continued.) Cf. § 52.

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u \tag{15}$$

$$\int - \frac{du}{\sqrt{1 - u^2}} = \cos^{-1} u \tag{16}$$

$$\int \frac{du}{1 + u^2} = \tan^{-1} u \tag{17}$$

$$\int - \frac{du}{1 + u^2} = \cot^{-1} u \tag{18}$$

$$\int \frac{du}{u\sqrt{u^2 - 1}} = \sec^{-1} u \tag{19}$$

$$\int - \frac{du}{u\sqrt{u^2 - 1}} = \csc^{-1} u \tag{20}$$

$$\int \frac{du}{\sqrt{2u - u^2}} = \text{vers}^{-1} u \tag{21}$$

$$\int - \frac{du}{\sqrt{2u - u^2}} = \text{covers}^{-1} u \qquad (22)$$

$$\int \frac{du}{u^2 - 1} = \frac{1}{2} \log \frac{u - 1}{u + 1} \qquad (23)$$

$$\int \frac{du}{\sqrt{u^2 \pm 1}} = \log (u + \sqrt{u^2 \pm 1}) \qquad (24)$$

Formulae (15) to (22), inclusive, may be obtained directly from the differential forms, § 44, *et seq.* Formulae (23) and (24) are derived as follows:

$$\int \frac{du}{u^2 - 1} = \frac{1}{2} \int \left\{ \frac{1}{u - 1} - \frac{1}{u + 1} \right\} du$$

$$= \frac{1}{2} \int \left\{ \frac{du}{u - 1} - \frac{du}{u + 1} \right\}$$

$$= \frac{1}{2} \{ \log (u - 1) - \log (u + 1) \}$$

$$= \frac{1}{2} \log \frac{u - 1}{u + 1}.$$

$$\int \frac{du}{\sqrt{u^2 \pm 1}} = \int \frac{du}{z},$$

where $$u^2 \pm 1 = z^2 \qquad (a)$$

But from (a), $2\,udu = 2\,zdz$;

$$\therefore \frac{du}{z} = \frac{dz}{u} = \frac{du + dz}{u + z};$$

$$\therefore \int \frac{du}{z} = \int \frac{du + dz}{u + z} = \log (u + z),$$

hence, $$\int \frac{du}{\sqrt{u^2 \pm 1}} = \log (u + \sqrt{u^2 \pm 1}).$$

EXAMPLES.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{\frac{du}{a}}{\sqrt{1 - \frac{u^2}{a^2}}} = \sin^{-1}\frac{u}{a} \text{ or } - \cos^{-1}\frac{u}{a}.$

2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\int \frac{\frac{du}{a}}{1 + \frac{u^2}{a^2}} = \frac{1}{a}\tan^{-1}\frac{u}{a} \text{ or } - \frac{1}{a}\cot^{-1}\frac{u}{a}.$

3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\int \frac{\frac{du}{a}}{\frac{u}{a}\sqrt{\frac{u^2}{a^2} - 1}} = \frac{1}{a}\sec^{-1}\frac{u}{a} \text{ or } - \frac{1}{a}\csc^{-1}\frac{u}{a}.$

4. $\int \frac{du}{\sqrt{2au - u^2}} = \int \frac{\frac{du}{a}}{\sqrt{2\frac{u}{a} - \frac{u^2}{a^2}}} = \text{vers}^{-1}\frac{u}{a} \text{ or } - \text{covers}^{-1}\frac{u}{a}.$

5. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u - a}{u + a}.$

6. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log(u + \sqrt{u^2 \pm a^2}).$

These six integrals are frequently termed auxiliary type forms.

7. $\int \frac{axdx}{\sqrt{1 - x^4}} = \frac{a}{2}\sin^{-1}x^2$

8. $\int \frac{axdx}{4 + x^4} = \frac{a}{4}\tan^{-1}\frac{x^2}{2}.$

9. $\int \frac{axdx}{x^4 - 4} = \frac{a}{8}\log\frac{x^2 - 2}{x^2 + 2}.$

10. $$\int \frac{5\,dx}{\sqrt{10\,x - 25\,x^2}} = \text{vers}^{-1}\,5\,x.$$

11. $$\int \frac{dx}{\sqrt{ax - x^2}} = 2\sin^{-1}\sqrt{\frac{x}{a}}.$$

12. $$\int \frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b}\,\text{vers}^{-1}\,\frac{2\,b^2x}{a^2}.$$

13. $$\int \frac{dx}{1 + 5\,x^2} = \frac{1}{\sqrt{5}}\tan^{-1}(x\sqrt{5}).$$

14. $$\int \frac{dx}{x\sqrt{c^2x^2 - a^2b^2}} = \frac{1}{ab}\sec^{-1}\frac{cx}{ab}.$$

15. $$\int \frac{x^4dx}{x^2 + 1} = \frac{x^3}{3} - x + \tan^{-1}x.$$

16. $$\int \frac{dx}{x^2 + 4\,x + 5} = \int \frac{dx}{1 + (x + 2)^2} = \tan^{-1}(x + 2).$$

17. $$\int \frac{dx}{\sqrt{x^2 - 4\,x + 13}} = \int \frac{dx}{\sqrt{(x - 2)^2 + 9}}$$
$$= \log\,(x - 2 + \sqrt{x^2 - 4\,x + 13}).$$

18. $$\int \frac{dx}{x^2 + x + 1} = 4\int \frac{dx}{3 + (2\,x + 1)^2} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\,x + 1}{\sqrt{3}}.$$

19. $$\int \frac{dx}{\sqrt{x^2 + x + 1}} = \int \frac{dx}{\sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}}$$
$$= \log\left\{x + \tfrac{1}{2} + \sqrt{x^2 + x + 1}\right\}.$$

20. $$\int \frac{dx}{\sqrt{5 - 4\,x - x^2}} = \sin^{-1}\frac{x + 2}{3}.$$

21. $\int \frac{dx}{x^2 - 2ax\cos\alpha + a^2} = \int \frac{dx}{(x - a\cos\alpha)^2 + a^2\sin^2\alpha}$

$= \frac{1}{a\sin\alpha}\tan^{-1}\frac{x - a\cos\alpha}{a\sin\alpha}.$

22. $\int \frac{x^2 + x + 1}{x^2 - x + 1}\,dx = x + \log(x^2 - x + 1) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x - 1}{\sqrt{3}}.$

23. $\int \frac{dx}{(x^2 + a^2)(x + b)} = \frac{1}{b^2 + a^2}\left\{\log\frac{x + b}{\sqrt{x^2 + a^2}} + \frac{b}{a}\tan^{-1}\frac{x}{a}\right\}.$

24. $\int \frac{m\,dx}{x\sqrt{4x^2 - 9}} = \frac{m}{3}\sec^{-1}\frac{2}{3}x.$

25. $\int \frac{dx}{\sqrt{5x^4 - 3x^2}} = \frac{1}{\sqrt{3}}\sec^{-1}\frac{\sqrt{5}}{\sqrt{3}}x.$

26. $\int \frac{dx}{\sqrt{3x^2 - 4x}} = \frac{1}{\sqrt{3}}\log\{3x - 2 + \sqrt{9x^2 - 12x}\}.$

27. $\int \frac{x^2 - 1}{x^2 - 4}\,dx = x + \frac{3}{4}\log\frac{x - 2}{x + 2}.$

28. $\int \frac{(a - x)^{\frac{1}{2}}}{(a + x)^{\frac{1}{2}}}\,dx = a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}.$

29. $\int \frac{dx}{x^2 - 2ax\sec\alpha + a^2} = \frac{1}{2a\tan\alpha}\log\frac{x - a(\sec\alpha + \tan\alpha)}{x - a(\sec\alpha - \tan\alpha)}.$

30. $\int \frac{2x - 5}{3x^2 - 2}\,dx = 2\int \frac{x\,dx}{3x^2 - 2} - 5\int \frac{dx}{3x^2 - 2}$

$= \frac{1}{3}\log(3x^2 - 2) - \frac{5}{2\sqrt{6}}\log\frac{x\sqrt{3} - \sqrt{2}}{x\sqrt{3} + \sqrt{2}}.$

31. $\int \frac{dx}{\sqrt{1 - 3x - x^2}} = \sin^{-1}\frac{3 + 2x}{\sqrt{13}}.$

32. $$\int \frac{dx}{e^x + e^{-x}} = \tan^{-1} e^x.$$

33. $$\int \sqrt{\frac{m + x}{x}}\, dx = \sqrt{mx + x^2} + m \log \{\sqrt{x} + \sqrt{m + x}\}.$$

34. $$\int \frac{xdx}{x^2 + x + 1} = \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$= \log \sqrt{x^2 + x + 1} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}.$$

35. $$\int \frac{dx}{1 + x - x^2} = \frac{1}{\sqrt{5}} \log \frac{2x - 1 + \sqrt{5}}{2x - 1 - \sqrt{5}}.$$

36. $$\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{cx + b}{\sqrt{ac - b^2}}.$$

37. $$\int \frac{dx}{\sqrt{c + bx - ax^2}} = \frac{1}{\sqrt{a}} \sin^{-1} \frac{2ax - b}{\sqrt{b^2 + 4ac}}.$$

165. Integration by Parts. — From equation 3, § 25, of the Differential Calculus, we have

$$d(uv) = udv + vdu.$$

Hence, $$uv = \int udv + \int vdu;$$

$$\therefore \int udv = uv - \int vdu \quad \ldots\ldots \quad (25)$$

Examining (25) we see that the required integral is *separated into two parts*, u and dv, and that the *first term* of the second member is obtained by integrating the first member, assuming u *constant*, and that the *second term* is obtained from the *first* *term* by differentiating that term, assuming v constant, and integrating the result. This process, known as integration by parts, is applicable whenever dv and vdu are integrable forms.

Let us apply the process to the example

$$\int x \sin x dx.$$

Let $u = x$ and $dv = \sin x dx$; then

$$\int x \sin x dx = x(-\cos x) - \int (-\cos x)\, dx$$
$$= -x \cos x + \sin x.$$

We might have assumed $u = \sin x$ and $dv = xdx$; then

$$\int x \sin x dx = \sin x \frac{x^2}{2} - \int \frac{x^2}{2} \cos x dx.$$

But the integral $\int \frac{x^2}{2} \cos x dx$ is more complicated than the given integral $\int x \sin x dx$; hence this assumption will not serve our purpose. In applying this process, therefore, we must determine the proper factor by trial.

EXAMPLES.

1. $$\int x \log x dx = \log x \frac{x^2}{2} - \int \frac{x^2}{2} \frac{dx}{x}$$
$$= \log x \frac{x^2}{2} - \frac{x^2}{4}$$
$$= \frac{x^2}{4}(\log x^2 - 1).$$

2. $$\int x \cos x dx = x \sin x + \cos x.$$

3. $$\int x \sec^2 x dx = x \tan x - \log \sec x.$$

4. $\int x \tan^2 x dx = \int (\sec^2 x - 1) x dx$

$= x \tan x - \log \sec x - \frac{x^2}{2}.$

5. $\int x \tan^{-1} x dx = \frac{x^2 + 1}{2} \tan^{-1} x - \frac{x}{2}.$

6. $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

7. $\int x \sin^{-1} (x)^2 dx = \frac{x^2}{2} \sin^{-1} (x)^2 + \frac{1}{2} \sqrt{1 - x^4}.$

8. $\int \frac{\sin^{-1} x dx}{\sqrt{1 - x^2}} = \frac{(\sin^{-1} x)^2}{2}.$

9. $\int \frac{x^2 \tan^{-1} x dx}{1 + x^2}$

$= \tan^{-1} x (x - \tan^{-1} x) - \int (x - \tan^{-1} x) \frac{dx}{1 + x^2},$

$= x \tan^{-1} x - (\tan^{-1} x)^2 - \log \sqrt{1 + x^2} + \frac{1}{2} (\tan^{-1} x)^2,$

$= x \tan^{-1} x - \frac{1}{2} (\tan^{-1} x)^2 - \log \sqrt{1 + x^2}.$

10. $\int \log x dx = x (\log x - 1).$

11. $\int x e^x dx = e^x (x - 1).$

12. $\int \frac{x dx}{e^x} = - e^{-x} (x + 1).$

13. $\int x e^{ax} dx = e^{ax} \left(\frac{ax - 1}{a^2} \right).$

14. $\int x^3 \sqrt{a - x^2} dx = - \frac{x^2 \sqrt{(a - x^2)^3}}{3} - \frac{2 \sqrt{(a - x^2)^5}}{15}.$

By repeating the process we may derive the following:

15. $$\int e^x x^2 dx = x^2 e^x - 2\int e^x x dx,$$
$$= x^2 e^x - 2\left\{x e^x - \int e^x dx\right\},$$
$$= e^x (x^2 - 2x + 2).$$

16. $$\int e^{ax} x^2 dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$

17. $$\int a^x x^2 dx = \frac{a^x}{\log a}\left(x^2 - \frac{2x}{\log a} + \frac{2}{\log^2 a}\right).$$

18. $$\int x^2 \log^2 x dx = \frac{x^3}{27}(9\log^2 x - 6\log x + 2).$$

19. $$\int x^n \log^2 x dx$$
$$= \frac{x^{n+1}}{(n+1)^3}\left\{(n+1)^2 \log^2 x - 2(n+1)\log x + 2\right\}.$$

20. $$\int x^2 \sin^{-1} x dx = \frac{x^3}{3}\sin^{-1} x + \frac{x^2+2}{9}\sqrt{1-x^2}.$$

21. $$\int x^2 \tan^{-1} x dx = \frac{x^3}{3}\tan^{-1} x + \frac{\log(1+x^2) - x^2}{6}.$$

22. $$\int x^2 \sec^{-1} x dx$$
$$= \frac{x^3}{3}\sec^{-1} x - \frac{\log(x + \sqrt{x^2-1}) + x\sqrt{x^2-1}}{6}.$$

23. $$\int e^{\frac{x}{2}} \cos\frac{x}{2} dx = e^{\frac{x}{2}}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right).$$

24. $$\int e^{ax} \sin nx dx = \frac{a \sin nx - n\cos nx}{a^2+n^2} e^{ax}.$$

25. $$\int e^{ax} \cos nx dx = \frac{n \sin nx + a\cos nx}{a^2+n^2} e^{ax}.$$

CHAPTER II.

RATIONAL FRACTIONS.

166. The fractional differential,

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + kx + l}{x^n + b_1x^{n-1} + c_1x^{n-2} + \dots + k_1x + l_1}dx \quad . \quad . \quad (a)$$

is rational when m and n are *positive* integers. To explain the method of integrating such differential forms is the object of this chapter.*

167. If $m > n$ or $m = n$.

In this case we can always by division change the fractional form into a mixed quantity composed of one or more monomial terms increased or diminished by a similar fractional form, in which $m < n$. The monomial terms are readily integrated by rules already explained. Thus

$$\int \frac{4x^3 + 2x^2 + 4}{2x^2 + 3x + 2}dx = \int \left\{ 2x - 2 + \frac{2x + 8}{2x^2 + 3x + 2} \right\} dx$$

$$= 2\int xdx - 2\int dx + 2\int \frac{x + 4}{2x^2+3x+2}dx$$

$$= x^2 - 2x + 2\int \frac{x + 4}{2x^2 + 3x + 2}dx.$$

To complete the integration of such expression therefore we are to obtain a rule applicable to rational differential forms (a) in which $m < n$.

* Leibnitz and John Bernouilli, in 1702 and 1703, showed that such integrals depended on the method of partial fractions. The simplified and general processes are due to Fuler.

168. When $m < n$. If p, s, t, . . . w are the roots of the equation

$$x^n + b_1x^{n-1} + c_1x^{n-2} + \ldots k_1x + l_1 = 0,$$

then by the General Theory of Equations,

$$x^n + b_1x^{n-1} + c_1x^{n-2} + \ldots k_1x + l_1 = (x-p)(x-s)(x-t)\ldots(x-w).$$

Substituting this value in the denominator of (a) § 166, we have

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \ldots kx + l}{(x-p)(x-s)(x-t)\ldots(x-w)}dx \quad . \quad . \quad . \quad . \quad . \quad (b)$$

By the method of Undetermined Coefficients we are enabled to decompose the fractional form (b) into a series of partial fractions of simpler forms. To do this, four cases present themselves, depending upon the value of the roots $p, s, t,$ . . . w.

CASE 1. When the factors of the denominator are *real* and *unequal.*

CASE 2. When the factors of the denominator are *real* and *equal.*

CASE 3. When the factors of the denominator are *imaginary* and *unequal.*

CASE 4. When the factors of the denominator are *imaginary* and *equal.*

169. CASE I. **Factors real and unequal.**

Here $$\int\frac{ax^m + bx^{m-1} + cx^{m-2}\ldots + kx + l}{(x-p)(x-s)(x-t)\ldots(x-w)}dx =$$

$$\int\left\{\frac{A}{x-p} + \frac{B}{x-s} + \frac{C}{x-t} + \ldots \frac{F}{x-w}\right\}dx =$$

$$A\log(x-p) + B\log(x-s) + C\log(x-t) + \ldots F\log(x-w)$$

in which A, B, C, . . . F are undetermined constants. The *method* of determining the *values* of the constants will appear in the process of integrating the following examples.

EXAMPLES.

1. $$\int \frac{3x^2 - 2x}{x^3 - 3x^2 + 2x} dx = \int \frac{3x^2 - 2x}{x(x-2)(x-1)} dx =$$

$$\int \left\{ \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1} \right\} dx.$$

Hence $$\frac{3x^2 - 2x}{x(x-2)(x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1};$$

clearing of fractions, we have (a)

$$\begin{aligned} 3x^2 - 2x &= A(x-2)(x-1) + B(x-1)x + C(x-2)x \\ &= A(x^2 - 3x + 2) + B(x^2 - x) + C(x^2 - 2x) \\ &= (A + B + C)x^2 - (3A + B + 2C)x + 2A. \end{aligned}$$

Since the members of this equation are finite series, and the equation is to be satisfied for all values of x, we must have, by the Theory of Undetermined Coefficients, the coefficients of like powers of x in the two members equal;

i.e.,
$$\begin{aligned} A + B + C &= 3, \\ 3A + B + 2C &= 2, \\ 2A &= 0. \end{aligned}$$

From the three equations we find the values of these constants to be $A = 0,\ B = 4,\ C = -1$;
hence, substituting, we have

$$\begin{aligned} \int \frac{3x^2 - 2x}{x^3 - 3x^2 + 2x} dx &= 4\int \frac{dx}{x-2} - \int \frac{dx}{x-1} \\ &= 4 \log(x-2) - \log(x-1) \\ &= \log \frac{(x-2)^4}{x-1}. \end{aligned}$$

A shorter and simpler process of obtaining the values of the constants is as follows:

Since (a) is true for all values of x, we may give x such values as will determine the constants at once: Thus

$$x = 0 \quad \therefore \quad 0 = 2A \quad \therefore \quad A = 0$$
$$x = 1 \quad \therefore \quad 1 = -C \quad \therefore \quad C = -1$$
$$x = 2 \quad \therefore \quad 8 = 2B \quad \therefore \quad B = 4.$$

2. $$\int \frac{x+4}{2x - x^2 - x^3}\,dx = \int \frac{x+4}{x(1-x)(2+x)}\,dx =$$
$$\int \left\{ \frac{A}{x} + \frac{B}{1-x} + \frac{C}{2+x} \right\} dx \cdot$$

Hence, $$\frac{x+4}{x(1-x)(2+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{2+x} \cdot$$

Clearing of fractions, we have

$$x + 4 = A(1-x)(2+x) + Bx(2+x) + Cx(1-x).$$

Here, $x = 0$ gives $4 = 2A$. $\therefore A = 2$.

$x = 1$ gives $5 = 3B$. $\therefore B = \frac{5}{3}$.

$x = -2$ gives $2 = -6C$. $\therefore C = -\frac{1}{3}$.

Hence,

$$\int \frac{x+4}{2x - x^2 - x^3}\,dx = \int \left\{ \frac{2}{x} + \frac{5}{3(1-x)} - \frac{1}{3(2+x)} \right\} dx$$
$$= 2\int \frac{dx}{x} - \frac{5}{3}\int \frac{-\,dx}{1-x} - \frac{1}{3}\int \frac{dx}{2+x}$$
$$= 2 \log x - \tfrac{5}{3} \log(1-x) - \tfrac{1}{3} \log(2+x).$$
$$= \log \frac{x^2}{\sqrt[3]{(1-x)^5(2+x)}} \cdot$$

3. $\int \frac{19x+1}{15x^2+x-2}dx = \int \left\{ \frac{A}{3x-1} + \frac{B}{5x+2} \right\} dx$

$= \log \sqrt[3]{(3x-1)^2}\sqrt[5]{(5x+2)^3}.$

4. $\int \frac{x^2+6x-8}{x^3-4x}dx = \log \frac{x^2(x-2)}{(x+2)^2}.$

5. $\int \frac{xdx}{x^2-4x+1} = \int \frac{xdx}{(x-2+\sqrt{3})(x-2-\sqrt{3})}$

$= \frac{2+\sqrt{3}}{2\sqrt{3}}\log(x-2-\sqrt{3}) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(x-2+\sqrt{3}).$

6. $\int \frac{x^2-1}{x^2-4}dx = x + \log\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}.$

7. $\int \frac{x^3dx}{x^2+7x+12} = \frac{x^2}{2} - 7x + 64\log(x+4) - 27\log(x+3).$

8. $\int \frac{x^2-3}{x^3-7x+6}dx$

$= \frac{1}{2}\log(x-1) + \frac{1}{5}\log(x-2) + \frac{3}{10}\log(x+3).$

9. $\int \frac{x^2+2x-\cos^2 a}{x^2+2x+\sin^2 a}dx = \int \left\{ 1 - \frac{1}{(x+1)^2-\cos^2 a} \right\} dx$

$= x + \frac{\sec a}{2}\log\frac{x+1+\cos a}{x+1-\cos a}.$

10. $\int \frac{x^2+8x+4}{x^3+x^2-4x-4}dx = \log\frac{(x+1)(x-2)^2}{(x+2)^2}.$

11. $\int \frac{adx}{x^2+mx} = \frac{a}{m}\log\frac{x}{x+m}.$

12. $\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab}\log\frac{a+bx}{a-bx}.$

170. Case II. **Factors real and equal.**

In this case, $p = s = t = \dots = w$. Therefore (*b*), § 168, becomes
$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + kx + l}{(x-p)^n}dx.$$

Following the method of Case I., we would write,

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + kx + l}{(x-p)^n}dx$$
$$= \left\{\frac{A}{x-p} + \frac{B}{x-p} + \frac{C}{x-p} + \dots + \frac{F}{x-p}\right\}dx$$
$$= \frac{A+B+C+\dots+F}{x-p}dx.$$

But this is impossible, for the given fraction cannot be reduced to an equivalent fraction having a variable denominator $(x-p)$ and a constant numerator $(A+B+C+\dots+F)$. To avoid this objection, we write,

$$\int\frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + kx + l}{(x-p)^n}dx$$
$$= \int\left\{\frac{A}{(x-p)^n} + \frac{B}{(x-p)^{n-1}} + \frac{C}{(x-p)^{n-2}} + \dots + \frac{F}{(x-p)}\right\}dx$$
$$= \frac{A}{(1-n)(x-p)^{n-1}} + \frac{B}{(2-n)(x-p)^{n-2}}$$
$$+ \frac{C}{(3-n)(x-p)^{n-3}} + \dots + F\log(x-p).$$

If *all* the factors of the denominator are not equal, we ascertain the partial fractions by combining Cases I. and II.

EXAMPLES.

1. $$\int\frac{x^2-11x+26}{(x-3)^3}dx=\int\left\{\frac{A}{(x-3)^3}+\frac{B}{(x-3)^2}+\frac{C}{x-3}\right\}dx$$

i.e., $$\frac{x^2-11x+26}{(x-3)^3}=\frac{A}{(x-3)^3}+\frac{B}{(x-3)^2}+\frac{C}{x-3}.$$

Clearing of fractions, we have,

$$\begin{aligned}x^2-11x+26&=A+B(x-3)+C(x-3)^2\\&=Cx^2+(B-6C)x+A-3B+9C.\end{aligned}$$

Equating coefficients of like powers of x, we have,

$$C=1,$$
$$B-6C=-11,$$
$$A-3B+9C=26.$$

Hence, $A=2,\ B=-5,\ C=1.$

$$\therefore\ \int\frac{x^2-11x+26}{(x-3)^3}dx=\int\left\{\frac{2}{(x-3)^3}-\frac{5}{(x-3)^2}+\frac{1}{x-3}\right\}dx$$

$$=2\int(x-3)^{-3}dx-5\int(x-3)^{-2}dx+\int\frac{dx}{x-3}$$

$$=-\frac{1}{(x-3)^2}+\frac{5}{x-3}+\log(x-3).$$

2. $$\int\frac{x^2+3x+4}{x^3+2x^2+x}dx=\int\frac{x^2+3x+4}{x(x+1)^2}dx$$

$$=\int\left\{\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{C}{x}\right\}dx.$$

It will be observed that this example affords an illustration of the combined methods of Cases I. and II.

Here, $$\frac{x^2+3x+4}{x^3+2x^2+x} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x}.$$

$$\therefore\ x^2+3x+4 = Ax + Bx(x+1) + C(x+1)^2$$
$$= (B+C)x^2 + (A+B+2C)x + C.$$

Equating coefficients of like powers, we have,

$$B + C = 1,$$
$$A + B + 2C = 3,$$
$$C = 4.$$

Hence, $$A = -2,\ B = -3,\ C = 4;$$

$$\therefore\ \int \frac{x^2+3x+4}{x^3+2x^2+x}dx = -2\int (x+1)^{-2}dx - 3\int\frac{dx}{x+1} + 4\int\frac{dx}{x}$$
$$= \frac{2}{x+1} - \log\frac{(x+1)^3}{x^4}.$$

3. $$\int \frac{x+2}{(x-2)(x-1)^2}dx = \frac{3}{x-1} + 4\log\frac{x-2}{x-1}.$$

4. $$\int \frac{3x+2}{x(x+1)^3}dx = \frac{4x+3}{2(x+1)^2} + \log\frac{x^2}{(x+1)^2}.$$

5. $$\int \frac{xdx}{(x+3)^2(x+2)} = \log\left(\frac{x+3}{x+2}\right)^2 - \frac{3}{x+3}.$$

6. $$\int \frac{dx}{x^3-x^2-x+1} = \log\sqrt[4]{\frac{x+1}{x-1}} - \frac{1}{2(x-1)}.$$

7. $$\int \frac{x^2}{(x^2-1)^2}dx = \frac{1}{4}\log\frac{x-1}{x+1} - \frac{x}{2(x^2-1)}.$$

8. $\int \frac{x^5 - 5x - 3}{x^2(x+1)^2}dx = \int \left\{ x - 2 + \frac{3x^3 + 2x^2 - 5x - 3}{x^2(x+1)^2} \right\} dx$

$$= \frac{x^2}{2} - 2x + \int \left\{ \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x+1)^2} + \frac{D}{x+1} \right\} dx$$

$$= \frac{x^2}{2} - 2x + \frac{2x+3}{x^2+x} + \log x(x+1)^2.$$

171. Case III. **Factors imaginary and unequal.**

In this case, p, s, t . . . w are imaginary and unequal. Following the method of Case I., we would write,

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \cdots kx + l}{(x-p)(x-s)(x-t)\cdots(x-w)}dx$$
$$= \left\{ \frac{A}{x-p} + \frac{B}{x-s} + \frac{C}{x-t} + \cdots \frac{F}{x-w} \right\} dx.$$

But, as the denominators of the partial fractions are imaginary, the application of this process cannot afford *real* results. To develop a process applicable to this case, let us resume the denominator of the general fraction, (a) § 166, and equate to zero, i.e.,

$$x^n + b_1x^{n-1} + c_1x^{n-2} + \cdots + k_1x + l_1 = 0 \quad . \quad . \quad (a)$$

By hypothesis, the roots of this equation, p, s, t, etc., are imaginary and unequal. We know, however, that imaginary roots enter equations by pairs, and that if $a + b\sqrt{-1}$ is a root of (a), its complex conjugate, $a - b\sqrt{-1}$, is also a root. Hence, n is an even number. Now let $p = a + b\sqrt{-1}$, $s = a - b\sqrt{-1}$, $t = c + d\sqrt{-1}$, $u = c - d\sqrt{-1}$, etc.; then,

$$(x-p)(x-s) = \{x - (a + b\sqrt{-1})\}\{x - (a - b\sqrt{-1})\}$$
$$= (x-a)^2 + b^2,$$
$$(x-t)(x-u) = \{x - (c + d\sqrt{-1})\}\{x - (c - d\sqrt{-1})\}$$
$$= (x-c)^2 + d^2, \text{ etc.}$$

$$\therefore\ x^n + b_1x^{n-1} + c_1x^{n-2} + \dots + k_1x + l_1$$
$$= (x-p)(x-s)(x-t)(x-u)\dots \text{ to } n \text{ factors}$$
$$= \{(x-a)^2 + b^2\}\{(x-c)^2 + d^2\}\dots \text{ to } \tfrac{n}{2} \text{ factors,}$$

i.e., every polynomial which, on decomposition, affords n *imaginary* binomial factors of the *first* degree affords also $\frac{n}{2}$ *real* binomial factors of the *second* degree. In this case, therefore, we may write (a) § 166 in the form

$$\int \frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + kx + l}{\{(x-a)^2 + b^2\}\{(x-c)^2 + d^2\}\dots \text{ to } \frac{n}{2} \text{ factors}}\,dx$$
$$= \int \left\{ \frac{Ax+B}{(x-a)^2+b^2} + \frac{Cx+D}{(x-c)^2+d^2} + \dots \text{ to } \frac{n}{2} \text{ fractions} \right\} dx \dots (b)$$

in which the numerators are determined by considering, (1), that, in general, $m = n - 1$, and that therefore the partial fractions, when reduced to a common denominator, must afford a polynomial of the same degree (m), and (2), as there are, in general, $m + 1\ (= n)$ terms in the numerator of the first member of (b) there will be $m + 1$ equations of condition between the constants, and therefore there must be $m + 1\ (= n)$ constants, in order that these equations may consist.

If, as generally happens, some of the factors of the denominator are real and of the first degree, we combine with this method the methods of Case I. or of Case II., according as these factors are not or are repeated.

EXAMPLES.

1. $$\int \frac{x^2 + 2x - 1}{(x^2+2)(x^2+1)}\,dx = \int \left\{ \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1} \right\} dx.$$

Here, $$\frac{x^2 + 2x - 1}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1};$$

hence, clearing of fractions we have,

$$x^2 + 2x - 1 = (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 2)$$
$$= (A + C)x^3 + (B + D)x^2 + (A + 2C)x + B + 2D.$$

Equating coefficients of like powers, we have,

$$A + C = 0,$$
$$B + D = 1,$$
$$A + 2C = 2,$$
$$B + 2D = -1.$$

$$\therefore\ A = -2,\ B = 3,\ C = 2,\ D = -2$$

Hence, $\displaystyle\int \frac{x^2 + 2x - 1}{(x^2+2)(x^2+1)}dx = \int \left\{ \frac{-2x+3}{x^2+2} + \frac{2x-2}{x^2+1} \right\} dx$

$$= -\int \frac{2\,xdx}{x^2+2} + 3\int \frac{dx}{2+x^2} + \int \frac{2\,xdx}{x^2+1} - 2\int \frac{dx}{1+x^2}$$

$$= -\log(x^2+2) + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \log(x^2+1) - 2\tan^{-1}x$$

$$= \log\frac{x^2+1}{x^2+2} + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - 2\tan^{-1}x.$$

2. $\displaystyle\int \frac{x^2dx}{x(x^3-1)} = \int \left\{ \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} \right\} dx,$

$$\therefore\ x^2 = A(x^3-1) + Bx(x^2+x+1) + (Cx+D)x(x-1)$$
$$= (A+B+C)x^3 + (B-C+D)x^2 + (B-D)x - A = 0.$$

$$\therefore\ A + B + C = 0,$$
$$B - C + D = 1,$$
$$B - D = 0,$$
$$A = 0.$$

$$\therefore\ A = 0,\ B = \frac{1}{3},\ C = -\frac{1}{3},\ D = \frac{1}{3}.$$

Hence, $\int \frac{x^2dx}{x(x^3-1)} = \int \left\{ \frac{dx}{3(x-1)} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1}dx \right\}$

$$= \log \sqrt[3]{x-1} - \frac{1}{6}\int \frac{(2x-2)\,dx}{x^2+x+1}$$

$$= \log \sqrt[3]{x-1} - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \log \sqrt[3]{x-1} - \frac{1}{6}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

3. $\int \frac{x^2dx}{(x^2-1)(x^2+2)} = \int \left\{ \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2} \right\} dx$

$$= \log \sqrt[6]{\frac{x-1}{x+1}} + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$$

4. $\int \frac{xdx}{(x+1)(x^2+1)} = \log \sqrt[4]{\frac{x^2+1}{(x+1)^2}} + \frac{1}{2}\tan^{-1}x.$

5. $\int \frac{5x+12}{x^3+4x}dx = 3\log\frac{x}{\sqrt{x^2+4}} + \frac{5}{2}\tan^{-1}\frac{x}{2}.$

6. $\int \frac{dx}{x^3+1} = \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$

7. $\int \frac{2\,xdx}{(x^2+1)(x^2+3)} = \log\sqrt{\frac{x^2+1}{x^2+3}}.$

8. $\int \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{1}{3}\left\{ 2\tan^{-1}\frac{x}{2} - \tan^{-1}x \right\}.$

9. $\int \frac{dx}{(x-1)(x^2+2)} = \log\sqrt[6]{\frac{(x-1)^2}{x^2+2}} - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$

10. $\int \frac{x^2 \cos 2a + 1}{x^4 + 2x^2 \cos 2a + 1} dx$

$$= \frac{\sin a}{4} \log \frac{x^2 + 2x \sin a + 1}{x^2 - 2x \sin a + 1} + \frac{\cos a}{2} \tan^{-1} \frac{2x \cos a}{1 - x^2}.$$

11. $\int \frac{dx}{x(1 + x)^2(1 + x + x^2)}$

$$= \frac{1}{1+x} + \log \frac{\sqrt{x^2 + x^3 + x^4}}{(1+x)^2} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}.$$

12. $\int \frac{1 - x + x^2}{1 + x + x^2 + x^3} dx = \frac{1}{2} \log \frac{(1+x)^3}{\sqrt{1+x^2}} - \frac{1}{2} \tan^{-1} x.$

172. Case IV. **Factors imaginary and equal.**

This case bears the same relation to Case III. that Case II. does to Case I. For the reasons indicated under Case II. we may write, therefore, equation (*b*) under Case III. in the following form:

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \cdots + kx + l}{\{(x-a)^2 + b^2\}^{\frac{n}{2}}} dx$$

$$= \left\{ \frac{Ax + B}{\{(x-a)^2 + b^2\}^{\frac{n}{2}}} + \frac{Cx + D}{\{(x-a)^2 + b^2\}^{\frac{n}{2}-1}} + \cdots \text{ to } \frac{n}{2} \text{ fractions} \right\} dx \quad \ldots \ldots \ldots \quad (a)$$

Note. — In the application of this method when the exponent of the denominator of a partial fraction is greater than 1, the numerator being constant, the student will find occasion to use the following formula. Cf. § 186, 2°.

$$\int x^m (a + bx^n)^p dx$$

$$= - \frac{x^{m+1}(a + bx^n)^{p+1}}{an(p+1)} + \frac{n(p+1) + m + 1}{an(p+1)} \int x^m (a + bx^n)^{p+1} dx \;. \quad (b)$$

EXAMPLES.

1. $$\int \frac{x^2 - x + 2}{(x^2+1)^2} dx = \int \left\{ \frac{Ax + B}{(x^2+1)^2} + \frac{Cx + D}{x^2 + 1} \right\} dx,$$

$$\therefore\ x^2 - x + 2 = Ax + B + (Cx + D)(x^2 + 1)$$

$$= Cx^3 + Dx^2 + (A + C)x + B + D.$$

Equating coefficients, we have,

$$C = 0,\ D = 1,\ A + C = -1,\ B + D = 2;$$

hence, $A = -1,\ B = 1,\ C = 0,\ D = 1.$

Hence,

$$\int \frac{x^2 - x + 2}{(x^2+1)^2} dx = \int \left\{ \frac{-x + 1}{(x^2+1)^2} + \frac{1}{x^2+1} \right\} dx$$

$$= -\int \frac{xdx}{(x^2+1)^2} + \int \frac{dx}{(x^2+1)^2} + \int \frac{dx}{x^2+1}$$

$$= \frac{1}{2(x^2+1)} + \tan^{-1} x + \int (1 + x^2)^{-2} dx.$$

To integrate the last term we find on comparing it with the first member of the reduction formula, *b*, § 172, that,

$$m = 0,\ a = 1,\ b = 1,\ n = 2,\ p = -2.$$

Substituting these values in the first and second members of (*b*), we have

$$\int (1 + x^2)^{-2} dx = -\frac{x(1 + x^2)^{-1}}{2(-1)} + \frac{-1}{2(-1)} \int (1 + x^2)^{-1} dx$$

$$= \frac{x}{2(1 + x^2)} + \frac{1}{2} \tan^{-1} x.$$

Hence

$$\int \frac{x^2 - x + 2}{(x^2+1)^2}\,dx = \frac{1}{2\,(x^2+1)} + \tan^{-1} x + \frac{x}{2\,(x^2+1)} + \frac{1}{2}\tan^{-1} x$$

$$= \frac{x+1}{2\,(x^2+1)} + \frac{3}{2}\tan^{-1} x.$$

2. $$\int \frac{x^3 + x^2 + 2}{(x^2+2)^2}\,dx = \frac{1}{x^2+2} + \log\sqrt{x^2+2} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

3. $$\int \frac{4x(x-2)}{(x-1)^2(x^2+1)^2}\,dx = \frac{x\,(3x-1)}{(x-1)(x^2+1)} + \log\frac{(x-1)^2}{x^2+1} + \tan^{-1} x.$$

4. $$\int \frac{3x+2}{(x^2-3x+3)^2}\,dx = \int \frac{3x+2}{\left\{\left(x-\frac{3}{2}\right)^2 + \frac{3}{4}\right\}^2}\,dx$$

$$= \frac{13x - 24}{3\,(x^2 - 3x + 3)} + \frac{26}{3\sqrt{3}}\tan^{-1}\frac{2x-3}{\sqrt{3}}.$$

5. $$\int \frac{dx}{x\,(a+bx^2)^2} = \int \left\{ \frac{A}{x} + \frac{Bx + C}{(a+bx^2)^2} + \frac{Dx+E}{a+bx^2} \right\} dx$$

$$= \frac{1}{2\,a\,(a+bx^2)} + \frac{1}{2\,a^2}\log\frac{x^2}{a+bx^2}.$$

6. $$\int \frac{x^5 dx}{(1+x^2)^3} = \frac{1}{x^2+1} - \frac{1}{4\,(x^2+1)^2} + \frac{1}{2}\log(x^2+1).$$

7. $$\int \frac{dx}{x\,(a+bx^n)^2} = \frac{1}{na\,(a+bx^n)} + \frac{1}{na^2}\log\frac{x^n}{a+bx^n}.$$

8. $$\int \frac{dx}{(x-1)^2(x^2+1)^2} = -\frac{1}{4\,(x-1)} - \frac{1}{2}\log(x-1)$$

$$+ \frac{1}{4}\tan^{-1}x - \frac{1}{4\,(x^2+1)} + \frac{1}{4}\log(x^2+1).$$

9. $$\int \frac{x^4 + 2x^3 + 3x^2 + 3}{(x^2 + 1)^3} dx$$

$$= \frac{2 + x}{4(x^2 + 1)^2} + \frac{7x - 8}{8(x^2 + 1)} + \frac{15}{8} \tan^{-1} x.$$

10. $$\int \frac{x^3 + 8x + 21}{(x^2 - 4x + 9)^2} dx$$

$$= \frac{3(x - 7)}{2(x^2 - 4x + 9)} + \frac{1}{2} \log(x^2 - 4x + 9) + \frac{3\sqrt{5}}{2} \tan^{-1} \frac{x - 2}{\sqrt{5}}.$$

IRRATIONAL FRACTIONS.

173. We have seen in the preceding discussions the methods of reducing rational differential fractions to one or more of the type integral forms, and hence the method of integrating such differential expressions. To integrate *irrational* fractional differentials, we have only to *reduce them to a rational form*, and then apply the methods previously developed.

174. Methods of Rationalization.

As irrational fractional forms occur in infinite variety, the method of rationalization will depend upon the particular form under consideration. As a general rule, however, we may state that the process in all cases is to *substitute for the variable in the given expression a new variable that will render the expression rational.* As illustrations of the method of rationalization, we shall consider certain groups of irrational forms, and explain the method applicable to each group.

175. Functions containing monomial surds only.

RULE: *Substitute for the old variable a new variable affected with an exponent equal to the least common multiple of the denominators of the fractional exponents of the old variable.*

EXAMPLES.

1. $\int \frac{x^{\frac{1}{2}} - x^{\frac{1}{3}}}{x^{\frac{1}{4}}} dx.$

The least common multiple of the denominators is in this case 12.

Let $$x = z^{12},$$

then $$x^{\frac{1}{2}} = z^6, \quad x^{\frac{1}{3}} = z^4, \quad x^{\frac{1}{4}} = z^3, \quad dx = 12\, z^{11} dz;$$

hence
$$\int \frac{x^{\frac{1}{2}} - x^{\frac{1}{3}}}{x^{\frac{1}{4}}} dx = \int \frac{z^6 - z^4}{z^3} 12\, z^{11} dz$$
$$= 12 \int \left\{ z^{14} - z^{12} \right\} dz$$
$$= \frac{12}{15} z^{15} - \frac{12}{13} z^{13}.$$

Substituting now for z its value $x^{\frac{1}{12}}$, we have

$$\int \frac{x^{\frac{1}{2}} - x^{\frac{1}{3}}}{x^{\frac{1}{4}}} dx = \frac{12}{15} x^{\frac{15}{12}} - \frac{12}{13} x^{\frac{13}{12}}.$$

By division, of course, the result in this case could be obtained immediately.

2. $\int \frac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{2}} + 1} dx.$

Here $$x = z^4,$$

$$\therefore \quad x^{\frac{1}{4}} = z, \quad x^{\frac{1}{2}} = z^2, \quad dx = 4\, z^3 dz;$$

hence
$$\int \frac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{2}} + 1} dx = \int \frac{z - 1}{z^2 + 1} 4\, z^3 dz$$
$$= 4 \int \frac{z^4 - z^3}{z^2 + 1} dz$$
$$= 4 \int \left\{ z^2 - z - 1 + \frac{z + 1}{1 + z^2} \right\} dz$$
$$= 4 \left\{ \frac{z^3}{3} - \frac{z^2}{2} - z + \frac{1}{2} \log (1 + z^2) + \tan^{-1} z \right\}$$

$$\therefore \int \frac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{2}} + 1}\, dx = 4\left\{ \frac{1}{3} x^{\frac{3}{4}} - \frac{1}{2} x^{\frac{1}{2}} - x^{\frac{1}{4}} + \frac{1}{2}\log(1 + \sqrt{x}) + \tan^{-1} x^{\frac{1}{4}}. \right\}$$

3. $$\int \frac{dx}{1 + \sqrt{x}} = 2\left\{ \sqrt{x} - \log(1 + \sqrt{x}) \right\}.$$

4. $$\int \frac{x^{\frac{1}{2}} dx}{x^{\frac{3}{4}} + 1} = \frac{4}{3} x^{\frac{3}{4}} - \frac{4}{3}\log(x^{\frac{3}{4}} + 1).$$

176. Fractions containing only binomial surds of the first degree.

The method of rationalization is the same as in the last article.

EXAMPLES.

1. $$\int \frac{dx}{(x-3)^{\frac{1}{2}} + (x-3)^{\frac{1}{3}}}.$$

Let $$x - 3 = z^6,$$

then $(x - 3)^{\frac{1}{2}} = z^3, \quad (x - 3)^{\frac{1}{3}} = z^2, \quad dx = 6 z^5 dz;$

$$\therefore \int \frac{dx}{(x-3)^{\frac{1}{2}} + (x-3)^{\frac{1}{3}}} = \int \frac{6 z^5 dz}{z^3 + z^2}$$

$$= 6\int \frac{z^3 dz}{z + 1}$$

$$= 6\left\{ \frac{z^3}{3} - \frac{z^2}{2} + z - \log(1 + z) \right\}.$$

Hence, substituting the value of $z = (x - 3)^{\frac{1}{6}}$, we have

$$\int \frac{dx}{(x-3)^{\frac{1}{2}} + (x-3)^{\frac{1}{3}}} = 2\sqrt{x-3} - 3\sqrt[3]{x-3} + 6\sqrt[6]{x-3} - 6\log(1 + \sqrt[6]{x-3}).$$

2. $$\int \frac{dx}{\sqrt{a + bx}}.$$

Here $$a + bx = z^2,$$

$$\therefore \quad \sqrt{a+bx} = z, \quad dx = \frac{2\,zdz}{b}.$$

Hence $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2}{b}\int dz = \frac{2}{b}z = \frac{2}{b}\sqrt{a+bx}.$$

3. $$\int \frac{(8x-8)^{\frac{1}{3}}}{(ax-a)^{\frac{1}{2}}}dx = \frac{2}{\sqrt{a}}\int \frac{(x-1)^{\frac{1}{3}}}{(x-1)^{\frac{1}{2}}}dx$$

$$= \frac{2}{\sqrt{a}}\int (x-1)^{-\frac{1}{6}}dx = \frac{12}{5\sqrt{a}}(x-1)^{\frac{5}{6}}.$$

We could rationalize by making $x - 1 = z^6$, but the process is obviously unnecessary in this case.

4. $$\int \frac{dx}{x\sqrt{x+2}} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{x+2} - \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}.$$

5. $$\int \frac{\sqrt[4]{2x-1}}{x - \sqrt{2x-1}}dx$$

$$= 4\sqrt[4]{2x-1} - \frac{2\sqrt[4]{2x-1}}{\sqrt{2x-1}-1} + 3\log\frac{\sqrt[4]{2x-1}-1}{\sqrt[4]{2x-1}+1}.$$

6. $$\int \frac{x^3dx}{\sqrt{(1+4x)^5}}$$

$$= \frac{1}{128}\left\{\frac{\sqrt{(1+4x)^3}}{3} - 3\sqrt{1+4x} - \frac{3}{\sqrt{1+4x}} + \frac{1}{3\sqrt{(1+4x)^3}}\right\}.$$

7. $$\int \frac{dx}{\sqrt[3]{x+1}+1}$$

$$= \frac{3}{2}\sqrt[3]{(x+1)^2} - 3\sqrt[3]{x+1} + 3\log(\sqrt[3]{x+1}+1).$$

8. $$\int \frac{x\,dx}{\sqrt{(1+x)^3}} = 2\sqrt{1+x} + \frac{2}{\sqrt{1+x}}.$$

177. Functions containing quadratic surds of the form

$$\sqrt{a + bx \pm x^2}.$$

1. When x^2 is *positive* let

$$\sqrt{a + bx + x^2} = z - x.$$

2. When x^2 is *negative* let

$$\sqrt{a + bx - x^2} = \sqrt{(x-c)(d-x)} = (x-c)z,$$

c and *d* being roots of the equation,

$$x^2 - bx - a = 0.$$

EXAMPLES.

1. $$\int \frac{dx}{\sqrt{2 + 3x + x^2}}.$$

Let $$\sqrt{2 + 3x + x^2} = z - x,$$

then, squaring, $$2 + 3x = z^2 - 2zx;$$

$$\therefore\ x = \frac{z^2 - 2}{2z + 3},$$

$$dx = \frac{2(z^2 + 3z + 2)}{(2z+3)^2}\,dz,$$

$$\sqrt{2 + 3x + x^2} = z - x = z - \frac{z^2 - 2}{2z + 3} = \frac{z^2 + 3z + 2}{2z + 3}.$$

Therefore,

$$\int \frac{dx}{\sqrt{2 + 3x + x^2}} = \int \frac{2\,dz}{2z + 3} = \log(2z + 3)$$

$$= \log(2\sqrt{2 + 3x + x^2} + 2x + 3).$$

2. $\displaystyle\int\frac{dx}{\sqrt{2+x-x^2}}.$

Here, $x^2 - x - 2 = (x+1)(x-2).$

$$\therefore\ \sqrt{2+x-x^2} = \sqrt{(x+1)(2-x)}.$$

Let $\sqrt{(x+1)(2-x)} = (x+1)z;$

then, $2 - x = (x+1)z^2 = xz^2 + z^2.$

$$\therefore\ x = \frac{2-z^2}{z^2+1},$$

$$dx = -\frac{6z}{(z^2+1)^2}dz,$$

$$\sqrt{2+x-x^2} = (x+1)z = \left(\frac{2-z^2}{z^2+1}+1\right)z = \frac{3z}{z^2+1}.$$

Therefore, $\displaystyle\int\frac{dx}{\sqrt{2+x-x^2}} = -\int\frac{2\,dz}{z^2+1} = 2\cot^{-1}z$

$$= 2\cot^{-1}\sqrt{\frac{2-x}{x+1}}.$$

3. $\displaystyle\int\frac{dx}{\sqrt{2-x-x^2}} = 2\cot^{-1}\sqrt{\frac{1-x}{x+2}}.$

4. $\displaystyle\int\frac{dx}{\sqrt{a+x^2}} = \log\left(x+\sqrt{a+x^2}\right).$

5. $\displaystyle\int\frac{dx}{x\sqrt{1+x+x^2}} = \log\frac{x-1+\sqrt{1+x+x^2}}{x+1+\sqrt{1+x+x^2}}.$

6. $\displaystyle\int\frac{dx}{x\sqrt{2-x+x^2}} = \frac{1}{\sqrt{2}}\log\frac{x-\sqrt{2}+\sqrt{2-x+x^2}}{x+\sqrt{2}+\sqrt{2-x+x^2}}.$

7. $\displaystyle\int\frac{dx}{x\sqrt{2+x-x^2}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2+2x}-\sqrt{2-x}}{\sqrt{2+2x}+\sqrt{2-x}}.$

8. $$\int \frac{dx}{\sqrt{2 - 2x - x^2}} = 2 \cot^{-1}\left(\frac{\sqrt{3} - 1 - x}{\sqrt{3} + 1 + x}\right)^{\frac{1}{2}}.$$

9. $$\int \frac{dx}{\sqrt{a + bx + x^2}} = \log\left(\frac{b}{2} + x + \sqrt{a + bx + x^2}\right).$$

10. $$\int \frac{dx}{(2 + 3x)\sqrt{4 - x^2}} = \frac{1}{4\sqrt{2}} \log \frac{\sqrt{4 + 2x} - \sqrt{2 - x}}{\sqrt{4 + 2x} + \sqrt{2 - x}}.$$

11. $$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log\left(\frac{b}{2\sqrt{c}} + x\sqrt{c} + \sqrt{a + bx + cx^2}\right).$$

12. $$\int \frac{dx}{(a+x)\sqrt{x^2+b^2}}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \log \frac{x + \sqrt{x^2+b^2} + a - \sqrt{a^2+b^2}}{x + \sqrt{x^2+b^2} + a + \sqrt{a^2+l^2}}.$$

178. Methods in Special Cases. As already stated (§ 174), no general rule can be given for the rationalization of irrational forms, other than the very general rule given in that article, viz., *to substitute in the given expression some variable that will effect the object in view.* We shall conclude this chapter by considering some forms as illustrative of the usual method of attacking the problem. It may be remarked that the process of substitution is also frequently used in simplifying irrational forms.

EXAMPLES.

1. $$\frac{x^3 dx}{\sqrt[3]{(x^2 + 1)^2}}.$$

Let $x^2 + 1 = z$;

then, $x^3 = (z - 1)^{\frac{3}{2}}$; $dx = \dfrac{dz}{2x} = \dfrac{dz}{2(z-1)^{\frac{1}{2}}}$, $\sqrt[3]{(x^2 + 1)^2} = z^{\frac{2}{3}}$;

$$\therefore \int \frac{x^3 dx}{\sqrt[3]{(x^2+1)^2}} = \tfrac{1}{2}\int \frac{(z-1)\,dz}{z^{\frac{2}{3}}}$$

$$= \tfrac{3}{2}\left(\frac{z^{\frac{4}{3}}}{4} - z^{\frac{1}{3}}\right)$$

$$= \tfrac{3}{8}(x^2+1)^{\frac{1}{3}}(x^2-3).$$

2. $\int x^3 \sqrt{a - x^2}\, dx.$

Let $$a - x^2 = z^2;$$

then, $x^3 = (a - z^2)^{\frac{3}{2}}$, $\sqrt{a - x^2} = z$, $dx = -\dfrac{zdz}{\sqrt{a - z^2}}$.

$$\therefore \int x^3 \sqrt{a - x^2}\, dx = -\int (a - z^2)\, z^2 dz = -\frac{az^3}{3} + \frac{z^5}{5}$$

$$= z^3\left(\frac{z^2}{5} - \frac{a}{3}\right) = \frac{z^3}{15}(3\,z^2 - 5\,a)$$

$$= \frac{-\sqrt{(a - x^2)^3}}{15}(3\,x^2 + 2\,a).$$

3. $\int \dfrac{dx}{x\sqrt{x^3 - a^3}}.$

Let $x^3 = y^2$. $\therefore dx = \tfrac{2}{3}\dfrac{ydy}{x^2} = \tfrac{2}{3}\dfrac{dy}{y^{\frac{1}{3}}}$,

$$x = y^{\frac{2}{3}},\quad \sqrt{x^3 - a^3} = \sqrt{y^2 - a^3}.$$

Hence, $$\int \frac{dx}{x\sqrt{x^3 - a^3}} = \tfrac{2}{3}\int \frac{dy}{y\sqrt{y^2 - a^3}}$$

$$= \frac{2}{3a^{\frac{3}{2}}}\sec^{-1}\frac{y}{a^{\frac{3}{2}}},$$

$$= \frac{2}{3a^{\frac{3}{2}}}\sec^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

4. $\int \frac{dx}{\sqrt{(ax^2+b)^3}}$.

Let $$x = \frac{1}{y}; \quad \therefore dx = -\frac{dy}{y^2},$$

$$\sqrt{(ax^2+b)^3} = \frac{\sqrt{(a+by^2)^3}}{y^3}.$$

Hence, $$\int \frac{dx}{\sqrt{(ax^2+b)^3}} = -\int \frac{ydy}{(a+by^2)^{\frac{3}{2}}} = -\int (a+by^2)^{-\frac{3}{2}} ydy$$

$$= \frac{1}{b\sqrt{a+by^2}}$$

$$= \frac{x}{b\sqrt{ax^2+b}}.$$

5. $\int \frac{x^2 - x}{(x-2)^3} dx.$

We may, of course, integrate in this case by decomposing into partial fractions. (See § 168.) A simpler process is to

let $$x - 2 = z,$$

then, $$\int \frac{x^2 - x}{(x-2)^3} dx = \int \frac{(z+2)^2 - (z+2)}{z^3} dz$$

$$= \int \frac{z^2 + 3z + 2}{z^3} dz$$

$$= \log z - \frac{3}{z} - \frac{1}{z^2}$$

$$= \log(x-2) - \frac{3}{x-2} - \frac{1}{(x-2)^2}.$$

6. $\int \frac{dx}{x(a+x^3)}$.

Let $$x^3 = z. \quad \therefore dx = \frac{dz}{3x^2} = \frac{1}{3}\frac{dz}{z^{\frac{2}{3}}}.$$

But $$x = z^{\frac{1}{3}} \text{ and } a + x^3 = a + z.$$

Hence, $\int \frac{dx}{x(a+x^3)} = \frac{1}{3}\int \frac{dz}{z(a+z)} = \frac{1}{3}\int \left(\frac{A}{z} + \frac{B}{a+z}\right) dz.$

By § 169, we find $A = \frac{1}{a}$ and $B = -\frac{1}{a}$; hence,

$$\frac{1}{3}\int \frac{dz}{z(a+z)} = \frac{1}{3}\left(\frac{1}{a}\log z - \frac{1}{a}\log(a+z)\right)$$

$$= \frac{1}{3a}\log\frac{z}{a+z};$$

$$\therefore \int \frac{dx}{x(a+x^3)} = \frac{1}{3a}\log\frac{x^3}{a+x^3}.$$

7. $\int \frac{e^x dx}{e^{2x}-4} = \int \frac{dz}{z^2-4}$ when $e^x = z$.

$$\int \frac{dz}{z^2-4} = \frac{1}{4}\log\frac{z-2}{z+2} \text{ (by § 164, 23).}$$

$$\therefore \int \frac{e^x dx}{e^{2x}-4} = \frac{1}{4}\log\frac{e^x-2}{e^x+2}.$$

8. $\int \frac{e^{2x}dx}{\sqrt[4]{e^x+1}} = \frac{4}{21}(3e^x-4)(e^x+1)^{\frac{3}{4}}.$

Let $e^x + 1 = z$.

9. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx = \frac{\sqrt{x^4+x^2+1}}{x}.$

Let $x^2 + \frac{1}{x^2} = z$.

10. $\int \frac{dx}{((a^2+x^2)^{\frac{1}{2}}+x)^{\frac{1}{4}}} = \frac{2}{3}((a^2+x^2)^{\frac{1}{2}}+x)^{\frac{3}{4}} - \frac{2}{5}\frac{a^2}{((a^2+x^2)^{\frac{1}{2}}+x)^{\frac{5}{4}}}.$

Let $(a^2+x^2)^{\frac{1}{2}} + x = z$.

11. $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt{a^3 - x^3}} = \tfrac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}}.$

Let $x^{\frac{3}{2}} = z$.

12. $\displaystyle\int \frac{dx}{\sqrt{(2a - x)(x - b)}} = \text{vers}^{-1}\frac{2(x-b)}{2a-b}.$

Let $x - b = z$.

CHAPTER III.

BINOMIAL DIFFERENTIALS.

179. The most general form of the binomial differential is

$$x^c(ax^d + bx^f)^h\,dx, \tag{1}$$

in which c, d, f, and h are any constants, positive or negative, entire or fractional. It is our purpose to explain the method of integrating these expressions.

180. *Every binomial differential may be reduced to the form*

$$x^m(a + bx^n)^p\,dx,$$

in which m and n are integers, n being positive.

For in (1), § 179, let $f > d$, and let us multiply and divide that expression by x^{dh}. We have

$$x^{c+dh}(a + bx^{f-d})^h\,dx,$$

in which $f - d$ is *a positive* whole number or fraction, and $c + dh$ is positive or negative, entire or fractional.

Let us suppose these exponents are fractional, and that

$$c + dh = \pm\frac{s}{t} \text{ and } f - d = +\frac{l}{k}.$$

Then the above expression takes the form

$$x^{\pm\frac{s}{t}}(a + bx^{\frac{l}{k}})^h\,dx.$$

Now let $x = z^{kt}$, then $x^{\pm\frac{s}{t}} = z^{\pm sk}$, $x^{\frac{l}{k}} = z^{lt}$, $dx = ktz^{kt-1}\,dz$;

hence, $x^{\pm\frac{s}{t}}(a + bx^{\frac{l}{k}})^h\,dx = ktz^{\pm sk + kt - 1}(a + bz^{lt})^h\,dz,$

in which the exponents of z are integers and lt positive.

Let $h = p$, $\pm sk + kt - 1 = m$, $lt = n$; then

$$x^c (ax^d + bx^f)^h \, dx = x^{\pm\frac{s}{t}} (a + bx^{\frac{l}{k}})^h \, dx$$
$$= ktz^{\pm sk + kt - 1} (a + bz^{lt})^h \, dz$$
$$= ktz^m (a + bz^n)^p \, dz,$$

which is of the required form. We shall confine our attention in what follows, therefore, to the form

$$x^m (a + bx^n)^p \, dx,$$

in which m and n are integers, n being positive, and p is whole or fractional, positive or negative.

RATIONALIZATION.

181. Case I. *If p is a positive integer* the form

$$x^m (a + bx^n)^p \, dx$$

is rational, and may be integrated by expansion and monomial integration.

Thus, $\int x^3 (1 + 2x^2)^2 \, dx = \int x^3 (1 + 4x^2 + 4x^4) \, dx$

$$= \int x^3 dx + 4\int x^5 dx + 4\int x^7 dx$$
$$= \frac{x^4}{4} + \frac{2}{3}x^6 + \frac{1}{2}x^8.$$

If p is a negative integer and greater than 1, we proceed as explained in § 172. Thus,

$$\int (1 + x^2)^{-2} \, dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x.$$

See latter part of Ex. 1, p. 260.

182. CASE II. *Where* $\frac{m+1}{n}$ *is an integer or zero and* $p = \frac{h}{k}$, *a fraction.*

In this case we have

$$\int x^m (a + bx^n)^p \, dx = \int x^m (a + bx^n)^{\frac{h}{k}} \, dx,$$

which may be rationalized, and hence integrated by putting

$$a + bx^n = z^k.$$

For under this assumption we have,

$$x = \left(\frac{z^k - a}{b}\right)^{\frac{1}{n}}, \quad x^m = \left(\frac{z^k - a}{b}\right)^{\frac{m}{n}}, \quad (a + bx^n)^{\frac{h}{k}} = z^h,$$

$$dx = \frac{k}{nb}\left(\frac{z^k - a}{b}\right)^{\frac{1}{n}-1} z^{k-1} \, dz;$$

hence,

$$\int x^m (a + bx^n)^{\frac{h}{k}} \, dx = \int \left(\frac{z^k - a}{b}\right)^{\frac{m}{n}} z^h \frac{k}{nb}\left(\frac{z^k - a}{b}\right)^{\frac{1}{n}-1} z^{k-1} dz$$

$$= \frac{k}{nb^{\frac{m+1}{n}}} \int (z^k - a)^{\frac{m+1}{n}-1} z^{h+k-1} \, dz,$$

— a form which is rational when $\frac{m+1}{n}$ is a integer or zero.

As an illustration let us find

$$\int x^3 (1 + 2x^2)^{\frac{3}{2}} \, dx.$$

Here, $\frac{m+1}{n} = \frac{3+1}{2} = 2$, an integer.

Therefore let $\quad 1 + 2x^2 = z^2.$

Hence, $x = \left(\frac{z^2 - 1}{2}\right)^{\frac{1}{2}}, \quad x^3 = \left(\frac{z^2 - 1}{2}\right)^{\frac{3}{2}}, \quad (1 + 2x^2)^{\frac{3}{2}} = z^3,$

$$dx = \tfrac{1}{2}\left(\frac{z^2 - 1}{2}\right)^{-\frac{1}{2}} \frac{2\,zdz}{2};$$

$$\therefore \int x^3 (1 + 2x^2)^{\frac{3}{2}} dx = \int \left(\frac{z^2 - 1}{2}\right)^{\frac{3}{2}} z^3 \frac{\sqrt{2}}{2} \frac{zdz}{(z^2 - 1)^{\frac{1}{2}}}$$

$$= \frac{\sqrt{2}}{4\sqrt{2}} \int z^3 (z^2 - 1)\, zdz$$

$$= \tfrac{1}{4}\left\{\frac{z^7}{7} - \frac{z^5}{5}\right\}.$$

Hence, $\displaystyle\int x^3 (1 + 2x^2)^{\frac{3}{2}} dx = \tfrac{1}{4}\left\{\frac{(1 + 2x^2)^{\frac{7}{2}}}{7} - \frac{(1 + 2x^2)^{\frac{5}{2}}}{5}\right\}.$

183. CASE III. *Where* $p = \frac{h}{k}$, *a fraction, and* $\frac{m + 1}{n} + \frac{h}{k}$ *is an integer or zero.*

In this case, as in Case II., we have

$$\int x^m (a + bx^n)^p dx = \int x^m (a + bx^n)^{\frac{h}{k}} dx.$$

This expression may be rationalized by putting

$$a + bx^n = x^n z^k.$$

For under this assumption we have,

$$x = \left(\frac{a}{z^k - b}\right)^{\frac{1}{n}}, \quad x^m = \left(\frac{a}{z^k - b}\right)^{\frac{m}{n}},$$

$$(a + bx^n)^{\frac{h}{k}} = (x^n z^k)^{\frac{h}{k}} = x^{\frac{nh}{k}} z^h = \left(\frac{a}{z^k - b}\right)^{\frac{h}{k}} z^h, \text{ and}$$

$$dx = -\frac{a}{n}\left(\frac{a}{z^k - b}\right)^{\frac{1}{n} - 1} \frac{kz^{k-1}\,dz}{(z^k - b)^2};$$

$$\therefore \int x^m (a + bx^n)^{\frac{h}{k}} dx$$

$$= -\int \left(\frac{a}{z^k - b}\right)^{\frac{m}{n}} \left(\frac{a}{z^k - b}\right)^{\frac{h}{k}} z^h \frac{a}{n}\left(\frac{a}{z^k - b}\right)^{\frac{1}{n}-1} \frac{kz^{k-1}dz}{(z^k - b)^2}$$

$$= -\frac{ka}{n}\int \left(\frac{a}{z^k - b}\right)^{\frac{m+1}{n}+\frac{h}{k}-1} \frac{z^{h+k-1}}{(z^k - b)^2} dz$$

$$= -\frac{k}{an}\int \left(\frac{a}{z^k - b}\right)^{\frac{m+1}{n}+\frac{h}{k}+1} z^{h+k-1} dz,$$

— a form which is rational and therefore integrable when $\frac{m+1}{n} + \frac{h}{k}$ is an integer or zero.

To illustrate this case let us write,

$$\int \frac{dx}{x^2\sqrt{2 + 3x^2}} = \int x^{-2}(2 + 3x^2)^{-\frac{1}{2}} dx.$$

Here, $\frac{m+1}{n} + \frac{h}{k} = \frac{-2+1}{2} - \frac{1}{2} = -1$, an integer;

$\therefore$ let $\qquad 2 + 3x^2 = x^2z^2.$

Hence, $x = \left(\frac{2}{z^2 - 3}\right)^{\frac{1}{2}}, \quad x^{-2} = \frac{z^2 - 3}{2},$

$$(2 + 3x^2)^{-\frac{1}{2}} = (xz)^{-1} = \left(\frac{z^2 - 3}{2}\right)^{\frac{1}{2}} \frac{1}{z},$$

$$dx = -\tfrac{1}{2}\left(\frac{2}{z^2 - 3}\right)^{-\frac{1}{2}} \frac{4zdz}{(z^2 - 3)^2} = -2\left(\frac{z^2 - 3}{2}\right)^{\frac{1}{2}} \frac{zdz}{(z^2 - 3)^2}.$$

$$\therefore \int x^{-2}(2 + 3x^2)^{-\frac{1}{2}} dx$$

$$= -2\int \frac{z^2 - 3}{2}\left(\frac{z^2 - 3}{2}\right)^{\frac{1}{2}} \frac{1}{z}\left(\frac{z^2 - 3}{2}\right)^{\frac{1}{2}} \frac{zdz}{(z^2 - 3)^2}$$

$$= -2\int \frac{dz}{4} = -\frac{z}{2};$$

$$\therefore \int \frac{dx}{x^2\sqrt{2 + 3x^2}} = -\tfrac{1}{2}\frac{\sqrt{2 + 3x^2}}{x}.$$

EXAMPLES.

1. $$\int x^2 (a + bx^3)^2\, dx = \frac{a^2x^3}{3} + \frac{abx^6}{3} + \frac{b^2x^9}{9}.$$

2. $$\int x^3 (a + bx^2)^{\frac{3}{2}} dx = \frac{(a + bx^2)^{\frac{7}{2}}}{7\, b^2} - \frac{a\,(a + bx^2)^{\frac{5}{2}}}{5\, b^2}.$$

3. $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log \left\{ x + \sqrt{a^2 + x^2} \right\}.$$

4. $$\int \frac{dx}{x^4 \sqrt{1 - x^2}} = -\frac{1 + 2\, x^2}{3\, x^3} \sqrt{1 - x^2}.$$

5. $$\int \frac{x^5 dx}{a^2 + x^2} = \frac{x^2 (x^2 - 2\, a^2)}{4} + a^4 \log \sqrt{a^2 + x^2}.$$

6. $$\int \frac{dx}{x^4 \sqrt{1 + x^2}} = \frac{\sqrt{1 + x^2}}{3\, x^3} (2\, x^2 - 1).$$

7. $$\int x^3 \sqrt{1 + x^2}\, dx = \frac{\sqrt{(1 + x^2)^3}}{15} (3\, x^2 - 2).$$

8. $$\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = -\frac{2\, a^2 + x^2}{3} \sqrt{a^2 - x^2}.$$

9. $$\int \frac{dx}{x^4 \sqrt{a^2 - x^2}} = -\frac{a^2 + 2\, x^2}{3\, a^4 x^3} \sqrt{a^2 - x^2}.$$

10. $$\int \frac{x^3 dx}{\sqrt{1 + 2\, x^2}} = \tfrac{1}{12} \sqrt{(1 + 2\, x^2)^3} - \tfrac{1}{4} \sqrt{1 + 2\, x^2}.$$

11. $$\int \frac{x^3\, dx}{(a + bx^2)^{\frac{3}{2}}} = \frac{2\, a + bx^2}{b^2 \sqrt{a + bx^2}}.$$

12. $\displaystyle\int \frac{x^2dx}{(a+bx^2)^{\frac{5}{2}}} = \frac{x^3}{3a(a+bx^2)^{\frac{3}{2}}}.$

13. $\displaystyle\int \frac{x^3dx}{\sqrt{a+bx^2}} = \frac{bx^2-2a}{3b^2}\sqrt{a+bx^2}.$

14. $\displaystyle\int \frac{dx}{x^2(a+bx^2)^{\frac{3}{2}}} = -\frac{a+2bx^2}{a^2x(a+bx^2)^{\frac{1}{2}}}.$

15. $\displaystyle\int \frac{dx}{x^2(1+x^2)^{\frac{3}{2}}} = -\frac{2x^2+1}{x\sqrt{1+x^2}}.$

16. $\displaystyle\int \frac{adx}{\sqrt{(1+x^2)^3}} = \frac{ax}{\sqrt{1+x^2}}.$

17. $\displaystyle\int \frac{x^5dx}{\sqrt{2x^2+1}} = \frac{3x^4-2x^2+2}{30}\sqrt{2x^2+1}.$

18. $\displaystyle\int x^3(a^2-x^2)^{\frac{1}{5}}dx = \frac{5}{132}(6x^4-a^2x^2-5a^4)(a^2-x^2)^{\frac{6}{5}}.$

REDUCTION FORMULAE.

184. It frequently happens that a given binomial differential is of such a form that the foregoing methods of rationalization are inapplicable. We proceed to derive a general method, by aid of which we are enabled to reduce *any* binomial differential to one of the type forms given in Chapter I.

185. *To deduce formulae for the reduction of the exponent of the variable without the parenthesis.*

Two cases present themselves according as m is *positive* or *negative* in the form

$$\int x^m(a+bx^n)^p\,dx.$$

1. **When m is positive.**

Multiplying and dividing the above form by x^{n-1}, we have

$$\int x^m (a + bx^n)^p \, dx = \int x^{m-n+1} (a + bx^n)^p \, x^{n-1} \, dx.$$

In the formula for integration by parts, § 165,

$$\int u dv = uv - \int v du,$$

let $u = x^{m-n+1}$ and $dv = (a + bx^n)^p \, x^{n-1} \, dx.$

Hence,

$$\int x^m (a + bx^n)^p \, dx$$

$$= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{nb (p + 1)} - \frac{m - n + 1}{nb (p + 1)} \int (a + bx^n)^{p+1} x^{m-n} dx$$

$$= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{nb (p + 1)} - \frac{(m - n + 1)}{nb (p + 1)} \left\{ \int ax^{m-n} (a + bx^n)^p \, dx \right.$$

$$\left. + \int bx^m (a + bx^n)^p \, dx \right\}.$$

Clearing of fractions and transposing,

$$b (m - n + 1) \int x^m (a + bx^n)^p \, dx$$

to the first member we have

$$(nbp + nb + mb - nb + b) \int x^m (a + bx^n)^p \, dx$$

$$= x^{m-n+1} (a + bx^n)^{p+1} - a (m - n + 1) \int x^{m-n} (a + bx^n)^p \, dx.$$

Hence,

$$\int x^m (a + bx^n)^p \, dx$$

$$= \frac{x^{m-n+1} (a + bx^n)^{p+1} - a (m - n + 1) \int x^{m-n} (a + bx^n)^p dx}{b (np + m + 1)} \quad . . (A)$$

which is the required form. Formula (A), it will be observed, enables us to make the required integration depend upon the integral of an expression in which the exponent of the variable without the parenthesis (m) is *decreased* by the exponent of the variable within the parenthesis (n).

2. **When m is negative.**

Let $m = m' + n$, and therefore $m' = m - n$.

Substituting these values in Formula A, and clearing of fractions, we have,

$$b(np + m' + n + 1)\int x^{m'+n}(a + bx^n)^p\,dx$$
$$= x^{m'+1}(a + bx^n)^{p+1} - a(m' + 1)\int x^{m'}(a + bx^n)^p\,dx.$$

Solving this equation for $\int x^{m'}(a + bx^n)^p\,dx$ and dropping accents as no longer necessary, we have,

$$\int x^m(a + bx^n)^p\,dx$$
$$= \frac{x^{m+1}(a+bx^n)^{p+1} - b(np+m+n+1)\int x^{m+n}(a+bx^n)^p\,dx}{a(m+1)}. \quad (C)$$

This formula, commonly called Formula C, enables us to make the required integral depend upon another in which the exponent of the variable without the parenthesis (m) is increased by the exponent (n) of the variable within the parenthesis.

186. *To deduce formulæ for the reduction of the exponent of the parenthesis.*

1. **When p is positive.**

In this case in the general form,

$$\int x^m(a + bx^n)^p\,dx,$$

let $u = (a + bx^n)^p$ and $dv = x^m\,dx$. Integrating by parts we have,

$$\int x^m (a + bx^n)^p\,dx$$
$$= \frac{x^{m+1}(a + bx^n)^p}{m+1} - \frac{nbp}{m+1}\int x^{m+n}(a + bx^n)^{p-1}\,dx \quad . \quad . \quad . \quad (1)$$

Applying formula (A) to the last term in the second member, we have,

$$\int x^{m+n}(a + bx^n)^{p-1}\,dx$$
$$= \frac{x^{m+1}(a + bx^n)^p - a(m+1)\int x^m (a + bx^n)^{p-1}\,dx}{b(np + m + 1)}.$$

Substituting this value in (1), we have after reduction,

$$\int x^m (a + bx^n)^p\,dx$$
$$= \frac{x^{m+1}(a + bx^n)^p + anp\int x^m (a + bx^n)^{p-1}\,dx}{np + m + 1} \quad . \quad . \quad . \quad (B)$$

Formula (B) enables us to *decrease* the exponent of the binomial by 1.

2. When is p negative.

Let $p = p' + 1$ and therefore, $p' = p - 1$.

Substituting these values in (B), we have, after clearing of fractions,

$$(np' + m + n + 1)\int x^m (a + bx^n)^{p'+1}\,dx$$
$$= x^{m+1}(a + bx^n)^{p'+1} + an(p' + 1)\int x^m (a + bx^n)^{p'}\,dx.$$

Dropping accents and solving we have,

$$\int x^m (a + bx^n)^p \, dx = \frac{-x^{m+1}(a+bx^n)^{p+1} + (np+m+n+1)\int x^m (a+bx^n)^{p+1} dx}{an(p+1)} \quad . \; . \; (D)$$

Formula (D) enables us to *increase* the exponent of the binomial by 1.

187. Summary. For ease of reference we collect here the foregoing formulae, and rearrange them in accordance with the letters used to designate them.

$$\int x^m (a + bx^n)^p \, dx = \frac{x^{m-n+1}(a+bx^n)^{p+1} - a(m-n+1)\int x^{m-n}(a+bx^n)^p dx}{b(np+m+1)} \quad . \; . \; . \; (A)$$

$$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1}(a+bx^n)^p + anp\int x^m (a+bx^n)^{p-1} dx}{np+m+1} \quad . \; . \; . \; . \; . \; . \; (B)$$

$$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1}(a+bx^n)^{p+1} - b(np+m+n+1)\int x^{m+n}(a+bx^n)^p dx}{a(m+1)} \quad (C)$$

$$\int x^m (a + bx^n)^p \, dx = \frac{-x^{m+1}(a+bx^n)^{p+1} + (np+m+n+1)\int x^m (a+bx^n)^{p+1} dx}{an(p+1)} \quad (D)$$

From an examination of the last term in the second member of each of these equations, we elicit the following:

(A) *decreases* the exponent of the *monomial* factor;

(B) *decreases* the exponent of the *binomial* factor;

(*C*) *increases* the exponent of the *monomial* factor;
(*D*) *increases* the exponent of the *binomial* factor.
Or, more generally,
(*A*) and (*B*) *decrease exponents;*
(*C*) and (*D*) *increase exponents.*

The terms *increase* and *decrease* are used of course in an algebraic sense.

EXAMPLES.

1. $\int \frac{x^3 dx}{\sqrt{a^2 - x^2}}$.

In this case we observe that the given expression can be integrated by (1) § 162 if we can make the integral depend upon the integral of $\frac{x dx}{\sqrt{a^2 - x^2}}$ $[= x (a^2 - x^2)^{-\frac{1}{2}} dx]$. This we are enabled to do by (*A*), as that formula decreases the exponent without the parenthesis by the exponent of the variable within the parenthesis. Writing, therefore, the expression in the general form,

$$\int x^3 (a^2 - x^2)^{-\frac{1}{2}} dx,$$

we see that $m = 3$, $a = a^2$, $b = -1$, $n = 2$, $p = -\frac{1}{2}$. Substituting these values in (*A*) we have,

$$\int x^3 (a^2 - x^2)^{-\frac{1}{2}} dx = \frac{x^2 (a^2 - x^2)^{\frac{1}{2}} - 2 a^2 \int x (a^2 - x^2)^{-\frac{1}{2}} dx}{-3}.$$

$$= -\frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{3} + \frac{a^2}{3} \int (a^2 - x^2)^{-\frac{1}{2}} 2 x dx.$$

$$= -\frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{3} - \frac{2 a^2}{3} (a^2 - x^2)^{\frac{1}{2}}$$

$$= -\frac{(x^2 + 2 a^2) \sqrt{a^2 - x^2}}{3}.$$

2. $\displaystyle\int \frac{x^2 dx}{\sqrt{1 - x^2}}$.

Here by (A) we can make the required integration depend upon $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$. In this case

$$\int \frac{x^2 dx}{\sqrt{1 - x^2}} = \int x^2 (1 - x^2)^{-\frac{1}{2}} dx;$$

$\therefore\ m = 2,\ a = 1,\ b = -1,\ n = 2,\ p = -\frac{1}{2}$.

Substituting in A, we have,

$$\int x^2 (1 - x^2)^{-\frac{1}{2}} dx = \frac{x (1 - x^2)^{\frac{1}{2}} - \int (1 - x^2)^{-\frac{1}{2}} dx}{-2}$$

$$= \frac{-x (1 - x^2)^{\frac{1}{2}} + \sin^{-1} x}{2}.$$

3. $\displaystyle\int \sqrt{1 - x^2} dx$.

Writing the example in the form $\int (1 - x^2)^{\frac{1}{2}} dx$ we see that the expression can be integrated if we can decrease the exponent of the binomial by 1, thus making the integration depend upon the $\displaystyle\int (1 - x^2)^{-\frac{1}{2}} dx = \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$. Formula ($B$) enables us to do this. In this case $m = 0$, $a = 1$, $b = -1$, $n = 2$, $p = \frac{1}{2}$. Substituting these values in (B), we have

$$\int (1 - x^2)^{\frac{1}{2}} dx = \frac{x (1 - x^2)^{\frac{1}{2}} + \int (1 - x^2)^{-\frac{1}{2}} dx}{2}$$

$$= \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x.$$

4. $\int \frac{dx}{x^3\sqrt{x^2-a^2}}$.

Here $\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \int x^{-3}(-a^2+x^2)^{-\frac{1}{2}}\,dx.$

In this case we wish to increase the exponent without the parenthesis by the exponent of the variable within the parenthesis, thus making the integration depend upon

$$\int x^{-1}(-a^2+x^2)^{-\frac{1}{2}}\,dx = \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}.$$

Applying (C) in order to accomplish this result, making

$m = -3,\ a = -a^2,\ b = 1,\ n = 2,\ p = -\frac{1}{2}$, we have

$$\int x^{-3}(-a^2+x^2)^{-\frac{1}{2}}\,dx = \frac{x^{-2}(-a^2+x^2)^{\frac{1}{2}} + \int x^{-1}(-a^2+x^2)^{-\frac{1}{2}}\,dx}{2a^2}$$

$$= \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{x}{a}.$$

5. $\int \frac{dx}{(a^2+x^2)^2}$.

Here $\int \frac{dx}{(a^2+x^2)^2} = \int (a^2+x^2)^{-2}\,dx.$

Formula D is here applicable making the integration depend upon

$$\int (a^2+x^2)^{-1}\,dx = \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$ Making

$m = 0,\ a = a^2,\ b = 1,\ n = 2,\ p = -2,$

in (D), we have,

$$\int (a^2+x^2)^{-2}\,dx = \frac{-x(a^2+x^2)^{-1} - \int (a^2+x^2)^{-1}\,dx}{-2a^2}$$

$$= \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a}.$$

In the following examples give the formulas which are applicable:

6. $\int \frac{x^4 dx}{\sqrt{1 - x^2}}$. *Ans.* Formula (*A*) twice successively.

7. $\int (a^2 - x^2)^{\frac{3}{2}}$. *Ans.* Formula (*B*) twice.

8. $\int \frac{dx}{\sqrt{(x^2 + a^2)^3}}$. *Ans.* Formula (*D*).

9. $\int \frac{x^2 dx}{\sqrt{2ax - x^2}} = \int x^{\frac{3}{2}}(2a - x)^{-\frac{1}{2}} dx.$

Ans. Formula (*A*) twice successively.

10. $\int \frac{dx}{x^4 \sqrt{x^2 + a^2}}$. *Ans.* Formula (*C*) twice.

11. $\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}}$. *Ans.* (*D*) once, (*A*) once.

12. $\int x^2 \sqrt{a^2 - x^2}\, dx.$ *Ans.* (*A*) once, (*B*) once.

Integrate the following:

13. $\int x^2 \sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}.$

14. $\int \sqrt{(x^2 + a^2)^3}\, dx$

$$= \frac{x}{8}(2x^2 + 5a^2)\sqrt{x^2 + a^2} + \frac{3a^4}{8}\log(x + \sqrt{x^2 + a^2}).$$

15. $\int \frac{dx}{x^3\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{x}{\sqrt{a^2 - x^2} + a}.$

16. $$\int \frac{xdx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \text{ vers}^{-1}\frac{x}{a}.$$

17. $$\int \frac{x^2 dx}{\sqrt{2ax - x^2}} = -\frac{x + 3a}{2}\sqrt{2ax - x^2} + \frac{3}{2}a^2 \text{ vers}^{-1}\frac{x}{a}.$$

18. $$\int \frac{x^3 dx}{\sqrt{x^2 + a^2}} = \frac{x^2 - 2a^2}{3}\sqrt{x^2 + a^2}.$$

19. $$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

20. $$\int \frac{x^2 dx}{(a + bx^2)^3}$$
$$= -\frac{x}{4b(a + bx^2)^2} + \frac{x}{8ab(a + bx^2)} + \frac{1}{8\sqrt{a^3b^3}}\tan^{-1}\sqrt{\frac{bx^2}{a}}.$$

21. $$\int \frac{xdx}{\sqrt{ax - x^2}} = -\sqrt{ax - x^2} + \frac{a}{2}\text{ vers}^{-1}\frac{2x}{a}.$$

CHAPTER IV.

TRIGONOMETRIC INTEGRALS.

188. Trigonometric formulae. The following trigonometric relations will be found of service in what follows:

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\csc^2 x = 1 + \cot^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
$$2 \sin^2 x = 1 - \cos 2x$$
$$2 \cos^2 x = 1 + \cos 2x.$$

189. General Rule. *Separate the given expression into two factors, the first being the* **differential** *of one of the trigonometric functions. Express the other factor in terms of the trigonometric function of which the first factor is the differential.*

If this rule is followed the resulting expression is, *in general*, in an integrable form.

For example: $\int \sin^2 x \cos^3 x dx.$

Set aside $\cos x dx$ and observe that it is the differential of $\sin x$. The remaining factor, $\sin^2 x \cos^2 x$, must now be expressed in terms of $\sin x$. Hence we write

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x (1 - \sin^2 x) \cos x dx$$
$$= \int \sin^2 x \cos x dx - \int \sin^4 x \cos x dx$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}.$$

Similarly,

$$\int \tan^3 x \sec^4 x dx = \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx$$
$$= \int \tan^3 x \sec^2 x dx + \int \tan^5 x \sec^2 x dx$$
$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6}.$$

Or, thus

$$\int \tan^3 x \sec^4 x dx = \int (\sec^2 x - 1) \sec^3 x . \sec x \tan x dx$$
$$= \int \sec^5 x . \sec x \tan x dx - \int \sec^3 x . \sec x \tan x dx$$
$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4}.$$

If the given expression *does not contain* the differential of one of the trigonometric functions we must attempt by aid of some trigonometric relation to reduce it to an equivalent expression that does contain such differential. Thus,

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx$$
$$= \int \sec^2 x dx - \int dx$$
$$= \tan x - x.$$

Let us now examine the various trigonometric forms in a general way.

190. $\int \tan^m x dx$ **and** $\int \cot^m x dx$.

These forms can be integrated when m is an integer, positive or negative. For, assuming *m positive*,

$$\int \tan^m x dx = \int \tan^{m-2} x (\sec^2 x - 1) dx = \frac{\tan^{m-1} x}{m - 1} - \int \tan^{m-2} x dx.$$

The required integral is thus made to depend upon an integral, $\int \tan^{m-2} x dx$, in which the exponent has been diminished by 2.

By repeating the process we find that the integral will ultimately depend upon $\int \tan x dx = \log \sec x$ or $\int dx = x$ according as m is *odd* or *even*.

$$\text{Similarly, } \int \cot^m x dx = \int \cot^{m-2} x (\csc^2 x - 1) dx$$

$$= -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x dx.$$

Ultimately the integral will depend upon

$$\int \cot x dx = \log \sin x, \text{ or } \int dx = x.$$

If *m is negative*, then

$$\int \tan^{-m} x dx = \int \cot^m x dx,$$

and

$$\int \cot^{-m} x dx = \int \tan^m x dx.$$

The integration is therefore always possible when m is an integer.

EXAMPLES.

1. $\int \tan^3 x dx = \int \tan x (\sec^2 x - 1) dx = \frac{\tan^2 x}{2} - \log \sec x.$

2. $\int \tan^4 x dx = \frac{\tan^3 x}{3} - \tan x + x.$

3. $\int \tan^5 x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x.$

4. $\displaystyle\int \frac{dx}{\cot^3 x} = \int \tan^3 x dx.$ Cf. 1.

5. $\displaystyle\int \cot^3 x dx = -\frac{\cot^2 x}{2} - \log \sin x.$

6. $\displaystyle\int \cot^4 \frac{x}{3} dx = -\cot^3 \frac{x}{3} + 3 \cot \frac{x}{3} + x.$

7. $\displaystyle\int (\tan^4 x + \tan^6 x)\, dx = \frac{\tan^5 x}{5}.$

8. $\displaystyle\int (\tan^m x + \tan^{m+2} x)\, dx = \frac{\tan^{m+1} x}{m+1}.$

9. $\displaystyle\int (\cot^5 x + \cot^7 x)\, dx = -\frac{\cot^6 x}{6}.$

10. $\displaystyle\int (\cot^n x + \cot^{n+2} x)\, dx = -\frac{\cot^{n+1} x}{n+1}.$

11. $\displaystyle\int \frac{dx}{\cot^5 \frac{x}{4}} = \tan^4 \frac{x}{4} - 2 \tan^2 \frac{x}{4} + 4 \log \sec \frac{x}{4}.$

12. $\displaystyle\int \frac{\tan^2 x}{\cot^3 x} dx = \int \tan^5 x dx.$ Cf. 3.

191. $\displaystyle\int \sec^n x dx$ **and** $\displaystyle\int \csc^n x dx.$

If n is an *even positive integer* these forms can be readily integrated, for

$$\int \sec^n x dx = \int (\tan^2 x + 1)^{\frac{n-2}{2}} \sec^2 x dx,$$

$$\int \csc^n x dx = \int (\cot^2 x + 1)^{\frac{n-2}{2}} \csc^2 x dx,$$

in which $\frac{n-2}{2}$ is a positive integer. The binomial factors may therefore be expanded, and the terms integrated separately.

When n is an *odd positive integer* this method does not apply, as $\frac{n-2}{2}$ is a fraction. For this case, see § 197. When n is a negative integer, even or odd, we have

$$\int \sec^{-n} x dx = \int \cos^n x dx,$$

$$\int \csc^{-n} x dx = \int \sin^n x dx.$$

For these cases see § 197.

EXAMPLES.

1. $$\int \sec^6 x dx = \int (\tan^2 x + 1)^2 \sec^2 x dx$$

$$= \int (\tan^4 x + 2 \tan^2 x + 1) \sec^2 x dx$$

$$= \frac{\tan^5 x}{5} + \tfrac{2}{3} \tan^3 x + \tan x.$$

2. $$\int \csc^6 x dx = \int (\cot^2 x + 1)^2 \csc^2 x dx$$

$$= -\frac{\cot^5 x}{5} - \tfrac{2}{3} \cot^3 x - \cot x.$$

3. $$\int \csc^6 \frac{x}{3} dx = -\tfrac{3}{5} \cot^5 \frac{x}{3} - 2 \cot^3 \frac{x}{3} - 3 \cot \frac{x}{3}.$$

4. $$\int \sec^8 2x dx = \tfrac{1}{2} \left\{ \frac{\tan^7 2x}{7} + \tfrac{3}{5} \tan^5 2x + \tan^3 2x + \tan 2x \right\}.$$

192. $\int \tan^m x \sec^n x\,dx$ and $\int \cot^m x \csc^n xax$.

If n is a *positive even number* the method of the preceding article is applicable. Thus,

$$\int \tan^3 x \sec^4 x dx = \int \tan^3 x (\tan^2 x + 1) \sec^2 x dx$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4}.$$

If m is a *positive odd* number then

$$\int \tan^m x \sec^n x dx = \int \sec^{n-1} x (\sec^2 x - 1)^{\frac{m-1}{2}} \sec x \tan x dx.$$

$$\int \cot^m x \csc^n x dx = \int \csc^{n-1} x (\csc^2 x - 1)^{\frac{m-1}{2}} \csc x \cot x dx,$$

which since $\frac{m-1}{2}$ is a positive integer the binomial may be expanded and the terms integrated separately.

Thus $$\int \tan^3 x \sec^5 x dx = \int \sec^4 x (\sec^2 x - 1) \sec x \tan x dx$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}.$$

EXAMPLES.

1. $$\int \frac{\sec^4 x}{\tan^4 x} dx = \int \tan^{-4} x (\tan^2 x + 1) \sec^2 x dx$$

$$= -\frac{1}{\tan x} - \frac{1}{3 \tan^3 x}.$$

2. $$\int \tan^{\frac{2}{3}} x \sec^4 x dx = \tfrac{3}{11} \tan^{\frac{11}{3}} x + \tfrac{3}{5} \tan^{\frac{5}{3}} x.$$

3. $$\int \tan^5 x \sec^{\frac{3}{2}} x dx = \tfrac{2}{11} \sec^{\frac{11}{2}} x - \tfrac{4}{7} \sec^{\frac{7}{2}} x + \tfrac{2}{3} \sec^{\frac{3}{2}} x.$$

4. $$\int \frac{\tan^5 x}{\sec^3 x}\,dx = \int \sec^{-4} x\,(\sec^2 x - 1)^2 \sec x \tan x dx$$
$$= \sec x + \frac{2}{\sec x} - \frac{1}{3\sec^3 x}.$$

5. $$\int \tan^5 x \sec^5 x dx = \frac{\sec^9 x}{9} - \frac{2\sec^7 x}{7} + \frac{\sec^5 x}{5}.$$

6. $$\int \cot^3 x \csc x dx = \csc x - \frac{\csc^3 x}{3}.$$

7. $$\int \cot^5 x \csc^4 x dx = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8}.$$

8. $$\int \tan^{-\frac{3}{2}} x \sec^4 x dx = \tfrac{2}{3}\tan^{\frac{3}{2}} x - \frac{2}{\tan^{\frac{1}{2}} x}.$$

9. $$\int \sec^4 (x + a)\,dx = \frac{\tan^3 (x + a)}{3} + \tan (x + a).$$

10. $$\int \tan^4 mx dx = \frac{\tan^3 mx}{3m} - \frac{\tan mx}{m} + x.$$

11. $$\int \cot^5 x \csc^5 x dx = -\tfrac{1}{9}\csc^9 x + \tfrac{2}{7}\csc^7 x - \tfrac{1}{5}\csc^5 x.$$

12. $$\int \frac{\sec^4 x dx}{\sqrt{\cot^3 x}} = \tfrac{2}{5}\tan^{\frac{5}{2}} x + \tfrac{2}{9}\tan^{\frac{9}{2}} x.$$

193. Since

$$\tan x = \frac{\sin x}{\cos x},\quad \cot x = \frac{\cos x}{\sin x},$$

$$\sec x = \frac{1}{\cos x},\quad \csc x = \frac{1}{\sin x},$$

it is obvious that all the foregoing trigonometric integrals may be reduced to equivalent integrals involving only $\sin x$, or $\cos x$, or both. Let us consider now integrals involving these functions.

194. $\int \sin^m x dx$ and $\int \cos^m x dx$.

1. If m is a *positive odd number* the integration is readily performed.

For
$$\int \sin^m x dx = -\int (1 - \cos^2 x)^{\frac{m-1}{2}} (-\sin x dx),$$
$$\int \cos^m x dx = \int (1 - \sin^2 x)^{\frac{m-1}{2}} \cos x dx,$$

in which $\frac{m-1}{2}$ is a positive integer.

Thus
$$\int \sin^3 x dx = -\int (1 - \cos^2 x)(-\sin x dx)$$
$$= -\cos x + \frac{\cos^3 x}{3}$$
$$\int \cos^5 x dx = \int (1 - \sin^2 x)^2 \cos x dx$$
$$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x dx$$
$$= \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x.$$

2. If m is an *even positive integer* the integration may be affected as follows:

$$\int \sin^m x dx = \int \left(\frac{1 - \cos 2x}{2}\right)^{\frac{m}{2}} dx,$$
$$\int \cos^m x dx = \int \left(\frac{1 + \cos 2x}{2}\right)^{\frac{m}{2}} dx.$$

Since $\frac{m}{2}$ is a positive integer the binomials may be expanded into a finite series of terms. Those terms involving *odd* exponents can be integrated by the method explained under 1 and those involving even exponents may be further reduced by repeating the process above. Thus

$$\int \sin^4 x dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$$

$$= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1 + \cos 4x}{2} dx$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32}$$

$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x.$$

3. When m is an *even negative integer*, we have

$$\int \sin^{-m} x dx = \int \frac{dx}{\sin^m x} = \int \csc^m x dx,$$

$$\int \cos^{-m} x dx = \int \frac{dx}{\cos^m x} = \int \sec^m x dx.$$

We proceed, therefore, as in § 191. Thus,

$$\int \frac{dx}{\sin^6 x} = \int \csc^6 x dx = -\frac{\cot^5 x}{5} - \tfrac{2}{3}\cot^3 x - \cot x. \quad \text{Ex. 2, p. 293.}$$

$$\int \frac{dx}{\cos^6 x} = \int \sec^6 x dx = \frac{\tan^5 x}{5} + \tfrac{2}{3}\tan^3 x + \tan x. \quad \text{Ex. 1, p. 293.}$$

4. When $m = -1$.

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tfrac{1}{2}\int \frac{\dfrac{dx}{\cos^2\frac{x}{2}}}{\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = \tfrac{1}{2}\int \frac{\sec^2\frac{x}{2}dx}{\tan\frac{x}{2}} = \log\tan\frac{x}{2}.$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2} - x\right)} = \log\tan\frac{\frac{\pi}{2} - x}{2} = \log\tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$$

5. When m is an *odd negative integer greater than* 1. See § 197.

195. $\int \sin^m x \cos^n x dx$.

1. If either m or n, or both, are *odd positive integers* the method of § 194, 1, is applicable. Thus,

$$\int \sin^4 x \cos^3 x dx = \int \sin^4 x (1 - \sin^2 x) \cos x dx$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}.$$

$$\int \sin^3 x \cos^4 x dx = -\int (1 - \cos^2 x) \cos^4 x (-\sin x dx)$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}.$$

If both m and n are odd, we may adopt either of the foregoing methods. Thus,

$$\int \sin^3 x \cos^3 x dx = \int \sin^3 x (1 - \sin^2 x) \cos x dx$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6};$$

or,

$$\int \sin^3 x \cos^3 x dx = -\int (1 - \cos^2 x) \cos^3 x (-\sin x dx)$$

$$= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6}.$$

2. When m and n are *even positive integers.*

Let $n > m$ and let $n = m + p$, p being an even positive integer.

Then, $\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{m+p} x dx$

$$= \int (\sin x \cos x)^m \cos^p x dx$$

$$= \int \left(\frac{\sin 2x}{2}\right)^m \left(\frac{1 + \cos 2x}{2}\right)^{\frac{p}{2}} dx$$

$$= \frac{1}{2^{m+\frac{p}{2}}} \int \sin^m 2x (1 + \cos 2x)^{\frac{p}{2}} dx.$$

Since $\frac{p}{2}$ is a positive integer the binomial can be expanded into a series, and the terms integrated separately.

If $m > n$ and $m = n + p$, we have, similarly,

$$\int \sin^m x \cos^n x dx = \frac{1}{2^{n+\frac{p}{2}}} \int \sin^n 2x (1 - \cos 2x)^{\frac{p}{2}} dx.$$

Thus,

$$\int \sin^4 x \cos^2 x dx = \int \left(\frac{\sin 2x}{2}\right)^2 \sin^2 x dx$$

$$= \int \left(\frac{\sin^2 2x}{4} \frac{1 - \cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int \sin^2 2x dx - \frac{1}{8} \int \sin^2 2x \cos 2x dx$$

$$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} dx - \frac{1}{16} \frac{\sin^3 2x}{3}$$

$$= \frac{1}{16} x - \frac{\sin 4x}{64} - \frac{1}{48} \sin^3 2x.$$

3. When $m + n$ is an *even negative integer.*

In this case we have,

$$\int \sin^m x \cos^n x dx = \int \frac{\sin^m x}{\cos^m x} \cos^{m+n} x dx$$

$$= \int \tan^m x \sec^{-(m+n)} x dx.$$

Since $(m + n)$ is by hypothesis an even negative integer, $-(m + n)$ is an even positive integer. Hence § 192, such trigonometric forms may be integrated.

Thus, $$\int \frac{\sin^3 x}{\cos^5 x} dx = \int \tan^3 x \sec^2 x dx$$

$$= \frac{\tan^4 x}{4}.$$

EXAMPLES.

1. $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2 x.$

2. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin 2 x.$

3. $\int \cos^3 x dx = \sin x - \frac{1}{3} \sin^3 x.$

4. $\int \sin^5 x dx = - \cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x.$

5. $\int \cos^4 x dx = \frac{3}{8} x + \frac{\sin 2 x}{4} + \frac{\sin 4 x}{32}.$

6. $\int \sin^5 x \cos^4 x dx = - \frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x.$

7. $\int \sin^4 x \cos^5 x dx = \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x.$

8. $\int \frac{dx}{\cos^4 x} = \tan x + \frac{1}{3}\tan^3 x.$

9. $\int \frac{dx}{\sin^4 x} = -\frac{1}{3}\cot^3 x - \cot x.$

10. $\int \sin^2 x \cos^4 x dx = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48}.$

11. $\int \frac{\sin^3 x}{\cos^2 x} dx = \frac{1}{\cos x} + \cos x.$

12. $\int \frac{\cos^5 x dx}{\sin^2 x} = \frac{1}{3}\sin^3 x - 2\sin x - \csc x.$

13. $\int \frac{\sin^{\frac{4}{5}} x dx}{\cos^{\frac{14}{5}} x} = \frac{5}{9}\tan^{\frac{9}{5}} x.$

14. $\int \frac{dx}{\sin^{\frac{3}{2}} x \cos^{\frac{5}{2}} x} = \frac{2}{3}\tan^{\frac{3}{2}} x - 2\cot^{\frac{1}{2}} x.$

15. $\int \frac{dx}{\sin^3 x \cos^3 x} = \frac{1}{2}\tan^2 x - \frac{1}{2}\cot^2 x + \log \tan^2 x.$

16. $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \tan x - \cot x.$

196. Formulae of Reduction.

When m and n are integers *positive* or *negative*, *even* or *odd*, we can by successive reduction make the expression

$$\int \sin^m x \cos^n x dx$$

depend ultimately upon an integrable form. It may be remarked that while the following formulæ are applicable to all cases where the exponents are *integral*, yet the preceding processes should be employed in all cases where they are applicable as being in general simpler.

197. *To deduce formulae for the reduction of the exponent* (m) *of* $\sin^m x$ *in the expression* $\int \sin^m x \cos^n x dx$.

1. *m positive.*

$$\int \sin^m x \cos^n x dx = -\int \sin^{m-1} x \cos^n x (-\sin x dx)$$

$$= -\sin^{m-1} x \frac{\cos^{n+1} x}{n+1} + \int \frac{m-1}{n+1} \sin^{m-2} x \cos^{n+2} x dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x + (m-1) \int \sin^{m-2} x \cos^n x (1 - \sin^2 x) dx}{n+1}$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x + (m-1) \int \sin^{m-2} x \cos^n x dx - (m-1) \int \sin^m x \cos^n x dx}{n+1}.$$

Clearing of fractions and transposing, we have

$$(m+n) \int \sin^m x \cos^n x dx$$

$$= -\sin^{m-1} x \cos^{n+1} x + (m-1) \int \sin^{m-2} x \cos^n x dx;$$

$$\therefore \int \sin^m x \cos^n x dx$$

$$= \frac{-\sin^{m-1} x \cos^{n+1} x + (m-1) \int \sin^{m-2} x \cos^n x dx}{m+n} \quad . \ . \ . \ (A)$$

Formula A reduces the exponent (m) of $\sin^m x$ by 2. By repeating the process the integration will depend ultimately upon

$$\int \sin x \cos^n x dx \text{ or } \int \cos^n x dx$$

according as m is *odd* or *even*.

2. *m positive and $n = 0$.*

Making this supposition in (A), we have,

$$\int \sin^m x dx = \frac{-\sin^{m-1} x \cos x + (m-1)\int \sin^{m-2} x dx}{m}. \quad (B)$$

3. *m negative.*

Let $\qquad m = -m' + 2, \quad m - 2 = -m',$

in which $m' > 2$. Substituting these values in (A), we have,

$$\int \sin^{-m'+2} x \cos^n x dx$$

$$= \frac{-\sin^{-m'+1} x \cos^{n+1} x + (-m'+1)\int \sin^{-m'} x \cos^n x dx}{-m' + n + 2}.$$

Clearing of fractions, transposing, solving, and dropping accents, we have,

$$\int \sin^{-m} x \cos^n x dx$$

$$= \frac{-\sin^{-m+1} x \cos^{n+1} x + (m-n-2)\int \sin^{-m+2} x \cos^n x dx}{m-1} \quad (C).$$

By repeating the process, the integration is made to depend ultimately upon

$$\int \frac{\cos^n x dx}{\sin x} \text{ or } \int \cos^n x dx,$$

according as m is *odd* or *even*.

4. *m negative and $n = 0$.*

In this case (C) becomes

$$\int \sin^{-m} x dx = \frac{-\sin^{-m+1} x \cos x + (m-2)\int \sin^{-m+2} x dx}{m-1}. \quad (D)$$

198. *To deduce formulae for the reduction of the exponent* (n) *of $\cos^n x$ in the expression $\int \sin^m x \cos^n x dx$.*

1. *n positive.*

$$\int \sin^m x \cos^n x dx = \int \cos^{n-1} x \sin^m x \cos x dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1} (n-1) \cos^{n-2} x (-\sin x dx)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^{m+2} x dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x + (n-1)\int \cos^{n-2} x \sin^m x (1 - \cos^2 x) dx}{m+1}.$$

Clearing of fractions, transposing, and solving, we have

$$\int \sin^m x \cos^n x dx = \frac{\sin^{m+1} x \cos^{n-1} x + (n-1)\int \sin^m x \cos^{n-2} x dx}{m+n} \quad (A').$$

(A'), after repeated use, makes the required integral depend upon

$$\int \sin^m x \cos x dx, \quad \text{or} \quad \int \sin^m x dx,$$

according as n is *odd* or *even.*

2. *n positive and m* = 0.

(A') under this supposition becomes

$$\int \cos^n x dx = \frac{\sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx}{n} . \quad (B')$$

3. *n negative.*

Let $\qquad n = -n' + 2, \quad n - 2 = -n',$

in which $n' > 2$. Substituting in (A') we have,

$$\int \sin^m x \cos^{-n'+2} x dx$$

$$= \frac{\sin^{m+1} x \cos^{-n'+1} x + (-n'+1)\int \sin^m x \cos^{-n'} x dx}{m - n' + 2}.$$

Reducing and dropping accents, we have,

$$\int \sin^m x \cos^{-n} x dx$$

$$= \frac{\sin^{m+1} x \cos^{-n+1} x + (n - m - 2)\int \sin^m x \cos^{-n+2} x dx}{n-1} . \quad (C')$$

Repeating the application of (C'), the integration will ultimately depend upon

$$\int \frac{\sin^m x}{\cos x} dx, \text{ or } \int \sin^m x dx,$$

according as n is *odd* or *even*. For further reduction we apply (A) or (B), § 197.

4. *n negative and m* = 0.

This supposition in (C') gives

$$\int \cos^{-n} x dx = \frac{\sin x \cos^{-n+1} x + (n-2)\int \cos^{-n+2} x dx}{n-1} . (D')$$

199. Reduction to Algebraic Forms.

Let $\sin x = z$; then,

$$\int \sin^m x \cos^n x dx = \int z^m (1 - z^2)^{\frac{n-1}{2}} dz \qquad (1)$$

For $\qquad x = \sin^{-1} z. \quad \therefore\ dx = \dfrac{dz}{\sqrt{1 - z^2}};$

also, $\qquad \cos^n x = (1 - \sin^2 x)^{\frac{n}{2}} = (1 - z^2)^{\frac{n}{2}},\ \sin^m x = z^m.$

Since the second member of (1) is of the form,

$$\int x^m (a + bx^n)^p\, dx,$$

formulae *A*, *B*, *C*, *D*, of §§185, 186, may be made applicable to trigonometric integrals.

EXAMPLES.

1. $\int \sin^2 x \cos^2 x dx.$

The process of § 195, 2, is obviously applicable. To reduce the expression, however, by the reduction formula, let us apply (A'), as it applies to this case. It will be observed that (A) is also applicable.

In this case $m = 2$ and $n = 2$,

$$\therefore \int \sin^2 x \cos^2 x dx = \frac{\sin^3 x \cos x + \int \sin^2 x dx}{4}$$

$$= \frac{\sin^3 x \cos x + \frac{x}{2} - \frac{1}{4}\sin 2x}{4} \quad \text{(Ex. 1, p. 300).}$$

$$= \tfrac{1}{4}\sin^3 x \cos x + \tfrac{1}{8}x - \tfrac{1}{16}\sin 2x.$$

2. $\int \sin^4 x \cos^2 x dx.$

Using (A) we have $m = 4$, $n = 2$. Substituting we have

$$\int \sin^4 x \cos^2 x dx = \frac{-\sin^3 x \cos^3 x + 3\int \sin^2 x \cos^2 x dx}{6}$$

$$= -\tfrac{1}{6}\sin^3 x \cos^3 x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin^3 x \cos x + \tfrac{1}{8}x - \tfrac{1}{16}\sin 2x\right)$$

$$= \frac{2x - \sin 2x}{32} + \frac{(3 - 4\cos^2 x)\sin^3 x \cos x}{24}. \qquad \text{(Ex. 1.)}$$

3. $\int \sec^3 x dx = \int \cos^{-3} x dx.$

Using (D') in which $n = 3$, we have

$$\int \sec^3 x dx = \frac{\sin x \cos^{-2} x + \int \frac{dx}{\cos x}}{2}$$

$$= \frac{\sin x}{2\cos^2 x} + \tfrac{1}{2}\log\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \qquad \text{§ 194, 4.}$$

$$= \tfrac{1}{2}\left\{\tan x \sec x + \log\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}.$$

4. $\int \sec^7 x dx = \tfrac{1}{2}\tan x \sec x \left\{\tfrac{1}{3}\sec^4 x + \tfrac{5}{12}\sec^2 x + \tfrac{5}{8}\right\}$
$+ \tfrac{5}{16}\log(\sec x + \tan x).$

5. $\int \csc^3 x dx = \int \sin^{-3} x dx$

$$= \frac{-\sin^{-2} x \cos x + \int \frac{dx}{\sin x}}{2}$$

$$= -\tfrac{1}{2}\cot x \csc x + \tfrac{1}{2}\log\tan\frac{x}{2}. \qquad \text{§ 194, 4.}$$

6. $\int \csc^5 x dx = -\tfrac{1}{4}\cot x \csc^3 x - \tfrac{3}{8}\cot x \csc x + \tfrac{3}{8}\log\tan\frac{x}{2}.$

7. $\int \frac{dx}{\csc^4 x} = -\frac{\sin^3 x \cos x}{4} - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x.$

8. $\displaystyle\int \frac{dx}{\sec^4 x} = \frac{\cos^3 x \sin x}{4} + \tfrac{3}{8} \sin x \cos x + \tfrac{3}{8} x.$

9. $\displaystyle\int \frac{dx}{\sin x \cos^3 x} = \tfrac{1}{2} \sec^2 x + \log \tan x.$

10. $\displaystyle\int \frac{\cos^4 x dx}{\sin x} = \tfrac{1}{3} \cos^3 x + \cos x + \log \tan \frac{x}{2}.$

11. $\displaystyle\int \frac{\cos^4 x dx}{\sin^2 x} = -\tfrac{1}{2} \cot x\,(3 - \cos^2 x) - \tfrac{3}{2} x.$

12. $\displaystyle\int \frac{dx}{\sin^5 x} = -\tfrac{1}{4} \cos x\,(\csc^4 x + \tfrac{3}{2} \csc^2 x) + \tfrac{3}{8} \log \tan \frac{x}{2}.$

Reduce the following to algebraic forms and integrate:

13. $\displaystyle\int \sin^2 x \cos^2 x dx.$

Let $\sin x = z$; then

$$\sin^2 x = z^2,\ \cos^2 x = 1 - z^2,\ dx = \frac{dz}{\sqrt{1 - z^2}}.$$

$$\therefore \int \sin^2 x \cos^2 x dx = \int z^2 (1 - z^2)^{\frac{1}{2}} dz.$$

Applying formula A, § 187, we have

$$x = z,\ m = 2,\ a = 1,\ b = -1,\ n = 2,\ p = \tfrac{1}{2};$$

hence

$$\int z^2 (1 - z^2)^{\frac{1}{2}} dz = \frac{z(1 - z^2)^{\frac{3}{2}} - \int (1 - z^2)^{\frac{1}{2}} dz}{-4}$$

$$= -\frac{z(1 - z^2)^{\frac{3}{2}}}{4} + \frac{1}{4}\left\{\frac{z}{2}\sqrt{1 - z^2} + \tfrac{1}{2}\sin^{-1} z\right\}$$

$$= -\frac{z(1 - z^2)^{\frac{3}{2}}}{4} + \frac{z}{8}(1 - z^2)^{\frac{1}{2}} + \tfrac{1}{8}\sin^{-1} z$$

$$= \frac{z}{8}(2z^2 - 1)(1 - z^2)^{\frac{1}{2}} + \tfrac{1}{8}\sin^{-1} z.$$

Ex. 3. p. 285. Substituting for z its value, we have

$$\int \sin^2 x \cos^2 x dx = \frac{\sin x}{8}(2\sin^2 x - 1)\cos x + \frac{x}{8}$$

$$= \tfrac{1}{4}\sin^3 x \cos x - \frac{\sin 2x}{16} + \frac{x}{8}. \quad \text{See Ex. 1.}$$

14. $\displaystyle\int \frac{dx}{\sin x \cos^3 x} = \tfrac{1}{2}\sec^2 x + \log \tan x.$

15. $\displaystyle\int \sin^2 x dx = \frac{x}{2} - \tfrac{1}{4}\sin 2x.$

16. $\displaystyle\int \sin^4 x dx = -\frac{\sin^3 x \cos x}{4} - \tfrac{3}{16}\sin 2x + \tfrac{3}{8}x.$

17. $\displaystyle\int \frac{dx}{\sin^5 x} = -\tfrac{1}{4}\cot x \csc^3 x - \tfrac{3}{8}\cot x \csc x + \tfrac{3}{8}\log\tan\frac{x}{2}.$

MISCELLANEOUS EXAMPLES.

1. $$\int \frac{dx}{a + b\sin x} = \int \frac{dx}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 2b\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\dfrac{dx}{\cos^2\frac{x}{2}}}{a + 2b\tan\frac{x}{2} + a\tan^2\frac{x}{2}}$$

$$= \int \frac{a\sec^2\frac{x}{2}dx}{a^2 + 2ab\tan\frac{x}{2} + a^2\tan^2\frac{x}{2} + b^2 - b^2}.$$

$$= 2\int \frac{a\sec^2\frac{x}{2}\frac{dx}{2}}{a^2 - b^2 + \left(a\tan\frac{x}{2} + b\right)^2} \qquad (c)$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}, \text{ when } a > b.$$

If $a < b$, then a^2-b^2 is negative, and we write

$$\int \frac{dx}{a+b\sin x} = 2\int \frac{a\sec^2\frac{x}{2}\frac{dx}{2}}{\left(a\tan\frac{x}{2}+b\right)^2-(b^2-a^2)}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}.$$

2. $$\int \frac{dx}{a+b\cos x} = \int \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$$

$$= 2\int \frac{\sec^2\frac{x}{2}\frac{dx}{2}}{a+b+(a-b)\tan^2\frac{x}{2}} \qquad (d)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan^2\frac{x}{2}, \text{ when } a > b.$$

If $a < b$, then

$$\int \frac{dx}{a+b\cos x} = -2\int \frac{\sec^2\frac{x}{2}\frac{dx}{2}}{(b-a)\tan^2\frac{x}{2}-(b+a)}$$

$$= -\frac{2}{b-a}\int \frac{\sec^2\frac{x}{2}\frac{dx}{2}}{\tan^2\frac{x}{2}-\frac{b+a}{b-a}}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}}.$$

3. $\int \frac{dx}{5 + 3 \sin 2x} = \frac{1}{4} \tan^{-1} \frac{5 \tan x + 3}{4}$. Ex. (1).

4. $\int \frac{dx}{4 + 5 \sin 2x} = \frac{1}{6} \log \frac{4 \tan x + 2}{4 \tan x + 8}$.

5. $\int \frac{dx}{3 + 5 \cos x} = \frac{1}{4} \log \frac{\tan \frac{x}{2} + 2}{\tan \frac{x}{2} - 2}$.

6. $\int \frac{dx}{5 - 3 \cos x} = \frac{1}{2} \tan^{-1} \left(2 \tan \frac{x}{2}\right)$.

CHAPTER V.

DEFINITE INTEGRALS.

200. Up to this point all the integrals derived have been *indefinite*, the *indefinite constant of integration*, C, having been understood to enter each integral expression. See § 160. If C is *determined* from given conditions, and its value substituted in the integral, or if it is *eliminated* altogether from the expression, the result is termed a **Definite Integral.**

201. First Method. *To determine the value of C from given conditions.*

Let
$$dy = f'(x)\,dx,$$
$$\therefore\ y = f(x) + C, \qquad (1)$$

$f(x)$ containing, of course, no constant term.

Now suppose the relation between x and its function y is such that when $x = a$, $y = 0$. Substituting these values in (1), we have,

$$0 = f(a) + C.$$
$$\therefore\ C = -f(a).$$

This value of C, which is now definite in (1), gives

$$y = f(x) - f(a),$$

a definite integral.

To illustrate, let
$$dy = x^2 dx.$$
$$\therefore\ y = \frac{x^3}{3} + C.$$

Suppose the relations of x and y are such that when $x = 2$, $y = 0$; then,

$$0 = \tfrac{8}{3} + C. \quad \therefore \; C = -\tfrac{8}{3}.$$

Hence, $$y = \frac{x^3}{3} - \frac{8}{3}.$$

Again, suppose a body in a vacuum falls from a state of rest, and it is required to ascertain the velocity it acquires and the distance it falls in a time t.

We know from mechanics that the acceleration of the velocity of a body falling in a vacuum is constant and $= 32.2$ ft. a second.

Let $g = 32.2$; then, § 82, we have,

$$\frac{dv}{dt} = g,$$

$$\therefore \; dv = gdt;$$

$$\therefore \; v = gt + C.$$

But, by condition, $v = 0$ when $t = 0$, $\therefore \; C = 0$.

Hence, $$v = gt. \qquad (a)$$

Also, § 17, we have,

$$\frac{ds}{dt} = v;$$

$$\therefore \; ds = gtdt;$$

$$\therefore \; s = \tfrac{1}{2}gt^2 + C'.$$

But $s = 0$ when $t = 0$; $\therefore \; C' = 0$.

Hence, $$s = \tfrac{1}{2}gt^2. \qquad (b)$$

Equations (a), (b), give the required velocity and distance.

202. Notation. The fact that a function (y) is zero for a particular value of the variable (x) that enters it is usually denoted by placing the value of the variable as a subscript to the integral sign. Thus,

$$y = \int_a f'(x)\, dx = 0$$

denotes that $y = 0$ when $x = a$.

203. Second Method. *To eliminate C.*

Let $$dy = f'(x)\, dx;$$
then, $$y = f(x) + C. \qquad (a)$$

Now, suppose we know, from given conditions, that

$$y = A \text{ when } x = a,$$
and $$y = B \text{ when } x = b.$$

Substituting these values in (a) we have,

$$A = f(a) + C,$$
$$B = f(b) + C.$$
Hence, $$A - B = f(a) - f(b).$$

But since A and B are values of y, $A - B$ is also a value of y; hence we may write,

$$y = f(a) - f(b).$$

This process is known as Integration between Limits, and is so called because it gives the value of y between certain limiting values (a and b) of x. The process eliminates C, and thus renders the result definite.

204. Notation. The above process is indicated by the notation, $\int_b^a$, in which a is called the *superior limit* and b the *inferior limit.* Thus, the expression

$$y = \int_b^a f'(x)\, dx$$

denotes that the integration is to be taken between the limits a and b of x.

EXAMPLES.

1. $\int_1^2 (x-3)\,dx = \left(\frac{x^2}{2} - 3x + C\right)\Big|_1^2 = 2 - 6 + C - (\tfrac{1}{2} - 3 + C) = -\tfrac{3}{2}.$

2. $\int_a \frac{dx}{x} = \log\frac{x}{a}.$

3. $\int_0^{\frac{\pi}{2}} \sin x\,dx = (-\cos x)\Big|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + C - (-\cos 0 + C) = 1.$

4. $\int_1 \frac{dx}{\sqrt{x^3}} = 2 - \frac{2}{\sqrt{x}}.$

5. $\int_2^3 \frac{x\,dx}{1 + x^2} = \log\sqrt{2}.$

6. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec x\,dx = \log\left(\frac{1+\sqrt{2}}{\sqrt{3}}\right).$

7. $\int_0^{\frac{\pi}{4}} \sec^4 x\,dx = \tfrac{4}{3}.$

8. $\int_0^a \frac{dx}{a^2 + x^2} = \frac{\pi}{4a}.$

9. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi}{2}.$

10. $\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx = \sqrt{2} - 1.$

11. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x\,dx = \tfrac{1}{12}.$

12. $\int_1^{\infty} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{2}.$

13. $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$

14. $\int_0^\infty \frac{dx}{1+x} = \infty.$

15. $\int_0^{2a} \frac{xdx}{\sqrt{2ax - x^2}} = \pi a.$

16. $\int_0^\infty \frac{\sin nxdx}{e^{nx}} = \frac{1}{2n}.$

17. $\int_0^\infty \frac{dx}{x^2 - 2x\cos\alpha + 1} = \frac{-\alpha}{\sin\alpha}.$

APPLICATIONS.

In order to illustrate the foregoing methods we shall give a few problems to which these methods are applicable.

1. Determine the curve whose subnormal is constant and equal to a.

From § 72, we have

$$\text{Subnormal} = y\frac{dy}{dx}.$$

Hence, $$y\frac{dy}{dx} = a,$$

∴ $$\int ydy = a\int dx,$$

i.e., $$y^2 = 2a(x + c).$$

The required locus is therefore a parabola the position of whose vertex is indeterminate. If, however, we suppose $y = 0$ when $x = 0$ we have $c = 0$ and the equation of the curve becomes

$$y^2 = 2ax.$$

As the position of the origin is arbitrary we may always so assume it as to determine c.

2. Determine the curve whose subtangent is equal to double the abscissa of the point of tangency.

From § 71 and by condition we have

$$y\frac{dx}{dy} = 2x,$$

i.e.,

$$\frac{dy}{y} = \frac{1}{2}\frac{dx}{x}.$$

Integrating

$$\log y = \tfrac{1}{2}(\log x + \log c)$$
$$= \log \sqrt{cx}.$$

Hence,

$$y = \sqrt{cx}$$

or

$$y^2 = cx,$$

i.e., a parabola with indeterminate parameter.

3. Required the curve in which the angle between the radius vector and tangent is m times the vectorial angle.

From Fig. 11 and § 77, 1, we have, since $\phi = m\theta$,

$$\frac{rd\theta}{dr} = \tan m\theta;$$

$$\therefore \quad \frac{dr}{r} = \frac{\cos m\theta}{\sin m\theta}d\theta.$$

Integrating,

$$\log r = \frac{1}{m}(\log \sin m\theta + \log c)$$

$$\therefore \quad r^m = c \sin m\theta.$$

When $m = 1$ this equation represents a family of circles touching the initial line at the pole.

4. Required the curve whose normal is constant and equal to a. From § 72 and by condition

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = a.$$

Hence $$x^2 + y^2 = a^2;$$

i.e., the circle is the required curve. The value of c is here determined as in Ex. 1 by assuming the position of the origin of coördinates.

5. Find the locus whose polar subtangent is constant and equal $-a$. $r\theta = a$.

6. Find the locus whose polar subnormal is constant and equal a. $r = a\theta$.

7. In the cardioid, $r = a(1 - \cos\theta)$, the angle between the tangent and radius vector is always $\frac{1}{2}$ the vectorial angle.

CHAPTER VI.

GEOMETRIC APPLICATIONS.

205. Definition. The process of determining the area bounded in whole or in part by a curve is termed **Quadrature.**

206. Quadrature of plane areas.

I. When bounded by Algebraic Curves.

Let $y = f(x)$ be the equation of any curve as *OPC*, Fig. 52, and let the area between the curve and the *X*-axis be generated by the ordinate (*PB*) of the curve moving parallel to itself from left to right. Let A = area $AP'PB$, and let *PB* be *any* position of the generating ordinate y; then dA, the increment that A would take on in any unit of time provided the change in A became uniform and so continued throughout the unit, is evidently the area that *PB* would describe if its *length and velocity* remained unaltered throughout the unit. But the velocity of *PB* is the same as the rate of change of the distance *OB* ($= x$), i.e., it is $= dx$. Hence the differential area (dA) is a rectangle whose altitude is y (*PB*) and whose base is dx (*BD*), i.e.,

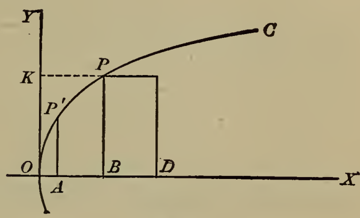


Fig. 52.

$$dA = ydx \qquad (a)$$

$$\therefore\ A = \int_a^b ydx \qquad (1)$$

in which b and a (OB and OA) are the limits of integration taken along the X-axis. Equation (1) is an expression for the area bounded by the curve, the X-axis, and terminal ordinates. Similarly, we find,

$$A = \int_a^b x\,dy \qquad (2)$$

the expression for the area bounded by the curve, the Y-axis and terminal abscissae, b and a being the limits of integration taken along the Y-axis.

To illustrate, let it be required to find the area of a parabolic segment.

Here $\qquad y^2 = 2px. \quad \therefore\ y = \sqrt{2p}x^{\frac{1}{2}};$

hence, $\qquad A = \int_a^b y\,dx$

$$= \int_0^x \sqrt{2p}x^{\frac{1}{2}}dx = \frac{\sqrt{2p}x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}xy,$$

i.e., the area of any segment as OBP is $\frac{2}{3}$ of the rectangle on the ordinate and abscissa, i.e., $\frac{2}{3}OBPK$.

II. *When bounded by Polar Curves.*

Let $r = f(\theta)$ be the equation of any curve as APC, Fig. 53, O being the pole and OX the initial line. Let $A = OP'P$, and let us suppose it to be generated by the radius vector revolving around O as an axis, and changing its length in obedience to the law expressed in the equation $r = f(\theta)$. Obviously the rate of change of A, i.e., dA, is the circular sector OPB described by OP in any unit of time with a constant angular velocity ($d\theta$).

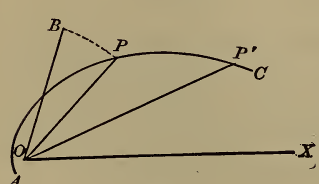


Fig. 53.

Hence, since $BP = rd\theta$ and $OP = r$, we have

$$dA = \tfrac{1}{2} r . r d\theta = \tfrac{1}{2} r^2 d\theta; \qquad (b)$$

$$\therefore A = \tfrac{1}{2} \int_{\psi}^{\phi} r^2 d\theta \qquad (3)$$

where $\phi = POX$ and $\psi = P'OX$. Equation (3), it will be observed, gives the area bounded by the curve and terminal radii-vectores. For example, let us find the area of one loop of the lemniscata,

$$r^2 = a^2 \cos 2\theta.$$

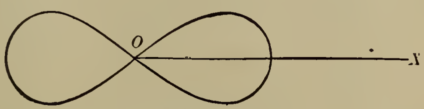


Fig. 54.

From tne equation we observe that the limiting values of θ are $45°$ and $-45°$;

$$\therefore \phi = 45° \text{ and } \psi = -45°.$$

Hence,

$$A = \tfrac{1}{2} \int_{\psi}^{\phi} r^2 d\theta$$

$$= \tfrac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta = \tfrac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta d\theta$$

$$= \tfrac{1}{4} a^2 \sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}};$$

hence, $$A = \frac{a^2}{2},$$

i.e., the area of the loop is $\frac{1}{2}$ the square constructed on a.

EXAMPLES.

1. Find area of the circle from its rectangular equation, $x^2 + y^2 = a^2$.

Here, $A = 4\int_0^a ydx = 4\int_0^a \sqrt{a^2 - x^2}\,dx$

$$= 4\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}\Bigg|_0^a.$$ See Ex. 3, p. 285.

$$= 4\left\{\frac{a^2}{2}\sin^{-1} 1 + C - \frac{a^2}{2}\sin^{-1} 0 - C\right\}$$

$$= 4\,\frac{a^2}{2}\,\frac{\pi}{2}$$

$$= \pi a^2.$$

2. Find the area of the circle from its polar equation $r = 2\,a\cos\theta$, the left-hand extremity of the horizontal diameter being the pole and the diameter being the initial line.

Here $A = \frac{1}{2}\int_\psi^\phi r^2 d\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\,a^2\cos^2\theta d\theta$$

$$= 2\,a^2\left\{\frac{\theta}{2} + \frac{\sin 2\,\theta}{4}\right\}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$ See Ex. 2, p. 300.

$$= 2\,a^2\left\{\frac{\pi}{4} + \frac{\sin\pi}{4} + C - \left(-\frac{\pi}{4} + \frac{\sin(-\pi)}{4} + C\right)\right\}$$

$$= 2\,a^2\left\{\frac{\pi}{4} + \frac{\pi}{4}\right\}$$

$$= \pi a^2.$$

3. Find the area in Ex. (2), (*a*), when the center is the pole and any diameter is the initial line; (*b*), when any point on the curve is the pole and the tangent at that point is taken as the initial line.

4. Find the area of the ellipse. πab.

5. Show that the entire area of the cissoid, $y^2(2a - x) = x^3$, is three times the area of its base circle.

6. Find the area of the first spire generated in the spiral of Archimedes, $r = a\theta$. $\frac{4}{3}\pi^3 a^2$.

7. Show that the area of the cardioid, $r = a(1 - \cos\theta)$, is six times the area of the generating circle.

8. Area between the Witch of Agnesi, $(x^2 + 4a^2)y = 8a^3$, and the X-axis. $4\pi a^2$.

9. Area between the cissoid, $(2a - x)y^2 = x^3$, and its asymptote, $x = 2a$. $3\pi a^2$.

10. Assuming the polar equation of the cissoid, $r = 2a \tan\theta \sin\theta$, and the polar equation of its asymptote, $r = 2a\sec\theta$, find the area.

11. The entire area of the hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, is $\frac{3}{8}$ the area of its base circle.

12.* The area of one arch of the cycloid, $\left.\begin{matrix} x = a\theta - a\sin\theta \\ y = a(1 - \cos\theta) \end{matrix}\right\}$ is three times the area of the rolling circle.

13. Area of one loop of $r = a\sin 2\theta$. $\frac{\pi a^2}{8}$.

14. Area of one loop of $r = a\cos n\theta$. $\frac{\pi a^2}{4n}$.

* This fact was first established by Roberval in 1634.

15. Entire area of $r = a(\cos 2\theta + \sin 2\theta)$. πa^2.

16. Area between $y^2(x^2 + a^2) = a^2x^2$ and its asymptote, $y = a$.
$2a^2$.

17. Area between $r = a(\sec\theta + \tan\theta)$ and its asymptote, $r = 2a\sec\theta$. $\left(\frac{\pi}{2} + 2\right)a^2$.

18. Area of one loop of $x^2(a^2 + y^2) = y^2(a^2 - y^2)$.
$\frac{\pi - 2}{2}a^2$.

19. Area between $y^2(a - x) = x^2(a + x)$ and its asymptote, $x = a$. $\left(\frac{\pi}{2} + 2\right)a^2$.

207. Definition. The process of determining the length of a curve is termed **Rectification.**

208. Rectification of Curves.

I. *Algebraic Curves.*

We have found, § 18, that,

$$ds = \sqrt{dx^2 + dy^2}.$$

This equation may be placed in either of the following forms:

$$ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx,$$

$$ds = \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy.$$

Hence,
$$S = \int_{x_2}^{x_1} \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \qquad (1)$$

$$S = \int_{y_2}^{y_1} \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy \qquad (2)$$

In (1) the limits of integration are taken along the X-axis; in (2) the limits are taken along the Y-axis.

To illustrate let us find the entire length of the circle $x^2+y^2=a^2$. From the equation we have $\frac{dy}{dx} = -\frac{x}{y}\cdot$ Substituting in (1), we have

$$S = \int_{x_2}^{x_1} \left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} dx.$$

If we integrate between the limits $x_1 = a$ and $x_2 = 0$, we obtain the length of a quadrant; hence for the entire length we have

$$S = 4\int_0^a \left(1 + \frac{x^2}{y^2}\right)^{\frac{1}{2}} dx = 4\int_0^a \left(\frac{a^2}{y^2}\right)^{\frac{1}{2}} dx$$

$$= 4\int_0^a \frac{a}{\sqrt{a^2 - x^2}} dx$$

$$= 4a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4a \sin^{-1}\frac{x}{a}\Big|_0^a$$

$$= 4a\left\{\frac{\pi}{2}\right\}$$

$$= 2\pi a.$$

II. *Polar Curves.*

From § 76, Cor., we have

$$ds = \sqrt{r^2 d\theta^2 + dr^2}.$$

Hence,
$$ds = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta;$$

or,
$$ds = \left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{1}{2}} dr.$$

Hence,
$$S = \int_\psi^\phi \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta \qquad (1)$$

or,
$$S = \int_{r_2}^{r_1} \left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{1}{2}} dr \qquad (2)$$

According as the limits of integration are limiting values of the vectorial angle (θ), or limiting values of the radius vector (r).

To illustrate, let us find the entire length of the cardioid $r = a(1 - \cos\theta)$.

From the equation we have,

$$\frac{dr}{d\theta} = a \sin\theta.$$

Substituting this value together with the value of r in (1), we have

$$S = \int_{\psi}^{\phi} \{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta\}^{\frac{1}{2}}\, d\theta.$$

The limiting values of θ are 2π and 0; hence

$$S = \int_0^{2\pi} \{a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta\}^{\frac{1}{2}}\, d\theta$$

$$= \sqrt{2}\, a \int_0^{2\pi} (1 - \cos\theta)^{\frac{1}{2}}\, d\theta$$

$$= \sqrt{2}\, a \int_0^{2\pi} \sqrt{2} \sin \tfrac{1}{2}\theta\, d\theta$$

$$= -4a \cos\tfrac{1}{2}\theta \Big|_0^{2\pi}$$

$$= -4a\{\cos\pi + c - (\cos 0 + c)\}$$

$$= 8a.$$

EXAMPLES.

1. Find the length of the circle, the pole being at the center. We have for the equation of the circle

$$r = a;$$

$$\therefore \frac{dr}{d\theta} = 0;$$

$$\therefore \S 208,\ \text{II., (1)},\ S = \int_0^{2\pi} \{a^2 + (0)^2\}^{\frac{1}{2}}\, d\theta$$

$$= a\int_0^{2\pi} d\theta = 2\pi a.$$

2. Find the length of one arch of the cycloid.

From the equations of the curve, Ex. 27, page 69, we have,

$$\frac{dy}{d\theta} = a \sin \theta,$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \qquad (a)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1 - \cos\theta} = \frac{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

Substituting in (1), § 208, I., the value of dx drawn from (a) and the value of $\frac{dy}{dx}$ and integrating between the limits 2π and 0, we have,

$$S = a\int_0^{2\pi} \left\{1 + \cot^2\frac{\theta}{2}\right\}^{\frac{1}{2}} (1 - \cos\theta)\, d\theta$$

$$= 2a \int_0^{2\pi} \csc\frac{\theta}{2} \sin^2\frac{\theta}{2}\, d\theta$$

$$= 2a \int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= -4a \cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= 8a.$$

3. Find the length of an arch of a cycloid, assuming the rectangular form of its equation, $x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}$.

Here, $$\frac{dx}{dy} = \frac{y}{\sqrt{2ay - y^2}}.$$

This value in (2), § 208, I., gives

$$S = \sqrt{2a}\int_{y_2}^{y_1} (2a - y)^{-\frac{1}{2}}\, dy.$$

Integrating between the limits $y_1 = 2a$ and $y_2 = 0$, and doubling the result, we have the length of one arch;

i.e.,
$$S = 2\sqrt{2a}\int_0^{2a}(2a - y)^{-\frac{1}{2}}dy$$
$$= -4\sqrt{2a}(2a - y)^{\frac{1}{2}}\Big|_0^{2a}$$
$$= 8a,$$

as before. See Ex. 2.

NOTE. — Sir C. Wren rectified the cycloid in 1673. It was the *second* curve rectified, the semi-cubic parabola having been previously (1660) rectified by William Neil.

4. Find the entire length of the hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$
$$S = 4\int_0^a\left(1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}}dx$$
$$= 4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}dx$$
$$= 6a^{\frac{1}{3}}x^{\frac{2}{3}}\Big|_0^a$$
$$= 6a.$$

5. Show that the entire length of the curve, $r = a\sin^3\frac{\theta}{3}$, is three-fourths of the circumference of the circle of radius a.

6. Find the length of an arc of the parabola, $y^2 = 4ax$, estimated from the origin. $\sqrt{ax + x^2} + a\log\frac{\sqrt{x} + \sqrt{a + x}}{\sqrt{a}}$.

7. Find the length of an arc of the catenary, $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, estimated from the vertex of the curve. $S = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

8. Find the length of an arc of the parabola, $r = \frac{p}{1 - \cos\theta}$.

9. Find the length of the curve, $8a^3y = x^4 + 6a^2x^2$, measured from the origin. $\frac{x}{8a^3}(x^2 + 4a^2)^{\frac{3}{2}}$.

10. Find the length of the logarithmic curve, $x = a \log \frac{y}{b}$.

$$S = a \log \frac{y}{a + \sqrt{a^2 + y^2}} + \sqrt{a^2 + y^2} + C.$$

11. Find the length of the logarithmic spiral, $\theta = a \log \frac{r}{b}$, between the limits r_1 and r_2. $S = \sqrt{(1 + a^2)}(r_2 - r_1)$.

12. Show that the cissoid is rectifiable.

13. Rectify $y = \log \frac{e^x + 1}{e^x - 1}$ between limits $x_1 = 2$, $x_2 = 1$.

$$S = \log\left(e + \frac{1}{e}\right).$$

14. St. Vincent, before the middle of the seventeenth century, proved that any arc of the spiral of Archimedes was equal to the corresponding arc of a parabola. Prove it.

209. Surfaces and Volumes of Revolution.

Let the curve $A'PC$ in the plane XY revolve around OX as an axis. Then the surface generated by the curve is a **surface of revolution** and the volume generated by the area $A'CC'A$ is a **volume of revolution.** As every point, as P in the curve, generates the circumference of a circle, and every ordinate of the curve, as $BP(=y)$, generates the area of a circle, we may conceive the surface and volume to be generated by the circumference and area of a circle whose center is in the X-axis moving in the direction of that axis, and changing its radius in obedience to the law expressed in the equation of the limiting curve $A'PC$. Assuming the latter mode of generation, let B be the position of the center of the generating circle at any instant; then the differential or rate of change of the surface (dS) is obviously the surface that *would be* generated in any interval of time if $PB = y$ remained constant throughout the interval, and the *velocity of each of the generating points of the circumference*, such as P, also remained unaltered. But such a surface is that of a cylinder $K'PB$, the circumference of whose base is $2\pi PB = 2\pi y$, and whose altitude $PK' = PK = ds =$ velocity of the generating point P of the curve $A'PC$. Hence

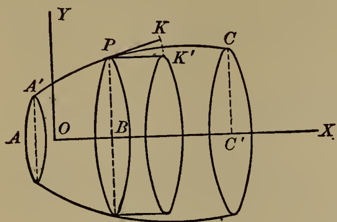


Fig. 55.

$$dS = 2\pi y ds.$$

But
$$ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx. \quad \text{See § 208.}$$

$$\therefore dS = 2\pi y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$

$$\therefore S = 2\pi \int_{x_2}^{x_1} y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \qquad (1)$$

Similarly the differential or rate of change of the volume (dV) is the volume that would be generated in any interval by the *area* of the circle πPB^2 ($= \pi y^2$) provided its radius remained unaltered and the velocity of *each point of that area* also remained unaltered throughout the interval. This differential volume is also that of a cylinder whose base is $\pi BP^2 = \pi y^2$ and whose altitude $PK' = dx$ (not ds since each point of the area is moving at the instant in the direction of the X-axis). Hence,

$$dV = \pi y^2 dx \qquad (a)$$

$$\therefore \ V = \pi \int_{x_2}^{x_1} y^2 dx \qquad (2)$$

To illustrate let us find the surface and volume of the sphere generated by the circle, $x^2 + y^2 = a^2$, revolving about the X-axis.

(1). To find the surface.

Here $\frac{dy}{dx} = -\frac{x}{y}$, and $y = (a^2 - x^2)^{\frac{1}{2}}$,

$$\therefore S = 2\pi \int_{x_2}^{x_1} y \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx$$

$$= 2\pi \int_{x_2}^{x_1} y \left\{ 1 + \frac{x^2}{y^2} \right\}^{\frac{1}{2}} dx$$

$$= 2\pi \int_{x_2}^{x_1} y \frac{a}{y} dx$$

$$= 2\pi a \int_{x_2}^{x_1} dx.$$

Taking the limits $x_1 = a$ and $x_2 = 0$ we obtain one-half of the surface. Hence for the whole surface we have

$$S = 4\pi a \int_0^a dx = 4\pi a^2.$$

(2). To find the volume.

Integrating between the limits $x_1 = a$ and $x_2 = 0$ and doubling the result we have

$$V = 2\pi \int_0^a y^2 dx$$

$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left\{ a^2 x - \frac{x^3}{3} \right\} \Big|_0^a$$

$$= \tfrac{4}{3} \pi a^3.$$

EXAMPLES.

1. Find the volume of the ellipsoid of revolution generated by revolving the area of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (1), about X, (2), about Y.

Ans. $\frac{4\,\pi ab^2}{3}$, $\frac{4\,\pi a^2 b}{3}$.

2. Find the volume generated by revolving the area bounded by the Witch of Agnesi, $(x^2+4a^2)\,y=8a^3$, about the X-axis.

Ans. $4\,\pi^2a^3$.

3. Show that the volume generated by revolving the area of the parabola, $y^2=2\,px$, about X is equal to one-half the volume of the circumscribing cylinder.

4. Find the surface and volume of the cone generated by revolving the line, $\frac{x}{a}+\frac{y}{b}=1$, and the area which it and the axes limit, about the X-axis. *Ans.* $V=\frac{\pi ab^2}{3}$, $S=\pi b\sqrt{a^2+b^2}$.

5. Find the volume and surface generated by revolving one arch of the cycloid and the area it bounds about X.

Ans. $V=5\,\pi^2a^3$, $S=\frac{64}{3}\,\pi a^2$.

6. Find the volume generated by the area of the cissoid $(2a-x)\,y^2=x^3$, as it revolves about its asymptote, $x=2\,a$.

Ans. $V=2\,\pi^2a^3$.

7. Find the surface generated by revolving the cycloid about its axis. *Ans.* $S=8\,\pi a^2\,(\pi-\frac{4}{3})$.

8. Find the volume generated by revolving the area of the hypocycloid, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, about X. *Ans.* $V=\frac{32}{105}\,\pi a^3$.

9. Show that the volume generated by revolving one branch of the equilateral hyperbola, $x^2-y^2=a^2$, about X, the limits of integration being $x_1=2a$, $x_2=a$, is equivalent to sphere of radius a.

10. Find the volume generated by revolving the area bounded by one branch of the sinusoid, $y = a \sin \frac{x}{b}$, about X. *Ans.* $\frac{\pi^2 a^2 b}{2}$.

11. Show that the expression, $S = 2\pi \int y \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx,$

becomes
$$S = 2\pi \int r \sin \theta \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta$$
when the coördinates are polar.

12. Find the entire surface generated by revolving the cardioid, $r = a\,(1 + \cos \theta)$, about the initial line.

Ans. $S = \frac{32}{5}\pi a^2$.

210. Surfaces and Volumes in General.

Let the surface $SBCS'p$ be generated by the curve SpS' as it moves in the direction of X, its plane being always parallel to ZY.

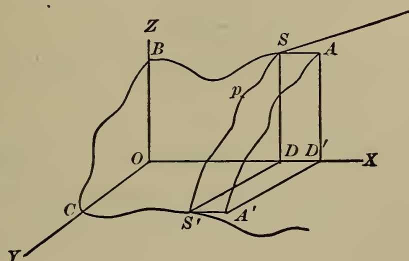


Fig. 56.

At the instant of reaching the position SpS' we have
$$dS = \text{area } ASpS'A'.$$
Let $SpS' = P$, $SA = ds$; then, since the surface is cylindrical,
$$dS = Pds. \qquad (a)$$
$$\therefore\ S = \int Pds. \qquad (1)$$

Similarly, $$dV = \text{volume } ASpS'A'D',$$
$$= Adx. \qquad (b)$$
$$\therefore\ V = \int Adx, \qquad (2)$$

in which A = area SDS' and $dx = SA$.

Formulæ (1) and (2) are obviously applicable to all cases where P can be expressed in terms of s and where A can be expressed as a function of x.

Cor. If SpS' is a circle with its center D in the X-axis; then

$$P = 2\,\pi y,\ A = \pi y^2,$$

and (1) and (2) become, respectively,

$$S = 2\int \pi y ds$$
$$V = \pi \int y^2 dx,$$

as heretofore determined. See § 209.

EXAMPLES.

1. To find the surface and volume of a regular pyramid or cone.

1. *To find the surface.*

Let P' = perimeter of base and $Oc = h'$ = slant height. Let mnd be the position of the *generating* perimeter P at any instant. Since P and P' are similar, we have

$$\frac{P}{P'} = \frac{Od}{Oc} = \frac{s}{h'};$$

hence $$P = \frac{P'}{h'} s.$$

This value of P in (1), § 210, gives

$$S = \frac{P'}{h'} \int_0^{h'} sds,$$

i.e., $$S = \frac{P'h'}{2}.$$

Hence the convex surface of any pyramid or cone is measured by $\frac{1}{2}$ product of perimeter of its base by its slant height.

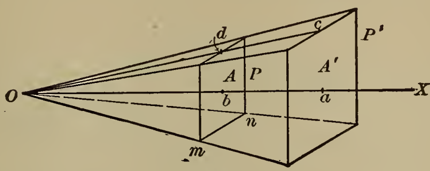


Fig. 57.

2. *To find the volume.*

Let A' = area of base and $Oa = h$ = altitude. Let $Ob = x$; then

$$\frac{A}{A'} = \frac{\overline{Ob}^2}{\overline{Oa}^2} = \frac{x^2}{h^2};$$

$$\therefore\ A = \frac{A'}{h^2}x^2.$$

Hence, § 210, (2),
$$V = \frac{A'}{h^2}\int_0^h x^2dx,$$

$$\therefore\ V = \frac{A'h}{3};$$

i.e., the volume of any cone or pyramid is measured by $\frac{1}{3}$ of the product of its base and altitude.

2. Show that the volume of the frustum of any pyramid or cone is equal to $\frac{h}{3}(A + A' + \sqrt{AA'})$ where A and A' are the bases, and h is its height.

3. Find the volume of a right conoid with circular base, the radius of base being a, and altitude h. *Ans.* $\frac{\pi a^2 h}{2}$.

4. Find the volume of the wedges cut from a tree (radius = a) in cutting it down, the faces of each wedge being inclined at an angle of 45°. *Ans.* $\frac{4}{3}a^3$.

CHAPTER VII.

SUCCESSIVE INTEGRATION.

211. Successive integration is a reversal of the process of successive differentiation. If, for example, x is equicrescent and

$$d^3y = x^2dx^3,$$

then, $$\int d^3y = dx^2\int x^2\,dx\,;$$

i.e., $$d^2y = dx^2\left\{\frac{x^3}{3}+C\right\}.$$

Again, $$\int d^2y = dx\int\left\{\frac{x^3dx}{3}+Cdx\right\};$$

i.e., $$dy = dx\left\{\frac{x^4}{12}+Cx+C_1\right\}.$$

And finally,

$$\int dy = \int\left\{\frac{x^4dx}{12}+Cxdx+C_1dx\right\};$$

i.e., $$y = \frac{x^5}{60}+\frac{Cx^2}{2}+C_1x+C_2.$$

In practical applications of this process the conditions given are, in general, sufficient to enable us to determine the values of C, C_1 . . . , and thus render the result definite. Thus, given the acceleration due to gravity (g), let it be required to find the distance s through which a body starting from rest will fall in a time t.

By condition,

$$\frac{d^2s}{dt^2} = g \text{ (see § 82)};$$

$$\therefore \frac{ds}{dt} = gt + C.$$

But $\frac{ds}{dt} = v$; and $v = 0$ when $t = 0$, $\therefore C = 0$.

Hence,
$$\frac{ds}{dt} = gt;$$

hence,
$$s = g\frac{t^2}{2} + C_1.$$

But $s = 0$ when $t = 0$, $\therefore C_1 = 0$; hence,

$$s = \tfrac{1}{2}gt^2.$$

212. Double Integration. Let

$$u = \frac{x^2y^3}{6}. \qquad (a)$$

Differentiating (*a*) regarding x as constant we have

$$\partial_y u = \frac{x^2y^2}{2}dy. \qquad (b)$$

Now differentiating (*b*), regarding y and its differential as constant, we have

$$\partial_x(\partial_y u) = \partial^2_{yx}u = xy^2dydx.$$

A reversal of this process is **Double Integration**, and is indicated by using a double integral sign. Thus, if

$$\partial^2_{yx}u = xy^2dydx.$$

then,
$$u = \int\int xy^2dydx.$$

In performing this operation the order of integration is denoted by the arrangement of the differentials proceeding from *right* to *left*. For example,

$$u = \int\int x^3y^4dydx$$

$$=\int\left(\frac{x^4}{4}+C\right)y^4dy$$

$$=\frac{x^4y^5}{20}+\frac{Cy^5}{5}+C_1.$$

213. Triple Integration. If

$$d^3_{zyx}u = x^2yz^3dzdydx$$

is the result obtained by a triple differentiation of a function of x, y, z, regarding only one variable to vary at a time, then

$$u=\int\int\int x^2yz^3dzdydx$$

$$=\int\int yz^3dzdy\left\{\frac{x^3}{3}+C\right\}$$

$$=\int z^3dz\left\{\frac{x^3y^2}{6}+\frac{Cy^2}{2}+C_1\right\}$$

$$=\frac{x^3y^2z^4}{24}+\frac{Cy^2z^4}{8}+\frac{C_1z^4}{4}+C_2,$$

and the process is termed **Triple Integration.**

214. Definite Double and Triple Integrals.

Where the limits of integration are given the result of the process is of course definite. Thus

$$u=\int_b^a\int_d^c x^2ydydx$$

$$=\int_b^a ydy\left\{\frac{c^3-d^3}{3}\right\}$$

$$=\frac{c^3-d^3}{3}\,\frac{a^2-b^2}{2}$$

$$=\frac{(c^3-d^3)(a^2-b^2)}{6}.$$

Again,
$$u=\int_0^a\int_0^b\int_0^c xyzdzdydx$$

$$=\int_0^a\int_0^b yzdzdy\frac{c^2}{2}$$

$$= \int_0^a zdz \frac{b^2c^2}{4}$$
$$= \frac{a^2b^2c^2}{8}.$$

It frequently happens in practical application that the limits of one integration are functions of the variable considered in a subsequent integration. Thus

$$u = \int_0^a \int_0^{a^2 - y^2} dydx$$
$$= \int_0^a (a^2 - y^2)\, dy$$
$$= \tfrac{2}{3} a^3.$$

EXAMPLES.

1. $\int_b^{2b} \int_0^a y^2 (a - x)\, dydx.$ *Ans.* $\frac{7}{6} a^2b^3.$

2. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x + y)\, dxdy.$ *Ans.* $\frac{2}{3} a^3.$

3. $4\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)\, dxdy.$ *Ans.* $\frac{\pi a^4}{2}.$

4. $2\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} rd\theta dr.$ *Ans.* $\pi a^2.$

5. $a\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dxdy}{\sqrt{a^2 - x^2}}.$ *Ans.* $a^2.$

6. $\frac{4}{3}\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} \int_0^{\frac{x^2 + y^2}{a}} dxdydz.$ *Ans.* $\pi a^3.$

7. $8\int_0^a \int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} \int_0^{c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dxdydz.$ *Ans.* $\frac{4}{3} \pi abc.$

CHAPTER VIII.

GEOMETRIC APPLICATIONS.

215. Quadrature of Plane Areas. From § 206 (*a*), we have

$$dA = ydx.$$

Differentiating this equation with respect to y we have

$$d^2A = dydx.$$

Hence, $$A = \int\int dydx. \qquad (1)$$

From § 206, (*b*), we have

$$dA = \tfrac{1}{2} r^2 d\theta;$$

hence $$d^2A = rdrd\theta;$$

$$\therefore A = \int\int rdrd\theta \qquad (2)$$

The order of the differentials in (1) and (2) are obviously immaterial.

Formulas (1) and (2) enable us to determine the areas bounded by curves by double integration.

EXAMPLES.

1. To find the area of the circle $x^2 + y^2 = a^2$ by double integration.

In this case (1) § 215 becomes for the area of the first quadrant

$$A = \int_0^a \int_0^{\sqrt{a^2-x^2}} dx\, dy$$

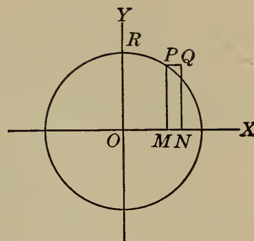


Fig. 58.

and it will be noted that the limits of y are from $y = 0$ (x-axis) to $y = \sqrt{a^2 - x^2}$ (any point on the curve). Thus the first integration (x and its differential being constant) would give the area contained in the strip *MNPQ*, which, itself, would be the differential area of the portion *OMPR*. Hence the limits of x are selected as in single integration from the origin to the limits of the curve on the x-axis, *i.e.*, from $x = 0$ to $x = a$, and

The entire area is then found from

$$A = 4\int_0 \int_0^{\sqrt{a^2-x^2}} dxdy$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left\{ \frac{a^2}{2} \cdot \frac{\pi}{2} \right\} \quad \text{Ex. 1, p. 322.}$$

$$= \pi a^2.$$

2. Find the area of the circle using a polar equation. Let *OX* be the polar axis and *O* the pole, then

$$r = 2\, a \cos \theta$$

will be its polar equation. Hence, (2) § 215, we have for the upper half

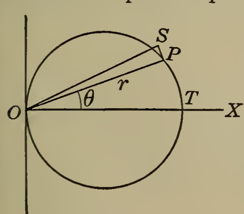


Fig. 58 *a*.

$$A = \int_0^{\frac{\pi}{2}} \int_0^{2a \cos \theta} d\vartheta\, rdr$$

The limits of r, taken first, are from the pole $r = 0$ to $r = 2\, a \cos \theta$ (any point on the curve) and when this integration has been performed (θ and its differential being constant) we will have the area of the sector *OPS* which forms the differential area of

OTP. Hence, the limits of θ are from 0 to $\frac{\pi}{2}$. And the complete integration would give the area of the upper half. For the entire area we have

$$A = 2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} d\theta\, rdr$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{4a^2\cos^2\theta}{2}\, d\theta = 4a^2\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$= 4a^2\left\{\frac{\pi}{4}\right\}$$ See Ex. 2, p. 322.

$$= \pi a^2.$$

3. Show that πab measures the area of the ellipse.

4. Find the area of the cardioid, $r = a(1 - \cos\theta)$.

Ans. $\frac{3}{2}\pi a^2$.

5. Find the area between the line, $ay = bx$, and the parabola, $ay^2 = bx$.

Ans. $\displaystyle\int_0^{\frac{a}{b}}\int_{\frac{bx}{a}}^{\sqrt{\frac{bx}{a}}} dxdy = \frac{a}{6b}$.

216. Surfaces and Volumes in General by Double and Triple Integration.

I. **Surfaces.** From § 210 (a), we have

$$dS = Pds.$$

Differentiating with respect to P, we have

$$d^2S = dPds.$$

Let SpS' be the position of the generating curve at any instant —p being any point of that curve. Suppose the surface to be generated by the curve moving in the direction of X and so changing that its coördinates always satisfy the equation of the surface, $y=f(x, z)$. Then, at the instant of reaching the position SpS' any point such as p has a motion (represented by pA) in a direction perpendicular to the tangent pB of the curve SpS' *by virtue of the motion* of

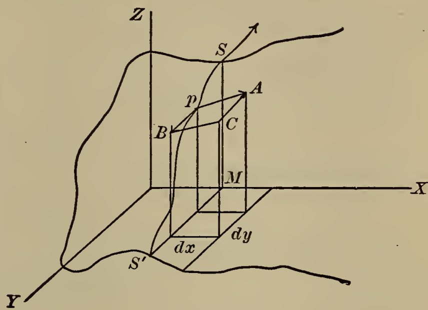


Fig. 59.

the plane of the curve in the direction of X, and a motion (represented by pB) in the direction of the tangent pB by virtue of the change in the curve as it conforms to the configuration of the surface, $y=f(x, z)$, it generates; and this whatever may be the absolute or resultant motion of p. But

$$pB = dP \text{ and } pA = ds.$$

Hence, $$d^2S = dPds = \text{area rectangle } pACB.$$

The projection of the rectangle $pACB$ on the coördinate planes XY, XZ, YZ, are obviously $dxdy$, $dxdz$, $dydz$, respectively. If we now let ϕ, θ, ψ represent the angles which the plane of the rectangle $pACB$ makes with XY, XZ, and YZ, respectively, we have

$$d^2S \cos \phi = dxdy,$$
$$d^2S \cos \theta = dxdz,$$
$$d^2S \cos \psi = dydz.$$

Squaring and adding these equations, remembering that

$$\cos^2 \phi + \cos^2 \theta + \cos^2 \psi = 1, \qquad \text{(Ana. Geom., p. 221.)}$$

we have, $$(d^2S)^2 = dx^2 dy^2 + dx^2 dz^2 + dy^2 dz^2.$$

Hence,

$$S = \int\int\left(1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}} dxdy \qquad (1)$$

in which $\frac{dz}{dy}$ and $\frac{dz}{dx}$ are partial derivatives drawn from the equation of the surface, $y = f(x, z)$.

II. **Volumes.**

From § 210 (*b*), we have

$$dV = Adx.$$

Differentiating with respect to A, we have

$$d^2V = dAdx.$$

But, § 206 (*a*), $dA = ydz$; since plane SMS' is ‖ to YZ; hence,

$$d^2V = ydxdz.$$

Differentiating with respect to y, we have

$$d^3V = dxdydz;$$

Hence,

$$V = \int\int\int dxdydz \qquad (2)$$

EXAMPLES.

1. To find the surface and volume of a sphere.

(*a*) *To find the surface.*

Let the origin of coördinates be taken at the center of the sphere, and let Fig. 60 represent the portion of the surface in the first angle, i.e., one-eighth of the surface. From the equation of the surface,

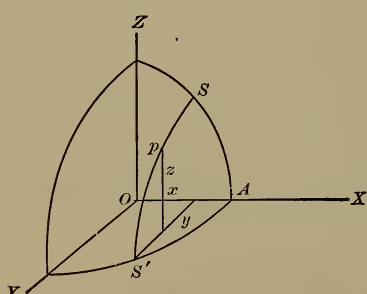


Fig. 60.

$$x^2 + y^2 + z^2 = a^2,$$

we obtain,

$$\frac{dz}{dy} = -\frac{y}{z}, \quad \frac{dz}{dx} = -\frac{x}{z}.$$

Hence,

$$S = 8\int\int\left(1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}}dxdy$$

$$= 8\int\int\left(1+\frac{y^2}{z^2}+\frac{x^2}{z^2}\right)^{\frac{1}{2}}dxdy$$

$$= 8\int\int\frac{a}{z}\,dxdy.$$

But, $\qquad z = \sqrt{a^2 - x^2 - y^2}\,;$

$$\therefore\ S = 8\,a\int\int\frac{dxdy}{\sqrt{a^2 - x^2 - y^2}}.$$

To determine the limits of integration we observe that XY cuts from the sphere a circle whose equation is $x^2 + y^2 = a^2$; $\therefore\ y = \sqrt{a^2 - x^2}$, as it everywhere measures the extension of SpS' in the direction of y, is the variable superior limit of y, its inferior limit being obviously zero. The limits in the direction of x are obviously $x = OA(= a)$ and $x = 0$.

Hence,

$$S = 8\,a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dxdy}{\sqrt{a^2 - x^2 - y^2}}$$

$$= 8\,a\int_0^a dx\left(\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right)\Bigg|_0^{\sqrt{a^2-x^2}}$$

$$= 8\,a\int_0^a\frac{\pi}{2}\,dx$$

$$= 4\,\pi a(x)\Big|_0^a.$$

$$\therefore\ S = 4\,\pi a^2.$$

(*b*) *To find the volume.*

Since $z = \sqrt{a^2 - x^2 - y^2}$ measures everywhere the extension of the surface in the direction of Z, we have, from § 216, (2),

$$V = 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} dxdydz$$

$$= 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} (a^2 - x^2 - y^2)^{\frac{1}{2}}\, dxdy$$

$$= 8\int_0^a dx \left\{ \frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2} \sin^{-1}\frac{y}{\sqrt{a^2 - x^2}} \right\} \Bigg|_0^{\sqrt{a^2 - x^2}}$$

Ex. 3, p. 285.

$$= 8\int_0^a dx \left(\frac{a^2 - x^2}{2} \cdot \frac{\pi}{2} \right)$$

$$= 2\pi\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a.$$

Hence, $V = \frac{4}{3}\pi a^3.$

2. Find the entire surface of the groin formed by the intersection of two equal semicircular cylinders whose radius is a.

Assuming the axes of the cylinders as those of Z and Y we have

$$S = 8\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{adxdy}{\sqrt{a^2 - x^2}} = 8\,a^2.$$

3. Show that the volume of the ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is two-thirds the volume of its circumscribing cylinder.

4. Find the volume common to the two cylinders given in Ex. 2. *Ans.* $\frac{8}{3}a^3$.

5. Find the volume cut from the paraboloid $z^2 + y^2 = 2\,ax$ by the cylinder $x^2 + y^2 = ax$.

Ans. $\left(\frac{2}{3} + \frac{\pi}{4}\right)a^3.$

CHAPTER IX.

DIFFERENTIAL EQUATIONS.

History. — Within the last half-century the theory of ordinary differential equations has become one of the most important branches of analysis.

Euler's memoirs, published in 1770, gave the first method of integrating *linear ordinary* differential equations with constant coefficients.

The science of *linear partial* differential equations may be said to have been created by Lagrange, in a series of memoirs published in 1779–1785, although Pfaff, in a paper read before the Berlin Academy in 1815, gave the first general method of integrating those of the first order.

Lie's labors in recent times have put the whole subject on a more satisfactory basis.

217. Definitions. A *differential equation* is one which contains one or more differentials or differential coefficients.

The *general solution* (also called the *complete integral* or *primitive*) of a differential equation is the *most general* equation free of differentials and differential coefficients, from which the former may be obtained by differentiation.

Thus, $$dy = 3x^2dx, \text{ and } \frac{dy}{dx} = \tan x,$$

are differential equations, while

$$y = x^3 + c, \text{ and } y = \log(c \sec x),$$

are their general solutions, or primitives.

With differential equations of the above forms, or, generally, of the form,

$$dy = f(x)\,dx,$$

we have had to do in preceding chapters. It is our purpose in this to deal with some of a more general character, and to indicate methods for their solution. We shall confine our attention, however, to *ordinary* differential equations, i.e., to those containing only one independent variable.

218. Orders and Degrees. The *order* of a differential equation is that of the highest differential or differential coefficient which enters it.

The *degree* is that of the highest differential or differential coefficient, *after* the equation is freed from radicals and fractions.

Thus, $$dy = 3\,x^2dx,\quad \frac{dy}{dx} = \tan x,$$

are of the first order and first degree;

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + x = 0,\quad ds = \sqrt{dx^2 + dy^2},$$

are of the first order and second degree;

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + x = 0,\quad d^2y = 6\,xdx^2,$$

are of the second order and first degree, and so on.

EQUATIONS OF FIRST ORDER AND FIRST DEGREE.

219. Form, $f(x)f_1(y)\,dx + \phi(x)\phi_1(y)\,dy = 0.$

Rule: *Separate the variables by division and integrate separately.*

EXAMPLES.

Solve:

1. $(1 + x)\,ydx + (1 - y)\,xdy = 0.$

Dividing through by xy, we have

$$\frac{1 + x}{x}\,dx + \frac{1 - y}{y}\,dy = 0.$$

Integrating separately, we find

$$\log x + x + \log y - y = c,$$

or,

$$xye^{x-y} = c_1,$$

to be the solution.

2. $\sin x \cos y dx = \sin y \cos x dy.$

Dividing by $\cos y \cos x$, we have

$$\tan x dx = \tan y dy.$$

Hence,

$$\log \sec x = \log \sec y + \log c;$$

or,

$$\sec x = c \sec y,$$

is the solution.

3. $(a - y) x \frac{dy}{dx} + 2 y = 0.$ $\qquad y^a x^2 = ce^y.$

4. $xy \frac{dy}{dx} = \frac{y^2 + 1}{x^2 + 1}.$ $\qquad (1 + y^2)(1 + x^2) = cx^2.$

5. The differential relation between volume (v) and pressure (p) of a gas, under condition that no heat leaves or enters it during expansion or compression, is $apdv + bvdp = 0$. Show that $pv^{\frac{a}{b}} =$ constant.

6. A source of constant electromotive force (E) is suddenly introduced into a circuit of resistance, R, and self-induction, L. The differential equation is $E = Ri + L\frac{di}{dt}$. Show that the strength of the electric current (i) is $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$, under the condition that when $t = 0$, $i = 0$.

7. The slope of a family of curves is $-\frac{y}{x}$; find the equation of the group. $\qquad xy = c.$

8. Given the family of curves, $y = sx$; find the curve which intersects each curve of the group at right angles, i.e., find their *orthogonal trajectory*, s being the variable parameter.

$$x^2 + y^2 = c^2.$$

9. Find the orthogonal trajectory of the hyperbolas $xy = m^2$, m being the variable parameter. $x^2 - y^2 = c^2$.

220. Homogeneous Equations of form,

$$f(x, y)\,dy + \phi(x, y)\,dx = 0.$$

Rule: *Let $y = vx$, and apply rule* § 219.

EXAMPLES.

1. $(x^2 - y^2)\,dy = 2\,xydx.$

Let $$y = vx,$$

and the equation becomes

$$(x^2 - v^2x^2)(vdx + xdv) = 2\,vx^2dx;$$

i.e., $$(1 - v^2)\,xdv = v(1 + v^2)\,dx.$$

Separating variables, we have

$$\frac{1 - v^2}{v(1 + v^2)}\,dv = \frac{dx}{x};$$

$$\therefore \text{ § 171, } \log v - \log(1 + v^2) + \log c = \log x;$$

or, $$\log \frac{cv}{1 + v^2} = \log x.$$

$\therefore$ Substituting for v its value, $\frac{y}{x}$, and reducing, we have

$$x^2 + y^2 = cy$$

for the required solution.

2. $(x^2 + y^2)\dfrac{dy}{dx} = xy.$ $\qquad x^2 = 2\,y^2 \log cy.$

3. $(x - 2\,y)\,dx + ydy = 0.$ $\qquad \log \dfrac{y - x}{c} = \dfrac{x}{x - y}.$

4. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}.$ $\qquad x^2 - 2\,cy = c^2.$

5. $x^2dy + y^2dx = -\,xydy.$ $\qquad xy^2 = c(x + 2\,y).$

6. $(xy - x^2)\dfrac{dy}{dx} = y^2.$ $\qquad y = ce^{\frac{y}{x}}.$

221. Form, $f(x, y)\,dy + \phi\,(x, y)\,dx = 0$, in which $f(x, y)$ and $\phi\,(x, y)$ are of the first degree.

Rule: *Reduce to an homogeneous equation and apply rule* § 220.

To show that such reduction is possible and to indicate the method of procedure let us assume the form

$$(a'x + b'y + c')\,dy + (ax + by + c)\,dx = 0. \qquad (1)$$

Let $x = x' + m$ and $y = y' + n$; then (1) becomes

$$(a'x' + b'y' + k')\,dy' + (ax' + by' + k)\,dx' = 0 \qquad (2)$$

in which

$$\left.\begin{aligned} k' &= a'm + b'n + c' \\ k &= am + bn + c \end{aligned}\right\}. \qquad (3)$$

If now we give such values to m and n as to reduce expressions (3) to zero, equation (2) becomes

$$(a'x' + b'y')\,dy' + (ax' + by')\,dx' = 0 \qquad (4)$$

which is homogeneous.

Equating (3) to zero we find

$$m = \frac{b'c - bc'}{a'b - ab'};\; n = \frac{ac' - a'c}{a'b - ab'} \qquad (5)$$

for the requirea values of m and n.

This process fails when

$$a'b - ab' = 0;$$

i.e., when

$$\frac{a'}{a} = \frac{b'}{b}.$$

If, however, we let these ratios $= m_{\prime}$ and substitute

$$a' = am_{\prime},\ b' = bm_{\prime},$$

in (1) we have

$$\{m_{\prime}(ax + by) + c'\}\,dy + (ax + by + c)\,dx = 0. \tag{6}$$

Now let

$$ax + by = z,$$

whence,

$$adx + bdy = dz.$$

Substituting in (6), eliminating y and its differential we have

$$dz = \left(a - \frac{b(z + c)}{m_{\prime}z + c'}\right)dx,$$

in which the variables can be separated.

EXAMPLES.

1. $(3x - 7y - 3)\,dy + (7x - 3y - 7)\,dx = 0.$

Substituting in (5) § 221, we find

$$m = 1,\ n = 0.$$

Letting $y' = v'x'$ in

$$(3x' - 7y')\,dy' + (7x' - 3y')\,dx' = 0,$$

(cf. (4) § 221) we have, after separating variables,

$$7\frac{dx'}{x'} = \frac{3 - 7v'}{v'^2 - 1}\,dv',$$

$$\therefore \log x'^7 + 2\log(v' - 1) + 5\log(v' + 1) = \log c.$$

From $x = x' + m$, $y = y' + n$, since $m = 1$, $n = 0$, we have

$$x' = x - 1,\ y = y'.$$

We have also,

$$v' = \frac{y'}{x'} = \frac{y}{x - 1}.$$

Substituting and reducing we find

$$(y - x + 1)^2 (y + x - 1)^5 = c$$

to be the required equation.

2. $(x + y - 2)\dfrac{dy}{dx} = y - 2x - 1.$

$$\log \{2(3x - 1)^2 + (3y - 5)^2\} = \sqrt{2} \tan^{-1} \frac{\sqrt{2}(3x - 1)}{3y - 5} + c.$$

3. $(2x + 3y - 8) - (x + y - 3)\dfrac{dy}{dx} = 0.$

$$\tfrac{1}{2} \log \{(y - x - 1)^2 - 3(x - 1)^2\} + \frac{1}{\sqrt{3}} \log \frac{y - x - 1 - (x - 1)\sqrt{3}}{y - x - 1 + (x - 1)\sqrt{3}} = c.$$

4. $(2x + 4y + 3)\,dy - (2y + x + 1)\,dx = 0.$

$$4x - 8y - \log(4x + 8y + 5) = c.$$

222. A *Linear* differential equation is an equation of the *first degree* with respect to the dependent variable and its derivatives.

223. Linear Equations of the First Order.

Form, $$dy + Py\,dx = Q\,dx. \tag{1}$$

in which P and Q are functions of x.

Rule: *Find value of* $\int P dx$ *and substitute in*

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + c. \tag{2}$$

The result obtained is the required solution.

For if we assume the form

$$dy + Py dx = 0,$$

separate the variables and integrate we have

$$\log y + \log e^{\int P dx} = \log c,$$

i.e., $$ye^{\int P dx} = c. \tag{3}$$

If we now differentiate (3) we obtain

$$e^{\int Pdx}(dy + Pydx) = 0.$$

Hence multiplying (1) by $e^{\int Pdx}$ and integrating we have (2).
As an example let us solve

$$xdy - aydx = (x + 1)\,dx.$$

Putting in form of (1), we have

$$dy - \frac{a}{x}ydx = \frac{x+1}{x}\,dx,$$

in which $$P = -\frac{a}{x},\ Q = \frac{x+1}{x}.$$

$$\therefore \int Pdx = -a\int\frac{dx}{x} = \log\frac{1}{x^a}.$$

Substituting these values in (2) we find

$$\frac{y}{x^a} = \int\frac{x+1}{x^{a+1}}\,dx + c.$$

$$\therefore y = \frac{x}{1-a} - \frac{1}{a} + cx^a$$

is the required solution.

224. Form, $$dy + Pydx = Qy^n dx. \quad (1)$$

Rule: *Reduce to linear form and apply rule for that form. Cf,* § 223.

This reduction may be effected by dividing through by y^n and then letting $y^{-n+1} = z$. The resulting equation will be linear in z and its derivatives.

For example, solve

$$dy - \frac{a}{3}ydx = \frac{x+1}{3y^2}\,dx.$$

Here $y^n = y^{-2}$. Dividing through by this we have

$$y^2dy - \frac{a}{3}y^3dx = \frac{x+1}{3}dx.$$

Now let $z = y^3$; $dz = 3y^2dy$. We have

$$dz - azdx = (x+1)\,dx,$$

which is linear in z and its derivative. Solving by the method of the preceding article we find

$$y^3 = ce^{ax} - \frac{x+1}{a} - \frac{1}{a^2}$$

for the solution.

EXAMPLES.

1. $x(1-x^2)\frac{dy}{dx} - ax^3 = (1-2x^2)y.$

$$y = x(a + c\sqrt{1-x^2}).$$

2. $(xy+a)\,dx = (1+x^2)\,dy.$ $\qquad y = ax + c\sqrt{1+x^2}.$

3. $xy(1+y)\,dx = (1-x^2)\,dy.$ $\qquad y(c\sqrt{1-x^2}+1) = 1.$

4. $e^xdy = (1-e^xy)\,dx.$ $\qquad ye^x = x + c.$

5. $dx - x^2y^3dy = xydy.$ $\qquad x(2-y^2) + \frac{cx}{\sqrt{e^{y^2}}} = 1.$

6. $(1+x^2)\,dy = 2x(2x-y)\,dx.$ $\qquad y = \frac{4}{3}x^3 - x^2y + c.$

7. $\frac{dy}{dx} + y\cos x = \sin 2x.$ $\qquad y = 2\sin x - 2 + ce^{-\sin x}.$

8. $\frac{dy}{dx} + y\cos x = y^n \sin 2x.$

$$y^{1-n} = 2\sin x + \frac{2}{n-1} + ce^{(n-1)\sin x}.$$

9. $\cos xdy + y\sin xdx = dx.$ $\qquad y = \sin x + c\cos x.$

10. $\cos x \frac{dy}{dx} + \sin x = 1 - y.$ $\quad y(\tan x + \sec x) = x + c.$

225. An *exact differential* of a function of two variables is a form that may result from the total differentiation of that function.

An *exact differential equation* is an equation formed by equating an exact differential to zero.

Thus if u represent a function of x and y, then

$$du = \frac{\partial u}{dx}dx + \frac{\partial u}{dy}dy \qquad \text{cf. § 121}$$

is an exact differential. Calling the partial derivatives $\frac{\partial u}{dx}$ and $\frac{\partial u}{dy}$, M and N respectively, we have

$$Mdx + Ndy = 0 \tag{1}$$

as a general form for an exact differential equation.

As a *criterion* of exactness, we have (cf. § 124),

$$\frac{\partial M}{dy} = \frac{\partial N}{dx}. \tag{2}$$

226. Exact Differential Equations.

Form, $$Mdx + Ndy = 0. \tag{1}$$

Rule: *Find the value of* $\int Mdx$, *regarding* y *as constant; substitute in the expression*

$$\int Mdx + \int\left(N - \frac{\partial}{dy}\int Mdx\right)dy = c \tag{2}$$

and perform the operations indicated. The result is the required solution.

For if we integrate (1), we have

$$\int Mdx + Y = c \tag{a}$$

in which Y is some function of y. Since (a) is a form of the primitive of (1) if we now differentiate with respect y, regarding x constant, we have

$$\frac{\partial}{dy}\int Mdx + \frac{\partial Y}{dy} = N \qquad (b)$$

since $\frac{\partial u}{dy} = N$. Solving for ∂Y and integrating, we find

$$Y = \int\left(N - \frac{\partial}{dy}\int Mdx\right)dy \qquad (c)$$

Substituting (c) in (a), we have (2). Hence the rule.

By a method entirely analogous we may prove that

$$\int Ndy + \int\left(M - \frac{\partial}{dx}\int Ndy\right)dx = c \qquad (3)$$

is the solution of (1).

To illustrate, let us solve

$$(x^3 + 3\,xy^2)\,dx + (y^3 + 3\,x^2y)\,dy = 0.$$

Here, $M = x^3 + 3\,xy^2$ and $N = y^3 + 3\,x^2y$; and since

$$\frac{\partial M}{dx} = 3\,(x^2 + y^2) = \frac{\partial N}{dy}$$

the equation is *exact*. Now applying the rule we find

$$\int Mdx = \int(x^3 + 3\,xy^2)\,dx = \frac{x^4}{4} + \frac{3}{2}x^2y^2.$$

$$\therefore \int Mdx + \int\left(N - \frac{\partial}{dy}\int Mdx\right)dy$$

$$= \frac{x^4}{4} + \frac{3}{2}x^2y^2 + \int(y^3 + 3\,x^2y - 3\,x^2y)\,dy = c;$$

$$\therefore \frac{x^4}{4} + \frac{3}{2}x^2y^2 + \frac{y^4}{4} = c$$

is the required solution. Since the given equation is homogeneous it may of course be solved by § 220.

EXAMPLES.

1. $(x^2 - y^2)\dfrac{dy}{dx} + x(x + 2y) = 0.$ $\quad \dfrac{x^3}{3} + x^2y - \dfrac{y^3}{3} = c.$

2. $(x^2 + y^2)\,dx - 2\,xydy = 0.$ $\quad x^2 - y^2 = cx.$

3. $y(1 + e^{\frac{x}{y}})\,dx + (y - x)\,e^{\frac{x}{y}}dy = 0.$ $\quad x + ye^{\frac{x}{y}} = c.$

4. $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0.$

$(x + 1)(x - y) + y^2 = c.$

227. Integrating Factor. A factor which converts a differential equation into an exact differential equation is called an *integrating factor.*

That such a factor exists for every equation of the form

$$Mdx + Ndy = 0 \qquad (1)$$

which *admits of solution* is evident from the following considerations:

If (1) is exact its primitive is of the form

$$u = c.$$

Cf. § 225. If (1) is *not* exact its primitive will contain a constant of integration (c) with respect to which it may always be solved. If, then, we differentiate the primitive thus solved for c we have an *exact* differential equation of the form

$$\mu\,(Mdx + Ndy) = 0 \qquad (2)$$

which will be satisfied for the same simultaneous values of x and y and their differentials as will satisfy the original inexact equation. Hence (2) is equivalent to that equation. Since (2) is of the first order and degree the factor μ may be a function of x or y, or both, but *not* of their derivatives or differentials. It is further evident, since forms of equivalent primitives are infinitely varied, that their exact differentials will also be infinitely varied; hence the number of values of μ in any given case is unlimited.

To illustrate: solving 2, p. 349, we have

$$c = \frac{\sec x}{\sec y} = \frac{\cos y}{\cos x};$$

hence, $$0 = \frac{\cos y \sin x dx - \cos x \sin y dy}{\cos^2 x}.$$

Therefore, comparing with the given differential equation, Ex. 2, p. 349, we see that $\frac{1}{\cos^2 x}$ is an integrating factor. We see also that $\frac{1}{\cos x \cos y}$, the factor by aid of which the variables were separated, is another value of μ.

Again, in Ex. 1, p. 348, we see that $\frac{1}{xy}$ is an integrating factor. Differentiating

$$xye^{x-y} = c,$$

we have, $$xye^{x-y}(dx - dy) + e^{x-y}(xdy + ydx) = 0.$$

i.e., $$e^{x-y}\{(1 + x)ydx + (1 - y)xdy\} = 0.$$

Hence e^{x-y} is another value of μ in this case.

228. The following methods may be observed in determining a value of μ in certain cases.

1. *By inspection.*

Thus if the equation,

$$x(1 - y^3)dy + ydx = 0,$$

is placed in the form,

$$xdy + ydx - xy^3dy = 0,$$

we readily see that $\frac{1}{xy}$ is an integrating factor and that

$$\log xy - \frac{y^3}{3} = c$$

is the solution.

2. *If* (1) *is homogeneous* $\mu = \frac{1}{Mx + Ny}$.

Take, for example, the equation,

$$(x^2 - y^2)\,dx + xydy = 0. \qquad (a)$$

In this case, $Mx + Ny = x^3 - xy^2 + xy^2 = x^3$; hence

$$\mu = \frac{1}{Mx + Ny} = \frac{1}{x^3}.$$

Multiplying (a) by this factor, we have

$$\frac{x^2 - y^2}{x^3}\,dx + \frac{y}{x^2}\,dy = 0;$$

which is *exact*, since

$$\frac{\partial M}{dy} = -\frac{2y}{x^3} = \frac{\partial N}{dx}; \text{ cf. § 225, (2).}$$

Hence, applying § 226, (2), we have

$$\log x + \frac{y^2}{2x^2} = c$$

for the solution.

If $Mx + Ny = 0$, this method fails, but the solution is $y = cx$.

3. *If* (1) *is of the form*

$$\phi(x, y)\,y\,dx + \psi(x, y)x\,dy = 0,$$

then

$$\mu = \frac{1}{Mx - Ny}.$$

Take as an example,

$$(1 + xy)\,ydx - (xy - 1)\,xdy = 0. \qquad (b)$$

Here, $Mx - Ny = xy + x^2y^2 + x^2y^2 - xy = 2\,x^2y^2;$

$$\therefore\ \mu = \frac{1}{Mx - Ny} = \frac{1}{2\,x^2y^2}.$$

Multiplying (*b*) by this factor, we have

$$\frac{1 + xy}{2\,x^2y}\,dx - \frac{xy - 1}{2\,y^2x}\,dy = 0.$$

Since
$$\frac{\partial M}{dy} = -\frac{1}{2\,x^2y^2} = \frac{\partial N}{dx},$$

the equation is exact. Applying § 226, (2), we find,

$$\frac{x}{y} = ce^{\frac{1}{xy}}$$

to be the solution.

If $Mx - Ny = 0$, this method fails, but the solution is $xy = c$.

4. If $\frac{1}{N}\left(\frac{\partial M}{dy} - \frac{\partial N}{dx}\right) = \phi(x)$, then $\mu = e^{\int \phi(x)dx}$;

or, if $\frac{1}{M}\left(\frac{\partial N}{dx} - \frac{\partial M}{dy}\right) = \psi(y)$, then $\mu = e^{\int \psi(y)dy}$.

Take for example, the equation,

$$(x^2 + y^2)\,dx - 2\,xydy = 0. \qquad (c)$$

Here,
$$\frac{1}{N}\left(\frac{\partial M}{dy} - \frac{\partial N}{dx}\right) = -\frac{2}{x} = \phi(x).$$

$$\therefore\ \mu = e^{\int \phi(x)dx} = e^{\int -\frac{2}{x}dx} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}.$$

Multiplying (*c*) by this value, we have

$$\frac{x^2 + y^2}{x^2}\,dx - 2\,\frac{y}{x}\,dy = 0,$$

which is exact, since

$$\frac{\partial M}{dy} = \frac{2\,y}{x^2} = \frac{\partial N}{dx}.$$

Applying § 226, (2), we find the solution to be

$$x^2 - y^2 = cx.$$

EXAMPLES.

1. $(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0.$ $\qquad \dfrac{x}{y} = \log\dfrac{cx^2}{y^3}.$

2. $(x^2y^2 + xy)y\,dx + (x^2y^2 - 1)x\,dy = 0.$ $\qquad xy = \log\dfrac{y}{c}.$

3. $x^2 + 2xy - y^2 = (x^2 - 2xy - y^2)\dfrac{dy}{dx}.$ $\qquad x^2 + y^2 = c(x + y).$

4. $x^2 + 2x + y^2 + 2y\dfrac{dy}{dx} = 0.$ $\qquad e^x(x^2 + y^2) = c.$

5. $2xy\,dx + (y^3 - 3x^2)\,dy = 0.$ $\qquad x^2 + y^3\log y = cy^3.$

EQUATIONS OF FIRST ORDER AND N^{TH} DEGREE.

229. **Form,** $\qquad f(p^n, p^{n-1}, \ldots x, y) = 0 \qquad (1)$

in which $p = \dfrac{dy}{dx}.$

1. *When the first member of* (1) *can be resolved into n rational binomial factors of the first degree with respect to p.*

Rule: *Factor, equate each factor to zero and solve separately.*

Take for example the equation

$$\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} = xy.$$

Letting $p = \dfrac{dy}{dx}$, transposing and factoring we have

$$(p + x)(p - y) = 0.$$

Equating each factor to zero and solving we find

$$2y + x^2 - c = 0,$$

and $$y = ce^x.$$

Hence, $(2y + x^2 - c)(y - ce^x) = 0$
is the general solution.

2. *When* (1) *can be put in the form*

$$y = f(p, x) \qquad (a)$$

Rule: *Differentiate* (*a*), *thus obtaining*

$$p = \phi\left(p, x, \frac{dp}{dx}\right) \qquad (b)$$

an equation of the first order between p and x. Solve (*b*), *and eliminate p between the resulting equation and* (*a*).

3. *When* (1) *can be put in the form*

$$x = f(p, y) \qquad (c)$$

Rule: *Differentiate* (*c*) *thus obtaining*

$$p' = \phi\left(p', y, \frac{dp'}{dy}\right) \qquad (d)$$

in which $p' = \frac{dx}{dy}$. *Solve* (*d*), *and eliminate* p' *between the resulting equation and* (*c*).

Take for example the equation

$$y = \tfrac{2}{3}p^3 + p^2.$$

Differentiating we have

$$p = (2p^2 + 2p)\frac{dp}{dx};$$

i.e., $$p\left(1 - 2(p+1)\frac{dp}{dx}\right) = 0.$$

Hence, $$p = 0,$$

and $$2(p+1)\,dp = dx.$$

Integrating the latter we find

$$p^2 + 2p = x + c;$$

hence, $$p = \sqrt{x + c} - 1.$$

This value of p in the given equation gives, after reducing and rationalizing,

$$(3x + 3y + 3c - 1)^2 = 4(x + c)^3$$

for the general solution. The first value, $p = 0$, when substituted in the given equation gives the *singular* solution $y = 0$.

The success of this method depends (1) on our ability to solve the derived equations (b) or (d), and (2) upon our ability to eliminate p.

The method can in general be successfully employed in equations in which one of the variables is absent, and in those where both enter but are of the first degree.

EXAMPLES.

1. $\dfrac{dy^2}{dx^2} = a^2y^2.$ $\qquad (y - ce^{ax})(y - ce^{-ax}) = 0.$

2. $p^2 - 5p + 6 = 0.$ $\qquad (y - 2x - c)(y - 3x - c) = 0.$

3. $p^2 - 3p + 2 = 0.$ $\qquad (y - 2x - c)(y - x - c) = 0.$

4. $p(p + 2x)(p - y^2) = 0.$
$\qquad (y - c)(y + x^2 - c)(xy + cy + 1) = 0.$

5. $y + p^2 = p(x + 1).$ $\qquad y + c^2 = c(x + 1).$

6. $p^2y - y + 2px = 0.$ $\qquad y^2 = 2cx + c^2.$

EQUATIONS OF THE N^{TH} ORDER.

230. Special Forms.

1. **Form,** $\qquad \dfrac{d^ny}{dx^n} = f(x).$

Rule: *Integrate n times with respect to x.* Cf. § 211.

2. **Form,** $$\frac{d^2y}{dx^2} = f(y).$$

Rule: *Multiply both members by 2 dy and integrate.*

Thus, let $$\frac{d^2y}{dx^2} = y;$$

then, $$\frac{2}{dx^2}\int dy d^2y = 2\int y dy.$$

Hence, $$\frac{dy^2}{dx^2} = y^2 + c.$$

Hence, $$dx = \pm \frac{dy}{\sqrt{y^2 + c}}.$$

Therefore, $$x = \pm \log(y + \sqrt{y^2 + c}) + c'.$$

3. **Form,** $$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \frac{dy}{dx}, x\right) = 0 \qquad (1)$$

Rule: *Put* $\frac{dy}{dx} = p$; *then* (1) *becomes*

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots p, x\right) = 0$$

which is of an order one lower than (1).

To illustrate, let
$$\left(\frac{d^2y}{dx^2}\right)^2 = a^2 - b^2\left(\frac{dy}{dx}\right)^2.$$

Letting $\frac{dy}{dx} = p$, and extracting square root of both members, we have

$$\frac{dp}{dx} = \sqrt{a^2 - b^2p^2};$$

$$\therefore\ x = \frac{1}{b}\sin^{-1}\frac{bp}{a} + c.$$

Hence, $$p = \frac{a}{b}\sin(bx - c).$$

Integrating again, we find

$$b^2y + a\cos(bx - c) = c_1$$

to be the general solution.

4. **Form,** $f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \frac{dy}{dx}, y\right) = 0$ (2)

Rule : *Put* $\frac{dy}{dx} = p$ *and change independent variable from x to y. Then* (2) *becomes*

$$f\left(\frac{d^{n-1}p}{dy^{n-1}}, \frac{d^{n-2}p}{dy^{n-2}}, \ldots p, y\right) = 0,$$

which reduces the order of (2).

Thus, let $$y\frac{d^2y}{dx^2} = a^2 + \left(\frac{dy}{dx}\right)^2;$$

Then since $$\frac{d^2y}{dx^2} = \frac{d}{dy}\frac{dy}{dx}\cdot\frac{dy}{dx} = \frac{dp}{dy}\cdot p,$$

we have $$py\frac{dp}{dy} = a^2 + p^2.$$

Hence, $$\frac{pdp}{a^2 + p^2} = \frac{dy}{y}.$$

Integrating, we find

$$a^2 + p^2 = c^2y^2.$$

Hence, $$dx = \frac{dy}{\sqrt{c^2y^2 - a^2}}.$$

Integrating again, and reducing to an exponential form, we have

$$e^{cx} = c_1(y + \sqrt{c^2y^2 - a^2})$$

for the general solution.

EXAMPLES.

1. $\frac{d^3y}{dx^3} = 5\,bx^2.$ $\qquad 12\,y = bx^5 + 6\,cx^2 + c_1x + c_2.$

2. $d^2y + a^2y\,dx^2 = 0.$ $\qquad y = c \sin(ax + c_1).$

3. $d^2y - a^2dx^2 = b^2dy^2.$ $\qquad e^{b^2y} = \sec\{ab(x + c)\}\,c_1.$

4. $\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 = 0.$ $\qquad e^{ay} = cax + c_1.$

5. $yd^2y + dy^2 = dx^2.$ $\qquad y^2 = x^2 + cx + c_1.$

6. $a^2\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$ $\qquad \frac{2\,y}{a} = c_1e^{\frac{x}{a}} + \frac{1}{c_1e^{\frac{x}{a}}} + c_2.$

7. $a = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$ $\qquad (x + c)^2 + (y + c_1)^2 = a^2.$

8. $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = y^2 \log y.$ $\qquad e^x \log y - ce^{2x} = c_1.$

CHAPTER X.

MECHANICAL APPLICATIONS.

RECTILINEAR MOTION.

231. Formulae. Let $v =$ velocity, $a =$ acceleration, and $s =$ distance traversed; then

$$v = \frac{ds}{dt} \therefore s = \int v\,dt, \text{ and } t = \int \frac{ds}{v} \tag{1}$$

$$a = \frac{dv}{dt} \therefore v = \int a\,dt, \text{ and } t = \int \frac{dv}{a} \tag{2}$$

Also $$a = \frac{dv}{dt} = \frac{d\frac{ds}{dt}}{dt} = \frac{d^2s}{dt^2} \therefore s = \int\int a\,dt^2 \tag{3}$$

232. *The acceleration of a body's velocity is constant; find the velocity of the body and distance traversed in any time t.*

From (3) we have $$\frac{d^2s}{dt^2} = a; \tag{a}$$

$\therefore$ by integration, $$\frac{ds}{dt} = at + C \tag{a'}$$

i.e., $$v = at + C.$$

Suppose $v = v_0$ when $t = 0$; then $C = v_0$.

Hence, $$v = v_0 + at, \tag{b}$$

is the required velocity expressed in terms of the initial velocity v_0, the acceleration a and time t.

Integrating (a'), remembering that $C = v_0$, we have

$$s = \frac{1}{2}at^2 + v_0t + C_1.$$

Let $s = s_0$ when $t = 0$; then $C_1 = s_0$.

Hence,
$$s = \frac{1}{2}at^2 + v_0t + s_0, \qquad (c)$$

is the required expression for the distance traversed.

COR. 1. If we suppose the body to move from *rest* then $v_0 = 0$ and $s_0 = 0$,

hence, (b) and (c) become

$$v = at \qquad (d)$$

$$s = \frac{1}{2}at^2 \qquad (e)$$

Eliminating t between (d) and (e) we have

$$v = \sqrt{2as} \qquad (f)$$

for the velocity acquired by the body in moving through the distance s.

233. Falling Bodies. We know from mechanics that the acceleration of the velocity of a body caused by the earth's attraction (force of gravity) is sensibly constant. Denoting this acceleration by g (= 32.2 ft. a second, nearly), and the distance fallen through by h, we have from (a), (b), (c), § 232.

$$\frac{d^2s}{dt^2} = g,$$

$$v = v_0 + gt \qquad (a)$$

$$h = v_0t + \frac{1}{2}gt^2 + h_0 \qquad (b)$$

in which v_0 = velocity, and h_0 = distance traversed at the beginning of the epoch.

Cor. If the body starts from rest then $v_0 = 0$, and $h_0 = 0$; hence,

$$v = gt,$$

$$h = \tfrac{1}{2}gt^2.$$

Eliminating t, $$v = \sqrt{2gh} \qquad (c)$$

234. Bodies projected vertically. If a body is projected vertically *downwards;* then (*a*), (*b*) § 233, give

$$v = v_0 + gt \qquad (a)$$

$$h = v_0 t + \tfrac{1}{2}gt^2 \qquad (\text{since } h_0 = 0) \qquad (b)$$

If projected vertically *upward*, then since $g = -g$, a retardation, we have

$$v = v_0 - gt \qquad (c)$$

$$h = v_0 t - \tfrac{1}{2}gt^2 \qquad (d)$$

Cor. If $v = 0$ in (*c*) we have

$$t = \frac{v_0}{g}$$

for the time it takes a body to rise to its highest point when projected upwards with a velocity v_0.

235. Body Falling in a Resisting Medium. Let us consider the case of a body falling in the air. It has been found by observation that the retarding effect of the air varies with the square of the velocity of the body; hence the acceleration due to gravity is at any instant less than g by an amount proportionate to v^2. We may therefore write

$$a = \frac{dv}{dt} = g - mv^2,$$

in which m is to be determined by observation. For convenience let $m = \frac{g}{n^2}$; then

$$\frac{dv}{dt} = g - \frac{g}{n^2} v^2 = g\left(\frac{n^2 - v^2}{n^2}\right).$$

$$\therefore \frac{g}{n^2} dt = \frac{dv}{n^2 - v^2}.$$

Integrating and suppose $v = 0$ when $t = 0$, we have

$$\frac{g}{n^2} t = \frac{1}{2n} \log \frac{n+v}{n-v};$$

or,
$$e^{\frac{2g}{n} t} = \frac{n+v}{n-v};$$

hence,
$$v = n \frac{e^{\frac{gt}{n}} - e^{-\frac{gt}{n}}}{e^{\frac{gt}{n}} + e^{-\frac{gt}{n}}} \qquad (1)$$

An expression which gives the velocity at the end of any time t.

Replacing v by its value $\frac{ds}{dt}$, § 231, and multiplying through by dt, we have

$$ds = \frac{n\left(e^{\frac{gt}{n}} - e^{-\frac{gt}{n}}\right) dt}{e^{\frac{gt}{n}} + e^{-\frac{gt}{n}}}$$

$$\therefore s = \frac{n^2}{g} \left\{ \log \left(e^{\frac{gt}{n}} + e^{-\frac{gt}{n}}\right) + C \right\}.$$

If $s = 0$ when $t = 0$; then $C = -\log 2 = \log \frac{1}{2}$. Hence,

$$s = \frac{n^2}{g} \left\{ \log \frac{e^{\frac{gt}{n}} + e^{-\frac{gt}{n}}}{2} \right\} \qquad (2)$$

Equa. (2) enables us to determine the distance traversed in a given time (t).

236. Body projected into a Resisting Medium. If we suppose the body is acted upon by no force other than the resistance of the medium into which it is projected, we have from the preceding article,

$$a = \frac{dv}{dt} = -mv^2,$$

in which m is to be determined by experiments made in the particular medium selected. From the above expression we have

$$dt = -\frac{1}{m} \cdot \frac{dv}{v^2};$$

hence,
$$t = \frac{1}{mv} + C.$$

Let $v = v_0$ when $t = 0$; then $C = -\frac{1}{mv_0}$;

$$\therefore \ t = \frac{1}{m}\left(\frac{1}{v} - \frac{1}{v_0}\right);$$

$$\therefore \ v = \frac{1}{mt + \frac{1}{v_0}},$$

an expression for the velocity at the end of any time t. Replacing v by its value $\frac{ds}{dt}$, and multiplying through by dt, we have

$$ds = \frac{dt}{mt + \frac{1}{v_0}};$$

$$\therefore \ s = \frac{1}{m} \log \left(mt + \frac{1}{v_0}\right) + C_1.$$

Let $s = 0$ when $t = 0$; then $C_1 = -\frac{1}{m} \log \frac{1}{v_0}$.

Hence,
$$s = \frac{1}{m} \log \{v_0 mt + 1\}$$

which gives a relation between the distance (s) and time (t).

237. Body Falling when Gravity is variable.

Let g = acceleration of a body at the earth surface, and let r = radius of the earth. Let $s\,(>r)$ = distance of a body from the earth center, and let a = its acceleration at that instant; then, according to Newton's law, viz., *that the acceleration of a body at different distances from the earth center varies inversely as the square of* its distance, we have

$$a : g :: r^2 : s^2.$$

Hence,
$$a = \frac{d^2s}{dt^2} = -\frac{gr^2}{s^2}.$$

Multiplying through by ds, and integrating, we have

$$\frac{1}{dt^2}\int ds\,d^2s = -gr^2\int\frac{ds}{s^2}.$$

Hence,
$$\frac{ds^2}{dt^2} = \frac{2\,gr^2}{s} + C,$$

i.e.,
$$v^2 = \frac{2\,gr^2}{s} + C.$$

Let $v = 0$ when $s = s_0$; then $C = -\dfrac{2\,gr^2}{s_0}$;

$$\therefore\; v^2 = 2\,gr^2\left\{\frac{1}{s} - \frac{1}{s_0}\right\} \qquad (1)$$

An expression for the velocity acquired by the body in falling from the height s_0.

Replacing v by its value $\dfrac{ds}{dt}$, and solving for dt, we have

$$dt = -\left(\frac{s_0}{2\,gr^2}\right)^{\frac{1}{2}}\frac{s\,ds}{\sqrt{s_0s - s^2}}.$$

Hence, $t = \left(\dfrac{s_0}{2\,gr^2}\right)^{\frac{1}{2}}\left\{(s_0\,s - s^2)^{\frac{1}{2}} - \dfrac{s_0}{2}\,\text{vers}^{-1}\dfrac{2\,s}{s_0}\right\} + C.$

Let $s = s_0$ when $t = 0$; then

$$C = \left(\frac{s_0}{2\,gr^2}\right)^{\frac{1}{2}} \frac{s_0}{2} \text{ vers}^{-1}\, 2 = \left(\frac{s_0}{2\,gr^2}\right)^{\frac{1}{2}} \frac{\pi s_0}{2}.$$

Hence,

$$t = \left(\frac{s_0}{2\,gr^2}\right)^{\frac{1}{2}} \left\{ (s_0 s - s^2)^{\frac{1}{2}} - \frac{s_0}{2} \text{vers}^{-1} \frac{2\,s}{s_0} + \frac{\pi s_0}{2} \right\} \qquad (2)$$

which gives the time t for a body to fall from height s_0 to height s.

Cor. 1. If in (1) we make $s_0 = \infty$ and $s = r$, we have

$$v = \sqrt{2\,gr}$$

for the velocity with which a body would strike the earth if it fell from an infinite distance in a vacuum.

Since $g = 32$ ft., nearly, and $r = 20{,}900{,}000$ ft., nearly, we find

$$v = 7 \text{ miles per second, nearly.}$$

Cor. 2. If in (2) we make $s = r$, we have the time for a body falling from the height s_0 to the earth.

CURVILINEAR MOTION.

238. Velocity of a body down a curve in a vertical plane.

Let ST be any curve in the plane YOX, referred to OY and OX as axes, OY being positive downwards. Let P be the position of the body at any instant and let $PA = (g)$ represent the acceleration due to gravity. Draw $AC \perp$ to the tangent PB, and let $PAC = \theta$; then

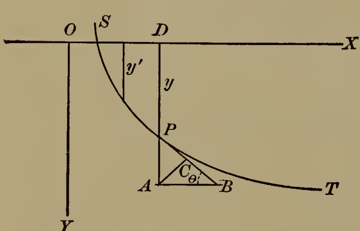


Fig. 61.

$$PC = g \sin\theta = \text{acceleration in direction of motion.}$$

But if we let $PB = ds$, then $PA = dy$; hence $\frac{dy}{ds} = \sin\theta$.

Hence, § 82, $$\frac{d^2s}{dt^2} = g\frac{dy}{ds};$$

$$\therefore \frac{dsd^2s}{dt^2} = gdy.$$

Integrating between limits y and y' we have

$$\frac{ds^2}{2\,dt^2} = g(y - y') \qquad (1)$$

$\therefore$ § 17, $$v^2 = 2g(y - y').$$

Comparing the last equation with (*c*) § 233, Cor., we see that the *velocity acquired by a body in rolling down a curve is the same as it would acquire in falling freely through the vertical height.*

Cor. From (1) we have

$$dt = \frac{ds}{\sqrt{2g(y - y')}} = \frac{ds}{dy} \cdot \frac{dy}{\sqrt{2g(y - y')}}$$

$$\therefore t = \int_{y'}^{y} \frac{ds}{dy} \cdot \frac{dy}{\sqrt{2g(y - y')}} \qquad (2)$$

is the time it takes the body to fall through the height, $y - y'$.

239. Time of descent down an inverted cycloid.

From § 238 (2), we have,

$$t = \int \frac{ds}{dy} \cdot \frac{dy}{\sqrt{2g(y - y')}} \qquad (a)$$

We are to find what this expression becomes when applied to the cycloid.

From the equation of the cycloid, $x = a \text{ vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2}$, we obtain

$$\frac{dx}{dy} = \frac{y}{\sqrt{2ay - y^2}}.$$

But § 18, (3),

$$\frac{ds}{dy} = \sqrt{\frac{dx^2}{dy^2} + 1}.$$

Hence,

$$\frac{ds}{dy} = \sqrt{\frac{2a}{2a - y}}.$$

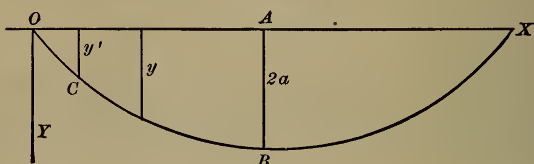


Fig. 62.

This value in (a) gives $t = \sqrt{\frac{a}{g}} \int \frac{dy}{\sqrt{(2a - y)(y - y')}}$.

Let $y - y' = z$; then $dy = dz$, and $2a - y = 2a - y' - z$.

Hence,

$$t = \sqrt{\frac{a}{g}} \int \frac{dz}{\sqrt{(2a - y')z - z^2}},$$

i.e.,

$$t = \sqrt{\frac{a}{g}} \text{ vers}^{-1} \frac{2z}{2a - y'} + C.$$

If we suppose the body to fall from C to B we have $z = 0$ at C, and $z = 2a - y'$ at B. Hence between these limits of z we have

$$t = \sqrt{\frac{a}{g}} \left\{ \text{vers}^{-1}\, 2\, \frac{2a - y'}{2a - y'} + C - \text{vers}^{-1} 0 - C \right\};$$

$$\therefore\ t = \pi \sqrt{\frac{a}{g}}$$

is the time it takes the body to fall from the position C to the lowest point B of the curve. Since y' is *any* value of y, the point C is *any* point of the cycloid; hence *the time required for a body to fall from any point of an inverted cycloid to its lowest point is constant.* Theoretically, therefore, the cycloidal arc is the path of a pendulum which vibrates in equal times.

240. *A projectile is thrown obliquely upward with a velocity v; find* (1), *the equation of its path;* (2), *the coördinates of its highest point;* (3), *the angle of projection in order that its range may be a maximum.*

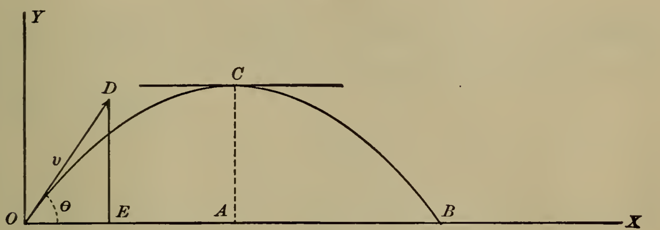


Fig. 63.

(1). Let the origin of coördinates O be the point of propulsion; $DOX =$ angle of projection and $OD = v =$ velocity of projection. Draw $DE \perp$ to OX; then

$$OE = v \cos\theta = \text{velocity in direction of } X,$$
$$DE = v \sin\theta = \text{velocity in direction of } Y.$$

Since no retarding force acts in the direction of X (the resistance of the air being neglected) the velocity in that direction is uniform. Denoting the distance traversed in that direction in any time t by x we have,

$$x = v \cos\theta\, t \qquad (a)$$

Denoting the distance traversed in the vertical direction in the same time t by y we have, § 234 (d),

$$y = v \sin\theta\, t - \tfrac{1}{2} g t^2 \qquad (b)$$

Eliminating t between (a) and (b) we have,

$$y = \tan\theta . x - \frac{gx^2}{2\, v^2 \cos^2\theta} \qquad (1)$$

which expresses the relation between x and y for all values of t; hence it is the equation of the trajectory. This curve is obviously a parabola. (Ana. Geom. p. 178.)

(2). At the highest point C, the tangent, is $\|$ to X; hence $\frac{dy}{dx} = 0$.

From (1),
$$\frac{dy}{dx} = \tan\theta - \frac{gx}{v^2\cos^2\theta} = 0,$$

$$\therefore\ x = \frac{v^2}{g}\sin\theta\cos\theta,$$

i.e.,
$$x = \frac{v^2}{2\,g}\sin 2\,\theta = OA \qquad (c)$$

is the abscissa of the highest point. This value of x in (1) gives

$$y = \frac{v^2}{2\,g}\sin^2\theta = AC$$

for the ordinate of the highest point.

(3) Denoting the range by R we have, since the curve is symmetrical with respect to AC,

$$R = 2\,OA = OB;$$

hence (c)
$$R = \frac{v^2}{g}\sin 2\,\theta.$$

Hence,
$$\frac{dR}{d\theta} = \frac{2\,v^2}{g}\cos 2\,\theta = 0;$$

$$\therefore\ \cos 2\,\theta = 0,$$

$$\therefore\ \theta = 45^\circ.$$

Since $f''(R)$ is negative for $\theta = 45°$ this value of θ corresponds to a maximum value of R.

Cor. Since

$$R \doteq \frac{v^2}{g} \sin 2\theta = \frac{v^2}{g} \sin (180° - 2\theta) = \frac{v^2}{g} \sin 2(90° - \theta)$$

we see that the same range (R) may be attained with a given initial velocity (v) under two angles of projection θ and $90° - \theta$. We see also that these angles are complementary.

CENTER OF GRAVITY.

241. Definition. *The center of gravity of a body is that point through which the line of action of the body's weight always passes.*

242. Formulae.

$$x_1 = \frac{\int xdv}{v}$$

$$y_1 = \frac{\int ydv}{v}$$

$$z_1 = \frac{\int zdv}{v}.$$

These formulae enable us to determine the coördinates (x_1, y_1, z_1) of the center of gravity of any given homogeneous body, of volume v.

If the body is symmetrical with reference to a plane, this plane may be taken as the XY plane; whence $z_1 = 0$.

If the body is symmetrical with reference to a straight line, this line may be taken as the X-axis; whence $y_1 = 0$ and $z_1 = 0$.

243. *To find the center of gravity of a circular arc.*

Let ABC be the arc and let OX be the axis of symmetry.

Let (x,y) [$= OD, DA$] be the coördinates of the extremity (A) of the arc. Since

$$dv = ds = \sqrt{dx^2 + dy^2}$$

we have, § 242,

$$x_1 = \frac{\int_{-y}^{y} x\sqrt{dx^2 + dy^2}}{s}$$

$$= \frac{\int_{-y}^{y} x\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}} dy}{s}.$$

From the equation of the circle, $x^2 + y^2 = a^2$, we have

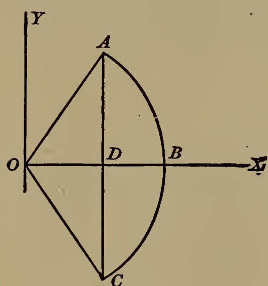


Fig. 64.

$$\frac{dx}{dy} = -\frac{y}{x};$$

$$\therefore \quad x_1 = \frac{\int_{-y}^{y} x\left(\frac{a^2}{x^2}\right)^{\frac{1}{2}} dy}{s} = \frac{2\,ya}{s}.$$

Since $2y =$ chord AC, we see that the center of gravity of a circular arc is on its radius of symmetry and at a distance from its center equal to the fourth proportional between the arc, radius and chord.

244. *To find the center of gravity of a circular segment.*

Here $dv = d^2A = dxdy$; hence,

$$x_1 = \frac{\int_{x'}^{a}\int_{-y}^{y} xdxdy}{A}$$

$$= \frac{\int_{x'}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} xdxdy}{A}$$

$$= \frac{2\int_{x'}^{a}\sqrt{a^2 - x^2}\,xdx}{A}$$

$$= \frac{-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}}{A}\Bigg|_{x'}^{a}$$

$$= \frac{\frac{2}{3}(a^2 - x'^2)^{\frac{3}{2}}}{A}$$

If the sector is a semicircle then $A = \dfrac{\pi a^2}{2}$, and $x' = OD = 0$.

$$\therefore \qquad x_1 = \frac{4a}{3\pi}.$$

245. *To find the center of gravity of the area bounded by a parabola, its axis and one of its ordinates.*

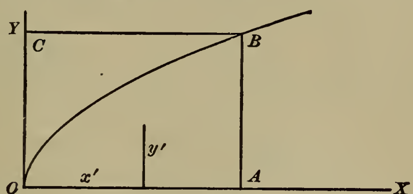


Fig. 65.

Let $y^2 = 2px$ be the equation of the parabola; then

$$x_1 = \frac{\int_0^x\int_0^y xdxdy}{A} = \frac{\int_0^x\int_0^{\sqrt{2px}} xdxdy}{A}$$

$$= \frac{\sqrt{2p}\int_0^x x^{\frac{3}{2}}dx}{A} = \frac{2\sqrt{2p}\,x^{\frac{5}{2}}}{5A},$$

and

$$y_1 = \frac{\int_0^x\int_0^y ydxdy}{A} = \frac{\int_0^x\int_0^{\sqrt{2px}} ydxdy}{A}$$

$$= \frac{p\int_0^x xdx}{A} = \frac{px^2}{2A}.$$

But, § 215,

$$A = \int_0^x \int_0^y dxdy = \int_0^x \int_0^{\sqrt{2px}} dxdy = \sqrt{2p} \int_0^x x^{\frac{1}{2}} dx$$

$$= \frac{2\sqrt{2p}x^{\frac{3}{2}}}{3} = \frac{2}{3} xy.$$

Hence, $x_1 = \frac{3}{5}x$ and $y_1 = \frac{3}{8}y$.

246. *To find the center of gravity of a parabolic spandrel.*

That is, to find the center of gravity of OBC, Fig. 65. Here,

$$x_1 = \frac{\int_0^y \int_0^x xdydx}{A} = \frac{\int_0^y \int_0^{\frac{y^2}{2p}} xdydx}{A} = \frac{\frac{1}{8p^2}\int_0^y y^4 dy}{A} = \frac{y^5}{40p^2A}.$$

$$y_1 = \frac{\int_0^y \int_0^x ydydx}{A} = \frac{\int_0^y \int_0^{\frac{y^2}{2p}} ydydx}{A} = \frac{\frac{1}{2p}\int_0^y y^3 dy}{A} = \frac{y^4}{8pA}.$$

But $\qquad A = OBC = OABC - OAB;$

i.e., $\qquad A = xy - \frac{2}{3}xy = \frac{1}{3}xy.$

Hence, $\qquad x_1 = \frac{3}{10}x$, and $y_1 = \frac{3}{4}y$.

It will be observed from the limits of integration that the area OBC is supposed to be generated by a line ‖ to X moving in the direction of Y, the line being limited by the Y-axis and the curve. The student may derive the same result by proceeding as in previous articles.

247. *To find the center of gravity of a Pyramid or Cone.*

From § 210 (b), we have

$$dV = Adx;$$

hence,
$$x_1 = \frac{\int xdV}{V} = \frac{\int Axdx}{\int Adx}.$$

Adopting the figure and notation of Ex. 1, p. 334, we have

$$\frac{A}{A'} = \frac{x^2}{h^2};$$

$$\therefore A = A'\frac{x^2}{h^2};$$

hence
$$x_1 = \frac{\int_0^h x^3dx}{\int_0^h x^2dx} = \frac{3}{4}h.$$

248. *To find the center of gravity of a paraboloid of revolution.*

Let $y^2 = 2px$ be the equation of the generating curve, and let $x = h$ and $x = h'$ be the equation of two planes $\parallel$ to YZ. We wish to find the center of gravity of that portion of the paraboloid included between the planes. Since X is an axis of symmetry we have

$$x_1 = \frac{\int xdV}{V} = \frac{\pi\int_{h'}^h xy^2dx}{\pi\int_{h'}^h y^2dx},$$

since $dV = \pi y^2dx$, § 209 (a);

hence,
$$x_1 = \frac{2p\int_{h'}^h x^2dx}{2p\int_{h'}^h xdx} = \frac{2}{3}\frac{h^3 - h'^3}{h^2 - h'^2}.$$

If $h' = 0$, then

$$x_1 = \tfrac{2}{3}h,$$

i.e., The center of gravity of a segment of a paraboloid of revolution estimated from its vertex is two-thirds of its altitude.

249. *To find the center of gravity of the semi-ellipsoid of revolution.*

Let
$$y^2 = \frac{b^2}{a^2}(2ax - x^2)$$

be the equation of the generating curve, then

$$x_1 = \frac{\int_0^a (2\,ax - x^2)\,xdx}{\int_0^a (2\,ax - x^2)\,dx} = \frac{5}{8}a.$$

MOMENTS OF INERTIA.

250. Definition. The moment of inertia of any area about an axis is the integral of the product arising by multiplying the differential of the area by the square of its distance from the axis.

251. Formula.

$$M = \int r^2 dA$$

in which M = moment of inertia, A = area, and r = distance of dA from the assumed axis.

252. *To find the moment of inertia of a rectangle.*

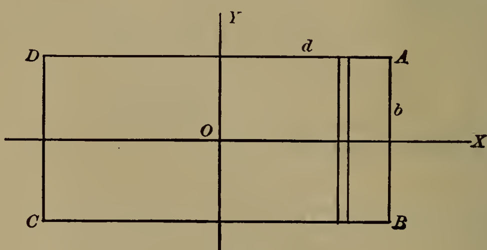


Fig. 66.

Let $AB = b$ and $AD = d$; then

(1), *OY being the axis.*

$$M = \int r^2 dA$$

$$= \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} x^2 dxdy$$

$$= b\int_{-\frac{d}{2}}^{\frac{d}{2}} x^2 dx = \frac{bx^3}{3}\Bigg|_{-\frac{d}{2}}^{\frac{d}{2}} = \frac{bd^3}{12} = \frac{Ad^2}{12}.$$

(2), *OX being the axis.*

$$M = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 dy dx$$

$$= d\int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 dy = \frac{db^3}{12} = \frac{Ab^2}{12}.$$

(3), *OZ being the axis.*

$$M = \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} (x^2 + y^2)\, dx dy$$

$$= \frac{d^3 b + db^3}{12} = \frac{A(d^2 + b^2)}{12}.$$

253. *Moments of inertia of hollow-girders, channel-bars, and I-iron.*

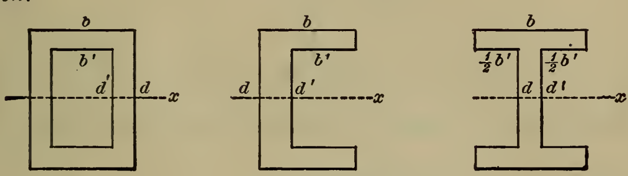


Fig. 67.

Let the X-axis, passing through the center of gravity of the section, be taken as the axis of moments; then, in all three cases, we readily deduce

$$M = \frac{bd^3 - b'd'^3}{12}.$$

254. *Moment of inertia of a circle about a diameter.*

In this case, a representing the radius of the circle,

$$M = \int r^2 dA$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} x^2 dx dy = 2\int_{-a}^{a} x^2 (a^2 - x^2)^{\frac{1}{2}}\, dx.$$

Using the reduction formula of §§ 185, 186, we find

$$M = \frac{\pi a^4}{4} = \frac{Aa^2}{4}.$$

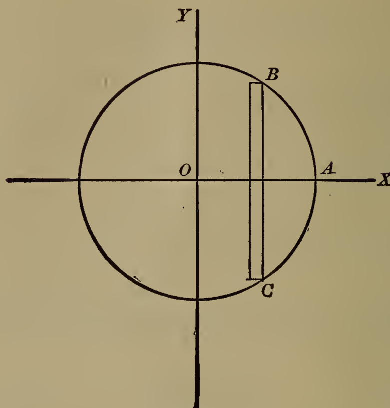


Fig. 68.

255. *Moment of inertia of an ellipse about its minor-axis.*

Here $$M = \int r^2 dA$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2 - x^2}}^{\frac{b}{a}\sqrt{a^2 - x^2}} x^2 dx dy$$

$$= \frac{\pi b a^3}{4} = \frac{Aa^2}{4}.$$

DEFLECTION AND SLOPE OF BEAMS.

256. Formula. From mechanics we have

$$M = \frac{EI}{\rho}, \tag{1}$$

for the relation between the moment of the extraneous forces

(M) and the moment of the internal resistance $\left(\frac{EI}{\rho}\right)$ about the neutral axis of any section. In this formula E = coefficient of elasticity of the material of which the beam is made; I = moment of inertia of section, and ρ = radius of curvature of the curve of mean fiber, at the point in which it pierces the section. But § 134,

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Hence,
$$M = \frac{EI\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

Since $\left(\frac{dy}{dx}\right)^2 = \tan^2 \alpha$, i.e., the square of the slope of the beam, and since in practice the value is small, we may omit it and write

$$M = EI\frac{d^2y}{dx^2}. \qquad (2)$$

Formula (2) is sufficiently accurate for all practical purposes. and is in general use.

257. *Slope and deflection of a beam loaded at one end and fixed at the other.*

Let l = length of beam, and W = weight applied at its end A. Let OA be the mean fiber, and S a plane ⊥ to OA at a distance x from O; then

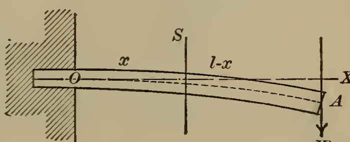


Fig. 69.

$$M = W(l - x).$$

Hence, § 256 (2), $\frac{d^2y}{dx^2} = \frac{W}{EI}(l - x)$;

$$\therefore \frac{dy}{dx} = \frac{W}{2EI}(2lx - x^2) + C.$$

When $x = 0$, $\frac{dy}{dx} = 0$, since the tangent at O is coincident with X; hence $C = 0$.

$$\therefore \frac{dy}{dx} = \frac{W}{2EI}(2lx - x^2).$$

Integrating, $$y = \frac{W}{6EI}(3lx^2 - x^3), \qquad (a)$$

since when $x = 0$, $y = 0$ and therefore $C' = 0$. Equa. (a) is the equation of the curve OA, i.e., the equation of the curve which the mean fiber takes under the action of the load W.

If in (a) we make $x = l$ and let $\delta =$ value of y when $x = l$, we have

$$\delta = \frac{Wl^3}{3EI}$$

for the maximum deflection of the beam.

258. *Shape and deflection of a beam fixed at one end and uniformly loaded.*

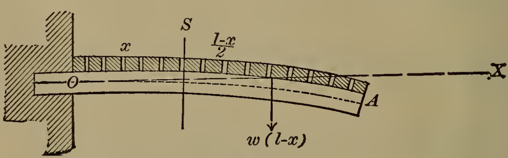


Fig. 70.

Let $w =$ load per unit of length of beam; then at any section S

$$M = w(l - x)\frac{l - x}{2} = \frac{w}{2}(l - x)^2.$$

Hence, $$\frac{d^2y}{dx^2} = \frac{w}{2\,EI}(l^2 - 2\,lx + x^2);$$

$$\therefore\ \frac{dy}{dx} = \frac{w}{2\,EI}\left(l^2x - lx^2 + \frac{x^3}{3}\right).$$

Since $\frac{dy}{dx} = 0$ when $x = 0$; $\therefore\ C = 0$.

Integrating again we have

$$y = \frac{w}{2\,EI}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right). \qquad (a)$$

Since $y = 0$ when $x = 0$; $\therefore\ C' = 0$. Equa. (a) gives the shape the beam assumes under the action of the load. Representing the maximum deflection by δ' which obviously occurs when $x = l$ we have

$$\delta' = \frac{wl^4}{8\,EI}.$$

Cor. Let $W = wl =$ load on beam; then

$$\delta' = \frac{Wl^3}{8\,EI}.$$

Comparing δ' with δ of § 257 we find

$$\delta = \tfrac{8}{3}\,\delta' = 3\,\delta', \text{ nearly.}$$

That is, the deflection is nearly three times as great when the load is concentrated at the end as it would be if uniformly distributed over the beam.

259. *Shape and deflection of a beam supported at both ends and loaded in center.*

In this case $$M = \frac{W}{2}x;$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{W}{2\,EI}x;$$

$$\therefore\ \frac{dy}{dx} = \frac{W}{4\,EI}x^2 + C.$$

If $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$, since at the middle of the beam the tangent is $\parallel$ to X; $\therefore$

$$C = -\frac{Wl^2}{16EI}.$$

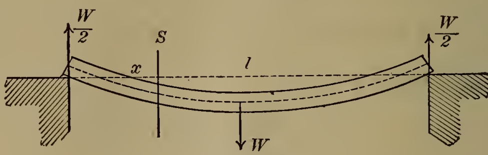


Fig. 71.

Hence, $$\frac{dy}{dx} = \frac{W}{4EI}\left(x^2 - \frac{l^2}{4}\right).$$

Integrating again, $$y = \frac{W}{4EI}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right).$$

Since $x = 0, y = 0$; $\therefore$ $C' = 0$.

When $x = \frac{l}{2}$ we have $$\delta = \frac{Wl^3}{48EI}.$$

260. *Shape and deflection of a beam supported at both ends and uniformly loaded.*

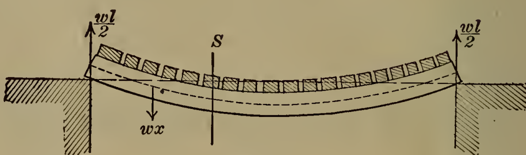


Fig. 72.

In this case we have

$$M = wx \cdot \frac{x}{2} - \frac{wl}{2} \cdot x = \frac{w}{2}(x^2 - lx).$$

Hence $$\frac{d^2y}{dx^2} = \frac{w}{2EI}(x^2 - lx);$$

hence $$\frac{dy}{dx} = \frac{w}{2\,EI}\left(\frac{x^3}{3} - \frac{lx^2}{2}\right) + C.$$

When $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; $\therefore\ C = \frac{1}{24}\frac{wl^3}{EI}$.

Hence $$\frac{dy}{dx} = \frac{w}{2\,EI}\left\{\frac{x^3}{3} - \frac{lx^2}{2} + \frac{l^3}{12}\right\};$$

$$\therefore\ y = \frac{w}{2\,EI}\left\{\frac{x^4}{12} - \frac{lx^3}{6} + \frac{l^3x}{12}\right\}.$$

Since $x = 0$, $y = 0$; $\therefore\ C' = 0$.

If $x = \frac{l}{2}$, then $$\delta' = \frac{5}{384}\frac{wl^4}{EI} = \frac{5}{384}\frac{Wl^3}{EI}.$$

Cor. Comparing the value of δ of § 259 with δ' of this article, we find

$$\delta = \tfrac{8}{5}\,\delta',$$

i.e., the deflection produced by a load concentrated at the center of a beam is $\frac{8}{5}$ of that produced by the same load when uniformly distributed.

261. *Shape and deflection of a beam fixed at both ends and uniformly loaded.*

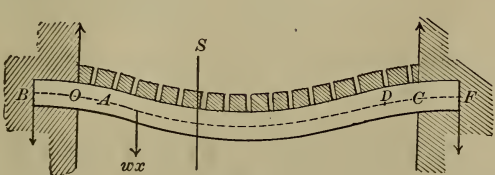


Fig. 73.

This case is similar to that of the preceding except that an unknown moment m acts on the portion of the beam OB; hence,

$$M = \frac{w}{2}(x^2 - lx) + m. \tag{1}$$

Hence
$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left\{\frac{wx^2}{2} - \frac{wlx}{2} + m\right\}. \qquad (a)$$

Integrating and noting that when $x = 0$, $\frac{dy}{dx} = 0$, and therefore $C = 0$, we have,

$$\frac{dy}{dx} = \frac{1}{EI}\left\{\frac{wx^3}{6} - \frac{wlx^2}{4} + mx\right\}. \qquad (b)$$

At the point C where $x = l$, $\frac{dy}{dx} = 0$. If we substitute these values in the last expression, we find after reduction

$$m = \frac{wl^2}{12}, \qquad (c)$$

for the moment of the unknown couple acting at the points of support. Substituting this value of m in (b) and integrating, we find,

$$y = \frac{1}{EI}\left\{\frac{wx^4}{24} - \frac{wlx^3}{12} + \frac{wl^2x^2}{24}\right\} \qquad (d)$$

since $x = 0$ gives $y = 0$, and therefore $C' = 0$.

Making $x = \frac{l}{2}$ in (d) we find

$$\delta = \frac{wl^4}{384EI} = \frac{Wl^3}{384EI}.$$

Comparing this value of δ with that of δ' in § 260, we find

$$\delta' = 5\,\delta,$$

that is, by fastening the ends of a beam at its points of support, the deflection caused by a uniform load is only one-fifth of what it would be if the beam merely rested on its supports.

Again, making $m = \frac{wl^2}{12}$ in (1), we have

$$M = \frac{wx^2}{2} - \frac{wlx}{2} + \frac{wl^2}{12}.$$

Making $x = \frac{l}{2}$ in this value we find,

$$M = -\frac{wl^2}{24}.$$

Comparing this value with the value of $m = \frac{wl^2}{12}$ we see that the bending moment at the point of support is twice that at the center, i.e., the beam is twice as strong at the center as it is at the points of support.

Making $\frac{d^2y}{dx^2} = 0$ in (*a*) and giving m its value in (*c*), we have

$$\frac{wx^2}{2} - \frac{wlx}{2} + \frac{wl^2}{12} = 0;$$

$$\therefore\ x^2 - lx + \frac{l^2}{6} = 0;$$

$$\therefore\ x = \frac{l}{2}\left\{1 \pm \sqrt{\frac{1}{3}}\right\};$$

are the abscissas of the points of inflexion A, D, Fig. 73.

262. *To find the strongest rectangular beam that can be cut from a cylindrical log.*

We have from mechanics

$$P = \frac{E}{\rho} \cdot \frac{d}{2}$$

for the stress on a unit of area at the distance $\frac{d}{2}$ from the neutral axis of a beam when under transverse strain.

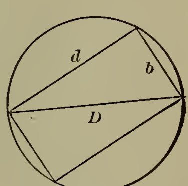


Fig. 74.

Hence §§ 256, 252, $P = \frac{Md}{I\,2} = \frac{M}{\frac{1}{6}bd^2}$, (*a*)

in which b and d are the breadth and depth of a rectangular beam. It is obvious that that beam strained by a moment M

will be strongest in which P is least. But P is least when bd^2 is greatest. Cf. (a).

Let D be the diameter of the log; then $d^2 = D^2 - b^2$,

$$\therefore\ bd^2 = bD^2 - b^3;$$

Differentiating, we have, $\dfrac{d\,(bd^2)}{db} = D^2 - 3\,b^2 = 0,$

hence $$b = D\sqrt{\frac{1}{3}}, \text{ and } d = D\sqrt{\frac{2}{3}}$$

are the dimensions of the strongest rectangular beam.